A generalization of perfectly clustering words via brick band modules of certain gentle algebras

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LaCIM (UQAM), Montréal

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LACIM
Laboratoire d'algèbre, de combinatoire et d'informatique mathématique

UQÀM

This is a joint work with Mélodie Lapointe, Yann Palu, Pierre-Guy Plamondon, Christophe Reutenauer and Hugh Thomas.

Underlying story

Word combinatorics	Representation theory of algebras

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Words	Modules over algebras

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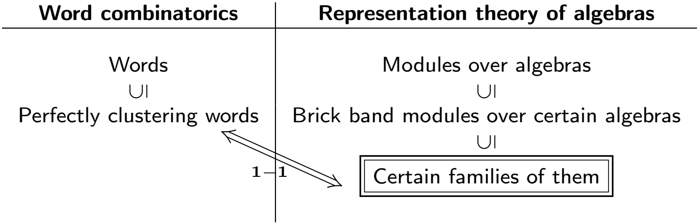

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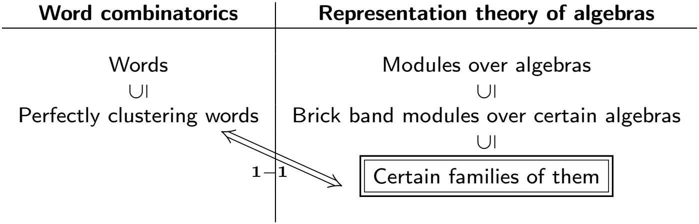

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The main point of this talk is to present this link, and how representation theoretic tools can be used for proving a conjecture over perflectly clustering words.

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1. Word universe

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    - Perfectly clustering words

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    - Using words and modules link

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  For instance, if $w = \mathbf{1321}$, then $|w| = 4$, $|w|_{\mathbf{1}} = 2$, $|w|_{\mathbf{2}} = 1$ and $|w|_{\mathbf{3}} = 1$.

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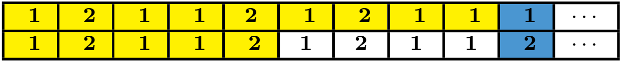
- Let $\leqslant$ be the *lexicographical order* (extended periodically to infinite words) on primitive words of Σ^*. For instance, $\mathbf{1211212111} < \mathbf{12112}$.



1	**2**	**1**	**1**	**2**	**1**	**2**	**1**	**1**	**1**	···
1	**2**	**1**	**1**	**2**	**1**	**2**	**1**	**1**	**2**	···

Burrows-Wheeler transformation

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1	**3**	**2**	**2**	**2**	**3**	**1**	**4**	**1**	**4**	**1**	**4**
1	**4**	**1**	**3**	**2**	**2**	**2**	**3**	**1**	**4**	**1**	**4**
1	**4**	**1**	**4**	**1**	**3**	**2**	**2**	**2**	**3**	**1**	**4**
1	**4**	**1**	**4**	**1**	**4**	**1**	**3**	**2**	**2**	**2**	**3**
2	**2**	**2**	**3**	**1**	**4**	**1**	**4**	**1**	**4**	**1**	**3**
2	**2**	**3**	**1**	**4**	**1**	**4**	**1**	**4**	**1**	**3**	**2**
2	**3**	**1**	**4**	**1**	**4**	**1**	**4**	**1**	**3**	**2**	**2**
3	**1**	**4**	**1**	**4**	**1**	**4**	**1**	**3**	**2**	**2**	**2**
3	**2**	**2**	**2**	**3**	**1**	**4**	**1**	**4**	**1**	**4**	**1**
4	**1**	**3**	**2**	**2**	**2**	**3**	**1**	**4**	**1**	**4**	**1**
4	**1**	**4**	**1**	**3**	**2**	**2**	**2**	**3**	**1**	**4**	**1**
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1	4	1	4	1	3	2	2	2	3	1	4
1	4	1	4	1	4	1	3	2	2	2	3
2	2	2	3	1	4	1	4	1	4	1	3
2	2	3	1	4	1	4	1	4	1	3	2
2	3	1	4	1	4	1	4	1	3	2	2
3	1	4	1	4	1	4	1	3	2	2	2
3	2	2	2	3	1	4	1	4	1	4	1
4	1	3	2	2	2	3	1	4	1	4	1
4	1	4	1	3	2	2	2	3	1	4	1
4	1	4	1	4	1	3	2	2	2	3	1

- Then we read the word obtained by taking the last column of this tableau. We get is the Burrows-Wheeler transform of w : $\mathrm{BW}(w) = \mathbf{444332221111}$

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(*ii*) Perfectly clustering words over an alphabet of two letters correspond exactly to Christoffel words.

(*iii*) The Burrows-Wheeler transform gives an injective map from conjugacy classes of primitive words to words.

Gessel-Reutenauer transformation

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The Gessel-Reutenauer['93,'12] tranformation gives a bijective map from multisets of conjugacy classes of primitive words over Σ^* to words over Σ^*. Let us explain how it works with an example. Let us take $s = \{(\mathbf{53512121}), (\mathbf{5343})\}$:

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- We consider all the conjugates of each word in the multiset and we order them with respect to the (extended version of the) lexicographical order.

1	**2**	**1**	**2**	**1**	**5**	**3**	**5**
1	**2**	**1**	**5**	**3**	**5**	**1**	**2**
1	**5**	**3**	**5**	**1**	**2**	**1**	**2**
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				3	**5**	**3**	**4**
				4	**3**	**5**	**3**
5	**1**	**2**	**1**	**3**	**1**	**5**	**3**
				5	**3**	**4**	**3**
5	**3**	**5**	**1**	**2**	**1**	**2**	**1**

- We get $\Psi(s) = \mathbf{522115543331} = w$.

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Theorem 3

There exists at most one perfectly clustering word (up to conjugation) with a given number of occurences of each letter in it.

Dyck path model

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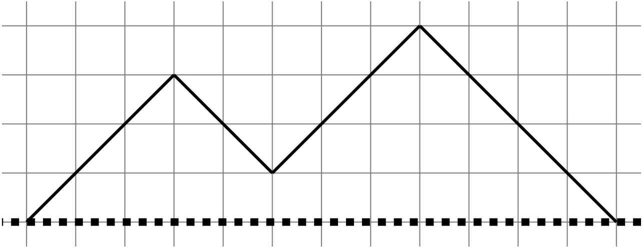
Now we will describe a morphism φ from g-vector to multiset of conjugacy classes of words.

- Given a g vector we can associate to it a Dyck path in a natural way.



Dyck path associated to $g = (3, -2, 3, -1, -3)$.

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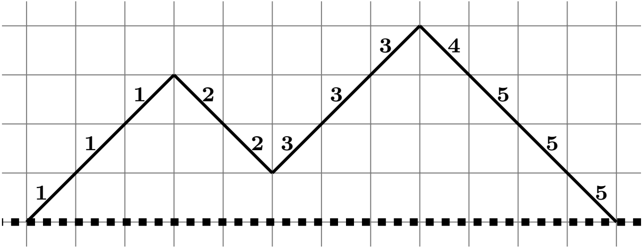
- Then we label the Dyck path thanks to the g-vector by induction : we label the $|g_1|$ first steps of the Dyck path by **1**, then the following $|g_2|$ steps by **2** and so on.



Labelling of the Dyck path associated to $g = (3, -2, 3, -1, -3)$.

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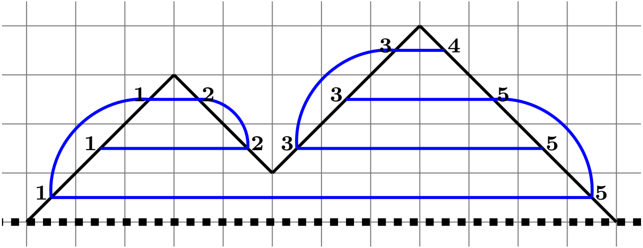
- We draw curves over the Dyck path as follows : we draw a horizontal line between two opposites steps of the Dyck path, and we draw rainbow arcs between steps with the same label out side of the surface delimited by the Dyck path and the dashed line.



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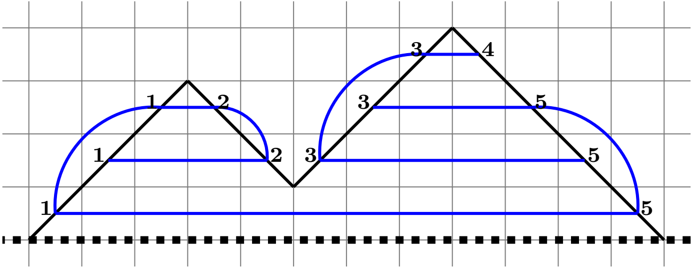

Curves over the Dyck path associated to $g = (3, -2, 3, -1, -3)$.

- This is what we can call *Dyck path model of g*. To get a multiset of conjugacy classes of words from this model, we proceed as follows :

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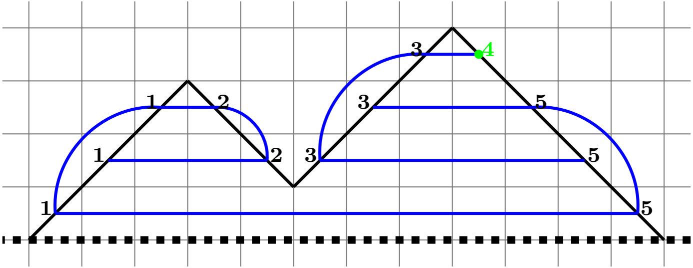

We start at 4.

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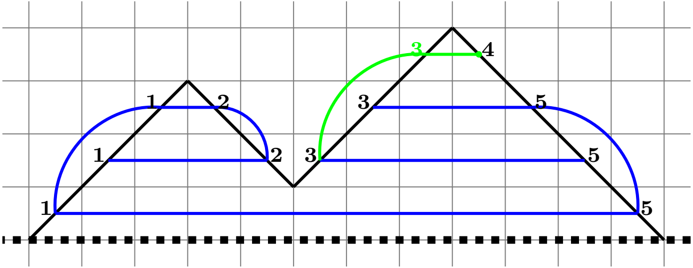
1) We start from an extremity of one of the curves over the Dyck path and we keep in mind the label of this extremity (if there are close curves, we can start from any step of the Dyck path)

Dyck path model



The recording of our travel : **43**

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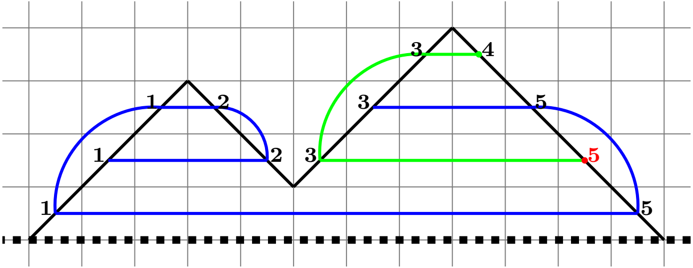
2) Then we follow the curve until we come back to where we start. We keep track of the labels of the Dyck path edge each time we go out of the surface delimited by the Dyck path and the dashed line (the label of the other extremity of the curve is included).

Dyck path model



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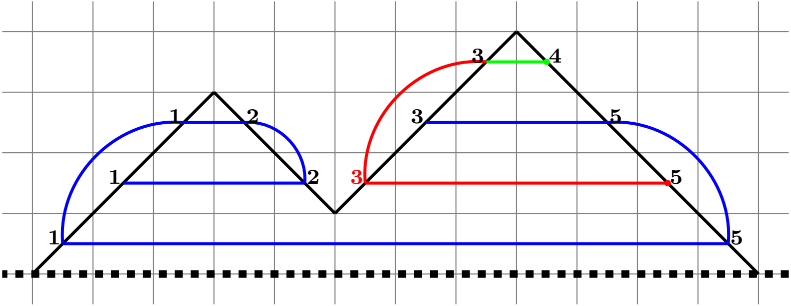

The recording of our travel : **4353**

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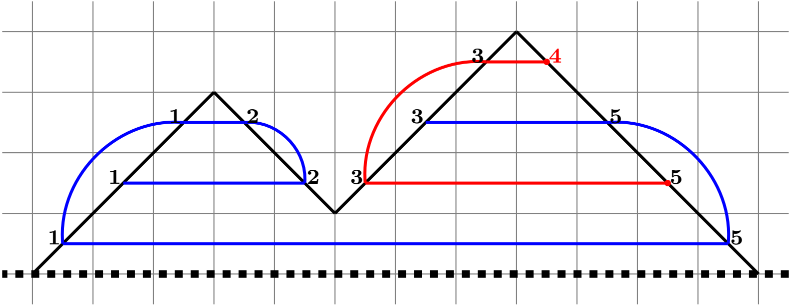
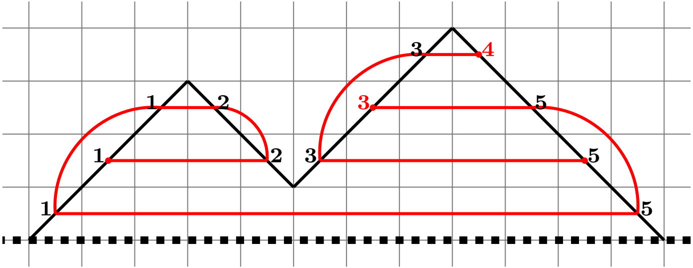


We ended the travel of this curve, we get $(\mathbf{4353})$.

- This is what we can call *Dyck path model of g*. To get a multiset of conjugacy classes of words from this model, we proceed as follows :

3) Once we end the travel of a curve, we record the conjugacy class of the word we obtained (if we followed a closed curve, we record two copies of it), and we start the traveling of another curve. We continue until we have done this for all the curves.

Dyck path model



- We give the result as a multiset of all conjugacy classes we got following the process.

$$\varphi((3,-2,3,-1,-3)) = \{(\mathbf{4353}), (\mathbf{35121215})\}$$

Correspondance between PCWs and g-vectors

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Proposition 5 [DLPPRT '22+]

Let $g = (g_1, \dots, g_n)$ be a g-vector with $g_1 > 0$ and $g_i \leqslant 0$ for $i > 1$. Then

$$f(\varphi(g)) = \Phi(\mathbf{n}^{|g_n|} \dots \mathbf{3}^{|g_3|}\mathbf{2}^{|g_2|})$$

where f is the erasing morphism of $\mathbf{1}$. Moreover each conjugacy class appearing is a conjugacy class of a perfectly clustering word.

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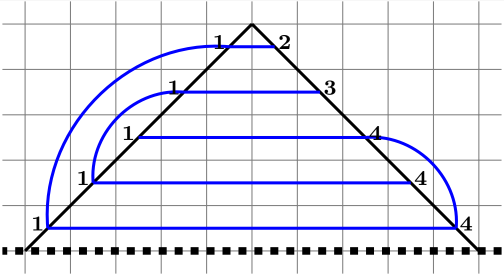

For $g = (5, -1, -3, -1)$, we get $\varphi(g) = \{(\mathbf{3141}), (\mathbf{141214})\}$ and so $f(\varphi(g)) = \{(\mathbf{34}), (\mathbf{424})\}$

Correspondance between PCWs and g-vectors

Proposition 5 [DLPPRT '22+]

Let $g = (g_1, \dots, g_n)$ be a g-vector with $g_1 > 0$ and $g_i \leqslant 0$ for $i > 1$. Then

$$f(\varphi(g)) = \Phi(\mathbf{n}^{|g_n|} \dots \mathbf{3}^{|g_3|}\mathbf{2}^{|g_2|})$$

where f is the erasing morphism of $\mathbf{1}$. Moreover each conjugacy class appearing is a conjugacy class of a perfectly clustering word.

2	**4**	**4**
	3	**4**
4	**2**	**4**
	4	**3**
4	**4**	**2**

We can check that $\Psi(\{(\mathbf{34}), (\mathbf{424})\}) = \mathbf{44432}$.

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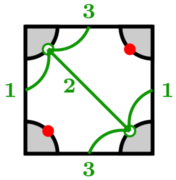

Surface model associated to the above algebra.

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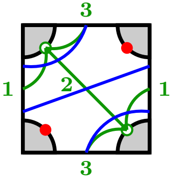

Example of a curve of this surface.

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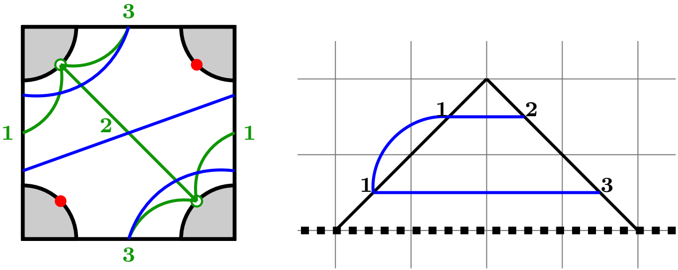
- In the Dyck path model, each curve can be associated to a (one-parameter family of) band modules. And so to $g-$vector, we can associated a module obtained a direct sum of those band modules. Let us denote it by M_g.



The Dyck path model simplifying the surface model.

Euler form

Euler form

Definition 6

The Euler form is a bilinear form defined on $\mathbb{R}^n$ by : for all $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$,

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- The name "Euler form" comes from representation theory. In particular, if g, h are two g-vectors, and if we denote by M_g and M_h the modules associated to them respectively, we get :

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- In particular, we can deduce that The Euler form is skew symmetric over g-vectors. We can also check it by calculus :

$$\langle g, h\rangle + \langle h, g\rangle = 2 \left(\sum_{i=1}^{n} g_i\right) \left(\sum_{j=1}^{n} h_j\right) = 0$$

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Let $g = (g_1, \ldots, g_n)$ be a g-vector. Then in $\Phi(\mathbf{n}^{|g_n|} \cdots \mathbf{3}^{|g_3|}\mathbf{2}^{|g_2|})$ there are at most $\lceil (n-1)/2 \rceil$ distinct circular primitive words.

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- A certain familly of g-vectors which gives a basis of this space,
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- Hence the number of conjugacy classes of words is bounded by $\lceil (n-1)/2 \rceil$.

$\square$

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Vielen Dank !

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Note that $\Psi = \Phi^{-1}$ coincides with BW for multisets made of an unique conjugacy class.