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MODELING SEASONAL TIME SERIES

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Abstract. The paper studies the seasonal time series as elements of a (finite dimensional) Hilbert space and proves that it is always better to consider a trend together with a seasonal component even the time series seams not to has one. We give a formula that determines the seasonal component in function of the considered trend that permits to compare the different kind of trends.

1 Preliminary notions

In the following we will consider $\mathcal{ST}_n$, the *vector space of n-time series* (the space $\mathbf{R}^n$ endowed with the canonical structure of $\mathbf{R}$ -vectorial space) ; an element of $\mathcal{ST}_n$ will be denoted by

$$\mathbf{X} = \{x_i\}_n = \{x_{1,\ldots,}x_n\} ,$$

and then

$$\begin{aligned} \mathbf{X}+\mathbf{Y} &= \{x_i\} + \{y_i\} := \{x_i + y_i\} , \\ \alpha\mathbf{X} &= \alpha\{x_i\} := \{\alpha x_i\} . \end{aligned}$$

Particularly, each real number k define a constant time series:

$$\mathbf{K} = \{k\}_n = \{k, ..., k\}$$

Definition 1. a) *If* $\mathbf{X}$ *and* $\mathbf{Y}$ *are two time series, we define the* product *by:*

$$\mathbf{X} \cdot \mathbf{Y} = \{x_i\}_n \{y_i\}_n := \{x_i y_i\}_n ,$$

and their inner product *by:*

$$\langle \mathbf{X}|\mathbf{Y}\rangle := \frac{1}{n}\sum_{i=1}^{n} x_i y_i.$$

The space $\mathcal{ST}_n$ of a n-time series endowed with this scalar product is a Hilbert space *.*

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b) *We associate to a given time series* $\mathbf{X} = \{x_i\}_n$ *:*

- *its* average

$$\overline{\mathbf{X}} := \langle \mathbf{X} | \mathbf{1} \rangle = \frac{1}{n} \sum_{i=1}^{n} x_i,$$

(we observe that scalar product between two time series is the average of their product),

-*its canonical* norm *(defined by the scalar product)*

$$\| \mathbf{X} \| = \sqrt{\langle \mathbf{X} | \mathbf{X} \rangle} := \sqrt{\overline{\mathbf{X}^2}} = \frac{1}{n} \sqrt{\sum_{i=1}^{n} x_i^2},$$

-*its* centred time series*:*

$$\mathbf{nX} := \mathbf{X} - \overline{\mathbf{X}},$$

-*its* variance*:*

$$Var\mathbf{X} := \| \mathbf{nX} \|^2 = \| \mathbf{X} \|^2 - \left(\overline{\mathbf{X}} \right)^2 = \| \mathbf{X} \|^2 - \langle \mathbf{X} | \mathbf{1} \rangle^2 .$$

c) *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be are two time series, we define their* covariance *by:*

$$Cov(\mathbf{X}, \mathbf{Y}) := \overline{\mathbf{X} \cdot \mathbf{Y}} - \overline{\mathbf{X}} \cdot \overline{\mathbf{Y}} = \langle \mathbf{X} | \mathbf{Y} \rangle - \langle \mathbf{X} | \mathbf{1} \rangle \langle \mathbf{Y} | \mathbf{1} \rangle = \langle \mathbf{X} | \mathbf{Y} \rangle - \langle \mathbf{X} | \langle \mathbf{Y} | \mathbf{1} \rangle \rangle = \langle \mathbf{X} | \mathbf{nY} \rangle ,$$

We observe that the covariance is a bilinear form and verifies the following conditions:

$$\begin{aligned} Cov(\mathbf{X}, \mathbf{Y}) &= \langle \mathbf{nX} | \mathbf{Y} \rangle = \overline{\mathbf{nX} \cdot \mathbf{Y}}, \\ Cov(\mathbf{X}, \mathbf{nY}) &= Cov(\mathbf{X}, \mathbf{Y}), \\ Cov(\mathbf{X}, \mathbf{X}) &= Var\mathbf{X}, \\ Var\,(\mathbf{X} \pm \mathbf{Y}) &= Var\mathbf{X} + Var\mathbf{Y} \pm 2Cov(\mathbf{X}, \mathbf{Y}). \end{aligned}$$

Definition 2. *If* $n = ks$*; we will call* s *the* period *and* k *the* numbers of periods*. The time series* $\mathbf{X}$ *will be called a* periodical time series *if* $x_i = x_{s+i} = x_{2s+i} = ... = x_{(k-1)s+i}$ *for every* $i = 1, ..., s$*; that means that a periodical time series is of the form:*

$$\mathbf{C} = \{c_1, ..., c_s\}_n := \{c_1, ..., c_s, c_1, ..., c_s, ... \qquad , c_1, ..., c_s\} .$$

The mean and the norm of such a series are:

$$\overline{\mathbf{C}} = \frac{1}{s} \sum_{i=1}^{s} c_i \quad , \quad \| \mathbf{C} \|^2 = \frac{1}{s} \sum_{i=1}^{s} c_i^2.$$

If $\overline{\mathbf{C}} = 0$ *, a periodical time series will be called a* seasonal time series*.*

Definition 3. *Suppose that $n = ks$. At at a given time series $\mathbf{X} \in \mathcal{ST}_n$ we will associate three time series :*

a) *its* periodical time series*:*

$$\mathbf{cX} := \{c_1(\mathbf{X}), ..., c_s(\mathbf{X})\}_n = \{c_1(\mathbf{X}), ..., c_s(\mathbf{X}), ..., c_1(\mathbf{X}), ..., c_s(\mathbf{X})\} \in \mathcal{ST}_n,$$

where

$$c_i(\mathbf{X}) := \frac{1}{k}\left(x_i + x_{s+i} + x_{2s+i} + ... + x_{(k-1)s+i}\right),$$

We observe that, generally, $\mathbf{c}(\mathbf{X})$ is not a seasonal time series (because $\overline{\mathbf{c}(\mathbf{X})} = \overline{\mathbf{X}}$),

b) *its* seasonal time series*: the normed time series $\mathbf{n}(\mathbf{cX})$ of its periodical time series,*

c) *its* deseasoned time series*:*

$$\mathbf{sX} := \mathbf{X} - \mathbf{cX}.$$

Proposition 4. *Suppose that $n = ks$, and consider a time series $\mathbf{X} \in \mathcal{ST}_n$. Then:*

a) *its periodical time series $\mathbf{cX}$ has the following properties:*

$$\begin{array}{ll}
\overline{\mathbf{c}(\mathbf{X})} = \overline{\mathbf{X}}, & \mathbf{c}(\mathbf{nX}) = \mathbf{n}(\mathbf{cX}), \\
\|\mathbf{cnX}\|^2 = Var(\mathbf{cX}) = \|\mathbf{cX}\|^2 - \left(\overline{\mathbf{X}}\right)^2, & \mathbf{c}(\mathbf{C}) = \mathbf{C}, \\
\overline{\mathbf{X}\cdot\mathbf{cY}} = \overline{\mathbf{cX}\cdot\mathbf{Y}} = \overline{\mathbf{cX}\cdot\mathbf{cY}}, & \overline{\mathbf{X}\cdot\mathbf{cX}} = \overline{\mathbf{cX}\cdot\mathbf{cX}} = \|\mathbf{cX}\|^2, \\
Cov(\mathbf{X}, \mathbf{cX}) = \quad \overline{\mathbf{X}\cdot\mathbf{cX}} - \overline{\mathbf{X}}\cdot\overline{\mathbf{cX}} \quad = & \|\mathbf{cX}\|^2 - \left(\overline{\mathbf{cX}}\right)^2 = Var(\mathbf{cX}).
\end{array}$$

b) *its deseasoned time series $\mathbf{sX}$ has the following properties:*

$$\begin{array}{ll}
\overline{\mathbf{sX}} = 0, & \mathbf{n}(\mathbf{sX}) = \mathbf{sX}, \\
\|\mathbf{sX}\|^2 = \quad Var\mathbf{X} - \mathbf{Var}(\mathbf{cX}) \quad = & \|\mathbf{X}\|^2 - \|\mathbf{cX}\|^2, \\
\mathbf{sC} = \mathbf{0}, & \mathbf{s}(\mathbf{nX}) = \mathbf{n}(\mathbf{sX}) = \mathbf{sX}, \\
\overline{\mathbf{sX}\cdot\mathbf{sY}} = \overline{\mathbf{X}\cdot\mathbf{Y}} - \overline{\mathbf{cX}\cdot\mathbf{cY}} \quad = & Cov(\mathbf{X}, \mathbf{Y}) - Cov(\mathbf{cX}, \mathbf{cY}).
\end{array}$$

Lemma 5. *Suppose that $n = ks$. A time series $\mathbf{X} \in \mathcal{ST}_n$ is a periodical (seasonal) time series if it coincides with its periodical (seasonal) time series.*

2 The problem

In this paragraph we will consider $\mathcal{F} = \{\mathbf{F}_1, ..., \mathbf{F}_k\}$ a finite system of time series and the constant time series $\mathbf{1}$ as elements of the Hilbert space $\mathcal{ST}_n$, then a time series from $v\mathcal{F}$ the subspace generated by $\{\mathbf{1}, \mathbf{F}_1, ..., \mathbf{F}_k\}$ will be of the form $\alpha_0 + \alpha_1\mathbf{F}_1 + ... + \alpha_k\mathbf{F}_k$ $(\alpha_i \in \mathbf{R})$.

At each time series $\mathbf{X}$ and each finite system of vectors $\mathcal{F}$ we will associate:

- $\mathcal{F}_{\mathbf{X}}$, the affine space generated by $\mathbf{n}\mathcal{F} = \{\mathbf{nF}_1, ..., \mathbf{nF}_k\}$ and the constant series $\overline{\mathbf{X}}$:

$$\mathcal{F}_{\mathbf{X}} = \{\overline{\mathbf{X}} + \alpha_1\mathbf{nF}_1 + ... + \alpha_k\mathbf{nF}_k | \alpha_i \in \mathbf{R}\}.$$

- $\boldsymbol{\alpha}$, the colon matrix defined by the coefficients of a time series from $\mathcal{F}_{\mathbf{X}}$:

$$\boldsymbol{\alpha} = (\alpha_1 ... \alpha_k)^t$$

- $Cov(\mathcal{F}, \mathcal{F})$, the matrix of which elements are $Cov(\mathbf{F}_i, \mathbf{F}_j)$, and $Cov(\mathbf{X}, \mathcal{F})$ the colon matrix of which elements are $Cov(\mathbf{X}, \mathbf{F}_i)$.

Lemma 6. *Let $\mathbf{X}$ be a time series, $\mathcal{F} = \{\mathbf{F}_1, ..., \mathbf{F}_k\}$ a finite system of time series and $\mathbf{Y} = \alpha_0 + \alpha_1\mathbf{F}_1 + ... + \alpha_k\mathbf{F}_k$ a time series from the subspace generated by $\{\mathbf{1}, \mathbf{F}_1, ..., \mathbf{F}_k\}$.*

a) *The following assertions are equivalent:*

> **a1)** *the time series $\mathbf{X} - \mathbf{Y}$ is orthogonal on $\mathbf{1}$,*
>
> **a2)** $\alpha_0 = \overline{\mathbf{X}} - \sum_1^k \alpha_i \overline{\mathbf{F}}_i$,
>
> **a3)** *$\mathbf{Y}$ is of the form: $\mathbf{Y} = \overline{\mathbf{X}} + \alpha_1\mathbf{nF}_1 + ... + \alpha_k\mathbf{nF}_k$ (i.e. $\mathbf{Y}$ is an element of the affine space $\mathcal{F}_{\mathbf{X}}$),*

b) *([1]) The Time series $\mathbf{X} - \mathbf{Y}$ is orthogonal on the system $\mathcal{F}$ if and only if $\boldsymbol{\alpha} = (\alpha_1 ... \alpha_k)^t$ verifies the system:*

$$\circledast \quad Cov(\mathbf{X}, \mathcal{F}) = Cov(\mathcal{F}, \mathcal{F}) \cdot \boldsymbol{\alpha}.$$

Proof. **a)** From the orthogonality of $\mathbf{X} - \mathbf{Y}$ on $\mathbf{1}$:

$$0 = \langle \mathbf{X} - \mathbf{Y} | \mathbf{1} \rangle = \overline{\mathbf{X}} - \overline{\mathbf{Y}} = \overline{\mathbf{X}} - \alpha_0 - \sum_1^k \alpha_i \overline{\mathbf{F}}_i,$$

it results that

$$\alpha_0 = \overline{\mathbf{X}} - \sum_1^k \alpha_i \overline{\mathbf{F}}_i,$$

and then $\mathbf{Y}$ is of the form:

$$\mathbf{Y} = \overline{\mathbf{X}} + \sum_{1}^{k} \alpha_i \mathbf{n} \mathbf{F}_i .$$

Obviously, if $\mathbf{Y}$ is of the last form we have

$$\langle \mathbf{X} - \mathbf{Y} | \mathbf{1} \rangle = \left\langle \mathbf{nX} + \sum_{\mathbf{1}}^{\mathbf{k}} \alpha_{\mathbf{i}} \mathbf{n} \mathbf{F}_{\mathbf{i}} | \mathbf{1} \right\rangle = 0.$$

□

Corollary 7. *$\widehat{\mathbf{X}}_{\mathcal{F}}$, the projection of the time series $\mathbf{X}$ onto the subspace generated by $\{\mathbf{1}, \mathbf{F}_1, ..., \mathbf{F}_k\}$, coincides with the projection of $\mathbf{X}$ onto $\mathcal{F}_{\mathbf{X}}$ and it is of the form*

$$\widehat{\mathbf{X}}_{\mathcal{F}} = \overline{\mathbf{X}} + \alpha_1 \mathbf{n} \mathbf{F}_1 + ... + \alpha_k \mathbf{n} \mathbf{F}_k ,$$

*where $\boldsymbol{\alpha} = (\alpha_1 ... \alpha_k)^t$ is the solution of the system ⊛ (Lemma 6.**b**).*

Proof. The time series $\mathbf{X} - \widehat{\mathbf{X}}_{\mathcal{F}}$ is orthogonal on $\mathbf{1}, \mathbf{F}_1, ..., \mathbf{F}_k$ ([4], V1 A), then where α_i are solutions of the system ⊛. □

Definition 8. a) *$\mathcal{F}_{\mathbf{X}}$ (the affine space generated by $\mathbf{n}\mathcal{F}$ and the constant series $\overline{\mathbf{X}}$) will be called a* class of models for X. *Their elements*

$$\mathbf{F} = \overline{\mathbf{X}} + \alpha_1 \mathbf{n} \mathbf{F}_1 + ... + \alpha_k \mathbf{n} \mathbf{F}_k = \alpha_0 + \alpha_1 \mathbf{F}_1 + ... + \alpha_k \mathbf{F}_k \quad , \quad \alpha_0 = \overline{\mathbf{X}} - \sum_{1}^{k} \alpha_i \overline{\mathbf{F}}_i ,$$

will called a models of X,*and the time series $\boldsymbol{\varepsilon}_{\mathbf{F}} = \mathbf{X} - \mathbf{F}$ will called the* error *of the model $\mathbf{F}$; the squared of his norm*

$$\|\varepsilon_{\mathbf{F}}\|^2 = \frac{1}{n} (x_i - f_i)^2 = d^2(\mathbf{X}, \mathbf{F}),$$

being the mean squared errors *for the model $\mathbf{F}$ and will measure the accuracy of the model.*

b) *$\widehat{\mathbf{X}}_{\mathcal{F}}$, the projection of $\mathbf{X}$ onto $\mathcal{F}_{\mathbf{X}}$, (or the the* regressor of X onto $F_{\mathbf{X}}$, *[2]),will be called* best model *of $\mathbf{X}$ from the class of models $\mathcal{F}_{\mathbf{X}}$, and its coefficients will be called the* best coefficients *of the decomposition:*

$$\mathbf{X} = \mathbf{F} + \boldsymbol{\varepsilon}_{\mathbf{F}}, \quad \mathbf{F} \in \mathcal{F}_{\mathbf{X}}.$$

The corresponding error will be denoted by $\widehat{\boldsymbol{\varepsilon}}_{\mathbf{F}}$:

$$\mathbf{X} = \widehat{\mathbf{X}}_{\mathcal{F}} + \widehat{\boldsymbol{\varepsilon}}_{\mathbf{F}}.$$

Remark 9. *The norm of* $\widehat{\varepsilon}_{\mathbf{F}}$ *is the distance from* $\mathbf{X}$ *to the affine space* $\mathcal{F}_{\mathbf{X}}$ *(in the Hilbert space of the* n*-time series).*

$$\|\widehat{\varepsilon}_{\mathbf{F}}\| = \min_{\mathbf{F}\in\mathcal{F}_{\mathbf{X}}} \|\varepsilon_{\mathbf{F}}\| = \min_{\mathbf{F}\in\mathcal{F}_{\mathbf{X}}} \|\mathbf{X}-\mathbf{F}\| = d(\mathbf{X},\mathcal{F}_{\mathbf{X}})$$

Proposition 10. *Let* $\mathbf{X}$ *be a time series,* $\mathcal{F}_{\mathbf{X}}$ *a class of models for* $\mathbf{X}$*, and let* $\mathbf{F}$ *be a model of this class:*

$$\mathbf{X} = \mathbf{F} + \varepsilon_{\mathbf{F}}.$$

Then:

a) *For every* $\mathbf{F} \in \mathcal{F}_{\mathbf{X}}$*, the norm of the error (the mean squared errors) of the decomposition*

$$\mathbf{X} = \mathbf{F} + \varepsilon_{\mathbf{F}}.$$

verifies:

$$\|\varepsilon_{\mathbf{F}}\|^2 = MSE = \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i^2 = Var\mathbf{X} + Var\mathbf{F} - 2Cov(\mathbf{X},\mathbf{F}),$$

b) *The minimum of all these norms (minimum mean squared errors) verifies:*

$$\|\widehat{\varepsilon}_{\mathcal{F}}\|^2 = \left\|\mathbf{X}-\widehat{\mathbf{X}}_{\mathcal{F}}\right\|^2 = \min\{\|\varepsilon_{\mathbf{F}}\|^2 \,|\mathbf{F}\in\mathcal{F}_{\mathbf{X}}\} = \min MSE = Var\mathbf{X} - Var\widehat{\mathbf{X}}_{\mathcal{F}}.$$

Proof. **a)** Because $\overline{\mathbf{X}} = \overline{\mathbf{F}}$:

$$\|\varepsilon_{\mathbf{F}}\|^2 = \|\mathbf{X}-\mathbf{F}\|^2 = Var\,(\mathbf{X}-\mathbf{F}) = Var\mathbf{X} + Var\mathbf{F} - 2Cov(\mathbf{X},\mathbf{F}).$$

b) Because $\mathbf{X}-\widehat{\mathbf{X}}_{\mathcal{F}}$ is orthogonal on $\mathcal{F}$ ([4], [3]), particularly on $\widehat{\mathbf{X}}_{\mathcal{F}}$, and then $0 = \overline{\left(\mathbf{X}-\widehat{\mathbf{X}}_{\mathcal{F}}\right)\cdot\widehat{\mathbf{X}}_{\mathcal{F}}} = \overline{\mathbf{X}\cdot\widehat{\mathbf{X}}_{\mathcal{F}}} - \left\|\widehat{\mathbf{X}}_{\mathcal{F}}\right\|^2$, it results that $Cov(\mathbf{X},\widehat{\mathbf{X}}_{\mathcal{F}}) = Var\widehat{\mathbf{X}}_{\mathcal{F}}$, and then

$$\|\widehat{\varepsilon}_{\mathcal{F}}(\mathbf{X})\|^2 = Var\mathbf{X} - Var\widehat{\mathbf{X}}_{\mathcal{F}}.$$

□

3 The three cases

In the following we will studies the class of the models that contain only a seasonal components $\mathbf{C}$, that contain only a trend $\mathbf{nU}$ and that contain a trend and a seasonal component $\mathbf{C}+\mathbf{bnU}$:

$$\mathcal{C}_{\mathbf{X}} = \{\overline{\mathbf{X}}+\mathbf{C}\} \quad , \quad \mathcal{U}_{\mathbf{X}} = \{\overline{\mathbf{X}}+\mathbf{bnU}\} \quad , \quad (\mathcal{C}+\mathcal{U})_{\mathbf{X}} = \{\overline{\mathbf{X}}+\mathbf{C}+\mathbf{bnU}\}.$$

Using 9 we will find $\widehat{\mathbf{X}}_{\mathcal{F}}$, its coefficients, and the norm of the associated errors for all the three models considered above .

In all the cases the idea is the same: we have to find the minimum of the function $\|\varepsilon_{\mathbf{F}}\|^2$ constrained by the fact that $\overline{\mathbf{C}} = 0$; that is to consider the function :

$$\Phi(\mathbf{b}, \mathbf{C}, \lambda) = \|\varepsilon_{\mathbf{F}}\|^2 - \lambda \overline{\mathbf{C}},$$

and solve the associate system

$$\bigstar \begin{cases} \frac{\partial \Phi}{\partial b} & = 0, \\ \frac{\partial \Phi}{\partial c_i} & = 0 \quad (i = 1, ..., s)\,, \\ \frac{\partial \Phi}{\partial \lambda} & = 0. \end{cases}$$

The solution of this system : $\widehat{\mathbf{b}}_{\mathcal{F}}(\mathbf{X})$,$\widehat{\mathbf{C}}\ _{\mathcal{F}}(\mathbf{X})$will be the *best coefficients* of $\mathbf{X}$ from the class of models $\mathcal{F}$.

Theorem 11. *The best coefficients, the regressors and the minimum squared errors of the three following decompositions are:*

a) *If* $\mathbf{X} = \overline{\mathbf{X}} + \mathbf{C} + \varepsilon$, *then*

$$\begin{aligned} \widehat{\mathbf{C}}_{\mathcal{C}} &= \mathbf{nc}\,(\mathbf{X})\,, \\ \widehat{\mathbf{X}}_{\mathcal{C}} &= \overline{\mathbf{X}} + \mathbf{nc}\,(\mathbf{X}) = \mathbf{cX}, \\ \|\widehat{\varepsilon}_{\mathcal{C}}\|^2 &= \|\mathbf{sX}\|^2 = Var\,(\mathbf{sX})\,. \end{aligned}$$

b) *If* $\mathbf{X} = \overline{\mathbf{X}} + \mathbf{bnU} + \varepsilon, \mathbf{U}$ *being non-constant and non periodical* ($\mathbf{U} \neq \mathbf{cU}$) *time series, then*

$$\begin{aligned} \widehat{\mathbf{b}}_{\mathcal{U}} &= \frac{Cov(\mathbf{U}, \mathbf{X})}{Var\,(\mathbf{U})}, \\ \widehat{\mathbf{X}}_{\mathcal{U}} &= \overline{\mathbf{X}} + \widehat{\mathbf{b}}_{\mathcal{U}} \mathbf{nU}, \\ \|\widehat{\varepsilon}_{\mathcal{U}}\|^2 &= Var\mathbf{X} - \left(\widehat{\mathbf{b}}_{\mathcal{U}}\right)^2 Var\mathbf{U}. \end{aligned}$$

c) *If* $\mathbf{X} = \overline{\mathbf{X}} + \mathbf{bnU} + \mathbf{C} + \varepsilon, \mathbf{U}$ *being non-constant and non periodical* ($\mathbf{U} \neq \mathbf{cU}$) *time series, then*

$$\begin{aligned} \widehat{\mathbf{b}}_{\mathcal{U}+\mathcal{C}} &= \frac{Cov(\mathbf{sU}, \mathbf{sX})}{Var\,(\mathbf{sU})}, \widehat{\mathbf{C}}_{\mathcal{U}+\mathcal{C}} = \mathbf{nc}(\mathbf{X} - \widehat{\mathbf{b}}_{\mathcal{U}+\mathcal{C}} \mathbf{U}), \\ \widehat{\mathbf{X}}_{\mathcal{U}+\mathcal{C}} &= \widehat{\mathbf{b}}_{\mathcal{U}+\mathcal{C}} \mathbf{sU} + \mathbf{cX}, \\ \|\widehat{\varepsilon}_{\mathcal{U}+\mathcal{C}}\|^2 &= Var\mathbf{sX} - \left(\widehat{\mathbf{b}}_{\mathcal{U}+\mathcal{C}}\right)^2 Var\mathbf{sU} = \mathbf{VarsX} - \frac{(\mathbf{Cov}(\mathbf{sU}, \mathbf{sX}))^{\mathbf{2}}}{\mathbf{Var}\,(\mathbf{sU})}. \end{aligned}$$

Proof. **a)** Because $\varepsilon = \mathbf{X} - \left(\overline{\mathbf{X}} + \mathbf{C}\right) = \mathbf{Y} - \mathbf{C}$ (where $\mathbf{Y} = \mathbf{nX} = \mathbf{X} - \overline{\mathbf{X}}$), the function to be considered is

$$\begin{aligned}\Phi &= \|\mathbf{Y}-\mathbf{C}\|^2 - \lambda\overline{\mathbf{C}} = \|\mathbf{Y}\|^2 + \|\mathbf{C}\|^2 - 2\overline{\mathbf{C}\cdot\mathbf{Y}} - \lambda\overline{\mathbf{C}} = \\ &= \|\mathbf{Y}\|^2 + \frac{1}{s}\sum_1^s c_i^2 - \frac{2}{s}\sum_1^s c_i c_i(\mathbf{Y}) - \frac{\lambda}{s}\sum_1^s c_i,\end{aligned}$$

and the system ★ becomes

$$★\begin{cases} \frac{2}{s}c_i - \frac{2}{s}c_i(\mathbf{Y}) - \frac{\lambda}{s} & = 0, \\ -\frac{1}{s}\sum\limits_1^s \quad c_i & = 0. \end{cases}.$$

From the first equation we obtain

$$\widehat{\mathbf{C}}_{\mathcal{C}} - \mathbf{c}\left(\mathbf{Y}\right) = \frac{\widehat{\lambda}}{2},$$

then $\widehat{\lambda}/2 = -\overline{\mathbf{c}\left(\mathbf{X}\right)}$ and

$$\widehat{\mathbf{C}}_{\mathcal{C}} = \mathbf{nc}\left(\mathbf{Y}\right) = \mathbf{nc}\left(\mathbf{nX}\right) = \mathbf{nc}\left(\mathbf{X}\right).$$

Evidently

$$\widehat{\mathbf{X}}_{\mathcal{C}} = \overline{\mathbf{X}} + \widehat{\mathbf{C}}_{\mathcal{C}} = \overline{\mathbf{X}} + \mathbf{nc}\left(\mathbf{X}\right) = \mathbf{cX},$$

and

$$\|\widehat{\varepsilon}_{\mathcal{C}}\|^2 = Var\mathbf{X} - Var\mathbf{cX} = Var\mathbf{sX}.$$

b) Because $\mathbf{Y} = \mathbf{nX} = \mathbf{bnU} + \varepsilon$, it results that the function to be considered is

$$\Phi = \|\mathbf{Y} - \mathbf{bnU}\|^2 = \|\mathbf{Y}\|^2 + \mathbf{b}^2 Var\mathbf{U} - \mathbf{2b}Cov(\mathbf{U},\mathbf{Y}),$$

and the system ★ has a single equation

$$★ \quad \mathbf{2b}Var\mathbf{U} - \mathbf{2}Cov(\mathbf{U},\mathbf{Y}) = 0.$$

$\mathbf{U}$ being non-constant, $Var\mathbf{U} \neq 0$, and then

$$\widehat{\mathbf{b}}_{\mathcal{U}} = \frac{Cov(\mathbf{U},\mathbf{Y})}{Var\mathbf{U}} = \frac{Cov(\mathbf{U},\mathbf{X})}{Var\mathbf{U}}.$$

Evidently, the regressor of $\mathbf{X}$ is

$$\widehat{\mathbf{X}}_{\mathcal{U}} = \overline{\mathbf{X}} + \widehat{\mathbf{b}}_{\mathcal{U}}\mathbf{nU},$$

and then

$$\|\widehat{\varepsilon}_{\mathcal{U}}\|^2 = Var\mathbf{X} - Var\left(\overline{\mathbf{X}} + \widehat{\mathbf{b}}_{\mathcal{U}}\mathbf{nU}\right) = Var\mathbf{X} - \left(\widehat{\mathbf{b}}_{\mathcal{U}}\right)^2 Var\mathbf{U}.$$

c) Because $\mathbf{Y} = \mathbf{nX} = \mathbf{bnU} + \mathbf{C} + \varepsilon$, the function Φ becomes

$$\begin{aligned}\Phi &= \|\mathbf{Y} - \mathbf{bnU} - \mathbf{C}\|^2 - \lambda\overline{\mathbf{C}} = \\ &= \|\mathbf{Y}\|^2 + \mathbf{b}^2 VarU + \|\mathbf{C}\|^2 - \mathbf{2b}Cov(\mathbf{U},\mathbf{Y}) - \mathbf{2}\overline{\mathbf{C}\cdot\mathbf{Y}} + \mathbf{2b}Cov(\mathbf{U},\mathbf{C}) - \lambda\overline{\mathbf{C}} = \\ &= \|\mathbf{Y}\|^2 + \mathbf{b}^2 VarU + \frac{1}{s}\sum_{i=1}^{s} c_i^2 - \mathbf{2b}Cov(\mathbf{U},\mathbf{Y}) - \frac{2}{s}\sum_{i=1}^{s} c_i c_i(\mathbf{Y}) + \\ &\quad +2\mathbf{b}\left(\frac{1}{s}\sum_{i=1}^{s} c_i c_i(\mathbf{U}) - \overline{\mathbf{N}}\frac{1}{s}\sum_{i=1}^{s} c_i\right) - \frac{\lambda}{s}\sum_{i=1}^{s} c_i,\end{aligned}$$

and the system $\bigstar$:

$$\bigstar\begin{cases} 2\mathbf{b}Var\mathbf{U} - 2Cov(\mathbf{U},\mathbf{Y}) + 2Cov(\mathbf{U},\mathbf{C}) & = 0, \\ \frac{2}{s}c_i - \frac{2}{s}c_i(\mathbf{Y}) + \frac{2\mathbf{b}}{s}\left(c_i(\mathbf{U}) - \overline{\mathbf{U}}\right) - \frac{\lambda}{s} & = 0 \quad (i = 1, ..., s), \\ -\frac{1}{s}\sum_{i=1}^{s} c_i & = 0. \end{cases}$$

From the first equation we obtain

$$\widehat{\mathbf{b}}_{\mathcal{U}+\mathcal{C}} Var\mathbf{U} = Cov(\mathbf{U},\mathbf{Y}) - Cov(\mathbf{U},\widehat{\mathbf{C}}_{\mathcal{U}+\mathcal{C}}),$$

and from the second

$$\widehat{\mathbf{C}}_{\mathcal{U}+\mathcal{C}} - \mathbf{c}(\mathbf{Y}) + \widehat{\mathbf{b}}_{\mathcal{U}+\mathcal{C}}\mathbf{ncU} = \frac{\widehat{\lambda}}{2}.$$

Then $\widehat{\lambda}/2 = -\overline{\mathbf{c}(\mathbf{Y})}$, and

$$\widehat{\mathbf{C}}_{\mathcal{U}+\mathcal{C}} = \mathbf{nc}(\mathbf{Y} - \widehat{\mathbf{b}}_{\mathcal{U}+\mathcal{C}}\mathbf{U}) = \mathbf{nc}(\mathbf{X} - \widehat{\mathbf{b}}_{\mathcal{U}+\mathcal{C}}\mathbf{U}).$$

Because

$$Cov(\mathbf{U},\widehat{\mathbf{C}}_{\mathcal{U}+\mathcal{C}}) = \overline{\left(\mathbf{nc}(\mathbf{Y} - \widehat{\mathbf{b}}_{\mathcal{U}+\mathcal{C}}\mathbf{U})\right)\cdot\mathbf{N}} = Cov(\mathbf{U},\mathbf{cY}) - \widehat{\mathbf{b}}_{\mathcal{U}+\mathcal{C}}(\mathbf{X})Var(\mathbf{cU}),$$

and

$$\widehat{\mathbf{b}}_{\mathcal{U}+\mathcal{C}}\left[Var\mathbf{U} - Var(\mathbf{cU})\right] = Cov(\mathbf{U},\mathbf{Y}) - Cov(\mathbf{U},\mathbf{cY}) = Cov(\mathbf{sU},\mathbf{sY}),$$

we have

$$\widehat{\mathbf{b}}_{\mathcal{U}+\mathcal{C}} = \frac{Cov(\mathbf{sU},\mathbf{sX})}{Var(\mathbf{sU})}.$$

We used the fact that $\mathbf{U}$ is nonseasonal and then $Var\,(\mathbf{sU}) = Var\,(\mathbf{sU}) - Var\,(\mathbf{sU})\,.$ Evidently, the regressor of $\mathbf{X}$ is

$$\widehat{\mathbf{X}}_{\mathcal{U}+\mathcal{C}}{=}\overline{\mathbf{X}}{+}\widehat{\mathbf{b}}_{\mathcal{U}+\mathcal{C}}\mathbf{nU} + \widehat{\mathbf{C}}_{\mathcal{U}+\mathcal{C}}{=}\widehat{\mathbf{b}}_{\mathcal{U}+\mathcal{C}}\mathbf{sU} + \mathbf{cX},$$

and then

$$\begin{aligned}\|\widehat{\varepsilon}_{\mathcal{U}+\mathcal{C}}\|^2 &= Var\mathbf{X}{-}Var\mathbf{cX}{-}\left(\widehat{\mathbf{b}}_{\mathcal{U}+\mathcal{C}}\right)^2 Var\mathbf{sU} = \\ &= Var\mathbf{X}{-}\left(\widehat{\mathbf{b}}_{\mathcal{U}+\mathcal{C}}\right)^2 Var\mathbf{sU} = Var\mathbf{sX}{-}\frac{(\mathbf{Cov}(\mathbf{sU},\mathbf{sX}))^{\mathbf{2}}}{\mathbf{Var}\,(\mathbf{sU})}.\end{aligned}$$

□

4 Comparisons

In this section we will determine the difference between:

- the regressions of a seasonal time series, if we consider or not a trend,
- the regression of a seasonal time series with a trend and the model obtained by the Cesus decomposition.

Theorem 12. *For a given time series* $\mathbf{X}$*, a regression to a model that considers a trend* $\mathbf{U}$ *and a seasonal component* $\mathbf{C}$ *is better that the one that considers only a seasonal component; that means that the error given by the decomposition*

$$\mathbf{X} = \overline{\mathbf{X}}{+}\mathbf{C}{+}\varepsilon,$$

is always greater than that given by

$$\mathbf{X} = \overline{\mathbf{X}}{+}\mathbf{bnU} + \mathbf{C}{+}\varepsilon.$$

The two errors are equal if and only if

$$Cov(\mathbf{sU},\mathbf{sX}) = 0,$$

or equivalent, if and only if $Cov(\mathbf{U},\mathbf{X}) = Cov(\mathbf{cU},\mathbf{cX})$*, or* $\overline{\mathbf{UX}}{=}\overline{\mathbf{cUcX}}$*.*

Proof. Using Theorem 11 we obtain:

$$\|\widehat{\varepsilon}_{\mathcal{U}+\mathcal{C}}(\mathbf{X})\|^2 = Var\mathbf{sX}{-}\frac{(\mathbf{Cov}(\mathbf{sU},\mathbf{sX}))^{\mathbf{2}}}{\mathbf{Var}\,(\mathbf{sU})}{\leq}Var\mathbf{sX} = \|\widehat{\varepsilon}_{\mathcal{C}}(\mathbf{X})\|^2\,,$$

and evidently $\|\widehat{\varepsilon}_{\mathcal{U}+\mathcal{C}}(\mathbf{X})\| = \|\widehat{\varepsilon}_{\mathcal{C}}(\mathbf{X})\|$ if and only if $Cov(\mathbf{sU},\mathbf{sX}) = Cov(\mathbf{U},\mathbf{X}){-}Cov(\mathbf{cU},\mathbf{cX}) = 0.$

We can also obtain the same results using the Hilbert space of n-time series and observe that the subspace $\mathcal{U}_{\mathbf{X}}{+}\mathcal{C}_{\mathbf{X}}$ is a direct sum (that means that $\mathcal{U}_{\mathbf{X}}{\cap}\mathcal{C}_{\mathbf{X}} = \left\{\overline{\mathbf{X}}\right\},$

or equivalent their elements are on the unique form: $\overline{\mathbf{X}}+\mathbf{bnU}+\mathbf{C})$: because $\mathcal{C}_{\mathbf{X}}$ is a affine subspace of $\mathcal{U}_{\mathbf{X}}+\mathcal{C}_{\mathbf{X}}$, the distance of $\mathbf{X}$ to $\mathcal{C}_{\mathbf{X}}$ is greater than the distance of $\mathbf{X}$ to $\mathcal{U}_{\mathbf{X}}+\mathcal{C}_{\mathbf{X}}$, and then:

$$\left\|\widehat{\varepsilon}_{\mathcal{U}+\mathcal{C}}\left(\mathbf{X}\right)\right\| = d(\mathbf{X},\mathcal{U}_{\mathbf{X}}+\mathcal{C}_{\mathbf{X}}) \leq d(\mathbf{X},\mathcal{C}_{\mathbf{X}}) = \left\|\widehat{\varepsilon}_{\mathcal{C}}\left(\mathbf{X}\right)\right\|.$$

□

Remark 13. *Let's now consider a time series* $\mathbf{X}$. *Its seasonal component* $\mathbf{C}'$ *(as is used in Census decomposition) is exactly the best seasonal components (as it results from 3.1.a)):*

$$\mathbf{C}' = \mathbf{ncX} = \widehat{\mathbf{C}}_{\mathcal{C}}(\mathbf{X}),$$

and his deseasoned time series will be $\mathbf{Z} = \mathbf{X} - \mathbf{C}' = \mathbf{X} - \mathbf{ncX} = \mathbf{sX} + \overline{\mathbf{X}}$. *The regressor* $\widehat{\mathbf{Z}}_{\mathcal{U}} = \overline{\mathbf{Z}}+\widehat{\beta}_{\mathcal{U}}(\mathbf{Z})\mathbf{U}$ *of* $\mathbf{Z}$ *to the subspace* $\mathcal{U}_{\mathbf{Z}}$ *gives us a model for* $\mathbf{X}$ *from* $\mathcal{U}_{\mathbf{X}}+\mathcal{C}_{\mathbf{X}}$*:*

$$\mathbf{X}' = \widehat{\mathbf{Z}}_{\mathcal{U}} + \widehat{\mathbf{C}}_{\mathcal{C}} = \overline{\mathbf{Z}}+\widehat{\beta}_{\mathcal{U}}(\mathbf{Z})\mathbf{nU} + \mathbf{ncX} = \overline{\mathbf{X}}+\widehat{\beta}_{\mathcal{U}}(\mathbf{Z})\mathbf{nU} + \mathbf{ncX} \in \mathcal{U}_{\mathbf{X}}+\mathcal{C}_{\mathbf{X}}.$$

Evidently, then the norm of the is greater than the norm of $\widehat{\varepsilon}_{\mathcal{U}+\mathcal{C}} = \mathbf{X}-\widehat{\mathbf{X}}_{\mathcal{U}+\mathcal{C}}$ *(the error associate at the best model):*

$$\left\|\widehat{\varepsilon}_{\mathcal{U}+\mathcal{C}}\left(\mathbf{X}\right)\right\| = \min_{\mathbf{F}\in\mathcal{U}+\mathcal{C}} \left\|\varepsilon_{\mathbf{F}}\right\| \leq \left\|\varepsilon_{\mathbf{X}'}\right\|.$$

In the following proposition we will calculate the norm obtained by the Census decomposition (the mean squared error obtained by the Census method) and determine the case when it coincide with the norm of the error obtained by the regression.

Proposition 14. *In the conditions of 13 - the mean of the squared error of the model obtained by the Census model is*

$$** \quad \left\|\varepsilon_{\mathbf{X}'}\right\|^2 = Var\mathbf{sX} - \frac{\left(Cov(\mathbf{sU},\mathbf{sX})\right)^2}{Var\left(\mathbf{U}\right)},$$

it is evidently greater that the error of the regressor of $\mathbf{X}$ *to the affine space* $\mathcal{U}_{\mathbf{X}}+\mathcal{C}_{\mathbf{X}}$ *obtained in the Proposition 3.1.c). The norm of these errors are equals if and only if the periodical time series* $\mathbf{cU}$ *associate at the trend* $\mathbf{U}$ *is a constant time series.*

Proof. First, we observe that $\mathbf{Z} = \mathbf{X} - \mathbf{C}' = \mathbf{X} - \mathbf{ncX} = \mathbf{sX} + \overline{\mathbf{X}}$ and then $\overline{\mathbf{Z}} = \overline{\mathbf{X}}$. The coefficient of the regressor of $\mathbf{Z}$ to the affine space $\mathcal{U}_{\mathbf{X}}$ becomes:

$$\widehat{\beta}_{\mathcal{U}}(\mathbf{Z}) = \frac{Cov(\mathbf{U},\mathbf{Z})}{Var\left(\mathbf{U}\right)} = \frac{Cov(\mathbf{U},\mathbf{sX}+\overline{\mathbf{X}})}{Var\left(\mathbf{U}\right)} = \frac{Cov(\mathbf{sU},\mathbf{sX})}{Var\left(\mathbf{U}\right)}.$$

The model

$$\mathbf{X}' = \widehat{\mathbf{Z}}_{\mathcal{U}} + \mathbf{C}' = \overline{\mathbf{X}} + \widehat{\beta}_{\mathcal{U}}(\mathbf{Z})\mathbf{nU} + \mathbf{ncX} \in \mathcal{U}_{\mathbf{X}} + \mathcal{C}_{\mathbf{X}}.$$

being an element of the subspace $\mathcal{U}+\mathcal{C}$ the norm of his error $\boldsymbol{\varepsilon}' = \mathbf{X} - \mathbf{X}'$ is evidently greater than that of the regressor of $\mathbf{X}$ to the subspace $\mathcal{U}+\mathcal{C}$ (Proposition 3.1.c):

$$\left\|\widehat{\varepsilon}_{\mathcal{U}+\mathcal{C}}\right\|^2 = \left\|\mathbf{X} - \widehat{\mathbf{X}}_{\mathcal{U}+\mathcal{C}}\right\|^2 = Var\mathbf{sX} - \frac{(Cov(\mathbf{sU}, \mathbf{sX}))^2}{Var(\mathbf{sU})}.$$

Evidently the two errors coincide if and only if $Var(\mathbf{U}) = Var(\mathbf{sU})$, that means if and only if $Var(\mathbf{cU}) = 0$. □

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