# LAVER TABLES

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# An interesting sequence

A098820

Not a moment sequence...

$$1, 2, 4, 4, 8, 8, 8, 8, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16,$$

$$16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16,$$

$$16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16,$$

$$16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16,$$

$$16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16,$$

$$16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16,$$

$$16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16,$$

$$16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16,$$

$$16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, \ldots$$

## The Laver table of size $8 \times 8$

| $\star$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 4 | 6 | 8 | 2 | 4 | 6 | 8 |
| 2 | 3 | 4 | 7 | 8 | 3 | 4 | 7 | 8 |
| 3 | 4 | 8 | 4 | 8 | 4 | 8 | 4 | 8 |
| 4 | 5 | 6 | 7 | 8 | 5 | 6 | 7 | 8 |
| 5 | 6 | 8 | 6 | 8 | 6 | 8 | 6 | 8 |
| 6 | 7 | 8 | 7 | 8 | 7 | 8 | 7 | 8 |
| 7 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| 8 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

On $[1, N]$ there exists a unique binary law

$$\star : [1, N] \times [1, N] \to [1, N]$$

such that, for all $p, q$

$$\begin{aligned} p \star 1 &= p + 1 \mod (N) \\ p \star (q \star 1) &= (p \star q) \star (p \star 1) \end{aligned}$$

**Theorem** (R. Laver)
If (and only if) $N = 2^n$ is some power of 2 then $\star$ is *left-distributive*: for all $p, q, r$

$$p \star (q \star r) = (p \star q) \star (p \star r)$$

The Laver Table of order $2^n$ is the left-distributive system generated by 1 with the relation

$$(\dots(1 \star 1) \star 1) \star 1) \dots \star 1) = 1^{[2^n+1]} = 1 \qquad (2^n + 1 \text{ terms})$$

Analogue of $\mathrm{Z}/2^{\mathrm{n}}\mathrm{Z}$.

A very simple definition but a great complexity.

# Some properties

| $\star$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 4 | 6 | 8 | 2 | 4 | 6 | 8 |
| 2 | 3 | 4 | 7 | 8 | 3 | 4 | 7 | 8 |
| 3 | 4 | 8 | 4 | 8 | 4 | 8 | 4 | 8 |
| 4 | 5 | 6 | 7 | 8 | 5 | 6 | 7 | 8 |
| — | — | — | — | — | — | — | — | — |
| 5 | 6 | 8 | 6 | 8 | 6 | 8 | 6 | 8 |
| 6 | 7 | 8 | 7 | 8 | 7 | 8 | 7 | 8 |
| 7 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| 8 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

- The projection $[1, 2^{n+1}] \to [1, 2^n]$ is a homomorphism. Consequently there exists a projective limit.
- $p \to p + 2^n : [1, 2^n] \to [1, 2^{n+1}]$ is a homomorphism. One can take an inductive limit by using the law
  $p * q = N - (N - p) \star (N - q)$

| $\star$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 4 | 6 | 8 | 2 | 4 | 6 | 8 | 4 |
| 2 | 3 | 4 | 7 | 8 | 3 | 4 | 7 | 8 | 4 |
| 3 | 4 | 8 | 4 | 8 | 4 | 8 | 4 | 8 | 2 |
| 4 | 5 | 6 | 7 | 8 | 5 | 6 | 7 | 8 | 4 |
| 5 | 6 | 8 | 6 | 8 | 6 | 8 | 6 | 8 | 2 |
| 6 | 7 | 8 | 7 | 8 | 7 | 8 | 7 | 8 | 2 |
| 7 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 1 |
| 8 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 8 |

- Each line $p \star q;\ q = 1, 2, \ldots$ is periodic with period $\pi_n(p) = 2^m$.
- $p \star q;\ q = 1, 2, \ldots, \pi(p) - 1$ is increasing
- $\pi_{n+1}(p) = \pi_n(p)$ or $2\pi_n(p)$.
- $p \sqsubset p \star q - 1$

$a \sqsubset b$ if the binary expansion of $a$ is contained in that of $b$.

## The law $*$

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 3 | 0 | 2 | 0 | 2 | 0 | 2 | 0 | 2 |
| 4 | 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 |
| 5 | 0 | 4 | 0 | 4 | 0 | 4 | 0 | 4 |
| 6 | 0 | 1 | 4 | 5 | 0 | 1 | 4 | 5 |
| 7 | 0 | 2 | 4 | 6 | 0 | 2 | 4 | 6 |

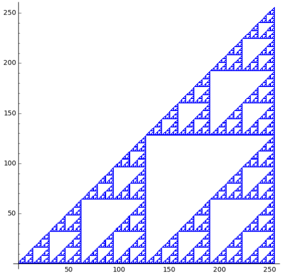
The law $*$ is determined by

- $p * (2^n - 1) = p - 1$ if $p < 2^n$
- $p * (q * r) = (p * q) * (p * r)$

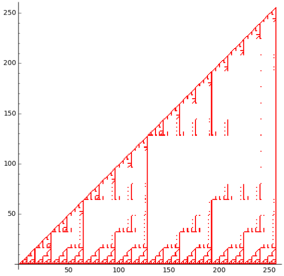
It satisfies

$$p * q \sqsubset p - 1$$

# The relation $\sqsubset$




# The Laver table of size 256



## Computing $p * q$

How to compute row $p+1$ knowing rows $0, 1, 2, \ldots, p$:

| * | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 3 | 0 | 2 | 0 | 2 | 0 | 2 | 0 | 2 | 0 |
| 4 | 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 | 0 |
| 5 | 0 | 4 | 0 | 4 | 0 | 4 | 0 | 4 | 0 |
| 6 | 0 | 1 | 4 | 5 | 0 | 1 | 4 | 5 | 0 |
| 7 | 0 | 2 | 4 | 6 | 0 | 2 | 4 | 6 | 0 |

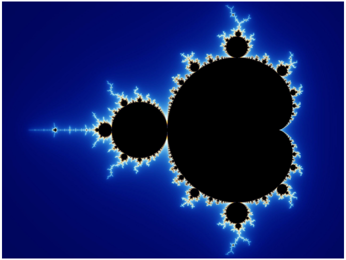
$7 * 7 = 6$; $7 * 6 = 6 * 6 = 4$; $7 * 5 = 4 * 6 = 2$; $7 * 4 = 2 * 6 = 0$.

In general computing $p * q$ is time consuming: one has to compute the rows of a large part of the $r < p$.

Laver tables are very recursive combinatorial objects quite different from usual structures in algebraic or enumerative combinatorics.

In general computing $p * q$ is time consuming: one has to compute the rows of a large part of the $r < p$.

Laver tables are very recursive combinatorial objects quite different from usual structures in algebraic or enumerative combinatorics. They have some similarity with the Mandelbrot set $z \to z^2 + c$



## Some properties: I. Homomorphisms

Left multiplication $\lambda_x : y \mapsto x * y$ is a homomorphism:

$$\lambda_x(y * z) = \lambda_x(y) * \lambda_x(z)$$

**Lemma**:
Every homomorphism of $*$ is of the form $\lambda_x$.

In particular $\lambda_x \circ \lambda_y = \lambda_{x \circ y}$ for some unique $x \circ y$.
The law $\circ$ is associative and one has

$$x \circ y = x * (y - 1) + 1$$

It is convenient to consider

$$x \bar{\circ} y = (x + 1) \circ (y + 1) - 1 = (x + 1) * y$$

# The law $\bar{\circ}$

| $\bar{\circ}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 2 | 0 | 2 | 0 | 2 | 0 | 2 | 0 | 2 |
| 3 | 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 |
| 4 | 0 | 4 | 0 | 4 | 0 | 4 | 0 | 4 |
| 5 | 0 | 1 | 4 | 5 | 0 | 1 | 4 | 5 |
| 6 | 0 | 2 | 4 | 6 | 0 | 2 | 4 | 6 |
| 7 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |

this is just the table for $*$ where the left column has been shifted by 1.
It is an associative law.

## Some properties: II. Thresholds

For fixed $p < 2^n$ the sequence $p * q;\ q = 1, 2, \ldots$ is increasing for $q < \pi(p)$.
The sequence $(p + 2^n) * q;\ q = 1, 2, \ldots$ is also increasing and reduces to $p * q$ modulo $2^n$.
$\rightarrow$ there exists $\theta(p)$ such that

$p * q = (p + 2^n) * q$ for $q < \pi_n(p) - \theta(p)$ and
$p * q = (p + 2^n) * q + 2^n$ fo $q \geq \pi_n(p) - \theta(p)$

$\theta(p)$ is the *threshold of* $p$.
knowing $\theta(p); p \leq 2^n$ allows to recover the whole Laver table of order $2^n$.

As an example we compute the row $p * q$ for
$p = 494 = 2^1 + 2^2 + 2^3 + 2^5 + 2^6 + 2^7 + 2^8$.
The first column shows the partial sums $p_i$
the thresholds are in the second column

| $p_i$ | $\theta$ | $p_i * q$ |
|---|---|---|
| 2 | 1 | $0, \underline{1}$ |
| 6 | 2 | $0, 1, \underline{4}, \underline{5}$ |
| 14 | 1 | $0, 1, 4, \underline{13}$ |
| 46 | 4 | $0, 1, 4, 13, \underline{32}, \underline{33}, \underline{36}, \underline{45}$ |
| 110 | 3 | $0, 1, 4, 13, 32, \underline{97}, \underline{100}, \underline{109}$ |
| 238 | 3 | $0, 1, 4, 13, 32, \underline{225}, \underline{228}, \underline{237}$ |
| 494 | 8 | $0, 1, 4, 13, 32, 225, 228, 237, \underline{256}, \underline{257}, \underline{260}, \underline{269}, \underline{288}, \underline{481}, \underline{484}, \underline{493}$ |

The knowledge of all Laver tables is equivalent to the knowledge of the sequence $\theta(p)$; $p = 1, 2, 3, \ldots$.

Repartition of thresholds and periods for $p \leq 2^{12}$

| | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 | 2048 | 4096 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 12 | 2103 | | | | | | | | | | |
| 2 | | 66 | | 30 | | | | | | | | |
| 3 | | | 398 | 213 | | | | | | | | |
| 4 | | | 58 | 761 | | | | | | | | |
| 7 | | | | 63 | | | | | | | | |
| 8 | | | | 121 | | | | | | | | |
| 16 | | | | | 12 | 110 | | | | | | |
| 32 | | | | | | 34 | | | | | | |
| 48 | | | | | | | 19 | 6 | | | | |
| 64 | | | | | | | 22 | 26 | | | | |
| 112 | | | | | | | | 1 | | | | |
| 128 | | | | | | | | 25 | | | | |
| 256 | | | | | | | | | 4 | 2 | | |
| 512 | | | | | | | | | | 6 | | |
| 1024 | | | | | | | | | | | 2 | |
| 2048 | | | | | | | | | | | | 1 |

**Conjecture** The numbers $\theta(p)$ all have the form $2^j - 2^j$.

## Where do Laver tables come from?

Let $f, g : X \to X$ be functions with graph

$$\{(x, g(x))\} \quad \{(x, f(x))\}$$

One can compose $f$ and $g$

$$\{(x, g(x))\} \to \{(x, f(g(x)))\}$$

One can also "apply"' $f$ to the graph of $g$:

$$\{(x, g(x))\} \to \{(f(x), f(g(x)))\}$$

It is a graph only if $f$ is invertible

$$f \star g = f \circ g \circ f^{-1}$$

$\star$ is left-distributive.
In fact for any group $G$ the conjugation

$$a \star b = aba^{-1}$$

is left-distributive

Let $\lambda$ be an infinite ordinal and $V_\lambda$ the associated rank ($V_{\lambda+1} = 2^{V_\lambda}$, $V_\lambda = \cup_{\mu<\lambda} V_\mu$ if $\lambda$ is a limit ordinal).
An *elementary embedding*

$$j : V_\lambda \to V_\lambda$$

is an injection which preserves the truth of every first order formula.

Laver's axiom is that there exists an ordinal $\lambda$ and a non trivial elementary embedding $j : V_\lambda \to V_\lambda$.
This axiom cannot be proved in ZFC.

This is analogous to the axiom of infinity but stronger.
An elementary embedding fixes all finite sets, countable sets, and all sets obtained from these by taking subsets, sets of subsets and increasing unions.

If $j$ is a nontrivial elementary embedding then there exists a smallest ordinal $\kappa$ with $j(\kappa) > \kappa$.

Then $\kappa$ is inaccessible: if $\mu < \kappa$ then $2^\mu < \kappa$.

Two elementary embeddings $j$ and $k$ can be composed $j \circ k$, but one can also "apply" $j$ to $k$ to get $j \star k$: one takes the image by $j$ of the graph of $k$.
This operation is left-distributive:

$$j \star (k \star l) = (j \star k) \star (j \star l)$$

The critical ordinals of the iterates of $j$ form an increasing sequence $\kappa_i; i = 1, 2, \ldots$.
The Laver table is obtained by taking the quotient of $j$ by equivalence modulo $\kappa_n$.
Laver tables were born from the theory of large cardinals.

# Asymptotics of periods and set theory

The periods $\pi_n(1)$ (for the $\star$ law; they are the numbers $\pi(2^n - 1)$ for the $*$ law) form an increasing sequence which is A098820 in and starts asOEIS

$$1, 2, 4, 4, 8, 8, 8, 8, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16,$$

$$16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, \ldots$$

Question: does $\pi_n(1) \to \infty$?

"Theorem" (Laver) $\pi_n(1) \to \infty$ as $n \to \infty$.

The proof uses Laver's axiom.
No proof inside ZFC is known.
In fact no number $n$ is known such that $\pi_n(1) \geq 32$.

Dougherty's lower bound is a HUGE number

## Ackerman function

$$\begin{aligned} f_0(n) &= n+1 \\ f_{p+1}(0) &= f_p(1) \\ f_{p+1}(n) &= f_p(f_{p+1}(n-1)) \quad n>0 \end{aligned}$$

| | 0 | 1 | 2 | ... | $n$ |
|---|---|---|---|---|---|
| $f_0$ : | 1 | 2 | 3 | ... | $n+1$ |
| $f_1$ : | 2 | 3 | 4 | ... | $n+2$ |
| $f_2$ : | 3 | 5 | 7 | ... | $2n+3$ |
| $f_3$ : | 5 | 13 | 29 | ... | $2^{n+3}-3$ |
| $f_4$ : | 13 | $2^{16}-3$ | $2^{2^{16}}-3$ | ... | ... |
| $f_5$ : | $2^{16}-3$ | ... | | | |
| $f_6$ : | ... | | | | |

## Dougherty's bound

If $\pi_n(1) \geq 32$ then $n > f_9(f_8(f_8(254)))$

## Asymptotic frequencies

$\Omega_n(k)$ = number of elements in the Laver table of order $2^n$ with period $k$.

$\omega_n(k) = \Omega_n(k)/2^n$ the corresponding frequency

**Theorem** The numbers $\omega_n(k)$ converge as $n \to \infty$ to $\omega(k)$.

Question : what is the value of $\omega(k)$?

Some values of the frequencies:

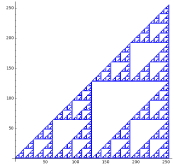
| k | 2 | 4 | 8 | 16 | 32 |
|---|---|---|---|---|---|
| $n = 22$ | 0.000572 | 52.936697 | 10.196733 | 30.197978 | 0.002623 |
| $n = 28$ | 0.000011 | 52.936601 | 10.193098 | 30.202101 | 0.001456 |
| $n = 31$ | 0.000002 | 52.936599 | 10.193012 | 30.202195 | 0.001429 |

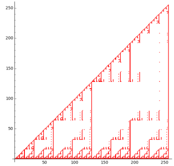
Question: do these numbers appear in other problems of mathematics and/or physics?

# Maximal elements

One has $p * q \sqsubset p - 1$.

$p$ is *maximal* if the numbers $p * q$ are all the numbers $r \sqsubset p - 1$.




These are the numbers where the blue and red columns are identical.

**Theorem** $p$ is maximal if and only if it is of the form

$$10^{b_0}1^{2^{a_1}}0^{b_1 2^{a_1}}1^{2^{a_2}}0^{b_2 2^{a_2}}\dots 1^{2^{a_r}}0^{b_r 2^{a_r}} \tag{0.1}$$

with $b_i \geq 0$ and $a_1 < a_2 < a_3 \dots$
The number of maximal $p$ in $[2^{n-2} + 2^{n-1}, 2^n]$ is equal to the number of *binary partitions* of $n-2$:

$$n - 2 = \sum_i \lambda_i \qquad \lambda_i \in \{1, 2, 4, 8, \dots\}$$

**Theoreme** If $p$ and $q$ are maximal then $p * q$ is maximal.
Computing $p * q$ for $p$ maximal is very easy.
Finding a formula for general $p, q$ seems very difficult.

THANK YOU!