Partitions, Kernels, and the Localization Method

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Partitions

Definition

For any $n \in \mathbb{Z}_{\geq 0}$, a partition of n is a representation of n as a sum of positive integers, called parts. The number of partitions of a given n is denoted $p(n)$.

For example, $p(4) = 5$:

- 4
- $3 + 1$
- $2 + 2$
- $2 + 1 + 1$
- $1 + 1 + 1 + 1$

Partitions

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^m}.$$

The sequence for $p(n)$ begins

$$(p(n))_{n\geq 0} = (1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 57, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, ...)$$

What kind of arithmetic properties does $p(n)$ have?

Ramanujan's Congruences

$$(p(n))_{n\geq 0} = (1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 57, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, ...)$$

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- $p(5n+4) \equiv 0 \pmod{5}$.

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- $p(5n+4) \equiv 0 \pmod{5}$.
- $p(25n+24) \equiv 0 \pmod{25}$.

Ramanujan's Congruences

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$$176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, ...)$$

- $p(5n+4) \equiv 0 \pmod{5}$.
- $p(25n+24) \equiv 0 \pmod{25}$.
- $p(125n+99) \equiv 0 \pmod{125}$.

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- $p(5n + 4) \equiv 0 \pmod 5$.
- $p(25n + 24) \equiv 0 \pmod{25}$.
- $p(125n + 99) \equiv 0 \pmod{125}$.

Theorem (Ramanujan, 1918)

Let $n, \alpha \in \mathbb{Z}_{\geq 0}$ such that $24n \equiv 1 \pmod{5^\alpha}$. Then

$$p(n) \equiv 0 \pmod{5^\alpha}.$$

$d_k(n)$: k-Elongated Plane Partitions of n

Define $D_k(q)$ by

$$D_k(q) := \sum_{n=0}^{\infty} d_k(n) q^n = \prod_{m=1}^{\infty} \frac{(1-q^{2m})^k}{(1-q^m)^{3k+1}}.$$

Here $d_k(n)$ counts the number of k-elongated plane partitions of n.

$d_k(n)$: k-Elongated Plane Partitions of n

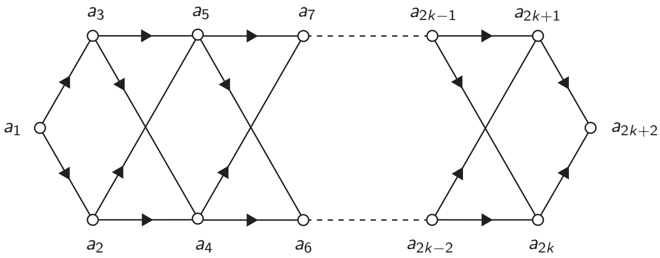

Figure: A length 1 k-elongated partition diamond.

- $a_j \in \mathbb{Z}_{\geq 0}$
- $a_b \to a_c$ indicates that $a_b \geq a_c$
- $a_1 + a_{2k+2} = n$

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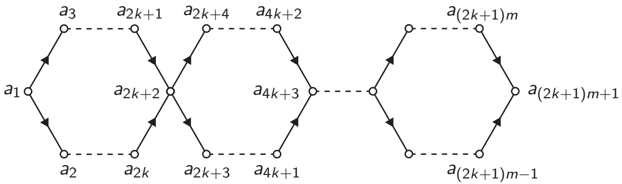

Figure: A length m k-elongated partition diamond.

- $a_j \in \mathbb{Z}_{\geq 0}$
- $a_b \to a_c$ indicates that $a_b \geq a_c$
- $a_1 + a_{2k+2} + ... + a_{(2k+1)m+1} = n$

Notice that $d_0(n) = p(n)$.

Congruences on $d_2(n)$

$$(d_2(n))_{n\geq 0} = (1, 7, 33, 126, 419, 1260, 3509, 9185, 22842, 54395, 124784, 277059, 597644, 1256341, 2580363, 5189185, 10236710, 19840410, 37832553, 71060190, 131610897, ...)$$

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Theorem (G.E. Andrews, P. Paule, 2021)

For all $n \geq 0$,

$$d_2(3n+2) \equiv 0 \pmod 3,$$
$$d_2(9n+8) \equiv 0 \pmod 9,$$
$$d_2(27n+17) \equiv 0 \pmod{27}.$$

Congruences on $d_2(n)$

Conjecture (G.E. Andrews, P. Paule, 2021)

Let $n, \alpha \in \mathbb{Z}_{\geq 0}$ such that $8n \equiv 1 \pmod{3^\alpha}$. Then

$$d_2\left(n\right) \equiv 0 \pmod{3^\alpha}.$$

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Theorem (Me, about a week after the conjecture was announced)

Let $n, \alpha \in \mathbb{Z}_{\geq 0}$ such that $8n \equiv 1 \pmod{3^\alpha}$. Then

$$d_2(n) \equiv 0 \pmod{3^{2\lfloor \alpha/2 \rfloor + 1}}.$$

Congruences on $d_5(n)$

$$(d_5(n))_{n\geq 0} = \big(1, 16, 147, 1008, 5705, 28080, 124054, 502336, 1892211, 6703200, 22519756, 72222192, 222280253, 659381856, 1892107005, 5268028752, 14268267146, \ldots\big)$$

Congruences on $d_5(n)$

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Theorem (da Silva, Hirschhorn, Sellers, 2021)

For all $j, n \geq 0$,

$$d_{5j+5}(5n+4) \equiv 0 \pmod 5.$$

Note that $d_5(5n+4) \equiv 0 \pmod 5$.

Banerjee's Congruences

This was conjectured by Koustav Banerjee:

Theorem (Banerjee, Me, 2022)

Let $n, \alpha \in \mathbb{Z}_{\geq 1}$ *such that* $4n \equiv 1 \pmod{5^{\alpha}}$*. Then*

$$d_5(n) \equiv 0 \pmod{5^{\lfloor \alpha/2 \rfloor + 1}}.$$

Ramanujan's Congruences

Theorem

Let $n, \alpha \in \mathbb{Z}_{\geq 0}$ *such that* $24n \equiv 1 \pmod{5^\alpha}$. *Then*

$$p(n) \equiv 0 \pmod{5^\alpha}.$$

$$L_\alpha := \Phi_\alpha \cdot \sum_{24n \equiv 1 \bmod 5^\alpha} p(n) q^{\lfloor n/5^\alpha \rfloor + 1}$$

$$L_1 = 5t$$

$$L_2 = 1575t + 162500t^2 + 4921875t^3 + 58593750t^4 + 244140625t^5$$

$$t = q \prod_{m=1}^{\infty} \left(\frac{1 - q^{5m}}{1 - q^m} \right)$$

Ramanujan's Congruences

Theorem

Let $n, \alpha \in \mathbb{Z}_{\geq 0}$ *such that* $24n \equiv 1 \pmod{5^\alpha}$. *Then*

$$p(n) \equiv 0 \pmod{5^\alpha}.$$

There exist operators $U^{(0)}$, $U^{(1)}$ such that

$$U^{(1)}(L_{2\alpha-1}) = L_{2\alpha},$$
$$U^{(0)}(L_{2\alpha}) = L_{2\alpha+1}.$$

By induction, we can prove

$$\frac{1}{5^\alpha} L_\alpha \in \mathbb{Z}[t].$$

Banerjee's Congruences

Theorem

Let $n, \alpha \in \mathbb{Z}_{\geq 1}$ such that $4n \equiv 1 \pmod{5^\alpha}$. Then

$$d_5(n) \equiv 0 \pmod{5^{\lfloor \alpha/2 \rfloor + 1}}.$$

$$L_\alpha := \Phi_\alpha \cdot \sum_{4n \equiv 1 \bmod 5^\alpha} d_5(n) q^{\lfloor n/5^\alpha \rfloor + 2/\gcd(\alpha,2)}.$$

First Example

$$
\begin{aligned}
L_1 = & \frac{1}{(1+5x)^6}\Big(5705x^2 + 6840120x^3 + 2034152125x^4 + 280484938650x^5 + 22921365211325x^6 + 1260917405154520x^7 \\
& + 50400843190048480x^8 + 1539115922208139200x^9 + 37183654303328448000x^{10} + 728924483359472640000x^{11} \\
& + 11816089262411136000000x^{12} + 16068144062805888000000x^{13} + 185329113419326464000000x^{14} \\
& + 1828416072736280985600000x^{15} + 1552867930100866252800000x^{16} + 114065722250547200000000000x^{17} \\
& + 726989442021507072000000000x^{18} + 4027764727740497920000000000x^{19} \\
& + 19409918786464645120000000000x^{20} + 81305458119372963840000000000x^{21} \\
& + 295454515024153804800000000000x^{22} + 928200573075849216000000000000x^{23} \\
& + 2508095187520061440000000000000x^{24} + 5787252595831603200000000000000x^{25} \\
& + 11291602030952448000000000000000x^{26} + 18381288507441152000000000000000x^{27} \\
& + 24508222899486720000000000000000x^{28} + 26072545283276800000000000000000x^{29} \\
& + 21283710435328000000000000000000x^{30} + 12519829667840000000000000000000x^{31} \\
& + 4724464025600000000000000000000x^{32} + 858993459200000000000000000000x^{33}\Big).
\end{aligned}
$$

$$
x = q\prod_{m=1}^{\infty}\frac{(1-q^{2m})(1-q^{10m})^3}{(1-q^m)^3(1-q^{5m})}
$$

Main Theorem

Theorem

Define

$$\psi := \psi(\alpha) := \left\lfloor \frac{5^{\alpha+1}}{4} \right\rfloor + 1 - \gcd(\alpha, 2),$$
$$\beta := \beta(\alpha) = \lfloor \alpha/2 \rfloor + 1.$$

Then for all $\alpha \geq 1$, we have

$$\frac{(1+5x)^{\psi}}{5^{\beta}} \cdot L_{\alpha} \in \mathbb{Z}[x].$$

From this, Banerjee's congruences follow.

L_α

$$L_1 = \frac{1}{(1+5x)^6}\cdot\Big(5705x^2 + 6840120x^3 + \ldots + 858993459200000000000000000000x^{33}\Big).$$

We want to express

$$L_\alpha = \sum_{m\geq 1} s(m)\cdot 5^{\theta_i(m)}\cdot\frac{x^m}{(1+5x)^n},$$

with $n\in\mathbb{Z}_{\geq 1}$ fixed, s, θ_i integer-valued functions, s discrete, and $i = 0, 1$ depending on the parity of α.

U Operator

$$L_\alpha = \sum_{m\geq 1} s(m)\cdot 5^{\theta_i(m)}\cdot \frac{x^m}{(1+5x)^n},$$

There exist operators $U^{(0)}$, $U^{(1)}$ such that

$$U^{(1)}\left(L_{2\alpha-1}\right) = L_{2\alpha},$$
$$U^{(0)}\left(L_{2\alpha}\right) = L_{2\alpha+1}.$$

We need to study

$$U^{(i)}\left(\frac{x^m}{(1+5x)^n}\right).$$

Inheritance Mapping

Definition

$$\mathcal{V}_n^{(1)} := \left\{ \frac{1}{(1+5x)^n} \sum_{m\geq 2} s(m) \cdot 5^{\theta_1(m)} \cdot x^m : (s(m))_{m\geq 2} \in \ker(\Omega) \right\},$$

$$\mathcal{V}_n^{(0)} := \left\{ \frac{1}{(1+5x)^n} \sum_{m\geq 1} s(m) \cdot 5^{\theta_0(m)} \cdot x^m \right\}.$$

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$$\Omega : \bigoplus_{m=2}^{\infty} \mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}^2,$$

$$: \mathbf{s} \mapsto \begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0\dots \\ 0 & 0 & 1 & 0 & 4 & 4 & 4 & 0 & 0 & 0\dots \end{pmatrix} \cdot \mathbf{s}.$$

Ω is the associated *inheritance mapping*.

Stability Within Inheritance Kernel

We have $L_1 \in \mathcal{V}_6^{(1)}$.

Theorem

Suppose $f \in \mathcal{V}_n^{(1)}$. Then

$$\frac{1}{5} \cdot U^{(1)}(f) \in \mathcal{V}_{5n}^{(0)},$$
$$\frac{1}{5} \cdot U^{(0)} \circ U^{(1)}(f) \in \mathcal{V}_{25n+6}^{(1)}.$$

From this, the Main theorem and Banerjee's congruences follow.

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- Analogues to Ω actually exist for every congruence family modulo powers of prime ℓ, e.g., the congruences for $p(n)$ by powers of 5.

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- When the level of the associated modular curve is ℓ, Ω is trivial.

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- When the level is $2\cdot\ell$, Ω has a form given in the family above.

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- Analogues to Ω actually exist for every congruence family modulo powers of prime ℓ, e.g., the congruences for $p(n)$ by powers of 5.
- When the level of the associated modular curve is ℓ, Ω is trivial.
- When the level is $2\cdot\ell$, Ω has a form given in the family above.
- When the level is $4\cdot\ell$, we don't yet understand Ω.

References

- G.E. Andrews, P. Paule, "MacMahon's Partition Analysis XIII: Schmidt Type Partitions and Modular Forms," *Journal of Number Theory* 234, pp. 95-119 (2022).
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