# Cumulants

## II. Some Combinatorics

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# Partition lattices

$\mathcal{P}_S$ lattice of set-partitions of $S$:

$$[\pi_1, \pi_2] \sim \prod_i \mathcal{P}_{S_i}$$

$$\mu([\pi_1, \pi_2]) = \prod_i (-1)^{|S_i|-1}(|S_i| - 1)!$$

$NC(n)$ lattice of non-crossing partitions of $\{1, 2, \ldots, n\}$.

$$[\pi_1, \pi_2] \sim \prod_i NC(k_i)$$

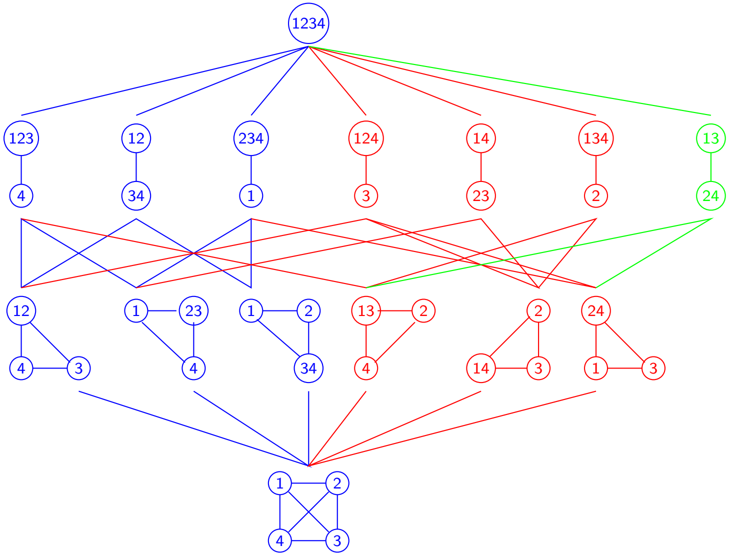
$$\mu([\pi_1, \pi_2]) = \prod_i (-1)^{k_i-1} C_{k_i-1}$$

$I(n)$ lattice of interval partitions of $\{1, 2, \ldots, n\}$

$$[\pi_1, \pi_2] \sim \prod_i I(k_i)$$

$$\mu([\pi_1, \pi_2]) = \prod_i (-1)^{k_i-1}$$

$I(4)$ $NC(4)$ $\mathcal{P}(4)$



One has

$$I(n) \subset NC(n) \subset \mathcal{P}_n$$

For $\pi \in \mathcal{P}_n$ let $\pi^*$ be the smallest non-crossing partition with $\pi \leq \pi^*$.

Analogously let $\pi^{**}$ be the smallest interval partition with $\pi \leq \pi^{**}$. One has

$$\pi \leq \pi^* \leq \pi^{**}$$

## Cumulants

$\mathcal{A}$ a $k$-algebra, $1 \in \mathcal{A}$

$$\varphi : \mathcal{A} \to k, \qquad \varphi(1) = 1$$

$$\varphi(a_1 a_2 \dots a_n) = \sum_{\pi \in \mathcal{P}_n} K_\pi(a_1, \dots, a_n)$$

$$\varphi(a_1 a_2 \dots a_n) = \sum_{\pi \in NC(n)} R_\pi(a_1, \dots, a_n)$$

$$\varphi(a_1 a_2 \dots a_n) = \sum_{\pi \in I(n)} B_\pi(a_1, \dots, a_n)$$

## Inversion formula

$$
\begin{aligned}
K_n(a_1, a_2, \ldots, a_n) &= \sum_{\pi \in \mathcal{P}_n} \varphi_\pi(a_1, \ldots, a_n)\mu^{\mathcal{P}}([\pi, 1_n]) \\
R_n(a_1, a_2, \ldots, a_n) &= \sum_{\pi \in NC(n)} \varphi_\pi(a_1, \ldots, a_n)\mu^{NC}([\pi, 1_n]) \\
B_n(a_1, a_2, \ldots, a_n) &= \sum_{\pi \in I(n)} \varphi_\pi(a_1, \ldots, a_n)\mu^{B}([\pi, 1_n])
\end{aligned}
$$

# Relations between cumulants

Consider

$$\begin{aligned} \varphi(a_1 \dots a_n) &= \sum_{\pi \in \mathcal{P}_n} K_\pi(a_1, \dots, a_n) \\ &= \sum_{\xi \in NC(n)} \left( \sum_{\pi : \pi^* = \xi} K_\pi(a_1, \dots, a_n) \right) \\ &= \sum_{\xi \in NC(n)} R_\xi(a_1, \dots, a_n) \end{aligned}$$

From this it is easy to deduce that

$$R_\xi(a_1, \dots, a_n) = \sum_{\pi : \pi^* = \xi} K_\pi(a_1, \dots, a_n)$$

In particular:

$$R_n(a_1, \dots, a_n) = \sum_{\pi : \pi^* = 1_n} K_\pi(a_1, \dots, a_n)$$

Similarly for $\xi \in I(n)$ one has

$$B_\xi(a_1, \ldots, a_n) = \sum_{\pi : \pi^{**} = \xi} K_\pi(a_1, \ldots, a_n)$$

and

$$B_\xi(a_1, \ldots, a_n) = \sum_{\pi \in NC(n) : \pi^{**} = \xi} R_\pi(a_1, \ldots, a_n)$$

## Γ-cumulants

Let $\Gamma = \gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_r$ be an interval partition with $r$ parts. One defines the Γ-cumulants as

$$K^\Gamma(a_1, a_2, \ldots, a_n) = K_r(A_1, A_2, \ldots, A_r)$$

where $A_i$ is the product of the $a_j$ for $j$ in $\gamma_i$.

One has

$$K^{0_n}(a_1, a_2, \ldots, a_n) = K_n(a_1, a_2, \ldots, a_n)$$

and

$$K^{1_n}(a_1, a_2, \ldots, a_n) = \varphi(a_1 a_2 \ldots a_n)$$

The $K^\Gamma$ interpolate between moments and cumulants

For $\pi$ a partition one can define

$$K_\pi^\Gamma(a_1,\ldots,a_n)=\prod_{p\in\pi}K_{|p|}^{\Gamma_{|p}}(a_{i_1},\ldots,a_{i_{|p|}})$$

Let the $a_i$ be Bernoulli variables ($a_i^2=a_i$) and $\Gamma$ is the partition of $[1,N]$ according to the value of $a_i$:

$a_i=a_j$ iff $i$ and $j$ are in the same part of $\Gamma$

then the $K_\pi^\Gamma$ are products of terms $K_k(a_{i_1},a_{i_2},\ldots,a_{i_k})$ with distinct variables.
There are $2^r-1$ such terms and they determine the joint distribution of the $a_i$.
In particular we would like to express any cumulant of the $a_i$ in terms of these cumulants.

One has

$$\varphi(A_1 A_2 \dots A_r) = \sum_{\xi \in \mathcal{P}_r} K_\xi(A_1, A_2, \dots, A_r)$$

and

$$\begin{array}{rcl}\varphi(a_1, \dots, a_n) & = & \sum_\pi K_\pi(a_1, \dots, a_n) \\ & = & \sum_{\xi \geq \Gamma} \left( \sum_{\pi : \pi \vee \Gamma = \xi} K_\pi(a_1, \dots, a_n) \right)\end{array}$$

One can identify intervals $[\Gamma, 1_n] \sim [0_r, 1_r]$.
Using this identification we get:

**Theorem** (Krawczyk, Speicher)

$$K^\Gamma(a_1, a_2, \dots, a_n) = \sum_{\pi \vee \Gamma = 1_n} K_\pi$$

One can define similar Γ-cumulants in the case of non-crossing or interval cumulants.
The formulas

$$R_\xi(a_1, \dots, a_n) = \sum_{\pi:\pi^*=\xi} K_\pi(a_1, \dots, a_n) \qquad \xi \in NC(n)$$

$$B_\xi(a_1, \dots, a_n) = \sum_{\pi:\pi^{**}=\xi} K_\pi(a_1, \dots, a_n) \qquad \xi \in I(n)$$

$$B_\xi(a_1, \dots, a_n) = \sum_{\pi \in NC(n):\pi^{**}=\xi} R_\pi(a_1, \dots, a_n) \qquad \xi \in NC(n)$$

$$K^\Gamma(a_1, a_2, \dots, a_n) = \sum_{\pi \vee \Gamma = 1_n} K_\pi$$

have a similar structure.
I will show how to invert them.

## $G$-partitions

Let $G = (V, E)$ be a finite graph.

The set $\mathcal{P}_G$ of partitions of $V$ whose parts are connected is a lattice for the refinement order.

It has a smallest element $0_V$ and a largest element, the partition $V = V_1 \cup V_2 \cup \ldots V_k$ into connected components of $G$.

One has $\mathcal{P}_G \sim \prod \mathcal{P}_{G_i}$ the $G_i$ being the connected components.

## Examples

If $G$ is the complete graph on $V$ then

$$\mathcal{P}_G = \mathcal{P}_V$$

If $G$ is an interval graph then

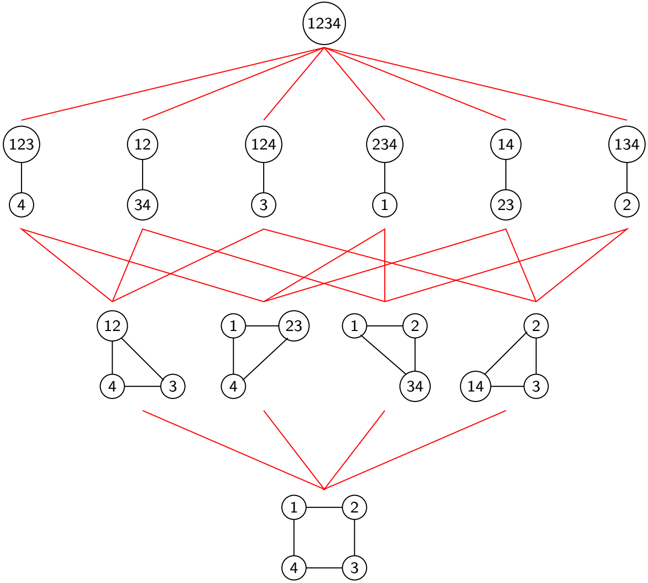
$$\mathcal{P}_G = I(V)$$

If $G$ is a tree then $\mathcal{P}_G$ is a Boolean lattice ($\sim$ the lattice of subsets of $E$).

Each partition $\pi \in \mathcal{P}_G$ induces a graph $G_\pi$ with vertices the parts of $\pi$ and an edge between $p_1$ and $p_2$ if $p_1 \cup p_2$ is connected in $G$.

Every interval $[\pi_1, \pi_2]$ of $\mathcal{P}_G$ is canonically isomorphic to a lattice $\mathcal{P}_{\mathcal{G}_{\pi_1,\pi_2}}$ where the number of connected components of $\mathcal{G}_{\pi_1,\pi_2}$ is the number of parts of $\pi_2$.

# $\mathcal{P}_G$ for $G$ a cycle of size 4



## The chromatic polynomial of a graph

A colouring of a graph with $k$ colours is a map $V \to [1, k]$.
The colouring is proper if for any edge the two extremities have a different colour.
Let $\omega_r$ be the number of proper colourings with exactly $r$ colours (i.e. the map $V \to [1, r]$ is surjective). Then $\omega_r = 0$ for $r > |V|$.
The number of proper colouring with $k$ colour is

$$\chi_G(k) = \sum_{r=1}^{|V|} \binom{k}{r} \omega_r$$

It is a polynomial function of $k$ called the chromatic polynomial.
One has $\chi_G = \chi_{G_1}\chi_{G_2}$ if $V = V_1 \cup V_2$ is a partition and there are no edges between $V_1$ and $V_2$.

**Theorem** The Möbius function on $\mathcal{P}_G$ is multiplicative and if $G$ is connected then

$$\mu([0_G, 1_G]) = [z]\chi_G(z)$$

For example the chromatic polynomial of the complete graph on $n$ vertices is

$$k(k-1)(k-2)\dots(k-n+1)$$

and

$$[z]\chi_G = (-1)^{n-1}(n-1)!$$

The chromatic polynomial of a tree with $n$ vertices is

$$k(k-1)^{n-1}$$

and

$$[z]\chi_G = (-1)^{n-1}$$

## Proof of the Möbius function evaluation (sketch)

The number of colourings (proper or improper) of $G$ with $k$ colours is $k^{|V|}$.
Each colouring defines a partition $\pi \in \mathcal{P}_G$ whose parts are the unicolour connected components of $G$. One thus gets a proper colouring of $G_\pi$ therefore

$$k^{|V|} = \sum_{\pi \in \mathcal{P}_G} \chi_{G_\pi}(k)$$

Note that this is an identity between polynomials and taking the degree one coefficient on both sides shows that $[z]\chi_{G_\pi}(z)$ satisfies the defining property of the Möbius function.

## Relations between cumulants

Recall the formulas

$$R_\xi(a_1,\ldots,a_n)=\sum_{\pi:\pi^*=\xi}K_\pi(a_1,\ldots,a_n)\qquad \xi\in NC(n)$$

$$B_\xi(a_1,\ldots,a_n)=\sum_{\pi:\pi^{**}=\xi}K_\pi(a_1,\ldots,a_n)\qquad \xi\in I(n)$$

$$B_\xi(a_1,\ldots,a_n)=\sum_{\pi\in NC(n):\pi^{**}=\xi}R_\pi(a_1,\ldots,a_n)\qquad \xi\in NC(n)$$

$$K^\Gamma(a_1,a_2,\ldots,a_n)=\sum_{\pi\vee\Gamma=1_n}K_\pi$$

We will invert these formulas using specific graphs and their chromatic polynomials.

## Graphs associated with a partition

Let $\pi \in \mathcal{P}_n$:

The crossing graph $G^c_\pi$ of $\pi$ has vertices the parts of $\pi$ and an edge between $p_1$ and $p_2$ if the partition $p_1 \cup p_2$ has a crossing.

The interval graph $G^I_\pi$ of $\pi$ has vertices the parts of $\pi$ and an edge between $p_1$ and $p_2$ if the partition $p_1 \cup p_2$ is not an interval partition.

The $\Gamma$-graph $G^\Gamma_\pi$ of $\pi$ has vertices the parts of $\pi$ and an edge between $p_1$ and $p_2$ if there is a part of $\Gamma$ which intersects both $p_1$ and $p_2$.

The inversion formula using the Möbius function on the partition lattices of the graphs $G_\pi^c$, $G_\pi^I$ and $G_\pi^\Gamma$ allows us to invert the relations between the different types of cumulants.
For example:

$$R_n(a_1 \dots a_n) = \sum_{\pi : \pi^* = 1_n} K_\pi(a_1, \dots, a_n)$$

Plugging in the formula

$$K_n(a_1, \dots, a_n) = \sum_{\pi : \pi^* = 1_n} R_\pi(a_1, \dots, a_n)[z]\chi_{G_\pi^c}$$

and using the multiplicativity and the Möbius function property one can easily recover

$$R_n(a_1, \dots, a_n) = R_n(a_1, \dots, a_n)$$

There are analogous formulas for the interval cumulants and for the Γ-cumulants.

This recovers several formulas from Belinschi-Nica, Josuat-Vergès, Arizmendi-Hasebe-Lehner-Vargas, with a different proof (the original proofs use rather recursive properties of Tutte polynomials).

In the case of Γ-cumulants this formula seems to be new.

In particular it allows to express cumulants of Bernoulli variables in terms of "fundamental cumulants" (the ones without repetition). This is useful for statistical physics for madels with Bernoulli variables (e.g. Ising model or the exclusion process) where the fundamental cumulants have a nice asymptotic behaviour.

THANK YOU!