Linear intervals in the Tamari and the Dyck lattices

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88th Séminaire Lotharingien de Combinatoire
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Posets, chains and intervals

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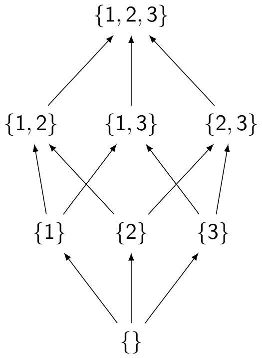

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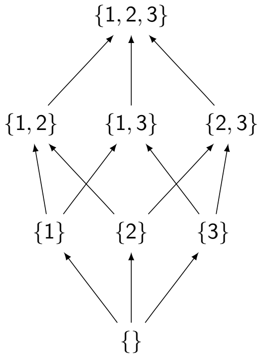
If $x \leq y$, *the interval* $[x, y]$ is the subset $\{t \in P | x \leq t \leq y\}$.



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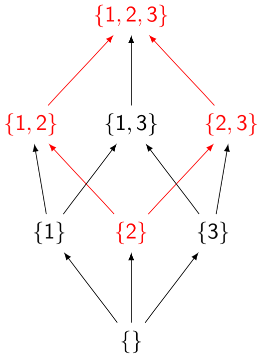

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A *chain* of length k from x to y is a sequence $x = x_0 < \cdots < x_k = y$. The *height* of an interval $[x, y]$ is the maximal length of a chain from x to y.

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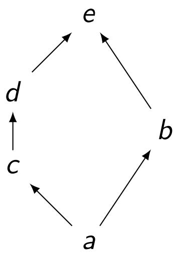
A *chain* of length k from x to y is a sequence $x = x_0 < \cdots < x_k = y$. The *height* of an interval $[x, y]$ is the maximal length of a chain from x to y. An interval is *linear* if it is totally ordered, *i.e.* if it is a chain.

Linear interval

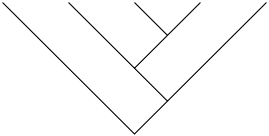


Non linear interval

Binary trees and the Tamari lattices

Definition

A (planar rooted) binary tree is a connected acyclic planar graph whose vertices have degree 3 or 1, with one marked vertex of degree 1 called the root.



Binary trees and the Tamari lattices

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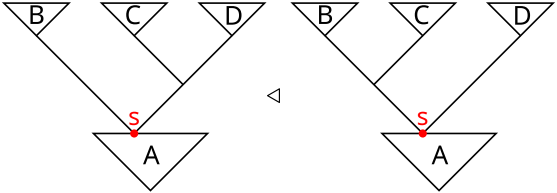
A (planar rooted) binary tree is a connected acyclic planar graph whose vertices have degree 3 or 1, with one marked vertex of degree 1 called the root. The Tamari lattice Tam_n [Tamari, 1962] is a poset on binary trees, described as the reflexive transitive closure of the left rotations.



A left rotation at the node s.

Dyck paths and the Dyck lattices

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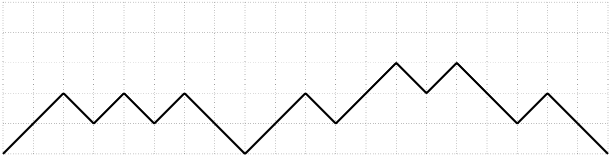
A Dyck path of length (or size) n is a path in $\mathbb{N}^2$ using up steps $(1,1)$ and down steps$(1,-1)$, starting at $(0,0)$ and ending at $(2n,0)$.



A Dyck path of length 10

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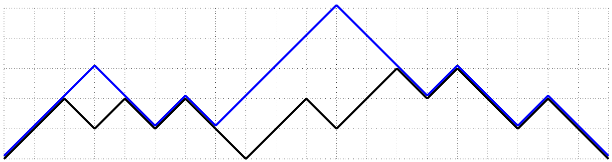
Definition

A Dyck path of length (or size) n is a path in $\mathbb{N}^2$ using up steps $(1, 1)$ and down steps$(1, -1)$, starting at $(0, 0)$ and ending at $(2n, 0)$. The Dyck lattice Dyck_n [Stanley, 1999] is a poset on Dyck paths where a path P is lower than a path Q if P is weakly under Q.



Two Dyck paths of length 10

The Tamari lattice on Dyck paths

Remark

The Tamari lattice Tam_n can also be defined on Dyck paths. The covering relations consist of swapping a down step with the excursion that follows it.

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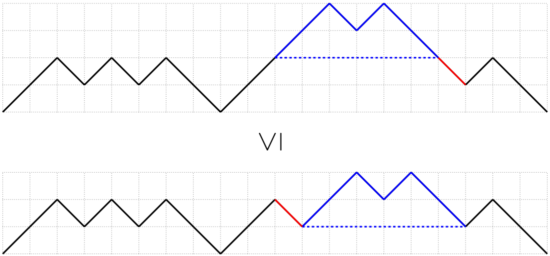

Main result

Theorem (C.)

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For any $n \geq 1$ and $k \geq 0$, the Tamari lattice Tam_n and the Dyck lattice Dyck_n have the same number of linear intervals of height k.
More precisely, both lattices have :

- $\frac{1}{n+1}\binom{2n}{n}$ linear intervals of height 0,
- $\binom{2n-1}{n-2}$ linear intervals of height 1,
- $2\binom{2n-k}{n-k-1}$ linear intervals of height k, for $2 \leq k < n$.

Furthermore, they have no linear interval of height $k \geq n$.

Linear intervals in the Tamari lattices

General fact

In any poset, the intervals of height 0 are those of the form $[x, x]$ with x some element of the poset and they are linear. We call them trivial intervals.

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Question

What are the linear intervals of height 2 or more in the Tamari lattice ?

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Questions

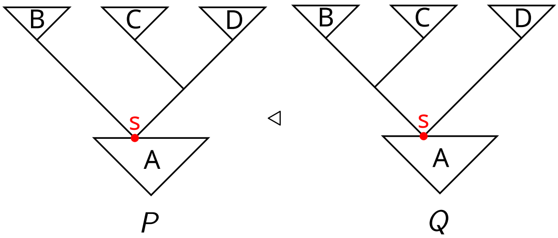
What are the linear intervals of height 2 or more in the Tamari lattice ?
How many of them are there ?

Intervals of height 2

Question

What are the linear intervals of height 2 in the Tamari lattice ?

Suppose that Q is obtained from a tree P by a rotation at the node s.

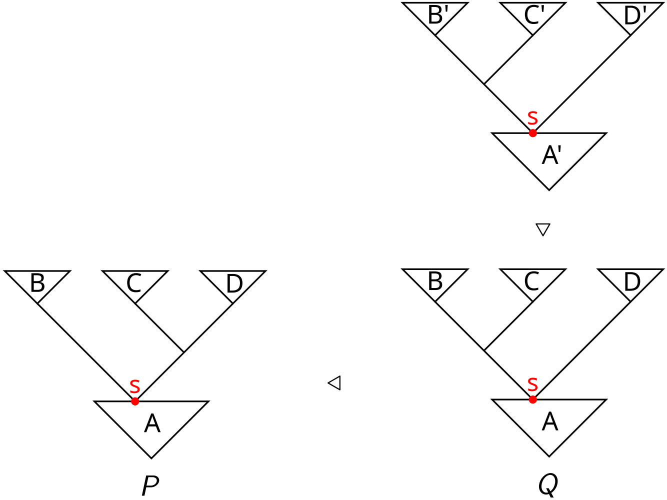


Problem

What covering relations $Q \lessdot R$ produce a linear interval $[P, R]$?

First case

A rotation within A (that preserves s), B, C or D?



First case: Non linear

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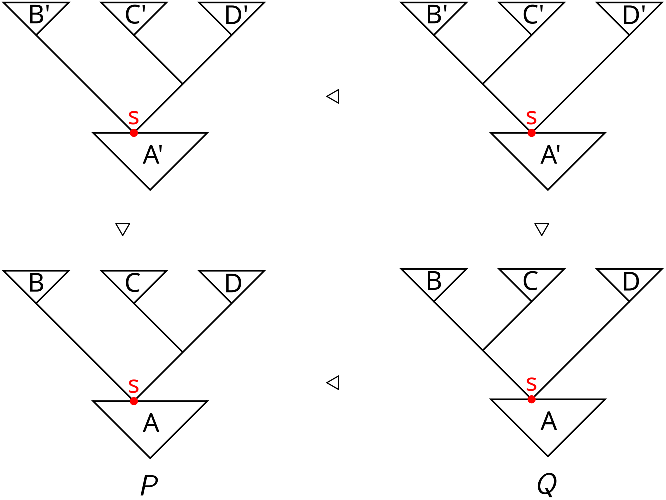
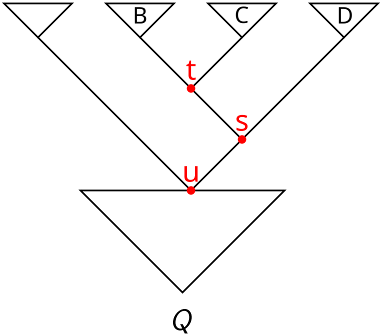
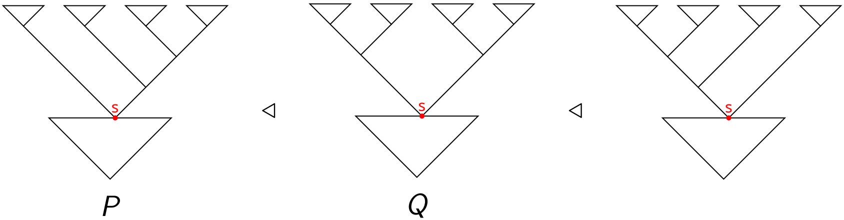


Remaining cases



Second case

Another rotation at the node s?



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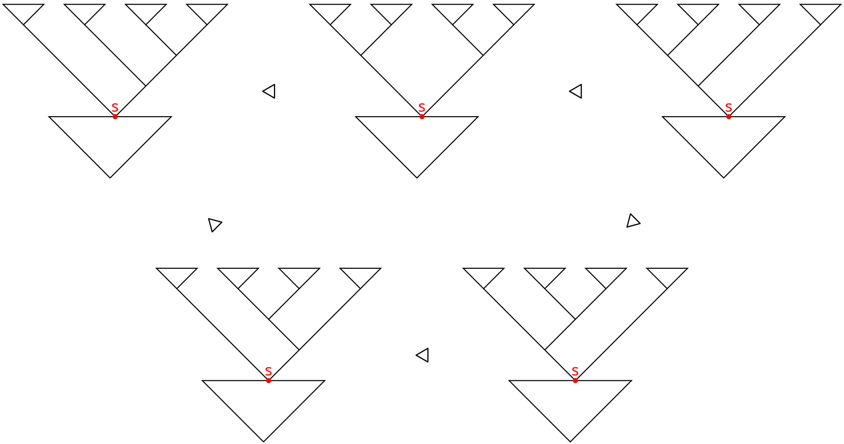
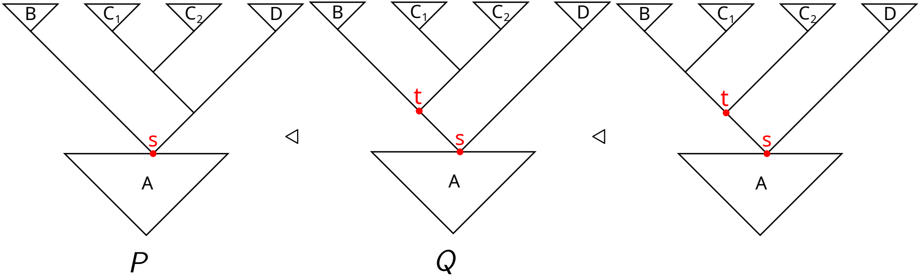


Third case

A rotation at the node t (if C is not trivial)?



Third case: Linear!

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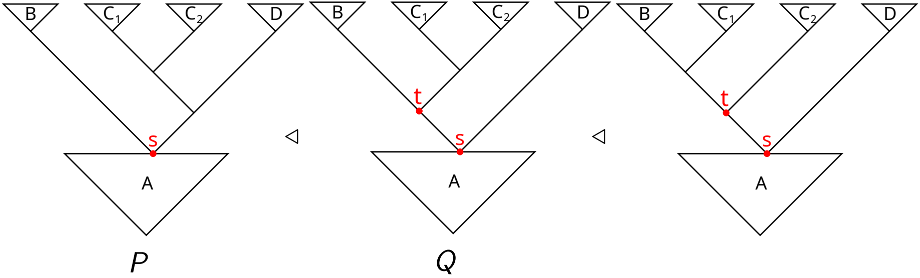

Fourth case

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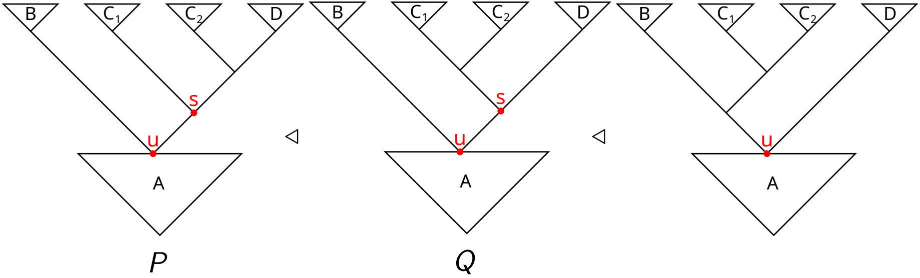

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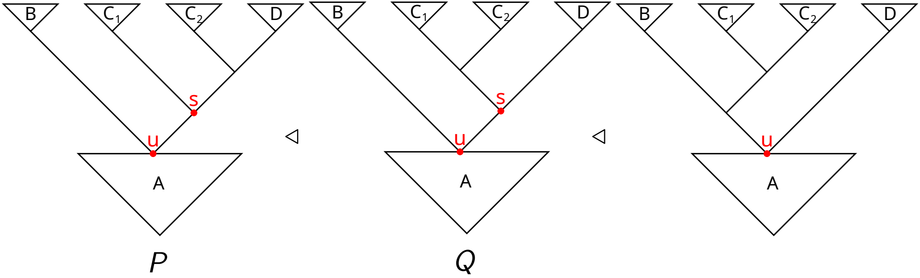

General case

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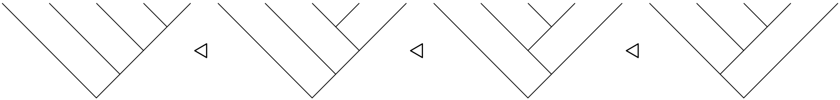
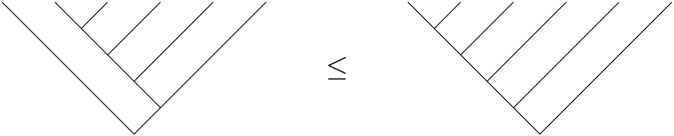
For $n \geq 2$, we can define intervals R_n and L_n with trees of size $n+1$. They are linear of height n.



The interval R_3 with its 4 elements



The interval L_4

General case

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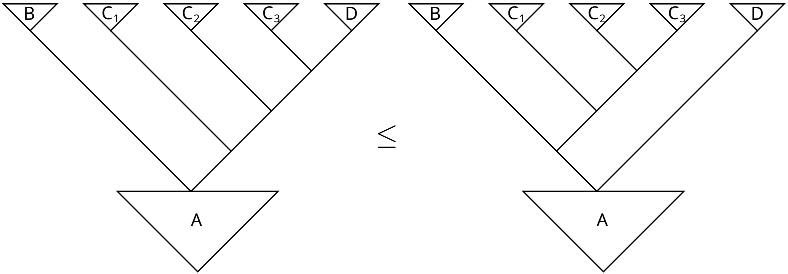
For $n \geq 2$, we can define intervals R_n and L_n with trees of size $n+1$. They are linear of height n. A right interval is an interval R_k with trees grafted on its leaves and the result grafted on a tree, and it is linear of height k.



A right interval of height 3

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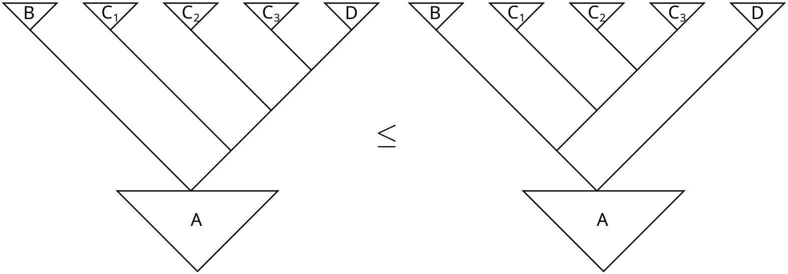

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Tools: combinatorial description, generating series, Lagrange inversion.

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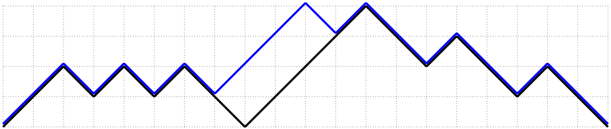
For $k \geq 2$, we can define two subsets of linear intervals of height k that we call left and right intervals of height k.



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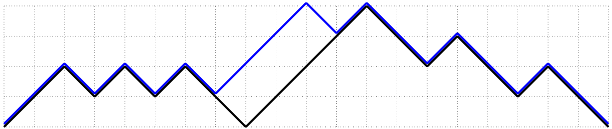

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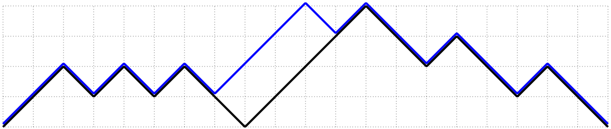

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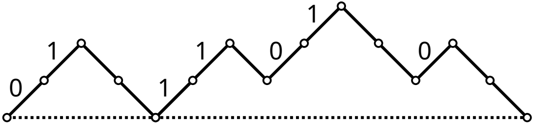

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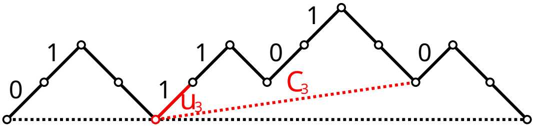

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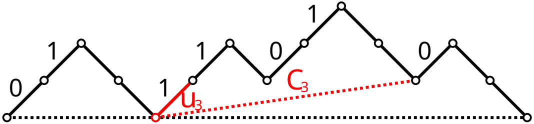

Proposition–Definition

The alt-Tamari poset Tam_n^δ is defined as the reflexive transitive closure of the δ-rotations on the set of Dyck paths of length n.

Linear intervals in the alt-Tamari posets

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What intervals are linear in Tam_n^{δ} ?

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In both cases, the interval is linear and k is its height.

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The proof is bijective!

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  2. Every linear intervals of height at least 2 is either a left or right interval.
  3. Left and right intervals decompose in similar ways in all alt-Tamari posets, which gives a bijective proof that they are counted by the same numbers.

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- Generalize the alt-Tamari posets to the m-Tamari or ν-Tamari lattices.

Takeouts and prospects and questions?

Thanks for your attention!

- We can define a whole new family of posets on Dyck paths which interpolate between the Tamari and the Dyck lattices.
- All of these posets have the same (nice !) numbers of linear intervals of any fixed height.

Prospects:

- Study the alt-Tamari posets: comparison criteria, lattice structure, changes of the Hasse diagram when we modify one entry of δ, etc..
- Count linear intervals in other posets: m-Tamari, ν-Tamari, Cambrian lattices, posets of tilting modules, etc..
- Generalize the alt-Tamari posets to the m-Tamari or ν-Tamari lattices.

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