Combinatorial Statistics, Probability and Moment Sequences

Einar Steingrímsson
University of Strathclyde

Joint work with

Natasha Blitvić and Slim Kammoun
Lancaster / Toulouse
$\rightarrow$ QMUL

Many combinatorial sequences are moment sequences of probability measures on the real line.

Example: The number of perfect matchings:

$$(2n-1)!! = \int_{\mathbb{R}} x^{2n} \cdot \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

Many combinatorial sequences are moment sequences of probability measures on the real line.

Example: The number of perfect matchings:

$$(2n-1)!! = \int_{\mathbb{R}} x^{2n} \cdot \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

Equivalently, the *Hankel determinants* of the sequence $1, 3, 15, 105, 945 \ldots$ are all positive, and the first one is 1:

$$\begin{pmatrix}1\end{pmatrix} \quad \begin{pmatrix}1 & 3\\ 3 & 15\end{pmatrix} \quad \begin{pmatrix}1 & 3 & 15\\ 3 & 15 & 105\\ 15 & 105 & 945\end{pmatrix} \quad \begin{pmatrix}1 & 3 & 15 & 105\\ 3 & 15 & 105 & 945\\ 15 & 105 & 945 & 10395\\ 105 & 945 & 10395 & 135135\end{pmatrix}$$

$$1 \qquad 6 \qquad 720 \qquad 3628800$$

Many combinatorial sequences are moment sequences of probability measures on the real line.

Example: The Catalan numbers:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \int_{-2}^{2} x^{2n} \cdot \frac{\sqrt{4-x^2}}{2\pi}$$

Many combinatorial sequences are moment sequences of probability measures on the real line.

Example: The Catalan numbers:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \int_{-2}^{2} x^{2n} \cdot \frac{\sqrt{4-x^2}}{2\pi}$$

Equivalently, the *Hankel determinants* of the sequence $1, 1, 2, 5, 14, 42, 132, 429, \ldots$ are all positive:

$$\begin{pmatrix}1\end{pmatrix} \quad \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix} \quad \begin{pmatrix}1 & 1 & 2\\ 1 & 2 & 5\\ 2 & 5 & 14\end{pmatrix} \quad \begin{pmatrix}1 & 1 & 2 & 5\\ 1 & 2 & 5 & 14\\ 2 & 5 & 14 & 42\\ 5 & 14 & 42 & 132\end{pmatrix}$$

$$1 \qquad 1 \qquad 1 \qquad 1$$

Andrew Elvey Price (June 2022): Of the 354,910 sequences in the OEIS, 16,595 (4.7%) may be moment sequences.

Andrew Elvey Price (June 2022): Of the 354,910 sequences in the OEIS, 16,595 (4.7%) may be moment sequences.

Which combinatorial sequences are moment sequences?

Andrew Elvey Price (June 2022): Of the 354,910 sequences in the OEIS, 16,595 (4.7%) may be moment sequences.

Which combinatorial sequences are moment sequences?

trees on labeled nodes (n^{n-2}):

1, 1, 3, 16, 125, 1296, 16807, 262144, 4782969, . . .

trees on unlabeled nodes:

1, 1, 2, 3, 6, 11, 23, 47, 106, 235, 551, 1301, 3159, . . .

Andrew Elvey Price (June 2022): Of the 354,910 sequences in the OEIS, 16,595 (4.7%) may be moment sequences.

Which combinatorial sequences are moment sequences?

trees on labeled nodes (n^{n-2}):

1, 1, 3, 16, 125, 1296, 16807, 262144, 4782969, ... ✓

trees on unlabeled nodes:

1, 1, 2, 3, 6, 11, 23, 47, 106, 235, 551, 1301, 3159, ... ✗

Andrew Elvey Price (June 2022): Of the 354,910 sequences in the OEIS, 16,595 (4.7%) may be moment sequences.

Which combinatorial sequences are moment sequences?

trees on labeled nodes (n^{n-2}):

1, 1, 3, 16, 125, 1296, 16807, 262144, 4782969, . . . ✓

trees on unlabeled nodes:

1, 1, 2, 3, 6, 11, 23, 47, 106, 235, 551, 1301, 3159, . . . ✗

permutations avoiding consecutive 123:

1, 1, 2, 5, 17, 70, 349, 2017, 13358, 99377, 822041, . . .

permutations avoiding consecutive 132:

1, 1, 2, 5, 16, 63, 296, 1623, 10176, 71793, 562848, . . .

Andrew Elvey Price (June 2022): Of the 354,910 sequences in the OEIS, 16,595 (4.7%) may be moment sequences.

Which combinatorial sequences are moment sequences?

trees on labeled nodes (n^{n-2}):

1, 1, 3, 16, 125, 1296, 16807, 262144, 4782969, ... ✓

trees on unlabeled nodes:

1, 1, 2, 3, 6, 11, 23, 47, 106, 235, 551, 1301, 3159, ... ✗

permutations avoiding consecutive 123:

1, 1, 2, 5, 17, 70, 349, 2017, 13358, 99377, 822041, ... ✓

permutations avoiding consecutive 132:

1, 1, 2, 5, 16, 63, 296, 1623, 10176, 71793, 562848, ... ✗

Andrew Elvey Price (June 2022): Of the 354,910 sequences in the OEIS, 16,595 (4.7%) may be moment sequences.

Which combinatorial sequences are moment sequences?

trees on labeled nodes (n^{n-2}):

1, 1, 3, 16, 125, 1296, 16807, 262144, 4782969, ... ✓

trees on unlabeled nodes:

1, 1, 2, 3, 6, 11, 23, 47, 106, 235, 551, 1301, 3159, ... ✗

permutations avoiding consecutive 123:

1, 1, 2, 5, 17, 70, 349, 2017, 13358, 99377, 822041, ... ✓

permutations avoiding consecutive 132:

1, 1, 2, 5, 16, 63, 296, 1623, 10176, 71793, 562848, ... ✗

C_n: 1, 1, 2, 5, 14, 42, 132, 329, 1430, 4862, 16796, ... ✓

Which combinatorial sequences are moment sequences?

Which combinatorial sequences are moment sequences?

Can we find structural properties of our combinatorial objects that determine positivity?

Which combinatorial sequences are moment sequences?

Can we find structural properties of our combinatorial objects that determine positivity?

Is this interesting?

Which combinatorial sequences are moment sequences?

Can we find structural properties of our combinatorial objects that determine positivity?

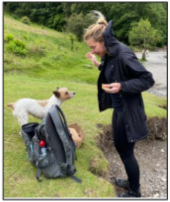
Is this interesting?

Depends on the answer(s) . . . ☺

Heilbronn Institute for Mathematical Research

Positivity Problems Associated to Permutation Patterns – June 6-10, 2022

Mathematics & Statistics | Lancaster University



There are two parts to these talks:

- A large and diverse family of combinatorial sequences, captured by a single multivariate continued fraction that guarantees they are all moment sequences:

  Permutations, set partitions, perfect matchings, colored permutations, . . .

  Bonus: A "new" family of combinatorial objects with many nice properties but mostly unstudied so far.

- A large uniform family of sequences we conjecture to be moment sequences:

  Permutations covered by occurrences of consecutive patterns.

In 1979 Françon and Viennot came up with a way to keep track of four statistics on permutations simultaneously:

peaks, valleys, double ascents, double descents

3 1 6 7 9 4 8 5 2

Flajolet (1980) then showed how the encoding of permutations F-W used for this was captured by a continued fraction.

In 1979 Françon and Viennot came up with a way to keep track of four statistics on permutations simultaneously:

peaks, valleys, double ascents, double descents

3 1 6 7 9 4 8 5 2

Flajolet (1980) then showed how the encoding of permutations F-W used for this was captured by a continued fraction.

Which is equivalent to putting certain weights on Motzkin paths.

He then gave a very general correspondence between labeled Motzkin paths and continued fractions.

In 1979 Françon and Viennot came up with a way to keep track of four statistics on permutations simultaneously:

peaks, valleys, double ascents, double descents

3 1 6 7 9 4 8 5 2

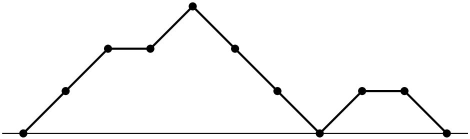
Flajolet (1980) then showed how the encoding of permutations F-W used for this was captured by a continued fraction.

Which is equivalent to putting certain weights on Motzkin paths.

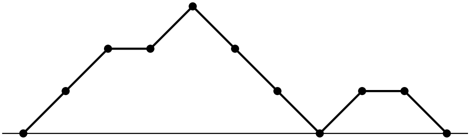
He then gave a very general correspondence between labeled Motzkin paths and continued fractions.

Flajolet's paper *Combinatorial aspects of continued fractions* is truly one of the great papers of combinatorics.

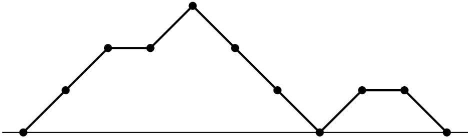
A *Motzkin path* is a sequence of up, down and level steps, starting at $(0,0)$, ending at $(n,0)$, never going below the x-axis:



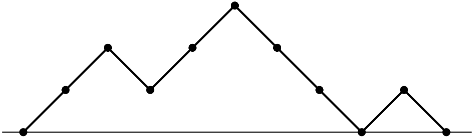
A *Motzkin path* is a sequence of (1,1), (1,0) and (1,-1) steps, starting at $(0, 0)$, ending at $(n, 0)$, never going below the x-axis:



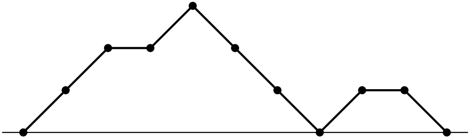
A *Motzkin path* is a sequence of up, down and level steps, starting at $(0,0)$, ending at $(n,0)$, never going below the x-axis:



A *Dyck path* is a Motzkin path with no level steps:

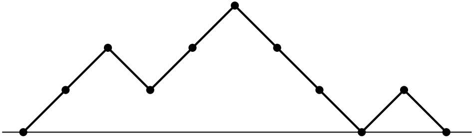


A *Motzkin path* is a sequence of up, down and level steps, starting at $(0, 0)$, ending at $(n, 0)$, never going below the x-axis:



$$\frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}$$

A *Dyck path* is a Motzkin path with no level steps:

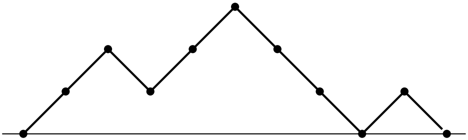


$$\frac{1 - \sqrt{1 - 4x^2}}{2x^2}$$

The Catalan numbers count Dyck paths, whose generating function is

$$C(x) = \frac{1-\sqrt{1-4x^2}}{2x^2} = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^{2n}$$

which satisfies $C = 1 + x^2 C^2$,

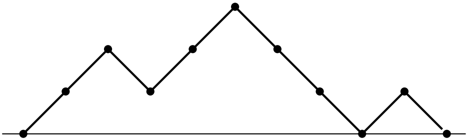


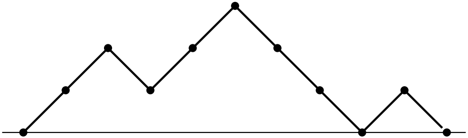
The Catalan numbers count Dyck paths, whose generating function is

$$C(x) \;=\; \frac{1-\sqrt{1-4x^2}}{2x^2} = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^{2n}$$

which satisfies $\quad C \;=\; 1 + x^2 C^2,$

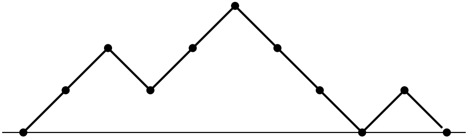
from which it follows that $\quad C(x) \;=\; \cfrac{1}{1-\cfrac{x^2}{1-\cfrac{x^2}{\ddots}}}$





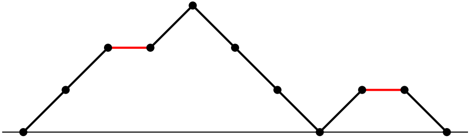
Dyck path

$$\cfrac{1}{1-\cfrac{x^2}{1-\cfrac{x^2}{\ddots}}}$$



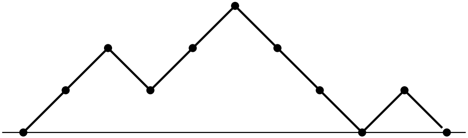
Dyck path

$$\cfrac{1}{1-\cfrac{x^2}{1-\cfrac{x^2}{\ddots}}}$$



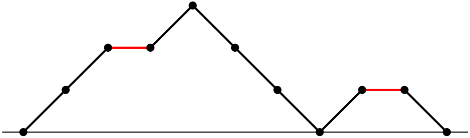
Motzkin path

$$\cfrac{1}{1-z-\cfrac{z^2}{1-z-\cfrac{z^2}{\ddots}}}$$



Dyck path

$$\cfrac{1}{1-\cfrac{x^2}{1-\cfrac{x^2}{\ddots}}}$$



Motzkin path

$$\cfrac{1}{1{-}z-\cfrac{z^2}{1{-}z-\cfrac{z^2}{\ddots}}}$$

In the continued fraction representation the level steps are directly visible.

$A = a_0, a_1, \ldots$ is a *Hamburger* moment sequence of a (positive) measure ρ on the real line if

$$a_n = \int_{\mathbb{R}} x^n \, d\rho(x)$$

$A = a_0, a_1, \ldots$ is a *Hamburger* moment sequence of a (positive) measure ρ on the real line if

$$a_n = \int_{\mathbb{R}} x^n \, d\rho(x)$$

Equivalently, the Hankel determinants are all positive (or positive for $n \leq N$ and 0 for $n > N$).

$A = a_0, a_1, \ldots$ is a *Hamburger* moment sequence of a (positive) measure ρ on the real line if

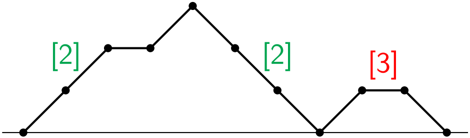
$$a_n = \int_{\mathbb{R}} x^n \, d\rho(x)$$

Equivalently, the Hankel determinants are all positive (or positive for $n \leq N$ and 0 for $n > N$).

Equivalently, there are real numbers β_i and α_i such that

$$\sum_{n \geq 0} a_n z^n = \cfrac{1}{1 - \alpha_0 z - \cfrac{\beta_1 z^2}{1 - \alpha_1 z - \cfrac{\beta_2 z^2}{\ddots}}}$$

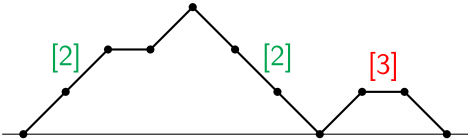
with $\beta_i > 0$ for all i (or all $i \leq N$ and 0 for $i > N$).



Weighted Motzkin path

$$\cfrac{1}{1 - z - \cfrac{z^2}{1 - 3z - \cfrac{2^2 z^2}{\ddots\, 1 - (2n+1)z - \cfrac{n^2 z^2}{\ddots}}}} = \sum_{n \geq 0} n! \cdot z^n$$

Special case of the general correspondence by Flajolet (1980).



Weighted Motzkin path

$$\cfrac{1}{1-\alpha_0 z-\cfrac{\beta_1 z^2}{1-\alpha_1 z-\cfrac{\beta_2 z^2}{\ddots\, 1-\alpha_n z-\cfrac{\beta_{n+1}z^2}{\ddots}}}}$$

where $\alpha_n(\cdot)$ has $\alpha_n(\mathbf{1}) = 2n+1$ and $\beta_n(\cdot)$ has $\beta_n(\mathbf{1}) = n^2$

Several papers have exploited Flajolet's 1980 correspondence to obtain distributions of various sets of permutations statistics:

Françon–Viennot 1979
Foata–Zeilberger 1990
Biane 1993
de Médicis–Viennot 1994
Simion–Stanton 1994
Clarke–Steingrímsson–Zeng 1996
Randrianarivony 1998
Elizalde 2018

Most recently:

Blitvić–Steingrímsson 2021
Sokal–Zeng 2022

Our Continued Fraction

For parameters $a, b, c, d, f, g, h, \ell, p, r, s, t, u, w \in \mathbb{R}$, let

$$\mathcal{C}(z) = \cfrac{1}{1 - \alpha_0 z - \cfrac{\beta_1 z^2}{1 - \alpha_1 z - \cfrac{\beta_2 z^2}{\ddots}}}$$

where

$$\alpha_n = u \cdot w^n + s\,[n]_{a,b} + t\,[n]_{f,g} \qquad \beta_n = p\,r\,[n]_{c,d}\,[n]_{h,\ell}$$

and $\quad [n]_{x,y} = x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}$

Our Continued Fraction

For parameters $a, b, c, d, f, g, h, \ell, p, r, s, t, u, w \in \mathbb{R}$, let

$$\mathcal{C}(z) = \cfrac{1}{1 - \alpha_0 z - \cfrac{\beta_1 z^2}{1 - \alpha_1 z - \cfrac{\beta_2 z^2}{\ddots}}}$$

where

$$\alpha_n = u \cdot w^n + s\,[n]_{a,b} + t\,[n]_{f,g} \qquad \beta_n = p\,r\,[n]_{c,d}\,[n]_{h,\ell}$$

The Plan: Find a bijection taking *permutations*, carrying lots of statistics, to Motzkin paths corresponding to $\mathcal{C}(z)$, using Flajolet's general correspondence.

Consider Motzkin paths labeled as follows, where $0 \leq i < k$

- Upsteps from height $k-1$ to k have labels $pc^i d^{k-1-i}$
- Downsteps from height k to $k-1$ have labels $rh^i \ell^{k-1-i}$
- Level steps at height k have labels in

$$\{u \cdot w^k\} \cup \{s\, a^i b^{k-1-i}\} \cup \{t\, f^i g^{k-1-i}\}.$$

Consider Motzkin paths labeled as follows, where $0 \leq i < k$

- Upsteps from height $k-1$ to k have labels $pc^i d^{k-1-i}$
- Downsteps from height k to $k-1$ have labels $rh^i \ell^{k-1-i}$
- Level steps at height k have labels in

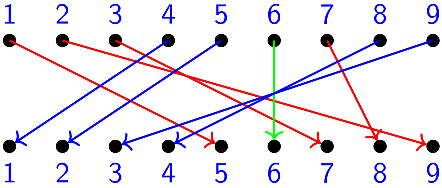
$$\{u \cdot w^k\} \;\cup\; \{s\, a^i b^{k-1-i}\} \;\cup\; \{t\, f^i g^{k-1-i}\}.$$

By Flajolet's correspondence, $\mathcal{C}(z)$ is the generating function for Motzkin paths thus labeled:

$$\mathcal{C}(z) = \cfrac{1}{\ddots\, 1 - (u \cdot w^n + s\,[n]_{a,b} + t\,[n]_{f,g})\, z - \cfrac{p\, r\, [n+1]_{c,d}\, [n+1]_{h,\ell}\, z^2}{\ddots}}$$

Fourteen statistics on permutations $\sigma(1)\sigma(2)\ldots\sigma(n)$, based on *excedances* and *inversions*:

$\sigma(i)$: 5 9 7 1 2 6 8 4 3
i: 1 2 3 4 5 6 7 8 9



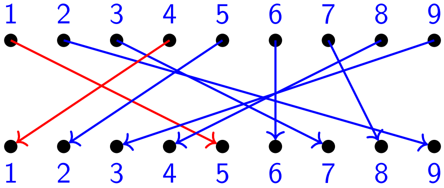
Excedances red

Anti-excedances blue

Fixed points green

Fourteen statistics on permutations $\sigma(1)\sigma(2)\ldots\sigma(n)$, based on *excedances* and *inversions*:

$\sigma(i)$: 5 9 7 1 2 6 8 4 3
i: 1 2 3 4 5 6 7 8 9



Excedances red

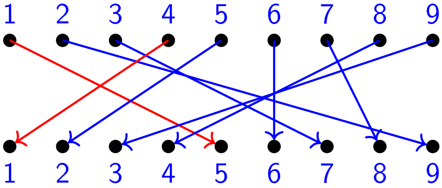
Anti-excedances blue

Fixed points green

One of the inversions red (crossing)

Fourteen statistics on permutations $\sigma(1)\sigma(2)\ldots\sigma(n)$, based on *excedances* and *inversions*:

$\sigma(i)$: 5 9 7 1 2 6 8 4 3
i: 1 2 3 4 5 6 7 8 9



Excedances red

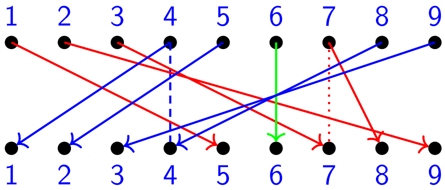
Anti-excedances blue

Fixed points green

One of the inversions red (crossing)

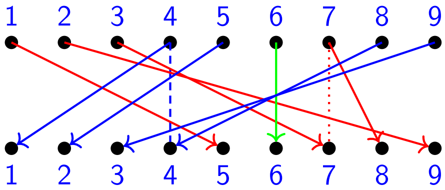
But this gets more complicated . . .

5 9 7 1 2 6 8 4 3
1 2 3 4 5 6 7 8 9



7 is a *linked* excedance: $8 = \sigma(7) > 7 > \sigma^{-1}(7) = 3$

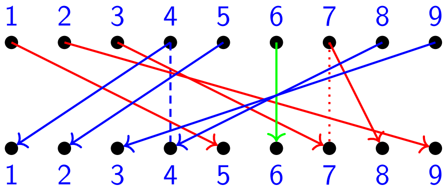
5 9 7 1 2 6 8 4 3
1 2 3 4 5 6 7 8 9



7 is a *linked* excedance: $8 = \sigma(7) > 7 > \sigma^{-1}(7) = 3$

4 is a *linked* anti-excedance: $1 = \sigma(4) < 4 < \sigma^{-1}(4) = 9$

5 9 7 1 2 6 8 4 3
1 2 3 4 5 6 7 8 9



7 is a *linked* excedance: $8 = \sigma(7) > 7 > \sigma^{-1}(7) = 3$

4 is a *linked* anti-excedance: $1 = \sigma(4) < 4 < \sigma^{-1}(4) = 9$

$9 \cdots 6$ is an inversion between excedance and fixed point

1. # excedances as $\mathrm{exc}(\sigma) := \#\{i \in [n] \mid i < \sigma(i)\}$,
2. # fixed points as $\mathrm{fp}(\sigma) := \#\{i \in [n] \mid i = \sigma(i)\}$,
3. # anti-excedances as $\mathrm{aexc}(\sigma) := \#\{i \in [n] \mid i > \sigma(i)\}$,
4. # linked excedances as $\mathrm{le}(\sigma) := \#\{i \in [n] \mid \sigma^{-1}(i) < i < \sigma(i)\}$,
5. # linked anti-excedances as $\mathrm{lae}(\sigma) := \#\{i \in [n] \mid \sigma^{-1}(i) > i > \sigma(i)\}$.
6. # inversions between excedances: $\mathrm{ie}(\sigma) := \#\{i,j \in [n] \mid i < j < \sigma(j) < \sigma(i)\}$.
7. # inversions between excedances where the greater excedance is linked:$\mathrm{ile}(\sigma) := \#\{i,j \in [n] \mid i < j < \sigma(j) < \sigma(i) \text{ and } \sigma^{-1}(j) < j\}$.
8. # restricted non-inversions between excedances:$\mathrm{nie}(\sigma) := \#\{i,j \in [n] \mid i < j < \sigma(i) < \sigma(j)\}$.
9. # restricted non-inversions between excedances where the rightmost excedance is linked: $\mathrm{nile}(\sigma) := \#\{i,j \in [n] \mid i < j < \sigma(i) < \sigma(j) \text{ and } \sigma^{-1}(j) < j\}$.
10. # inversions between anti-excedances: $\mathrm{iae}(\sigma) := \#\{i,j \in [n] \mid j > i > \sigma(i) > \sigma(j)\}$.
11. # inversions between anti-excedances where the smaller anti-excedance is linked: $\mathrm{ilae}(\sigma) := \#\{i,j \in [n] \mid j > i > \sigma(i) > \sigma(j) \text{ and } \sigma^{-1}(i) > i\}$.
12. # restricted non-inversions between anti-excedances: $\mathrm{niae}(\sigma) := \#\{i,j \in [n] \mid j > i > \sigma(j) > \sigma(i)\}$.
13. # restricted non-inversions between anti-excedances where the smaller anti-excedance is linked: $\mathrm{nilae}(\sigma) := \#\{i,j \in [n] \mid j > i > \sigma(j) > \sigma(i) \text{ and } \sigma^{-1}(i) > i\}$.
14. # inversions between excedances and fixed points: $\mathrm{iefp}(\sigma) := \#\{i,j \in [n] \mid i < j = \sigma(j) < \sigma(i)\}$.

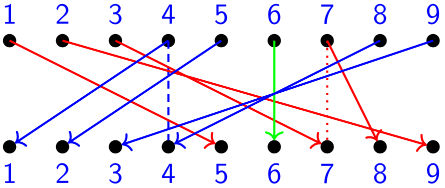
1. # excedances as $\mathrm{exc}(\sigma) := \#\{i \in [n] \mid i < \sigma(i)\}$,
2. # fixed points as $\mathrm{fp}(\sigma) := \#\{i \in [n] \mid i = \sigma(i)\}$,
3. # anti-excedances as $\mathrm{aexc}(\sigma) := \#\{i \in [n] \mid i > \sigma(i)\}$,
4. # linked excedances as $\mathrm{le}(\sigma) := \#\{i \in [n] \mid \sigma^{-1}(i) < i < \sigma(i)\}$,
5. # linked anti-excedances as $\mathrm{lae}(\sigma) := \#\{i \in [n] \mid \sigma^{-1}(i) > i > \sigma(i)\}$.
6. # inversions between excedances: $\mathrm{ie}(\sigma) := \#\{i, j \in [n] \mid i < j < \sigma(j) < \sigma(i)\}$.
7. # inversions between excedances where the greater excedance is linked:$\mathrm{ile}(\sigma) := \#\{i, j \in [n] \mid i < j < \sigma(j) < \sigma(i)$ and $\sigma^{-1}(j) < j\}$.
8. # restricted non-inversions between excedances:$\mathrm{nie}(\sigma) := \#\{i, j \in [n] \mid i < j < \sigma(i) < \sigma(j)\}$.
9. # restricted non-inversions between excedances where the rightmost excedance is linked: $\mathrm{nile}(\sigma) := \#\{i, j \in [n] \mid i < j < \sigma(i) < \sigma(j)$ and $\sigma^{-1}(j) < j\}$.
10. # inversions between anti-excedances: $\mathrm{iae}(\sigma) := \#\{i, j \in [n] \mid j > i > \sigma(i) > \sigma(j)\}$.
11. # inversions between anti-excedances where the smaller anti-excedance is linked: $\mathrm{ilae}(\sigma) := \#\{i, j \in [n] \mid j > i > \sigma(i) > \sigma(j)$ and $\sigma^{-1}(i) > i\}$.
12. # restricted non-inversions between anti-excedances: $\mathrm{niae}(\sigma) := \#\{i, j \in [n] \mid j > i > \sigma(j) > \sigma(i)\}$.
13. # restricted non-inversions between anti-excedances where the smaller anti-excedance is linked: $\mathrm{nilae}(\sigma) := \#\{i, j \in [n] \mid j > i > \sigma(j) > \sigma(i)$ and $\sigma^{-1}(i) > i\}$.
14. # inversions between excedances and fixed points: $\mathrm{iefp}(\sigma) := \#\{i, j \in [n] \mid i < j = \sigma(j) < \sigma(i)\}$.

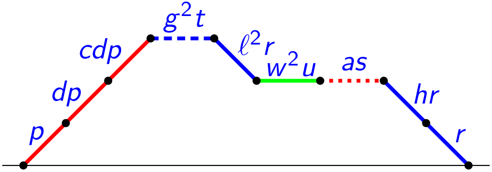
1. # excedances as $\text{exc}(\sigma) := \#\{i \in [n] \mid i < \sigma(i)\}$,
2. # fixed points as $\text{fp}(\sigma) := \#\{i \in [n] \mid i = \sigma(i)\}$,
3. # anti-excedances as $\text{aexc}(\sigma) := \#\{i \in [n] \mid i > \sigma(i)\}$,
4. # linked excedances as $\text{le}(\sigma) := \#\{i \in [n] \mid \sigma^{-1}(i) < i < \sigma(i)\}$,
5. # linked anti-excedances as $\text{lae}(\sigma) := \#\{i \in [n] \mid \sigma^{-1}(i) > i > \sigma(i)\}$.
6. # inversions between excedances: $\text{ie}(\sigma) := \#\{i,j \in [n] \mid i < j < \sigma(j) < \sigma(i)\}$.
7. # inversions between excedances where the greater excedance is linked:$\text{ile}(\sigma) := \#\{i,j \in [n] \mid i < j < \sigma(j) < \sigma(i) \text{ and } \sigma^{-1}(j) < j\}$.
8. # restricted non-inversions between excedances:$\text{nie}(\sigma) := \#\{i,j \in [n] \mid i < j < \sigma(i) < \sigma(j)\}$.
9. # restricted non-inversions between excedances where the rightmost excedance is linked:
   $\text{nile}(\sigma) := \#\{i,j \in [n] \mid i < j < \sigma(i) < \sigma(j) \text{ and } \sigma^{-1}(j) < j\}$.
10. # inversions between anti-excedances:
    $\text{iae}(\sigma) := \#\{i,j \in [n] \mid j > i > \sigma(i) > \sigma(j)\}$.
11. # inversions between anti-excedances where the smaller anti-excedance is linked: $\text{ilae}(\sigma) := \#\{i,j \in [n] \mid j > i > \sigma(i) > \sigma(j) \text{ and } \sigma^{-1}(i) > i\}$.
12. # restricted non-inversions between anti-excedances:
    $\text{niae}(\sigma) := \#\{i,j \in [n] \mid j > i > \sigma(j) > \sigma(i)\}$.
13. # restricted non-inversions between anti-excedances where the smaller anti-excedance is linked:
    $\text{nilae}(\sigma) := \#\{i,j \in [n] \mid j > i > \sigma(j) > \sigma(i) \text{ and } \sigma^{-1}(i) > i\}$.
14. # inversions between excedances and fixed points:
    $\text{iefp}(\sigma) := \#\{i,j \in [n] \mid i < j = \sigma(j) < \sigma(i)\}$.

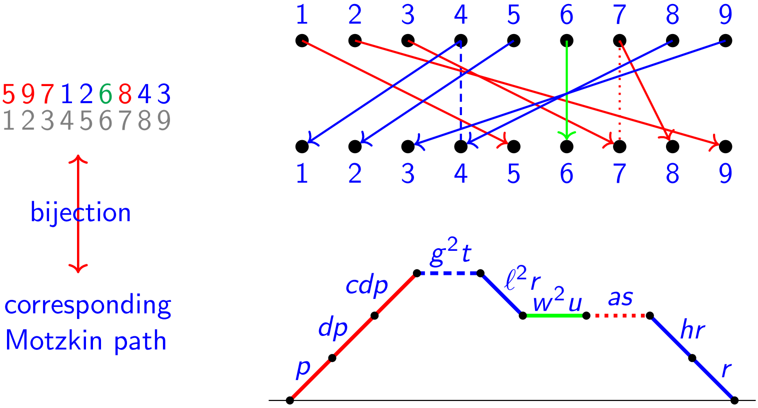
5 9 7 1 2 6 8 4 3
1 2 3 4 5 6 7 8 9

bijection

corresponding Motzkin path

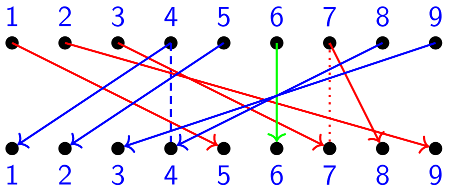






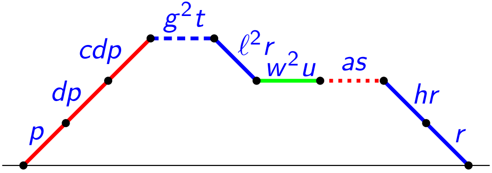
Weight of labeled Motzkin path, wt(M): Product of its labels

5 9 7 1 2 6 8 4 3
1 2 3 4 5 6 7 8 9



bijection

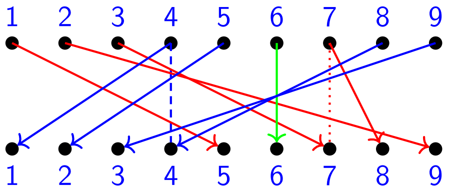
corresponding Motzkin path



wt: $a \cdot c \cdot d^2 \cdot g^2 \cdot h \cdot \ell^2 \cdot p^3 \cdot r^3 \cdot s \cdot t \cdot u \cdot w^2$

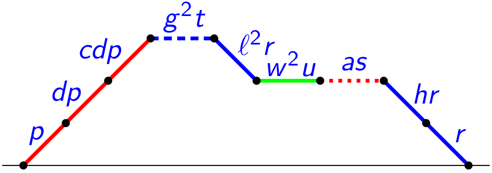
Weight of labeled Motzkin path, wt(M): Product of its labels

5 9 7 1 2 6 8 4 3
1 2 3 4 5 6 7 8 9



bijection

corresponding Motzkin path



wt: $a \cdot c \cdot d^2 \cdot g^2 \cdot h \cdot \ell^2 \cdot p^3 \cdot r^3 \cdot s \cdot t \cdot u \cdot w^2$

Above wt is one term in $[z^9]\mathcal{C}(z)$

The *weight* of a labeled Motzkin path M, $\mathrm{wt}(M)$, is the product of its labels.

Theorem: There is a bijection $\eta : \mathcal{S}_n \to \mathcal{M}_n$ such that if $M = \eta(\sigma)$ then $\mathrm{wt}(M)$ equals

$$\begin{aligned}\mathrm{stat}(\sigma) = \quad & a^{\mathrm{ile}(\sigma)} b^{\mathrm{nile}(\sigma)} c^{\mathrm{ie}(\sigma)-\mathrm{ile}(\sigma)} d^{\mathrm{nie}(\sigma)-\mathrm{nile}(\sigma)} \\ & \times f^{\mathrm{ilae}(\sigma)} g^{\mathrm{nilae}(\sigma)} h^{\mathrm{iae}(\sigma)-\mathrm{ilae}(\sigma)} \ell^{\mathrm{niae}(\sigma)-\mathrm{nilae}(\sigma)} \\ & \times p^{\mathrm{exc}(\sigma)-\mathrm{le}(\sigma)} r^{\mathrm{aexc}(\sigma)-\mathrm{lae}(\sigma)} s^{\mathrm{le}(\sigma)} t^{\mathrm{lae}(\sigma)} u^{\mathrm{fp}(\sigma)} w^{\mathrm{iefp}(\sigma)}\end{aligned}$$

Corollary: $\mathcal{C}(z) = \sum_{n \geq 0} \sum_{\sigma \in \mathcal{S}_n} \mathrm{stat}(\sigma) z^n.$

The *weight* of a labeled Motzkin path M, $\mathrm{wt}(M)$, is the product of its labels.

Theorem: There is a bijection $\eta : \mathcal{S}_n \to \mathcal{M}_n$ such that if $M = \eta(\sigma)$ then $\mathrm{wt}(M)$ equals

$$\begin{aligned}\mathrm{stat}(\sigma) = \quad & a^{\mathrm{ile}(\sigma)} b^{\mathrm{nile}(\sigma)} c^{\mathrm{ie}(\sigma)-\mathrm{ile}(\sigma)} d^{\mathrm{nie}(\sigma)-\mathrm{nile}(\sigma)} \\ & \times f^{\mathrm{ilae}(\sigma)} g^{\mathrm{nilae}(\sigma)} h^{\mathrm{iae}(\sigma)-\mathrm{ilae}(\sigma)} \ell^{\mathrm{niae}(\sigma)-\mathrm{nilae}(\sigma)} \\ & \times p^{\mathrm{exc}(\sigma)-\mathrm{le}(\sigma)} r^{\mathrm{aexc}(\sigma)-\mathrm{lae}(\sigma)} s^{\mathrm{le}(\sigma)} t^{\mathrm{lae}(\sigma)} u^{\mathrm{fp}(\sigma)} w^{\mathrm{iefp}(\sigma)}\end{aligned}$$

Corollary: $\mathcal{C}(z) = \sum_{n \geq 0} \sum_{\sigma \in \mathcal{S}_n} \mathrm{stat}(\sigma) z^n.$

In short: Weight of Motzkin path goes to 14-parameter statistic on corresponding permutation

There are several related bijections in earlier literature by

Françon-Viennot 1979
Foata-Zeilberger 1990
Biane 1993
de Médicis-Viennot 1994
Simion-Stanton 1994
Clarke-Steingrímsson-Zeng 1996
Randrianarivony 1998
Elizalde 2018

Our results generalize most of these, some modulo a bijection interchanging excedances and descents.

There are several related bijections in earlier literature by

Françon-Viennot 1979
Foata-Zeilberger 1990
Biane 1993
de Médicis-Viennot 1994
Simion-Stanton 1994
Clarke-Steingrímsson-Zeng 1996
Randrianarivony 1998
Elizalde 2018

Our results generalize most of these, some modulo a bijection interchanging excedances and descents.

In a contemporaneous paper, Sokal and Zeng (2022) present a framework similar to ours, but with an additional four statistics, including the *crossings* and *alignments* defined by Corteel.

There are several related bijections in earlier literature by

Françon-Viennot 1979
Foata-Zeilberger 1990
Biane 1993
de Médicis-Viennot 1994
Simion-Stanton 1994
Clarke-Steingrímsson-Zeng 1996
Randrianarivony 1998
Elizalde 2018

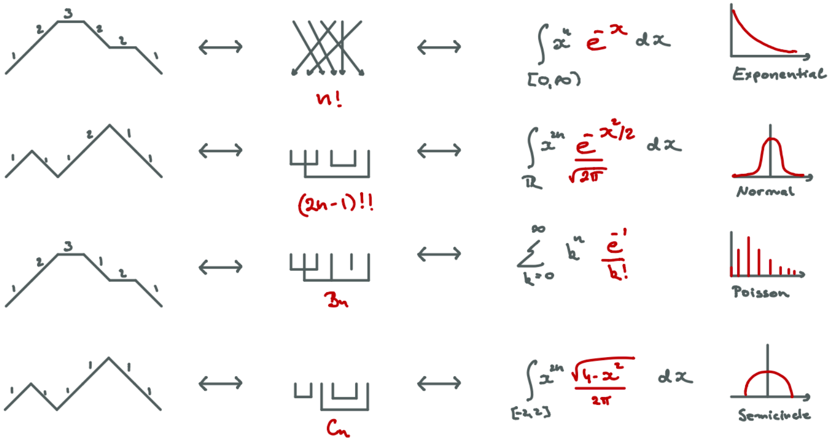
Our results generalize most of these, some modulo a bijection interchanging excedances and descents.

In a contemporaneous paper, Sokal and Zeng (2022) present a framework similar to ours, but with an additional four statistics, including the *crossings* and *alignments* defined by Corteel.

Of the above, only Biane, Elizalde and Sokal-Zeng separate fixed points from anti-excedances, as we do. This leads to greater symmetry in the continued fraction, and to results not otherwise obtainable.

The number sequences arising from $\mathcal{C}$ enumerate many different combinatorial structures, such as permutations, perfect matchings and set partitions.

These basic examples happen to be moment sequences of important distributions from probability theory.



Some refinements of these objects also have meaning in probability theory.

Which structures give something probabilistically meaningful?

Parameter settings	Combinatorial objects	Moment seq. (OEIS [OEI])	Measure
	Permutations	$n!$ (A000142)	Exponential: $e^{-x}\mathbb{1}_{[0,\infty)}\,dx$
$h,s,t,u=0$	Perfect matchings	$(2n-1)!!$ (A001147)	Gaussian*: $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx$
$c,h,s,t,u=0$	Non-crossing perfect matchings	$\frac{1}{n+1}\binom{2n}{n}$ (A000108) Catalan numbers	Wigner semicircle*: $\frac{1}{2\pi}\sqrt{4-x^2}\mathbb{1}_{[\lambda_-,\lambda_+]}\,dx$
$h,s,t,u=0;\ c=q$	Perfect matchings by #crossings	$\sum_{\pi\in\mathcal{P}_2(2n)} q^{\mathrm{cr}(\pi)}$ (A067311)	q-Gaussian* [BS91][Spe92]
$h,s,t,u=0;\ c=q;\ d=t$	Perfect matchings by #crossings & nestings	$\sum_{\pi\in\mathcal{P}_2(2n)} q^{\mathrm{cr}(\pi)}t^{\mathrm{nest}(\pi)}$	(q,t)-Gaussian* [Bli12][Bli14]
$h,t=0;\ p,u=\lambda$	Set partitions by #blocks	Stirling 2^{nd}: $\sum_{\pi\in\mathcal{P}(n)}\lambda^{\lvert\pi\rvert}$ (A008277)	Poisson, rate λ: $e^{-\lambda}\lambda^k/k!$
$a,c,h,t=0,\ p,u=\lambda$	Non-crossing set partitions of $[n]$ into k blocks	$\sum_k \frac{1}{n}\binom{n}{k}\binom{n}{k-1}\lambda^k$ Narayana numbers (A001263)	Free Poisson: $\lambda_\pm=(1\pm\sqrt{\lambda})^2, \lambda\geq 1$, $\frac{\sqrt{(\lambda_+-x)(x-\lambda_-)}}{2\pi x}\mathbb{1}_{[\lambda_-,\lambda_+]}\,dx$
$h,t=0;\ a,c=q;\ p,u=\lambda$	Restricted crossings in partitions [Bia97]	$\sum_{\pi\in\mathcal{P}(n)} q^{\mathrm{cr}(\pi)}\lambda^{\lvert\pi\rvert}$	q-Poisson, rate λ [Ans01]
$h,t,u=0;\ b,d=x;\ a,c=q$	Restricted cross/nest in partitions [KZ06]	$\sum_{\pi\in\mathcal{P}(n)} q^{\mathrm{cr}(\pi)}x^{\mathrm{nest}(\pi)}$	(q,t)-Poisson [Ejs20]
$u=0$	Derangements	A000166	e.g. [MK15]
$s,t,u=0$	Alternating permutations of $[2n]$	A000364	e.g. [Sok18]*
$a,c,f,h=0;\ p=2$	Little Schröder numbers	A001003	[MP13]
$a,u=0;\ t=2$	Permutations, no strong fixed points	A052186	[MK15]
$p,s=x$	Eulerian polynomials	$\sum_{\sigma\in\mathcal{S}_n} x^{\mathsf{des}(\sigma)}$ (A008292)	[Bar18][BM16]
$p,s=2x;\ r,t=2;$ $u=x+1$	Eulerian polynomials for hyperoctahedral groups	$\sum_{\sigma\in B_n} x^{\mathsf{des}(\sigma)}$ (A060187)	[Bar18][BM16]

Moment sequences

A sequence $a_0, a_1, a_2, \ldots$ is a moment sequence of a positive measure on the real line *if and only if* all principal minors of

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ & & \vdots & \\ a_n & a_{n+1} & \cdots & a_{2n} \end{pmatrix}$$

are non-negative for any n. (Hamburger, a 100 years ago)

Moment sequences

A sequence $a_0, a_1, a_2, \ldots$ is a moment sequence of a positive measure on the real line *if and only if* all principal minors of

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ & & \vdots & \\ a_n & a_{n+1} & \cdots & a_{2n} \end{pmatrix}$$

are non-negative for any n. (Hamburger, a 100 years ago)

In particular, $(a_n)_{n\geq 0}$ is then log-convex $\left(a_{n-1}a_{n+1} \geq a_n^2\right)$.

A sequence $a_0, a_1, a_2, \ldots$ is a moment sequence of a positive measure on the real line *if and only if* all principal minors of

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ & & \vdots & \\ a_n & a_{n+1} & \cdots & a_{2n} \end{pmatrix}$$

are non-negative for any n. (Hamburger, a 100 years ago)

In particular, $(a_n)_{n\geq 0}$ is then log-convex $\left(a_{n-1}a_{n+1} \geq a_n^2\right)$.

Can get strong lower bounds on growth rates of moment sequences (provided the α_i are positive).

(Haagerup–Haagerup–Ramirez-Solano,
Elvey Price, Clisby–Conway–Guttmann)

Moment sequences

$$\sum_{n\geq 0} m_n z^n \quad = \quad \mathcal{C}(z) = \cfrac{1}{1-\alpha_0 z - \cfrac{\beta_1 z^2}{1-\alpha_1 z - \cfrac{\beta_2 z^2}{\ddots}}}$$

$$\alpha_n = u\cdot w^n + s\,[n]_{a,b} + t\,[n]_{f,g} \qquad \beta_n = p\,r\,[n]_{c,d}\,[n]_{h,\ell}$$

Theorem: For $a,b,c,d,f,g,h,\ell,p,r,s,t,u,w \in \mathbb{R}$ with $pr>0$ and c,d,h,ℓ satisfying

$$c=-d \quad \textbf{or} \quad h=-\ell \quad \textbf{or}$$
$$(c>-d \text{ and } h>-\ell) \quad \textbf{or} \quad (c<-d \text{ and } h<-\ell),$$

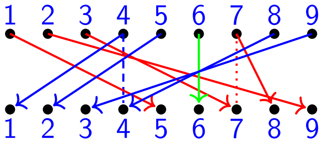
the sequence (m_n) is the moment sequence of some probability measure on $\mathbb{R}$. In particular if all non-negative and $pr>0$.

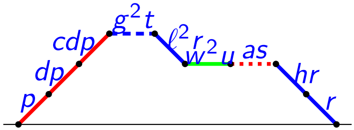
Moment sequences

$$\sum_{n\geq 0} m_n z^n \quad = \quad \mathcal{C}(z) = \cfrac{1}{1-\alpha_0 z - \cfrac{\beta_1 z^2}{1-\alpha_1 z - \cfrac{\beta_2 z^2}{\ddots}}}$$

$$\alpha_n = u\cdot w^n + s\,[n]_{a,b} + t\,[n]_{f,g} \qquad \beta_n = p\,r\,[n]_{c,d}\,[n]_{h,\ell}$$

With mild conditions on the parameters of $\mathcal{C}(z)$, which are easy to check, we get moment sequences.





$$\mathcal{C}(z) = \cfrac{1}{1 - \alpha_0 z - \cfrac{\beta_1 z^2}{1 - \alpha_1 z - \cfrac{\beta_2 z^2}{\ddots}}}$$

where

$$\alpha_n = u \cdot w^n + s\,[n]_{a,b} + t\,[n]_{f,g} \qquad \beta_n = p\,r\,[n]_{c,d}\,[n]_{h,\ell}$$

Here, u carries #fixed points, s carries #linked excedances, a carries #inversions among linked excedances, . . .

$$\mathcal{C}(z) = \cfrac{1}{\ddots\,1 - (u \cdot w^n + s\,[n]_{a,b} + t\,[n]_{f,g})\,z - \cfrac{p\,r\,[n+1]_{c,d}\,[n+1]_{h,\ell}\,z^2}{\ddots}}$$

With $s = qx$, $p = x$, all other parameters $= 1$, we get

$$\mathcal{C}(z) = \sum_{n \geq 0} \sum_{\sigma \in \mathcal{S}_n} x^{\mathrm{DES}(\sigma)} q^{\mathrm{occ}_{321}(\sigma)} z^n,$$

where occ_{321} is #occurrences of the consecutive pattern 321

occurrence: 356412 not consecutive: 356412

First shown by Elizalde 2018, using a different continued fraction.

$$\mathcal{C}(z) = \cfrac{1}{\ddots \; 1 - (u \cdot w^n + s\,[n]_{a,b} + t\,[n]_{f,g})\, z - \cfrac{p\, r\,[n+1]_{c,d}\,[n+1]_{h,\ell}\, z^2}{\ddots}}$$

With $s = 0$, all other parameters $= 1$, we get

$$\mathcal{C}(z) = \sum_{n \geq 0} \mathrm{Av}_{321}(n) z^n,$$

$\mathrm{Av}_{321}(n) = \#$ n-permutations *avoiding* consecutive pattern 321

occurrence: 356412 not consecutive: 356412

First shown by Elizalde 2018, using a different continued fraction.

With $s = 0$, all other parameters $= 1$, we get

$$\mathcal{C}(z) = \sum_{n \geq 0} \mathrm{Av}_{321}(n) z^n,$$

$\mathrm{Av}_{321}(n) = \#$ n-permutations avoiding consecutive pattern 321

With $s = 0$, all other parameters $= 1$, we get

$$\mathcal{C}(z) = \sum_{n \geq 0} \mathrm{Av}_{321}(n) z^n,$$

$\mathrm{Av}_{321}(n) = \#$ n-permutations avoiding consecutive pattern 321

If $b, d, g, \ell = q$, $s = xq$, $p, u = x$, others $= 1$:

$$\mathcal{C}(z) = \sum_{n \geq 0} \sum_{\sigma \in \mathcal{S}_n} x^{\mathrm{DES}(\sigma)+1} q^{\mathrm{occ}_{2\text{-}31}(\sigma)} z^n.$$

where $\mathrm{occ}_{2\text{-}31}$ is #occurrences of the *vincular* pattern 2–31

2–31 occurrence: 416523 62 not adjacent: 416523

First shown by Claesson-Mansour 2002, using different continued fraction.

With $s = 0$, all other parameters $= 1$, we get

$$\mathcal{C}(z) = \sum_{n \geq 0} \mathrm{Av}_{321}(n) z^n,$$

$\mathrm{Av}_{321}(n) = \#$ n-permutations avoiding consecutive pattern 321

If $b, d, g, \ell, s = 0$, others $= 1$:

$$\mathcal{C}(z) = \sum_{n \geq 0} \mathrm{Av}_{2\text{-}31}(n) z^n.$$

$\mathrm{Av}_{2\text{-}31}(n) = \#$ n-permutations avoiding vincular pattern 2–31

2–31 occurrence: 416523 62 not adjacent: 416523

First shown by Claesson-Mansour 2002, using different continued fraction.

With $s = 0$, all other parameters $= 1$, we get

$$\mathcal{C}(z) = \sum_{n \geq 0} \mathrm{Av}_{321}(n) z^n,$$

$\mathrm{Av}_{321}(n) = \#$ n-permutations avoiding consecutive pattern 321

If $b, d, g, \ell, s = 0$, others $= 1$:

$$\mathcal{C}(z) = \sum_{n \geq 0} \mathrm{Av}_{2\text{-}31}(n) z^n.$$

$\mathrm{Av}_{2\text{-}31}(n) = \#$ n-permutations avoiding vincular pattern 2–31

2–31 occurrence: 416523 62 not adjacent: 416523

Two more cases: Catalan and Bell numbers, both moment sequences

1–2–3 1–23

The only 3-pattern whose avoiders don't give a moment sequence is the consecutive pattern 132 (equivalently 213, 231, 312).

This is the only 3-pattern whose avoidance is not captured in $\mathcal{C}(z)$. (Trying to fit the β_i to this sequence leads to a contradiction.)

Theorem: The sequence of numbers of avoiders of a pattern of length 3 is a moment sequence *iff* it is a special case of $\mathcal{C}(\mathrm{z})$.

Of the three sequences for classical patterns of length 4, two are known to be moment sequences. Elvey Price conjectures the same is true of the third, the enigmatic 1324.

Of the three sequences for classical patterns of length 4, two are known to be moment sequences. Elvey Price conjectures the same is true of the third, the enigmatic 1324.

Clisby-Conway-Guttmann-Inoue conjecture that the same is true for all 16 Wilf classes of length 5, and prove this in the case of 12345.

Of the three sequences for classical patterns of length 4, two are known to be moment sequences. Elvey Price conjectures the same is true of the third, the enigmatic 1324.

Clisby-Conway-Guttmann-Inoue conjecture that the same is true for all 16 Wilf classes of length 5, and prove this in the case of 12345.

Conjecture: The numbers of permutations avoiding any single classical pattern form a moment sequences.

Of the three sequences for classical patterns of length 4, two are known to be moment sequences. Elvey Price conjectures the same is true of the third, the enigmatic 1324.

Clisby-Conway-Guttmann-Inoue conjecture that the same is true for all 16 Wilf classes of length 5, and prove this in the case of 12345.

Conjecture: The numbers of permutations avoiding any single classical pattern form a moment sequences.

Which combinatorial sequences are moment sequences?

Of the three sequences for classical patterns of length 4, two are known to be moment sequences. Elvey Price conjectures the same is true of the third, the enigmatic 1324.

Clisby-Conway-Guttmann-Inoue conjecture that the same is true for all 16 Wilf classes of length 5, and prove this in the case of 12345.

Conjecture: The numbers of permutations avoiding any single classical pattern form a moment sequences.

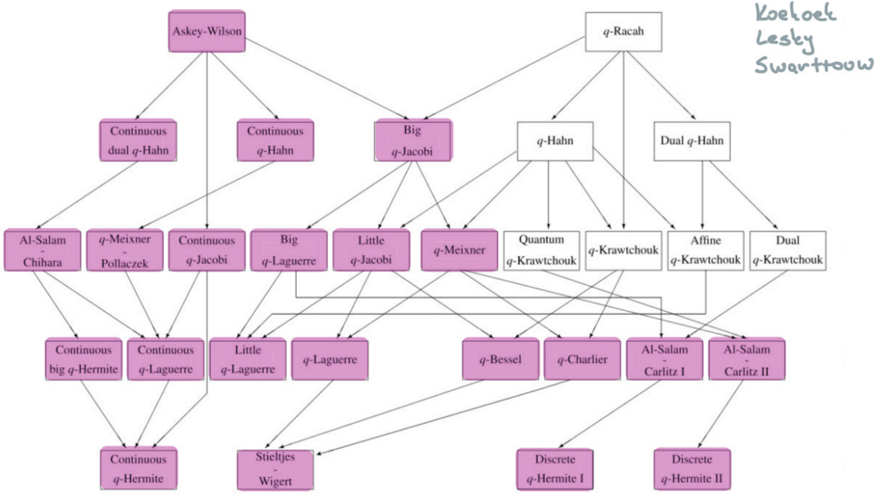
Which combinatorial sequences are moment sequences?

Which tools from probability/analysis would that let us use?

Specializations of $\mathcal{C}(z)$ also capture a large part of the q-Askey scheme of orthogonal polynomials, here interpreted in terms of the simple concepts of excedances and inversions in permutations.

Specializations of $\mathcal{C}(z)$ also capture a large part of the q-Askey scheme of orthogonal polynomials, here interpreted in terms of the simple concepts of excedances and inversions in permutations.

Corteel and Williams have a combinatorial interpretation with statistics on different objects (staircase tableaux) for all polynomials that specialize from the Askey-Wilson family.

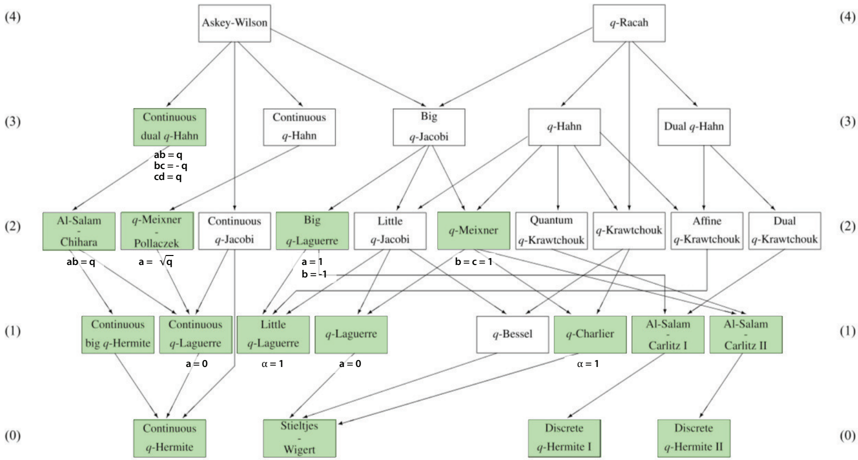


Corteel & Williams '11/'12:

$$m_n = \frac{(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty} \sum_{\ell=0}^{n} (-1)^{n-\ell} \binom{n}{\ell} \left(\frac{1-q}{2}\right)^{\ell} \frac{Z_\ell}{\prod_{i=0}^{\ell-1}(\alpha\beta - \gamma\delta q^i)}.$$

Specializations of our $\mathcal{C}(z)$ do not capture the entire q-Askey scheme, but our underlying statistics are somewhat simpler.

SCHEME
OF
BASIC HYPERGEOMETRIC ORTHOGONAL POLYNOMIALS



A very open problem

Sokal and Zeng have a continued fraction with another four parameters, carrying statistics on alignments and crossings in permutations, first defined by Corteel.

(They also have multivariate continued fractions carrying lots of statistics on set partitions and perfect matchings. Recommended!)

A very open problem

Sokal and Zeng have a continued fraction with another four parameters, carrying statistics on alignments and crossings in permutations, first defined by Corteel.

(They also have multivariate continued fractions carrying lots of statistics on set partitions and perfect matchings. Recommended!)

Is it possible to add further parameters carrying even more permutation statistics?

In particular, is it possible to expand these continued fractions to encompass all of the q-Askey scheme?

Generalizations

Via simple substitutions of parameters, many of the permutation statistics carried by $\mathcal{C}(z)$ generalize to the k*-colored permutations* $\mathcal{S}_n^k$ — each letter gets one of k colors — in particular the signed permutations of the type B Coxeter groups ($k = 2$).

3 2 6 4 5 1

Generalizations

Via simple substitutions of parameters, many of the permutation statistics carried by $\mathcal{C}(z)$ generalize to the k*-colored permutations* $\mathcal{S}_n^k$ — each letter gets one of k colors — in particular the signed permutations of the type B Coxeter groups ($k = 2$).

$$3_1 2_5 6_2 4_2 5_0 1_3$$

Generalizations

Via simple substitutions of parameters, many of the permutation statistics carried by $\mathcal{C}(z)$ generalize to the k*-colored permutations* $\mathcal{S}_n^k$ — each letter gets one of k colors — in particular the signed permutations of the type B Coxeter groups ($k = 2$).

$$3_1 2_5 6_2 4_2 5_0 1_3$$
$$1\ 2\ 3\ 4\ 5\ 6$$

Generalizations

Via simple substitutions of parameters, many of the permutation statistics carried by $\mathcal{C}(z)$ generalize to the *k-colored permutations* $\mathcal{S}_n^k$ — each letter gets one of k colors — in particular the signed permutations of the type B Coxeter groups ($k = 2$).

$$3_1 2_5 6_2 4_2 5_0 1_3$$
$$1\ 2\ 3\ 4\ 5\ 6$$

Let c_i be the color of the i-th letter.

An *excedance* in a colored permutation $a_1 a_2 \ldots a_n$ is an i such that

$$a_i > i \quad \text{OR} \quad (a_i = i \ \text{ AND } \ c_i > 0)$$

Via simple substitutions of parameters, many of the permutation statistics carried by $\mathcal{C}(z)$ generalize to the k-*colored permutations* $\mathcal{S}_n^k$ — each letter gets one of k colors — in particular the signed permutations of the type B Coxeter groups ($k = 2$).

$$3_1 2_5 6_2 4_2 5_0 1_3$$
$$1\ 2\ 3\ 4\ 5\ 6$$

Let c_i be the color of the i-th letter.

An *excedance* in a colored permutation $a_1 a_2 \ldots a_n$ is an i such that

$$a_i > i \quad \text{OR} \quad (a_i = i \ \text{ AND } \ c_i > 0)$$

A *fixed point* is an i such that $a_i = i$ AND $c_i = 0$

Generalizations

Via simple substitutions of parameters, many of the permutation statistics carried by $\mathcal{C}(z)$ generalize to the k*-colored permutations* $\mathcal{S}_n^k$ — each letter gets one of k colors — in particular the signed permutations of the type B Coxeter groups ($k = 2$).

$$3_1 2_5 6_2 4_2 5_0 1_3$$
$$1\ 2\ 3\ 4\ 5\ 6$$

Let c_i be the color of the i-th letter.

An *excedance* in a colored permutation $a_1 a_2 \dots a_n$ is an i such that

$$a_i > i \quad \text{OR} \quad (a_i = i \ \text{ AND } \ c_i > 0)$$

A *fixed point* is an i such that $a_i = i$ AND $c_i = 0$

An *anti-excedance* is an i such that $a_i < i$

- An *excedance* in a colored permutation is an i such that

$$a_i > i \quad \text{OR} \quad (a_i = i \;\; \text{AND} \;\; c_i > 0)$$

- A *fixed point* is an i such that $a_i = i$ AND $c_i = 0$

- An *anti-excedance* is an i such that $a_i < i$

- An *excedance* in a colored permutation is an i such that

$$a_i > i \quad \text{OR} \quad (a_i = i \;\; \text{AND} \;\; c_i > 0)$$

- A *fixed point* is an i such that $a_i = i \;\; \text{AND} \;\; c_i = 0$

- An *anti-excedance* is an i such that $a_i < i$

Proposition: With $s, p = kx,\; t, r = ky,\; u = (k-1)x + q$, and all other parameters set to 1, we get

$$\mathcal{C}(z) \;=\; \sum_{n \geq 0} \sum_{\sigma \in \mathcal{S}_n^k} x^{\mathrm{exc}(\sigma)} y^{\mathrm{aexc}(\sigma)} q^{\mathrm{FIX}(\sigma)} z^n .$$

- An *excedance* in a colored permutation is an i such that

$$a_i > i \quad \text{OR} \quad (a_i = i \;\; \text{AND} \;\; c_i > 0)$$

- A *fixed point* is an i such that $a_i = i$ AND $c_i = 0$

- An *anti-excedance* is an i such that $a_i < i$

Proposition: With $s, p = kx,\ t, r = ky,\ u = (k-1)x + q$, and all other parameters set to 1, we get

$$\mathcal{C}(z) \;=\; \sum_{n\geq 0} \sum_{\sigma\in\mathcal{S}_n^k} x^{\mathrm{exc}(\sigma)} y^{\mathrm{aexc}(\sigma)} q^{\mathrm{FIX}(\sigma)} z^n .$$

Easy to refine this to distinguish linked/unlinked (anti-)excedances, because the colors embed naturally in $\mathcal{C}(z)$.

- An *excedance* in a colored permutation is an i such that

$$a_i > i \quad \text{OR} \quad (a_i = i \ \ \text{AND} \ \ c_i > 0)$$

- A *fixed point* is an i such that $a_i = i$ AND $c_i = 0$

- An *anti-excedance* is an i such that $a_i < i$

Proposition: With $s, p = kx,\ t, r = ky,\ u = (k-1)x + q$, and all other parameters set to 1, we get

$$\mathcal{C}(z) \;=\; \sum_{n \geq 0} \sum_{\sigma \in \mathcal{S}_n^k} x^{\mathrm{exc}(\sigma)} y^{\mathrm{aexc}(\sigma)} q^{\mathrm{FIX}(\sigma)} z^n .$$

Easy to refine this to distinguish linked/unlinked (anti-)excedances, because the colors embed naturally in $\mathcal{C}(z)$.

(There are quite a few papers on various statistics on the colored permutations.)

- An *inversion* is a pair (i, j) where $i < j$ and

$$c_i > c_j \quad \text{OR} \quad (c_i = c_j \ \text{AND} \ a_i > a_j)$$

With

$a, c, h, r = q, \quad b, f, d, \ell, t = q^2, \quad g, w = 0, \quad p, u = 1, \quad s = 2q,$
we get the distribution of inversions over $\mathcal{S}_n$ from $\mathcal{C}$:

$$\mathcal{C}(z) = \sum_{n \geq 0} \sum_{\pi \in \mathcal{S}_n} q^{\mathrm{INV}(\pi)} z^n.$$

- An *inversion* is a pair (i, j) where $i < j$ and

$$c_i > c_j \quad \text{OR} \quad (c_i = c_j \ \text{ AND } \ a_i > a_j)$$

With

$a, c, h, r = q, \quad b, f, d, \ell, t = q^2, \quad g, w = 0, \quad p, u = 1, \quad s = 2q,$
we get the distribution of inversions over $\mathcal{S}_n$ from $\mathcal{C}$:

$$\mathcal{C}(z) \ = \ \sum_{n \geq 0} \sum_{\pi \in \mathcal{S}_n} q^{\mathrm{INV}(\pi)} z^n.$$

Biane already obtained this in 1993, starting from a different continued fraction.

- An *inversion* is a pair (i,j) where $i<j$ and

$$c_i > c_j \quad \text{OR} \quad (c_i = c_j \;\; \text{AND} \;\; a_i > a_j)$$

With

$a,c,h,r=q,\quad b,f,d,\ell,t=q^2,\quad g,w=0,\quad p,u=1,\quad s=2q,$
we get the distribution of inversions over $\mathcal{S}_n$ from $\mathcal{C}$:

$$\mathcal{C}(z) \;=\; \sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n} q^{\mathrm{INV}(\pi)} z^n.$$

Biane already obtained this in 1993, starting from a different continued fraction.

If we replace z by $z((k-1)q+1)$ above, we get the distribution of inversions over the k-colored permutations $\mathcal{S}_n^k$ for $k>1$.

- An *inversion* is a pair (i, j) where $i < j$ and

$$c_i > c_j \quad \text{OR} \quad (c_i = c_j \;\; \text{AND} \;\; a_i > a_j)$$

With

$a, c, h, r = q, \quad b, f, d, \ell, t = q^2, \quad g, w = 0, \quad p, u = 1, \quad s = 2q,$
we get the distribution of inversions over $\mathcal{S}_n$ from $\mathcal{C}$:

$$\mathcal{C}(z) \;=\; \sum_{n \geq 0} \sum_{\pi \in \mathcal{S}_n} q^{\mathrm{INV}(\pi)} z^n.$$

Biane already obtained this in 1993, starting from a different continued fraction.

If we replace z by $z((k-1)q+1)$ above, we get the distribution of inversions over the k-colored permutations $\mathcal{S}_n^k$ for $k > 1$.

Further, setting $p = x$, $s = (1+x)q$, we get

$$\mathcal{C}(z) \;=\; \sum_{n \geq 0} \sum_{\pi \in \mathcal{S}_n} x^{\mathrm{exc}(\pi)} q^{\mathrm{INV}\,\pi} z^n.$$

- An *inversion* is a pair (i, j) where $i < j$ and

$$c_i > c_j \quad \text{OR} \quad (c_i = c_j \;\; \text{AND} \;\; a_i > a_j)$$

With

$a, c, h, r = q, \quad b, f, d, \ell, t = q^2, \quad g, w = 0, \quad p, u = 1, \quad s = 2q,$
we get the distribution of inversions over $\mathcal{S}_n$ from $\mathcal{C}$:

$$\mathcal{C}(z) \;=\; \sum_{n \geq 0} \sum_{\pi \in \mathcal{S}_n} q^{\mathrm{INV}(\pi)} z^n.$$

Biane already obtained this in 1993, starting from a different continued fraction.

If we replace z by $z((k-1)q+1)$ above, we get the distribution of inversions over the k-colored permutations $\mathcal{S}_n^k$ for $k > 1$.

Further, setting $p = x$, $s = (1+x)q$, we get

$$\mathcal{C}(z) \;=\; \sum_{n \geq 0} \sum_{\pi \in \mathcal{S}_n} x^{\mathrm{exc}(\pi)} q^{\mathrm{INV}\,\pi} z^n.$$

Unclear whether that can be extended to $\mathcal{S}_n^k$ via $\mathcal{C}$

- An *inversion* is a pair (i, j) where $i < j$ and

$$c_i > c_j \quad \text{OR} \quad (c_i = c_j \ \text{ AND } \ a_i > a_j)$$

With

$a, c, h, r = q, \quad b, f, d, \ell, t = q^2, \quad g, w = 0, \quad p, u = 1, \quad s = 2q,$
we get the distribution of inversions over $\mathcal{S}_n$ from $\mathcal{C}$:

$$\mathcal{C}(z) \ = \ \sum_{n \geq 0} \sum_{\pi \in \mathcal{S}_n} q^{\mathrm{INV}(\pi)} z^n.$$

Biane already obtained this in 1993, starting from a different continued fraction.

If we replace z by $z((k-1)q+1)$ above, we get the distribution of inversions over the k-colored permutations $\mathcal{S}_n^k$ for $k > 1$.

Further, setting $p = x$, $s = (1+x)q$, we get

$$\mathcal{C}(z) \ = \ \sum_{n \geq 0} \sum_{\pi \in \mathcal{S}_n} x^{\mathrm{exc}(\pi)} q^{\mathrm{INV}\,\pi} z^n.$$

Unclear whether that can be extended to $\mathcal{S}_n^k$ via $\mathcal{C}$ and whether other Euler-Mahonian pairs can be obtained from $\mathcal{C}$.

Coloring only fixed points

Because fixed points live independently in $\mathcal{C}(z)$, the following generalization is obvious:

k-arrangements: Permutations with k-colored fixed points

- 0-arrangements are derangements (no fixed points)
- 1-arrangements are permutations
- 2-arrangements were called just *arrangements* by Comtet, and coincide with Postnikov's *decorated permutations*, which are in bijection with 'certain non-negative Grassmann cells'.

Coloring only fixed points

Because fixed points live independently in $\mathcal{C}(z)$, the following generalization is obvious:

k-arrangements: Permutations with k-colored fixed points

- 0-arrangements are derangements (no fixed points)
- 1-arrangements are permutations
- 2-arrangements were called just *arrangements* by Comtet, and coincide with Postnikov's *decorated permutations*, which are in bijection with 'certain non-negative Grassmann cells'.

For $k > 2$ the k-arrangements do not seem to have been studied.

Coloring only fixed points

Because fixed points live independently in $\mathcal{C}(z)$, the following generalization is obvious:

k-arrangements: Permutations with k-colored fixed points

- 0-arrangements are derangements (no fixed points)
- 1-arrangements are permutations
- 2-arrangements were called just *arrangements* by Comtet, and coincide with Postnikov's *decorated permutations*, which are in bijection with 'certain non-negative Grassmann cells'.

For $k > 2$ the k-arrangements do not seem to have been studied.

But they have many nice properties, and doubtless many more to be discovered.

Proposition: Let $A_k(n)$ be the number of k-arrangements of $[n]$. Then

- $A_k(0) = 1$. For $n > 0$: $A_k(n) = n \cdot A_k(n-1) + (k-1)^n$

Proposition: Let $A_k(n)$ be the number of k-arrangements of $[n]$. Then

- $A_k(0) = 1$. For $n > 0$: $A_k(n) = n \cdot A_k(n-1) + (k-1)^n$
- $\displaystyle\sum_{n \geq 0} A_k(n) \frac{x^n}{n!} = \frac{e^{(k-1)x}}{(1-x)}$

Proposition: Let $A_k(n)$ be the number of k-arrangements of $[n]$. Then

- $A_k(0) = 1$. For $n > 0$: $A_k(n) = n \cdot A_k(n-1) + (k-1)^n$
- $\displaystyle\sum_{n\geq 0} A_k(n)\frac{x^n}{n!} = \frac{e^{(k-1)x}}{(1-x)}$
- $\displaystyle A_k(n) = \sum_{i\geq 0}\binom{n}{i}A_{k-1}(i)$ (successive binomial transforms)

Proposition: Let $A_k(n)$ be the number of k-arrangements of $[n]$. Then

- $A_k(0) = 1$. For $n > 0$: $A_k(n) = n \cdot A_k(n-1) + (k-1)^n$

- $\displaystyle\sum_{n \geq 0} A_k(n) \frac{x^n}{n!} = \frac{e^{(k-1)x}}{(1-x)}$

- $\displaystyle A_k(n) = \sum_{i \geq 0} \binom{n}{i} A_{k-1}(i)$ (successive binomial transforms)

- $A_k(n)$ equals the permanent of the $n \times n$ matrix with k on the diagonal and 1s elsewhere.

Proposition: Let $A_k(n)$ be the number of k-arrangements of $[n]$. Then

- $A_k(0) = 1$. For $n > 0$: $A_k(n) = n \cdot A_k(n-1) + (k-1)^n$

- $\displaystyle\sum_{n\geq 0} A_k(n)\frac{x^n}{n!} = \frac{e^{(k-1)x}}{(1-x)}$

- $\displaystyle A_k(n) = \sum_{i\geq 0}\binom{n}{i}A_{k-1}(i)$ (successive binomial transforms)

- $A_k(n)$ equals the permanent of the $n \times n$ matrix with k on the diagonal and 1s elsewhere.

All of this holds for $k < 0$.

Proposition: Let $A_k(n)$ be the number of k-arrangements of $[n]$. Then

- $A_k(0) = 1$. For $n > 0$: $A_k(n) = n \cdot A_k(n-1) + (k-1)^n$

- $$\sum_{n \geq 0} A_k(n) \frac{x^n}{n!} = \frac{e^{(k-1)x}}{(1-x)}$$

- $A_k(n) = \sum_{i \geq 0} \binom{n}{i} A_{k-1}(i)$ (successive binomial transforms)

- $A_k(n)$ equals the permanent of the $n \times n$ matrix with k on the diagonal and 1s elsewhere.

All of this holds for $k < 0$. Seems that $A_k(n) > 0$ for $n >> 0$.

Proposition: Let $A_k(n)$ be the number of k-arrangements of $[n]$. Then

- $A_k(0) = 1$. For $n > 0$: $A_k(n) = n \cdot A_k(n-1) + (k-1)^n$

- $\displaystyle\sum_{n \geq 0} A_k(n) \frac{x^n}{n!} = \frac{e^{(k-1)x}}{(1-x)}$

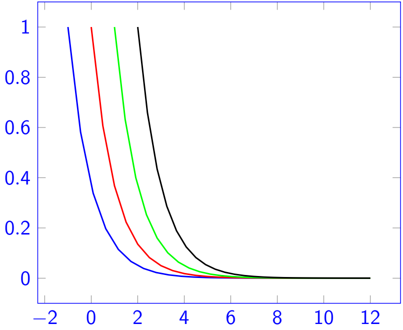
- $\displaystyle A_k(n) = \sum_{i \geq 0} \binom{n}{i} A_{k-1}(i)$ (successive binomial transforms)

- $A_k(n)$ equals the permanent of the $n \times n$ matrix with k on the diagonal and 1s elsewhere.

All of this holds for $k < 0$. Seems that $A_k(n) > 0$ for $n >> 0$.

What does that count?

$$\#k\text{-arrangements on } [n] = \int_{k-1}^{\infty} x^n e^{-x+(k-1)} dx$$



Positivity previously observed for:

- $k = 0$: Martin & Kearney '15
- $k = 2$: Ardila, Rincón, Williams '16 (# positroids)

Encoding k-arrangements

Replacing fixed points colored i by $-i$ gives the *derangement form* of a k-arrangement. Example:

$$\begin{array}{ccccccccc} 6 & 2_3 & 1 & 4_5 & 9 & 8 & 7_2 & 5 & 3 \\ 6 & -3 & 1 & -5 & 9 & 8 & -2 & 5 & 3 \end{array}$$

Encoding k-arrangements

Replacing fixed points colored i by $-i$ gives the *derangement form* of a k-arrangement. Example:

$$\begin{array}{ccccccccc} 6 & 2_3 & 1 & 4_5 & 9 & 8 & 7_2 & 5 & 3 \\ 6 & -3 & 1 & -5 & 9 & 8 & -2 & 5 & 3 \end{array}$$

Replacing fixed points colored $i < k$ by $-i$ gives the *permutation form* of a k-arrangement. Example for a 5-arrangement:

$$\begin{array}{ccccccccc} 6 & 2_3 & 1 & 4_5 & 9 & 8 & 7_2 & 5 & 3 \\ 6 & -3 & 1 & 4 & 9 & 8 & -2 & 5 & 3 \end{array}$$

Replacing fixed points colored i by $-i$ gives the *derangement form* of a k-arrangement. Example:

$$\begin{array}{ccccccccc} 6 & 2_3 & 1 & 4_5 & 9 & 8 & 7_2 & 5 & 3 \\ 6 & -3 & 1 & -5 & 9 & 8 & -2 & 5 & 3 \end{array}$$

Replacing fixed points colored i by $-i$ gives the *derangement form* of a k-arrangement. Example:

$$\begin{matrix} 6 & 2_3 & 1 & 4_5 & 9 & 8 & 7_2 & 5 & 3 \\ 6 & -3 & 1 & -5 & 9 & 8 & -2 & 5 & 3 \end{matrix}$$

Replacing fixed points colored $i < k$ by $-i$ gives the *permutation form* of a k-arrangement. Example for a 5-arrangement:

$$\begin{matrix} 6 & 2_3 & 1 & 4_5 & 9 & 8 & 7_2 & 5 & 3 \\ 6 & -3 & 1 & 4 & 9 & 8 & -2 & 5 & 3 \end{matrix}$$

Replacing fixed points colored i by $-i$ gives the *derangement form* of a k-arrangement. Example:

$$\begin{array}{ccccccccc} 6 & 2_3 & 1 & 4_5 & 9 & 8 & 7_2 & 5 & 3 \\ 6 & -3 & 1 & -5 & 9 & 8 & -2 & 5 & 3 \end{array}$$

Replacing fixed points colored $i < k$ by $-i$ gives the *permutation form* of a k-arrangement. Example for a 5-arrangement:

$$\begin{array}{ccccccccc} 6 & 2_3 & 1 & 4_5 & 9 & 8 & 7_2 & 5 & 3 \\ 6 & -3 & 1 & 4 & 9 & 8 & -2 & 5 & 3 \end{array}$$

Given a k-arrangement A as a permutation π with fixed points colored with $\{1, 2, \ldots, k\}$, letting its non-fixed points have color 0 we can regard A as a k-colored permutation in $\mathcal{S}_n^k$.

Replacing fixed points colored i by $-i$ gives the *derangement form* of a k-arrangement. Example:

$$\begin{array}{ccccccccc} 6 & 2_{3} & 1 & 4_{5} & 9 & 8 & 7_{2} & 5 & 3 \\ 6 & -3 & 1 & -5 & 9 & 8 & -2 & 5 & 3 \end{array}$$

Replacing fixed points colored $i < k$ by $-i$ gives the *permutation form* of a k-arrangement. Example for a 5-arrangement:

$$\begin{array}{ccccccccc} 6 & 2_{3} & 1 & 4_{5} & 9 & 8 & 7_{2} & 5 & 3 \\ 6 & -3 & 1 & 4 & 9 & 8 & -2 & 5 & 3 \end{array}$$

Given a k-arrangement A as a permutation π with fixed points colored with $\{1, 2, \ldots, k\}$, letting its non-fixed points have color 0 we can regard A as a k-colored permutation in $\mathcal{S}_n^k$.

Conjecture: INV and MAJ are equidistributed on k-arrangements as colored permutations.

Replacing fixed points colored i by $-i$ gives the *derangement form* of a k-arrangement. Example:

$$\begin{array}{ccccccccc} 6 & 2_3 & 1 & 4_5 & 9 & 8 & 7_2 & 5 & 3 \\ 6 & -3 & 1 & -5 & 9 & 8 & -2 & 5 & 3 \end{array}$$

Replacing fixed points colored $i < k$ by $-i$ gives the *permutation form* of a k-arrangement. Example for a 5-arrangement:

$$\begin{array}{ccccccccc} 6 & 2_3 & 1 & 4_5 & 9 & 8 & 7_2 & 5 & 3 \\ 6 & -3 & 1 & 4 & 9 & 8 & -2 & 5 & 3 \end{array}$$

Given a k-arrangement A as a permutation π with fixed points colored with $\{1, 2, \ldots, k\}$, letting its non-fixed points have color 0 we can regard A as a k-colored permutation in $\mathcal{S}_n^k$.

Conjecture: INV and MAJ are equidistributed on k-arrangements as colored permutations.

Conjecture: DES has the same distribution on k-arrangements as colored permutations as it does on the permutation or derangement form.

Proposition: $\textsc{exc}$ and $\textsc{des}$ are equidistributed on the permutation form of k-arrangements of $[n]$ for any n and k, as are $\textsc{inv}$ and $\textsc{maj}$.

Proposition: The number of 2-arrangements of $[n]$ whose permutation form avoids a classical 3-pattern is C_{n+1}.
Those with k negative entries: the ballot number $\frac{k+1}{n+1}\binom{2n-k}{n}$.

Conjecture: DES has the same distribution on the derangement and permutation forms for k-arrangements of $[n]$.

Conjecture: DES has the same distribution on the derangement and permutation forms for k-arrangements of $[n]$.

Conjecture: The number of 3-arrangements of $[n]$ whose permutation form avoids any given classical 3-pattern is $C_{n+2} - 2^n$.
(2-arrangements: C_{n+1})

Conjecture: DES has the same distribution on the derangement and permutation forms for k-arrangements of $[n]$.

Conjecture: The number of 3-arrangements of $[n]$ whose permutation form avoids any given classical 3-pattern is $C_{n+2} - 2^n$. (2-arrangements: C_{n+1})

Conjecture: The number of 2-arrangements of $[n-1]$ with k descents whose permutation form avoids any given classical 3-pattern, "equals" the number of rooted ordered trees with n non-root nodes and k leaves (A108838 in OEIS).

Conjecture: DES has the same distribution on the derangement and permutation forms for k-arrangements of $[n]$.

Conjecture: The number of 3-arrangements of $[n]$ whose permutation form avoids any given classical 3-pattern is $C_{n+2} - 2^n$. (2-arrangements: C_{n+1})

Conjecture: The number of 2-arrangements of $[n-1]$ with k descents whose permutation form avoids any given classical 3-pattern, "equals" the number of rooted ordered trees with n non-root nodes and k leaves (`A108838` in OEIS).

Conjecture: The number 2-arrangements of $[n-1]$ with k ascents whose permutation form avoids 123 "equals" the number of 123-avoiding permutations of $[n]$ with k peaks. (`A236406`).

Proposition: DES has the same distribution on the derangement and permutation forms for k-arrangements of $[n]$.

Proposition: The number of 3-arrangements of $[n]$ whose permutation form avoids any given classical 3-pattern is $C_{n+2} - 2^n$. (2-arrangements: C_{n+1})

Proposition: The number of 2-arrangements of $[n-1]$ with k descents whose permutation form avoids any given classical 3-pattern, "equals" the number of rooted ordered trees with n non-root nodes and k leaves (A108838 in OEIS).

Proposition: The number 2-arrangements of $[n-1]$ with k ascents whose permutation form avoids 123 "equals" the number of 123-avoiding permutations of $[n]$ with k peaks. (A236406).

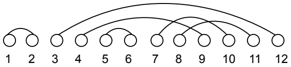
Proved by Fu-Han-Lin. Surprisingly non-trivial.

Classical CLT

Theorem

Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{E}(X_i) = 0$ and $\mathbb{E}(X_i^2) = 1$. Then $S_N := \frac{1}{\sqrt{N}} \sum_{i=1}^N X_i \xrightarrow{d} \mathcal{N}(0,1)$. Equivalently,

$$
\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left(S_N^{2n-1}\right) &= 0, \\
\lim_{N\to\infty} \mathbb{E}\left(S_N^{2n}\right) &= (2n-1)!! := (2n-1)(2n-3)\cdots 5\cdot 3\cdot 1 \\
&= \sum_{\pi\in\mathcal{P}_2(2n)} 1
\end{aligned}
$$



Proof.

Product of sums as a sum of products:

$$\mathbb{E}(S_N^k) = \frac{1}{N^{k/2}} \sum_{i(1),\ldots,i(k)\in[N]} \mathbb{E}(X_{i(1)} \cdots X_{i(k)}).$$

$$\mathbb{E}(S_N^k) = \frac{1}{N^{k/2}} \sum_{i(1),\dots,i(k)\in[N]} \mathbb{E}(X_{i(1)} \cdots X_{i(k)}).$$

- Independence $g \implies$ factorization. E.g.

$$\mathbb{E}(X_1 X_2 X_2 X_1 X_1) = \mathbb{E}(X_1^3)\mathbb{E}(X_2^2)$$

- Independence + identical distribution $\implies$ same repetition patterns yield identical mixed moments. E.g.

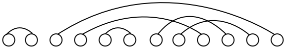
$$\mathbb{E}(X_1 X_2 X_2 X_1 X_1) = \mathbb{E}(X_5 X_3 X_3 X_5 X_5)$$

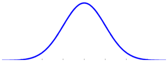
- $\mathbb{E}(X_i) = 0 \implies$ partitions with a singleton don't contribute.
- Remaining partitions with a block of size ≥ 3 are too few ($o(N^{k/2})$). Hence, only pair partitions ($\Theta(N^{k/2})$ for k even) appear in the limit and

$$\lim_{N\to\infty} \mathbb{E}\left(S_N^{2n-1}\right) = 0, \quad \lim_{N\to\infty} \mathbb{E}\left(S_N^{2n}\right) = \sum_{\pi\in\mathcal{P}_2(2n)} 1.$$

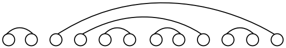
$\square$

Statistics on perfect matchings — Measure

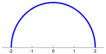




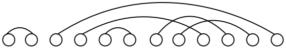
Gaussian N(0,1)



non-crossing perfect matchings

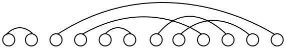


"free Gaussian"

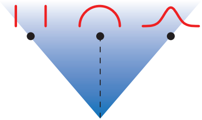


$q^{\#\text{crossings}}$

Bożejko & Speicher '91



$q^{\#\text{crossings}} t^{\#\text{nestings}}$



Blitvić '12

And now a different look at positivity for permutation patterns

Joint work with Natasha Blitvić and Slim Kammoun

The *descent set* of a permutation $\pi = a_1 a_2 \ldots a_n$ is

$$\mathrm{Dset}(\pi) = \{i \mid a_i > a_{i+1}\}.$$

$\mathrm{Dset}(31452) = \{1, 4\}$.

The *descent set* of a permutation $\pi = a_1 a_2 \dots a_n$ is

$$\mathrm{Dset}(\pi) = \{i \mid a_i > a_{i+1}\}.$$

$\mathrm{Dset}(31452) = \{1, 4\}$.

Theorem (Gessel–Viennot 1985): The number of n-permutations with a given descent set is a minor of the binomial matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 0 & 1 & 3 & 6 & 10 & \cdots \\ 0 & 0 & 1 & 4 & 10 & \cdots \\ 0 & 0 & 0 & 1 & 5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

The *descent set* of a permutation $\pi = a_1 a_2 \ldots a_n$ is

$$\mathrm{Dset}(\pi) = \{i \mid a_i > a_{i+1}\}.$$

$\mathrm{Dset}(31452) = \{1, 4\}$.

Theorem (Gessel–Viennot 1985): The number of n-permutations with a given descent set is a minor of the binomial matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 0 & 1 & 3 & 6 & 10 & \cdots \\ 0 & 0 & 1 & 4 & 10 & \cdots \\ 0 & 0 & 0 & 1 & 5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

The *descent set* of a permutation $\pi = a_1 a_2 \ldots a_n$ is

$$\mathrm{Dset}(\pi) = \{i \mid a_i > a_{i+1}\}.$$

$\mathrm{Dset}(31452) = \{1, 4\}$.

Theorem (Gessel–Viennot 1985): The number of n-permutations with a given descent set is a minor of the binomial matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 0 & 1 & 3 & 6 & 10 & \cdots \\ 0 & 0 & 1 & 4 & 10 & \cdots \\ 0 & 0 & 0 & 1 & 5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

The *descent set* of a permutation $\pi = a_1 a_2 \ldots a_n$ is

$$\mathrm{Dset}(\pi) = \{i \mid a_i > a_{i+1}\}.$$

$\mathrm{Dset}(31452) = \{1, 4\}$.

Theorem (Gessel–Viennot 1985): The number of n-permutations with a given descent set is a minor of the binomial matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 0 & 1 & 3 & 6 & 10 & \cdots \\ 0 & 0 & 1 & 4 & 10 & \cdots \\ 0 & 0 & 0 & 1 & 5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

A special case of the Lindström–Gessel–Viennot Lemma, counting non-intersecting lattice paths.

The *descent set* of a permutation $\pi = a_1 a_2 \ldots a_n$ is

$$\mathrm{Dset}(\pi) = \{i \mid a_i > a_{i+1}\}.$$

$\mathrm{Dset}(31452) = \{1, 4\}$.

Theorem (Gessel–Viennot 1985): The number of n-permutations with a given descent set is a minor of the binomial matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 0 & 1 & 3 & 6 & 10 & \cdots \\ 0 & 0 & 1 & 4 & 10 & \cdots \\ 0 & 0 & 0 & 1 & 5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

A special case of the Lindström–Gessel–Viennot Lemma, counting non-intersecting lattice paths.

A descent is an occurrence of the *consecutive pattern* 21.

What about arbitrary consecutive patterns?

Occurrence of the consecutive pattern 1324:

1 4 2 6 3 7 5 8

Four consecutive letters, in the same order of size as 1,3,2,4.

Occurrence of the consecutive pattern 1324:

1 4 2 6 3 7 5 8

Occurrence of the consecutive pattern 1324:

1	3	2	4				
1	4	2	6	3	7	5	8

Occurrence of the consecutive pattern 1324:

1 3 2 4
1 4 2 6 3 7 5 8

Occurrence of the consecutive pattern 1324:

1	3	2	4				
1	4	2	6	3	7	5	8
		1	3	2	4		

Occurrence of the consecutive pattern 1324:

1	3	2	4				
1	4	2	6	3	7	5	8
		1	3	2	4		

Occurrence of the consecutive pattern 1324:

1	3	2	4	1	3	2	4
1	4	2	6	3	7	5	8
		1	3	2	4		

1	3	2	4	1	3	2	4
1	4	2	6	3	7	5	8
		1	3	2	4		

1	3	2	4	1	3	2	4
1	4	2	6	3	7	5	8
		1	3	2	4		

This permutation has occurrences of 1324 starting at positions 1, 3 and 5. Equivalently, it is *covered by* 1324 *with overlap 2*.

1	3	2	4	1	3	2	4
1	4	2	6	3	7	5	8
		1	3	2	4		

This permutation has occurrences of 1324 starting at positions 1, 3 and 5. Equivalently, it is *covered by* 1324 *with overlap 2*.

There is one permutation of length 4 covered by 1324, two such of length 6 with overlap 2, five of length 8:

1 3 2 4	1 3 2 5 4 6	1 3 2 5 4 7 6 8
	1 4 2 5 3 6	1 3 2 6 4 7 5 8
		1 4 2 5 3 7 6 8
		1 4 2 6 3 7 5 8
		1 5 2 6 3 7 4 8

1	3	2	4	1	3	2	4
1	4	2	6	3	7	5	8
		1	3	2	4		

This permutation has occurrences of 1324 starting at positions 1, 3 and 5. Equivalently, it is *covered by* 1324 *with overlap 2*.

Fact: For $n \geq 1$, the number of permutations of length $2n + 2$ covered by 1324 with overlap 2 is the Catalan number C_n.

We only deal with cases where a pattern P *covers* a permutation π: every letter of π belongs to an occurrence of P.

Allowing gaps between occurrences introduces a trivially computed factor to enumeration formulas.

We only deal with cases where a pattern P *covers* a permutation π: every letter of π belongs to an occurrence of P.

Allowing gaps between occurrences introduces a trivially computed factor to enumeration formulas.

We count permutations in which a pattern P occurs *at least* in prescribed places, disregarding whether P may occur elsewhere.

We only deal with cases where a pattern P *covers* a permutation π: every letter of π belongs to an occurrence of P.

Allowing gaps between occurrences introduces a trivially computed factor to enumeration formulas.

We count permutations in which a pattern P occurs *at least* in prescribed places, disregarding whether P may occur elsewhere.

In many cases, if a pattern covers a permutation π with fixed overlaps then it can't occur in other places in π.

Example: 2143 can overlap in one or two letters. In either case it can't occur in other positions than those prescribed by the overlap.

We only deal with cases where a pattern P *covers* a permutation π: every letter of π belongs to an occurrence of P.

Allowing gaps between occurrences introduces a trivially computed factor to enumeration formulas.

We count permutations in which a pattern P occurs *at least* in prescribed places, disregarding whether P may occur elsewhere.

In many cases, if a pattern covers a permutation π with fixed overlaps then it can't occur in other places in π.

Example: 2143 can overlap in one or two letters. In either case it can't occur in other positions than those prescribed by the overlap.

The permutation 132547698 is covered by 13254 with overlap 1, but also has it occurring in the middle: 132547698.

We only deal with cases where a pattern P *covers* a permutation π: every letter of π belongs to an occurrence of P.

Allowing gaps between occurrences introduces a trivially computed factor to enumeration formulas.

We count permutations in which a pattern P occurs *at least* in prescribed places, disregarding whether P may occur elsewhere.

In many cases, if a pattern covers a permutation π with fixed overlaps then it can't occur in other places in π.

Example: 2143 can overlap in one or two letters. In either case it can't occur in other positions than those prescribed by the overlap.

The permutation 132547698 is covered by 13254 with overlap 1, but also has it occurring in the middle: 132547698.

Posssible because 13254 has *autocorrelation* $3 > 5/2$:

```
1 3 2
1 3 2 5 4
    1 3 2
```

For patterns of different lengths and different overlaps we get lots of different counting sequences. These depend only on the first j and last j letters in a pattern, where m is the size of the overlap.

Pattern	Enumeration	
$1 \cdots (k\text{-}1):$	$n!_j := (n-j)(n-2j)(n-3j)\cdots$	
$1 \cdots 2:$	$\dfrac{(n-1)!}{(j+1)! \cdot (3!)^{j+1}}$ $=$ # partitions of $[kn]$, block sizes k	
$1 \cdots (k\text{-}d):$	$\displaystyle\prod_{i=0}^{j} \binom{i(k-1)+d}{d}$	
$2\,k \cdots 1\,3:$	$\dfrac{((k-2)j+k-2))_j}{(j+1)!}$	*overlap 2, those above have overlap 1*

The numbers of permutations of length $4 + 3j$ covered by 2143 with overlap 1:

$$1, 9, 234, 12204, 1067040, 140641920, 26053347600, \ \ldots$$

The numbers of permutations of length $4 + 3j$ covered by 2143 with overlap 1:

$$1, 9, 234, 12204, 1067040, 140641920, 26053347600, \ldots$$

No simple expression, but there is a general recursive formula.

Let $p = p_1 p_2 \dots p_m \cdots p_{m+1} p_{m+2} \dots p_{2m}$ be a K-pattern where $p_1 < p_2 < \cdots < p_m$ & $p_{m+1} < p_{m+2} < \cdots < p_{2m}$.

Let π_i be the place of the i-th smallest among $p_1, p_2, \dots, p_{2m}$.

Let $g_j(L)$ be the number of permutations of length $K + j(K - m)$ with p-overlap m and ending with $L = \ell_1, \ell_2, \dots, \ell_m$.

Then $g_0(\mathrm{p}) = 1$, $g_0(L) = 0$ if $L \neq \mathrm{p}$, and for $j \geq 0$ we have

$$g_{j+1}(L) = \sum_{\ell_1 < \ell_2 < \cdots < \ell_m} g_j(\ell_1, \ell_2, \dots, \ell_m) \prod_{i=0}^{2m} \binom{\ell_{\pi_{i+1}} - \ell_{\pi_i} - 1}{p_{\pi_{i+1}} - p_{\pi_i} - 1}.$$

Let $p = p_1 p_2 \ldots p_m \cdots p_{m+1} p_{m+2} \ldots p_{2m}$ be a K-pattern where $p_1 < p_2 < \cdots < p_m$ & $p_{m+1} < p_{m+2} < \cdots < p_{2m}$.

Let π_i be the place of the i-th smallest among $p_1, p_2, \ldots, p_{2m}$.

Let $g_j(L)$ be the number of permutations of length $K + j(K - m)$ with p-overlap m and ending with $L = \ell_1, \ell_2, \ldots, \ell_m$.

Then $g_0(\mathrm{p}) = 1$, $g_0(L) = 0$ if $L \neq \mathrm{p}$, and for $j \geq 0$ we have

$$g_{j+1}(L) = \sum_{\ell_1 < \ell_2 < \cdots < \ell_m} g_j(\ell_1, \ell_2, \ldots, \ell_m) \prod_{i=0}^{2m} \binom{\ell_{\pi_{i+1}} - \ell_{\pi_i} - 1}{p_{\pi_{i+1}} - p_{\pi_i} - 1}.$$

Now sum over all L.

A simple lemma (bijection) removes the requirement of increasing prefix and suffix.

Let $p = p_1 p_2 \dots p_m \cdots p_{m+1} p_{m+2} \dots p_{2m}$ be a K-pattern where $p_1 < p_2 < \cdots < p_m$ & $p_{m+1} < p_{m+2} < \cdots < p_{2m}$.

Let π_i be the place of the i-th smallest among $p_1, p_2, \dots, p_{2m}$.

Let $g_j(L)$ be the number of permutations of length $K + j(K - m)$ with p-overlap m and ending with $L = \ell_1, \ell_2, \dots, \ell_m$.

Then $g_0(\mathrm{p}) = 1$, $g_0(L) = 0$ if $L \neq \mathrm{p}$, and for $j \geq 0$ we have

$$g_{j+1}(L) = \sum_{\ell_1 < \ell_2 < \cdots < \ell_m} g_j(\ell_1, \ell_2, \dots, \ell_m) \prod_{i=0}^{2m} \binom{\ell_{\pi_{i+1}} - \ell_{\pi_i} - 1}{p_{\pi_{i+1}} - p_{\pi_i} - 1}.$$

Idea of proof: To construct a permutation ending in L, look at all possible prefixes L' of the last occurrence of p. In how many ways can we choose the letters between L' and L?

Conjecture: *For any pattern and any size overlap, the counting sequence is a moment sequence.*

Conjecture: *For any pattern and any size overlap, the counting sequence is a moment sequence.*

Verified for:

- all patterns of length ≤ 20, overlap 1, enumerating sequences of length 50 (25×25 Hankel determinants),
- all patterns of length ≤ 9, overlap 2, sequences of length 20
- several cases for overlap 3 ...

Conjecture: *For any pattern and any size overlap, the counting sequence is a moment sequence.*

Verified for:

- all patterns of length ≤ 20, overlap 1, enumerating sequences of length 50 (25×25 Hankel determinants),
- all patterns of length ≤ 9, overlap 2, sequences of length 20
- several cases for overlap 3 ...

In some cases we can determine the corresponding measure.

A bolder conjecture:

For any periodic overlap sequence we also get moment sequences.

Example: First two occurrences overlap by 2, second and third by 3, third and fourth by 1, then by 2, 3, 1, 2, 3, 1, . . .

A bolder conjecture:

For any periodic overlap sequence we also get moment sequences.

Example: First two occurrences overlap by 2, second and third by 3, third and fourth by 1, then by 2, 3, 1, 2, 3, 1, . . .

We have confirmed this for a variety of examples.

A bolder conjecture:

For any periodic overlap sequence we also get moment sequences.

Example: First two occurrences overlap by 2, second and third by 3, third and fourth by 1, then by 2, 3, 1, 2, 3, 1, . . .

We have confirmed this for a variety of examples.

What about arbitrary (non-periodic) overlap sequences?

What kind of tools do we have for showing positivity of combinatorial sequences?

What kind of tools do we have for showing positivity of combinatorial sequences?

- Continued fractions

What kind of tools do we have for showing positivity of combinatorial sequences?

- Continued fractions
- Fast growth

What kind of tools do we have for showing positivity of combinatorial sequences?

- Continued fractions
- Fast growth
- Turning the problem into a graph and ...

Continued fractions

$A = a_0, a_1, \ldots$ is a *Hamburger* moment sequence of a (positive) measure ρ on the real line if

$$a_n = \int_{\mathbb{R}} x^n \, d\rho(x)$$

Equivalently, there are real numbers β_i and α_i such that

$$\sum_{n\geq 0} a_n z^n = \cfrac{1}{1 - \alpha_0 z - \cfrac{\beta_1 z^2}{1 - \alpha_1 z - \cfrac{\beta_2 z^2}{\ddots}}}$$

with $\beta_i > 0$ for all i (or all $i \leq N$ and 0 for $i > N$).

Fast growth

Theorem(Katkova-Vishnyakova, 2006): Let $M = (a_{ij})$ be a $k \times k$ matrix with positive entries such that, for all i, j,

$$a_{i,j} \cdot a_{i+1,j+1} \; > \; 4 \cdot \cos^2 \frac{\pi}{k+1} \cdot a_{i,j+1} \cdot a_{i+1,j}$$

Then $\det M > 0$.

Fast growth

Theorem(Katkova-Vishnyakova, 2006): Let $M = (a_{ij})$ be a $k \times k$ matrix with positive entries such that, for all i, j,

$$a_{i,j} \cdot a_{i+1,j+1} \ > \ 4 \cdot \cos^2 \frac{\pi}{k+1} \cdot a_{i,j+1} \cdot a_{i+1,j}$$

Then $\det M > 0$.

Corollary: Let $A = a_0, a_1, a_2, \ldots$ be an infinite sequence such that, for all i,

$$a_{i-1} \cdot a_{i+1} \ \geq \ 4 \cdot a_i^2$$

Then all the Hankel determinants of A are positive.

Theorem(Katkova-Vishnyakova, 2006): Let $M = (a_{ij})$ be a $k \times k$ matrix with positive entries such that, for all i, j,

$$a_{i,j} \cdot a_{i+1,j+1} \; > \; 4 \cdot \cos^2 \frac{\pi}{k+1} \cdot a_{i,j+1} \cdot a_{i+1,j}$$

Then $\det M > 0$.

Corollary: Let $A = a_0, a_1, a_2, \ldots$ be an infinite sequence such that, for all i,

$$a_{i-1} \cdot a_{i+1} \; \geq \; 4 \cdot a_i^2$$

Then all the Hankel determinants of A are positive.

$$1, \; a,$$

Fast growth

Theorem(Katkova-Vishnyakova, 2006): Let $M = (a_{ij})$ be a $k \times k$ matrix with positive entries such that, for all i, j,

$$a_{i,j} \cdot a_{i+1,j+1} \; > \; 4 \cdot \cos^2 \frac{\pi}{k+1} \cdot a_{i,j+1} \cdot a_{i+1,j}$$

Then $\det M > 0$.

Corollary: Let $A = a_0, a_1, a_2, \ldots$ be an infinite sequence such that, for all i,

$$a_{i-1} \cdot a_{i+1} \; \geq \; 4 \cdot a_i^2$$

Then all the Hankel determinants of A are positive.

$$1, \; a, \; 4a^2,$$

Theorem(Katkova-Vishnyakova, 2006): Let $M = (a_{ij})$ be a $k \times k$ matrix with positive entries such that, for all i, j,

$$a_{i,j} \cdot a_{i+1,j+1} \; > \; 4 \cdot \cos^2 \frac{\pi}{k+1} \cdot a_{i,j+1} \cdot a_{i+1,j}$$

Then $\det M > 0$.

Corollary: Let $A = a_0, a_1, a_2, \ldots$ be an infinite sequence such that, for all i,

$$a_{i-1} \cdot a_{i+1} \; \geq \; 4 \cdot a_i^2$$

Then all the Hankel determinants of A are positive.

$$1, \; a, \; 4a^2, \; \ldots, \; 4^{\binom{n}{2}} a^n, \; \ldots$$

Fast growth

Theorem(Katkova-Vishnyakova, 2006): Let $M = (a_{ij})$ be a $k \times k$ matrix with positive entries such that, for all i, j,

$$a_{i,j} \cdot a_{i+1,j+1} \; > \; 4 \cdot \cos^2 \frac{\pi}{k+1} \cdot a_{i,j+1} \cdot a_{i+1,j}$$

Then $\det M > 0$.

Corollary: Let $A = a_0, a_1, a_2, \ldots$ be an infinite sequence such that, for all i,

$$a_{i-1} \cdot a_{i+1} \; \geq \; 4 \cdot a_i^2$$

Then all the Hankel determinants of A are positive.

$$1, \; a, \; 4a^2, \; \ldots, \; 4^{\binom{n}{2}} a^n, \; \ldots$$

This does not seem to apply to any of the "pattern cover" sequences we have seen.

Turning the problem into a graph . . .

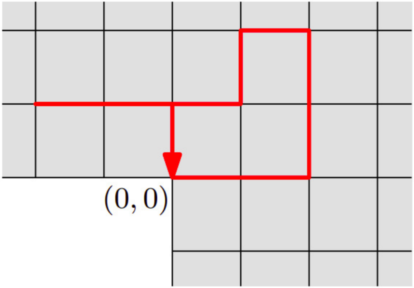
Theorem (Elvey Price–Guttmann, 2019): G a locally finite graph, v a vertex of G, L_n number of loops of length n starting and ending at v. Then $L_0, L_1, L_2, \ldots$ is a moment sequence.

Theorem (Elvey Price–Guttmann, 2019): G a locally finite graph, v a vertex of G, L_n number of loops of length n starting and ending at v. Then $L_0, L_1, L_2, \ldots$ is a moment sequence.

EXAMPLE: 3/4-PLANE EXCURSIONS (A060898)

Definition: Let a_n be the number of $2n$ step walks on the square lattice from $(0,0)$ to $(0,0)$ avoiding the negative quadrant.

By our result, $a_0, a_1, \ldots$ is a Stieltjes moment sequence.



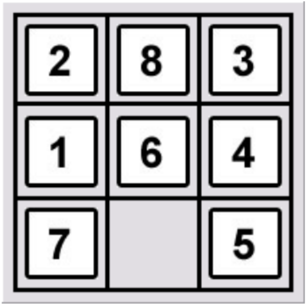
Slide borrowed from Andrew Elvey Price, with kind permission

8-PUZZLE MOVE SEQUENCES (A343146)

Definition: Let a_n be the number of move sequences of the 8 puzzle of length $2n$ leaving final state unchanged.
Claim: This sequence is Stieltjes.
Proof: Consider graph

- One vertex for each possible position of the 8-puzzle
- Edge between vertices for each possible move.

Then a_n is the number of excursions in this graph of length $2n$.



Slide borrowed from Andrew Elvey Price, with kind permission

Thanks!

N. Blitvić and E. Steingrímsson: *Permutations, moments, measures*

Transactions of the AMS, 374 (8) 2021, 5473–5508.

N. Blitvić, S. Kammoun and E. Steingrímsson: Permutations covered by a consecutive pattern, in preparation.