Sequences in Overpartitions

joint work with George E. Andrews

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Partitions

A *partition* π is a finite sequence of *non-increasing* positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_{\#(\pi)})$.

For a given partition $\pi = (\lambda_1, \lambda_2, \ldots, \lambda_{\#(\pi)})$ the sum $\lambda_1 + \lambda_2 + \cdots + \lambda_{\#(\pi)}$ is the *size* of the partition π and it is denoted by $|\pi|$.

<u>Ex:</u>

- $\pi = (5, 1, 1)$ is a partition of $|\pi| = 7$.
- $\pi = \emptyset$ is the unique partition of 0.

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OverPartitions

An *overpartition* π is a finite sequence of *non-increasing* positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_{\#(\pi)})$ where the first instance of a part size may be overlined.

For a given overpartition $\pi = (\lambda_1, \lambda_2, \ldots, \lambda_{\#(\pi)})$ the sum $\lambda_1 + \lambda_2 + \cdots + \lambda_{\#(\pi)}$ is the *size* of the overpartition π and it is denoted by $|\pi|$.

Ex:

- $\pi = (5, \overline{1}, 1)$ is an overpartition of $|\pi| = 7$.
- $\pi = \emptyset$ is the unique overpartition of 0.

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Generating Functions

For a sequence $\{a_n\}_{n=0}^{\infty}$, the series

$$\sum_{n\geq 0} a_n q^n$$

is called a *generating function*.

Let $\mathcal{D}$ be the set of all partitions into non-repeating parts.

$$\sum_{\pi\in\mathcal{D}} q^{|\pi|} = 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 4q^6 + 5q^7 + 6q^8 + 8q^9 \ldots.$$

$\emptyset$	$(2,1),\ (3)$	$(3,2,1),\ (5,1),\ (4,2),\ (6)$
(1)	$(3,1),\ (4)$	$(4,2,1),\ (6,1),\ (5,2),\ (4,3),\ (7)$
(2)	$(4,1),\ (3,2),\ (5)$	$\ldots$

q-Pochhammer Symbol

$$(a;q)_L := \prod_{i=0}^{L-1}(1-aq^i), \text{ and } (a;q)_\infty := \lim_{L\to\infty}(a;q)_L.$$

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$$\sum_{\pi\in\mathcal{D}} q^{|\pi|} = (-q;q)_\infty$$

where $\mathcal{D}$ is the set of all partitions into non-repeating parts.

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where $\mathcal{D}$ is the set of all partitions into non-repeating parts.
Similarly,

$$\sum_{\pi\in\mathcal{U}} q^{|\pi|} = \frac{1}{(q;q)_\infty},$$

where $\mathcal{U}$ is the set of partitions.

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q-Pochhammer Symbol

Basics
- Partitions
- OverPartitions
- Generating Functions
- Rogers–Ramanujan Identities
- Schur's Identity

Sequences in Overpartitions

$$(a;q)_L := \prod_{i=0}^{L-1}(1-aq^i), \text{ and } (a;q)_\infty := \lim_{L\to\infty}(a;q)_L.$$

$$\sum_{\pi\in\mathcal{D}} q^{|\pi|} = (-q;q)_\infty$$

where $\mathcal{D}$ is the set of all partitions into non-repeating parts.
Similarly,

$$\sum_{\pi\in\mathcal{U}_{r,s}} q^{|\pi|} = \frac{1}{(q^r;q^s)_\infty},$$

where $\mathcal{U}_{r,s}$ is the set of partitions where each part is r modulo s.

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q-Pochhammer Symbol

$$(a;q)_L := \prod_{i=0}^{L-1}(1-aq^i), \text{ and } (a;q)_\infty := \lim_{L\to\infty}(a;q)_L.$$

Also

$$\sum_{\pi\in\mathcal{O}} q^{|\pi|} = \frac{(-q;q)_\infty}{(q;q)_\infty}$$

where $\mathcal{O}$ is the set of all overpartitions.

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q-Binomial Coefficients

$$(a;q)_L := \prod_{i=0}^{L-1}(1-aq^i), \text{ and } (a;q)_\infty := \lim_{L\to\infty}(a;q)_L.$$

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q-Binomial Coefficients

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$$(a_1,a_2,\ldots,a_k;q)_L := (a_1;q)_L(a_2;q)_L\ldots(a_k;q)_L.$$

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q-Binomial Coefficients

$$(a;q)_L := \prod_{i=0}^{L-1}(1-aq^i), \text{ and } (a;q)_\infty := \lim_{L\to\infty}(a;q)_L.$$

$$(a_1,a_2,\dots,a_k;q)_L := (a_1;q)_L(a_2;q)_L\dots(a_k;q)_L.$$

We define the q-binomial coefficients as

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q := \begin{cases} \frac{(q;q)_{m+n}}{(q;q)_m(q;q)_n}, & \text{for } m,n\geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Partition Identities

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Partition Identities

Theorem (Rogers–Ramanujan Identities)

For $m = 1, 2$ and $n \in \mathbb{Z}_{\geq 0}$, the number of partitions of n with gaps between parts ≥ 2, all $\geq m$

$=$

the number of partitions of n into $\pm m$ mod 5 parts.

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Theorem (Rogers–Ramanujan Identities)

For $m = 1, 2$, we have

$$\sum_{n\geq 0} \frac{q^{n^2+(m-1)n}}{(q;q)_n} = \frac{1}{(q^m, q^{5-m}; q^5)_\infty}.$$

G. E. Andrews, *The Theory of Partitions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998. Reprint of the 1976 original.

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Partition Identities

Theorem (The First Rogers–Ramanujan Identity)

For any $n \in \mathbb{Z}_{\geq 0}$, the number of partitions of n with gaps between parts ≥ 2

$=$

the number of partitions of n into ± 1 mod 5 parts.

Example: $n = 10$

(10)	$(9,1)$
$(9,1)$	$(6,4)$
$(8,2)$	$(6,1,1,1,1)$
$(7,3)$	$(4,4,1,1)$
$(6,4)$	$(4,1,1,1,1,1,1)$
$(6,3,1)$	$(1,1,1,1,1,1,1,1,1,1)$

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Partition Identities

Theorem (Schur's Partition Identity)

Let $n \in \mathbb{Z}_{\geq 0}$, the number of partitions of n with gaps between parts ≥ 3, with no consecutive multiples of 3 appears

$=$

the number of partitions of n into distinct ± 1 mod 3 parts.

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Partition Identities

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Theorem (Schur's Partition Identity)

$$? = (-q, -q^2; q^3)_\infty .$$

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Basics
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Sequences in Overpartitions

Theorem (Schur's Partition Identity)

Let $n \in \mathbb{Z}_{\geq 0}$, the number of partitions of n with gaps between parts ≥ 3, with no consecutive multiples of 3 appears

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the number of partitions of n into distinct ± 1 mod 3 parts.

Theorem (Schur's Partition Identity)

$$\sum_{m,n\geq 0} (-1)^n \frac{q^{3n(3n+2m)+m(3m-1)/2}}{(q;q)_m(q^6;q^6)_\infty} = (-q,-q^2;q^3)_\infty.$$

G. E. Andrews, K. Bringmann, and K. Mahlburg, *Double Series Representations for Schur's Partition Function and Related Identities*, JCT-A 132, pg 102- 119, (2015).

Two Conjectures

After a computerized search I found these two:

$$\sum_{m,n\geq 0} (-1)^n \frac{q^{\frac{3n(3n+1)}{2}+m^2+3mn}}{(q;q)_m(q^3;q^3)_n} = \frac{1}{(q;q^3)_\infty},$$

$$\sum_{m,n\geq 0} (-1)^n \frac{q^{\frac{3n(3n+1)}{2}+m^2+3mn+m+n}}{(q;q)_m(q^3;q^3)_n} = \frac{1}{(q^2,q^3;q^6)_\infty}.$$

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One Theorem & One Conjecture

After a computerized search I found these two:

Theorem (Andrews-U, 2021)

$$\sum_{m,n\geq 0} (-1)^n \frac{q^{\frac{3n(3n+1)}{2}+m^2+3mn}}{(q;q)_m(q^3;q^3)_n} = \frac{1}{(q;q^3)_\infty}.$$

Conjecture

$$\sum_{m,n\geq 0} (-1)^n \frac{q^{\frac{3n(3n+1)}{2}+m^2+3mn+m+n}}{(q;q)_m(q^3;q^3)_n} = \frac{1}{(q^2,q^3;q^6)_\infty}.$$

G. E. Andrews, and A.K. Uncu *Sequences in Overpartitions*, arXiv:2111.15003

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Theorem (Chern, 2022)

$$\sum_{m,n\geq 0} (-1)^n \frac{q^{\frac{3n(3n+1)}{2}+m^2+3mn+m+n}}{(q;q)_m(q^3;q^3)_n} = \frac{1}{(q^2,q^3;q^6)_\infty}.$$

S. Chern, *Asymmetric Rogers–Ramanujan type identities. I. The Andrews–Uncu Conjecture*, arXiv:2203.15168

Connection to Overpartitions

In the paper, we studied

$$F(i,k;x) = \sum_{m,n\geq 0} \frac{(-1)^n q^{\binom{(2k+1)n+1}{2}+m^2+(2k+1)mn+i(m+n)} x^{m+(2k+1)n}}{(q;q)_m (q^{2k+1};q^{2k+1})_n},$$

and its applications to overpartitions.

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and its applications to overpartitions.The theorems before are related to $i=0,1$, $k=1$, and $x=1$ cases.

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and its applications to overpartitions.The theorems before are related to $i=0,1$, $k=1$, and $x=1$ cases. In particular,

$$F(0,1;1)=\frac{1}{(q;q^3)_\infty}.$$

Connection to Overpartitions

Theorem

The number of overpartitions of n, where for any $k \geq 1$, where $\overline{k} + \overline{(k+1)}$ or $\overline{1} + 2 + \overline{3} + 4 + \overline{5} + \cdots + (2k) + \overline{(2k+1)}$, does not appear

$=$

number of partitions of n into red and green parts where green parts $\equiv 1 \pmod 3$.

Connection to Overpartitions

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The number of overpartitions of n, where for any $k \geq 1$, where $\overline{k} + \overline{(k+1)}$ or $\overline{1} + 2 + \overline{3} + 4 + \overline{5} + \cdots + (2k) + \overline{(2k+1)}$, does not appear

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number of partitions of n into red and green parts where green parts $\equiv 1 \pmod 3$.

For example, when $n = 4$, the 13 overpartitions in the first class are

$$4,\ \overline{4},\ 3+1,\ \overline{3}+1,\ 3+\overline{1},\ \overline{3}+\overline{1},\ 2+2,\ \overline{2}+2,\ 2+1+1,$$
$$\overline{2}+1+1,\ 2+\overline{1}+1,\ 1+1+1+1,\ \overline{1}+1+1+1,$$

and the 13 colored partitions in the second class are

$$4_r,\ 4_g,\ 3_r+1_r,\ 3_r+1_g,\ 2_r+2_r,\ 2_r+1_r+1_r,\ 1_r+1_g+1_r,$$
$$2_r+1_g+1_g,\ 1_r+1_r+1_r+1_r,\ 1_g+1_r+1_r+1_r,$$
$$1_g+1_g+1_r+1_r,\ 1_g+1_g+1_g+1_r,\ 1_g+1_g+1_g+1_g.$$

Connection to Overpartitions

Indeed, for $k, j, N \geq 0$, we will focus on

$$F_N(i,j,k;x,q) = F_N(i,j,k;x) = F_N(i,j,k)$$

$$= \begin{cases} \displaystyle\sum_{m,n\geq 0} (-1)^n q^{\binom{(2k+1)n+1}{2}+m^2+(2k+1)mn+i(m+n)} x^{m+(2k+1)n} \\ \qquad\qquad \times \begin{bmatrix} N-(2k+1)n-m+j \\ m \end{bmatrix}_q \begin{bmatrix} N-2kn-m \\ n \end{bmatrix}_{q^{2k+1}}, & \text{if } N \geq 0, \\ 0, & \text{if } N < 0, \end{cases}$$

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$$\lim_{N\to\infty} F_N(i,j,k;x,q) = F(i,k;x,q).$$

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$$\lim_{N\to\infty} F_N(i,j,k;x,q) = F(i,k;x,q).$$

In the limit, j becomes irrelevant.

What do we know about F_N?

Theorem

$$F_N(i,j,k;x) = F_{N-1}(i,j,k;x) + xq^{N+j+i-1}F_{N-2}(i,j,k;x) - x^{2k+1}q^{(2k+1)(N-k)+i}F_{N-(2k+1)}(i,j,k;x),$$

$$F_N(i,j,k;x) = F_N(i,j-1,k;x) + xq^{N+i+j-1}F_{N-1}(i,j-1,k;x).$$

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$$F_N(i,j,k;x) = F_N(i,j-1,k;x) + xq^{N+i+j-1}F_{N-1}(i,j-1,k;x).$$

Corollary

For $N \geq 1$,

$$F_N(0,1,1;x) = (1+xq^N)F_{N-1}(0,1,1;x) - x^2q^{2N-1}F_{N-2}(0,1,1;x).$$

Proof of the Main Theorem

Theorem

For non-negative integers N, let

$$f_N(q) := \sum_{j\geq 0} q^{3j^2-2j} \begin{bmatrix} N \\ 3j \end{bmatrix}_q (q^2, q^3)_j,$$

then

$$f_{N+1}(q) = F_N(0,1,1;1) - q^N F_{N-1}(0,1,1;1).$$

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Proof follows from holonomic closure properties, Zeilberger's algorithm and checking some initial conditions.

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then

$$f_{N+1}(q) = F_N(0,1,1;1) - q^N F_{N-1}(0,1,1;1).$$

$$\lim_{N\to\infty} f_N(q) = \sum_{j\geq 0} \frac{(q^2;q^3)_j}{(q;q)_{3j}} q^{3j^2-2j} = \sum_{m,n\geq 0} \frac{(-1)^n q^{\frac{3n(3n+1)}{2}+m^2+3mn}}{(q)_m (q^3;q^3)_n}.$$

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More on F_N

We can also prove the following q-difference equations

Theorem

$$F_N(i,0,k;x) = F_{N-1}(i,0,k;xq) + xqF_{N-2}(i,0,k;xq^2) - x^{2k+1}q^{\binom{2k+2}{2}+i}F_{N-(2k+1)}(i,0,k;xq^{2k+1}),$$

$$F_N(0,1,1;x) = (1+xq)F_{N-1}(0,1,1;xq) - x^2q^3F_{N-2}(0,1,1;xq^2).$$

Determinants!

$$F_N(i,0,k;x) =$$

$$\begin{vmatrix}
1 & xq^{1+i} & 0 & 0 & \cdots & 0 & -x^{2k+1}q^{(2k+2)\cdot 1+i} & 0 & \cdots & 0 \\
-1 & 1 & xq^{2+i} & 0 & \cdots & & 0 & -x^{2k+1}q^{(2k+2)\cdot 2+i} & & \vdots \\
0 & -1 & 1 & xq^{3+i} & \ddots & & & & & \\
0 & 0 & -1 & \ddots & \ddots & & & & & \\
\vdots & \vdots & & \ddots & & & & & & \vdots \\
 & & & & & & & & \ddots & 0 \\
 & & & & & & & & & -x^{2k+1}q^{(2k+1)(N-k)+i} \\
 & & & & & & & & \ddots & 0 \\
 & & & & & & & & \ddots & \vdots \\
 & & & & & & \ddots & \ddots & \ddots & 0 \\
 & & & & & & \cdots & 1 & xq^{N-2} & 0 \\
\vdots & & & & & & \cdots & -1 & 1 & xq^{N-1+i} \\
0 & \cdots & & & & & \cdots & 0 & -1 & 1
\end{vmatrix}$$

Determinants!

$$F_N(0,1,1;x) =$$

$$\begin{vmatrix}
1+xq & -x^2q^3 & 0 & \dots & 0 & & & \\
-1 & 1+xq^2 & -x^2q^5 & 0 & \dots & & & \vdots \\
0 & -1 & 1+xq^3 & \ddots & & & & \\
\vdots & \vdots & 0 & \ddots & & & & \\
 & & \vdots & \ddots & & & & \\
 & & & & & & & \vdots \\
 & & & & & \ddots & \ddots & 0 \\
\vdots & & & \ddots & 0 & -1 & 1+xq^{N-1} & -x^2q^{2N-1} \\
0 & \dots & & & \dots & 0 & -1 & 1+xq^N
\end{vmatrix}$$

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Sequences in Overpartitions

Theorem

$$\frac{F(0,k;x)}{(xq;q)_\infty}$$

is the generating function for the overpartitions, where the exponent of x keeps track of the number of parts, in which

i. $\overline{j} + \overline{(j+1)}$ *does not appear,*

ii. *there are no sequences of the form* $\overline{1} + 2 + \overline{3} + 4 + \overline{5} + \cdots + (2k) + \overline{(2k+1)}$.

Sequences in Overpartitions

Theorem

If $i > 0$,

$$\frac{F(i,k;x)}{(xq;q)_\infty}$$

is the generating function for the overpartitions,in which

i. $\overline{j} + \overline{(j+1)}$ *does not appear,*

ii. *the smallest overlined part is* $> i$,

iii. *sequences of the form* $2+3+\cdots+i+\overline{(i+1)}+(i+1)+(i+2)+\overline{(i+3)}+(i+4)+\overline{(i+5)}+\cdots+\overline{(2k)}+(2k+1)$ *if* i *is odd, and* $2+3+\cdots+i+\overline{(i+1)}+(i+1)+(i+2)+\overline{(i+3)}+(i+4)+\overline{(i+5)}+\cdots+(2k)+\overline{(2k+1)}$ *if* i *is even, are excluded.*

Sequences in Overpartitions

Sketch of Proof

$$F_N(i,0,k;x) = F_{N-1}(i,0,k;xq) + xqF_{N-2}(i,0,k;xq^2) - x^{2k+1}q^{\binom{2k+2}{2}+i}F_{N-(2k+1)}(i,0,k;xq^{2k+1})$$

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Sequences in Overpartitions
Sketch of Proof

$$F_N(i,0,k;x) = F_{N-1}(i,0,k;xq) + xqF_{N-2}(i,0,k;xq^2) - x^{2k+1}q^{\binom{2k+2}{2}+i}F_{N-(2k+1)}(i,0,k;xq^{2k+1})$$

$$f(i,k;x) := \frac{F(i,0,k;x)}{(xq;q)_\infty},$$

Sequences in Overpartitions
Sketch of Proof

$$F_N(i,0,k;x) = F_{N-1}(i,0,k;xq) + xqF_{N-2}(i,0,k;xq^2) - x^{2k+1}q^{\binom{2k+2}{2}+i}F_{N-(2k+1)}(i,0,k;xq^{2k+1})$$

$$f(i,k;x) := \frac{F(i,0,k;x)}{(xq;q)_\infty},$$

then

$$f(i,k;x) = \frac{1}{(1-xq)}f(i,k;xq) + \frac{xq^{i+1}}{(1-xq)(1-xq^2)}f(i,k;xq^2) - \frac{x^{2k+1}q^{\binom{2k+2}{2}+i}}{(xq;q)_{2k+1}}f(i,k;xq^{2k+1}).$$

Quick Recall

Recall that we saw these two 3-term relations:

$$F_N(0,1,1;x) = (1+xq^N)F_{N-1}(0,1,1;x) - x^2q^{2N-1}F_{N-2}(0,1,1;x),$$
$$F_N(0,1,1;x) = (1+xq)F_{N-1}(0,1,1;xq) - x^2q^3F_{N-2}(0,1,1;xq^2).$$

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Continued Fractions

$$F_N(0,1,1;x) = (1+xq^N)F_{N-1}(0,1,1;x) - x^2q^{2N-1}F_{N-2}(0,1,1;x)$$

Theorem

For $N \geq 1$,

$$\frac{F_N(0,1,1;x)}{F_{N-1}(0,1,1;x)} = 1 + xq^N - \cfrac{x^2q^{N-1}}{1 + xq^{N-1} - \cfrac{x^2q^{N-2}}{\ddots - \cfrac{\ddots}{1 + xq^2 - \cfrac{x^2q}{1+xq}}}}.$$

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Continued Fractions

$$F_N(0,1,1;x) = (1+xq)F_{N-1}(0,1,1;xq) - x^2q^3F_{N-2}(0,1,1;xq^2)$$

Theorem

For $N \geq 1$,

$$\frac{F_N(0,1,1;x)}{F_{N-1}(0,1,1;xq)} = 1 + xq + \cfrac{x^2q^3}{1+xq^2 - \cfrac{x^2q^5}{\ddots - \cfrac{\ddots}{1+xq^{N-1}+\cfrac{x^2q^{2N-1}}{1+xq^N}}}}.$$

Continued Fractions

$$F(0,1,1;x) = (1+xq)F(0,1,1;xq) - x^2q^3F(0,1,1;xq^2)$$

and a bit of algebra:

Theorem

$$(q;q^3)_\infty \sum_{m,n\geq 0} \frac{(-1)^n q^{\frac{3n(3n+1)}{2}+m^2+3mn+m+3n+1}}{(q)_m(q^3;q^3)_n} = \cfrac{q}{1+q-\cfrac{q^3}{1+q^2-\cfrac{q^5}{1+q^3-\cfrac{q^7}{\ddots}}}}.$$

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Continued Fractions

Theorem (Ramanujan)

$$\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty} = \cfrac{1}{1-\cfrac{q}{1+q-\cfrac{q^3}{1+q^2-\cfrac{q^5}{1+q^3-\cfrac{q^7}{\ddots}}}}}.$$

Some more identities

Theorem

$$\sum_{m,n\geq 0} \frac{(-1)^n q^{\frac{3n(3n+1)}{2}+m^2+3mn+m+3n+1}}{(q;q)_m(q^3;q^3)_n} = \frac{1}{(q;q^3)_\infty} - \frac{1}{(q^2;q^3)_\infty}$$

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Some more identities

Theorem

$$\sum_{m,n\geq 0} \frac{(-1)^n q^{\frac{3n(3n+1)}{2}+m^2+3mn+m+3n+1}}{(q;q)_m(q^3;q^3)_n} = \frac{1}{(q;q^3)_\infty} - \frac{1}{(q^2;q^3)_\infty}$$

Theorem

$$\sum_{m,n\geq 0} \frac{(-1)^n q^{\frac{3n(3n+1)}{2}+m^2+3mn}(1-q^{m+3n+1})}{(q;q)_m(q^3;q^3)_n} = \frac{1}{(q^2;q^3)_\infty}$$

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Thank you for your time

Sequences in Overpartitions

joint work with George E. Andrews

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