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SPECIAL FEATURES IN TURBULENT MIXING. COMPARISON BETWEEN PERIODIC AND NON PERIODIC CASE

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Abstract. After hundreds of years of stability study, the problems of flow kinematics are far from complete solving. A modern theory appears in this field: the mixing theory. Its mathematical methods and techniques developed the significant relation between turbulence and chaos. The turbulence is an important feature of dynamic systems with few freedom degrees, the so-called "far from equilibrium systems". These are widespread between the models of excitable media.

Studying a mixing for a flow implies the analysis of successive stretching and folding phenomena for its particles, the influence of parameters and initial conditions. In the previous works, the study of the 3D non-periodic models exhibited a quite complicated behavior, involving some significant events - the so-called "rare events". The variation of parameters had a great influence on the length and surface deformations. The 2D (periodic) case is simpler, but significant events can also issue for irrational values of the length and surface versors, as was the situation in 3D case.

The comparison between 2D and 3D case revealed interesting properties; therefore a modified 2D (periodic) model is tested. The numerical simulations were realized in MapleVI, for searching special mathematical events. Continuing this work both from analytical and numeric standpoint would relieve useful properties for the turbulent mixing. A proximal target is to test some special functions for the periodic model, and to study the behavior of the structures realized by the model.

1 Introduction

The turbulence term can be defined as "chaotic behavior of far from equilibrium systems, with very few freedom degrees". In this area there are two important theories:

a) The transition theory from smooth laminar flows to chaotic flows, characteristic to turbulence.

b) Statistic studies of the complete turbulent systems.

The statistical idea of flow is represented by the map:

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(1) $x = \Phi_t(X),\ \ with\ X = \Phi_t(t=0)(X)\,.$

We say that X *is mapped in* x after a time t.

In the continuum mechanics the relation (1) is named *flow*, and it must be of class C^k. From the dynamic standpoint we have a diffeomorphism of class C^k. Moreover, (1) must satisfy the relation:

(2) $0\ \langle J\ \langle \infty,\ J = \det\left(\frac{\partial x_i}{\partial X_j}\right),$

where D denotes the derivation with respect to the reference configuration, in this case X. The relation (2) implies two particles, X_1 and X_2, which occupy the same position x at a moment. Non-topological behavior (like break up, for example) *is not allowed*.

The basic measure for the deformation with respect to X is the deformation gradient, F:

(3) $\mathbf{F} = \left(\nabla_X \Phi_t(\mathbf{X})\right)^T,\ \ F_{ij} = \left(\frac{\partial x_i}{\partial X_j}\right)$

where ∇_X denotes differentiation with respect to X. According to (2), $\mathbf{F}$ is non singular. The basic measure for the deformation with respect to x is the *velocity gradient* (∇_x denote differentiation with respect to x).

The above relations allow the definition of the basic deformation for a material filament and for the area of an infinitesimal material surface.

Let us define the basic deformation measures: the *length deformation* λ and *surface deformation* η, with the relations [3], [4].

(4) $\lambda = \lim\limits_{|d\mathbf{X}|\to 0} \frac{|d\mathbf{x}|}{|d\mathbf{X}|},\ \ \eta = \lim\limits_{|d\mathbf{A}|\to 0} \frac{|d\mathbf{a}|}{|d\mathbf{A}|}$

which are obtained from

(5) $\lambda = (C : MM)^{\frac{1}{2}}\,,\ \eta = (\det F)\cdot\left(C^{-1} : NN\right)^{\frac{1}{2}}$

with $\mathbf{C}(= \mathbf{F}^T\cdot\mathbf{F})$ the Cauchy-Green deformation tensor, and the vectors M, N defined by

(6) $\mathbf{M} = d\mathbf{X}/\,|d\mathbf{X}|\,,\ \mathbf{N} = d\mathbf{A}/\,|d\mathbf{A}|$

The relation (5) has the scalar form:

(7) $\lambda = C_{ij}\cdot M_i\cdot N_j,\quad \eta = (\det F)\cdot\left(C_{ij}^{-1}\cdot N_i\cdot N_j\right),\ \ with\ \sum M_i^2 = 1,\ \sum N_j^2 = 1$

In this framework the mixing concept implies the stretching and folding of the material elements. If in an initial location P there is a material filament dX and an area element dA, the specific length and surface deformations are given by the relations:

(8) $\frac{D(\ln\lambda)}{Dt} = \mathbf{D} : \mathbf{mm},\ \ \frac{D(\ln\eta)}{Dt} = \nabla\mathbf{v} - \mathbf{D} : \mathbf{nn}$

where D is the deformation tensor, obtained by decomposing the velocity gradient in its symmetric and non-symmetric part.

We say that the flow $\mathbf{x} = \Phi_t(\mathbf{X})$ has *a good mixing* if the mean values $D(ln\lambda)/Dt$ and $D(ln\eta)/Dt$ are not decreasing to zero, for any initial position P and any initial orientations $\mathbf{M}$ and $\mathbf{N}$.

As the above two quantities are bounded, the deformation efficiency can be naturally quantified. Thus, there is defined [3] the *deformation efficiency in length*, $e_\lambda = e_\lambda(X, M, t)$ of the material element dX, as:

(9) $e_\lambda = \frac{D(ln\lambda)/Dt}{(\mathbf{D}{:}\mathbf{D})^{1/2}} \leq 1,$

and similarly, the *deformation efficiency in surface*, $e_\eta = e_\eta(X, N, t)$ of the area element dA: in the case of an isochoric flow (the jacobian equal 1), we have:

(10) $e_\eta = \frac{D(ln\eta)/Dt}{(\mathbf{D}{:}\mathbf{D})^{1/2}} \leq 1.$

2 The tendril-whorl flow. The model and results

As shown in [3] and [4], two-dimensional flows increase their length by forming two basic kinds of structures: tendrils and whorls and their combinations. In complex two-dimensional fluid flows we can encounter tendrils within tendrils, whorls within whorls, and all other possible combinations. The tendril-whorl flow (TW) introduced by Khakhar, Rising and Ottino (1987) is a discontinuous succession of extensional flows and twist maps. In the simplest case all the flows are identical and the period of alternation extensional/ rotational is also constant. But even the simplest case is complex enough and, on the other hand, it can be considered as the point of departure for several generalizations (smooth variation, distribution of time periods, etc.).

In the simplest case of the TW model, the velocity field over a single period is given by its extensional part:

(11) $v_x = -\varepsilon \cdot x,$

$v_y = \varepsilon \cdot y, \quad 0 < t < T_{ext}$

and its rotational part:

$(11)_b$ $v_r = 0,$

$v_\theta = -\omega(r), \; T_{ext} < t < T_{ext} + T_{rot},$

where T_{ext} denotes the duration of the extensional component and T_{rot} the duration of rotational component.

The model consists of vortices producing whorls which are periodically squeezed by the hyperbolic flow leading to the formation of tendrils, and the process repeats. The function $\omega(r)$ is positive and specifies the rate of rotation. Its form is quite arbitrary and an important aspect is that it has a maximum, that is, $d\omega(r)/dr = 0$ for some r.

It were studied the deformation efficiencies in length and surface, e_λ and e_η only for the extensional component, for the moment. For the extensional part (11)a of TW model, the tensors $\mathbf{F}$ and $\mathbf{C}^{-1}$ have quite simple form:

(12) $\mathbf{F} = \begin{pmatrix} \exp(-\varepsilon \cdot T_{ext}) & 0 \\ 0 & \exp(\varepsilon \cdot T_{ext}) \end{pmatrix},$

$$\mathbf{C}^{-1} = \begin{pmatrix} \exp(2\varepsilon \cdot T_{ext}) & 0 \\ 0 & \exp(-2\varepsilon \cdot T_{ext}) \end{pmatrix}$$

It is useful to note that in three dimensions, there were found rather complicated expressions [1], depending on few parameters.

Therefore, the deformations λ^2 and η^2, in length and surface, have a similar form. It was found:

$$(13)\quad \lambda^2 = \exp(-2\varepsilon \cdot T_{ext}) \cdot M_1^2 + \exp(2\varepsilon \cdot T_{ext}) \cdot M_2^2, \\ \eta^2 = \exp(2\varepsilon \cdot T_{ext}) \cdot N_1^2 + \exp(-2\varepsilon \cdot T_{ext}) \cdot N_2^2.$$

In this context, the deformation efficiencies have the following expressions:

$$(14)\ e_\lambda = 2\varepsilon \cdot \left(1 - \frac{2\exp(-2\varepsilon T_{ext}) \cdot M_1^2}{\exp(-2\varepsilon T_{ext}) \cdot M_1^2 + \exp(2\varepsilon T_{ext}) \cdot M_2^2}\right)$$

$$(15)\quad e_\eta = 2\varepsilon \cdot \left(1 - \frac{2\exp(-2\varepsilon T_{ext}) \cdot N_2^2}{\exp(-2\varepsilon T_{ext}) \cdot N_2^2 + \exp(2\varepsilon T_{ext}) \cdot N_1^2}\right)$$

Let us note that in order to put in practice the relations (9), (10) the following formula was used [3]:

$$e_\lambda = \frac{1}{2\lambda^2} \cdot \frac{d\lambda^2}{dt}, e_\eta = \frac{1}{2\eta^2} \cdot \frac{d\eta^2}{dt}, with\ \sum M_i^2 = 1,\ \sum N_j^2 = 1$$

for the versors.

The relations (14) and (15) give two functions of time, depending on the parameters ε, M_i, N_j, $0 < \varepsilon < 1$. T_{ext} represents the duration of the extensional component of TW model. It is a time period which can vary in a discrete range. Also, the relations show similar forms for the efficiencies, since λ^2 and η^2 are similar.

3 Graphical analysis

The central target is to exhibit some analytical features of 2D (periodic) flow, in comparison with 3D (non-periodic) flow. In [1] it was pointed out that the deformation efficiencies in length and surface represent *strongly nonlinear phenomena*. In order to see the behavior in the 2D case, there were considered few *irrational* (equal) cases for the length and surface versors:

(a) $(M_1, M_2) = (N_1, N_2) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$; (b) $(M_1, M_2) = (N_1, N_2) = \left(\frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}\right)$; (c) $(M_1, M_2) = (N_1, N_2) = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$

and for the parameter $0\langle\varepsilon\langle 1$ there were considered two values: $\varepsilon_1 = 0.05$, $\varepsilon_2 = 0.08$. Thus, there, were identified six cases, and calculating e_λ and e_η for each of them involved some differential equations in twelve situations. For the moment we are interested only in comparing the surface deformations. Therefore we shall expose

four of these cases, all nonlinear. The differential equations are the following, with the index i corresponding to the parameter ε_i:

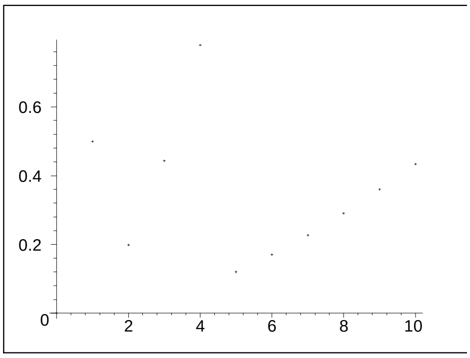
$$a1)\ e_\eta = 0.1 \cdot \left(1 - \frac{2 \cdot \exp(-0.1 \cdot T)}{\exp(-0.1 \cdot T) + \exp(0.1 \cdot T)}\right);$$

$$b2)\ e_\eta = 1.6 \cdot \left(1 - \frac{\frac{4}{3}\cdot\exp(-1.6\cdot T)}{\frac{1}{3}\exp(1.6\cdot T)+\frac{2}{3}\exp(-1.6\cdot T)}\right);$$

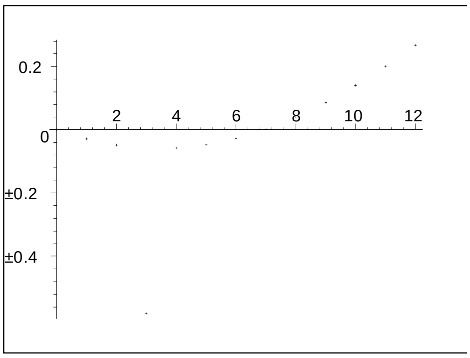
$$c1)\ e_\eta = 0.1 \cdot \left(1 - \frac{\frac{8}{5}\cdot\cdot\exp(-0.1\cdot T)}{\frac{1}{5}\cdot\exp(0.1\cdot T)+\frac{4}{5}\cdot\exp(-0.1\cdot T)}\right);$$

$$c2)\ e_\eta = 1.6 \cdot \left(1 - \frac{\frac{8}{5}\cdot\exp(-1.6\cdot T)}{\frac{1}{5}\exp(1.6\cdot T)+\frac{4}{5}\exp(-1.6\cdot T)}\right).$$

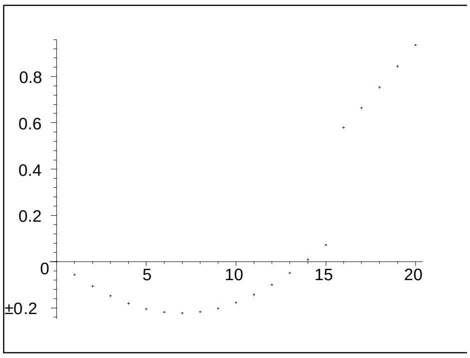
The other differential equations can be found in [2]. For solving these differential equations, the Maple numeric procedure *Dsolve* was used, obtaining in each case a list of pairs $(t_i, x(t_i))$. The index i goes from 1 to 15, for the moment, as we are not interested to give very few values for the period T_{ext}. Only few cases contain more time period, where it was necessary to outline the nonlinearity. Finally, using the plot lists there were realized discrete plots with the Maple function *Pointplot*. Between the plots obtained, three are nonlinear. We shall expose the plots for the deformation efficiency in surface, corresponding to the above cases, in the order a1, b2, c1, c2:



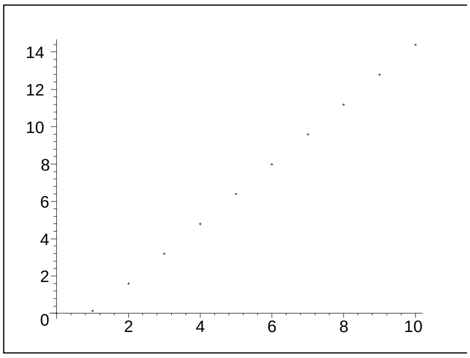
Picture 1 - a1



Picture 2 - b2



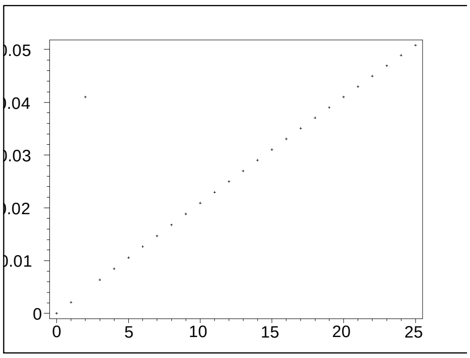
Picture 3 - c1



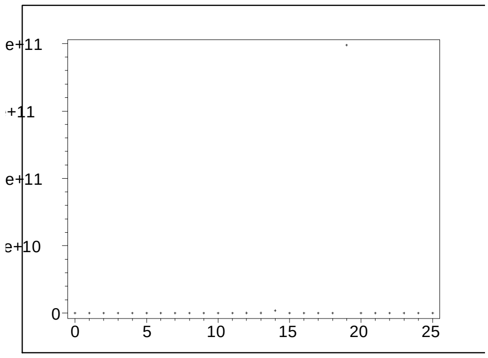
Picture 4 - c2

In order to compare this behavior with the non- periodic case, let us extract from the 3D analysis, the surface deformation for the two-dimensional component of the model [1]. Since the non periodic model contains a lot of (non-linear) parameters, it suffices to focus only on a simple case of the surface versors, namely $(1, 0)$. We note that this case is presented only for comparison, in [1] and [5] are studied much more (about 60) statistical cases, for revealing the non linear behavior.

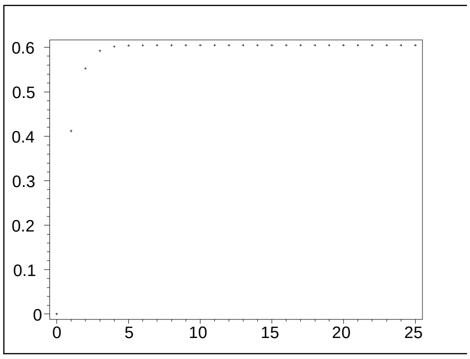
We shall expose only four of these plots:



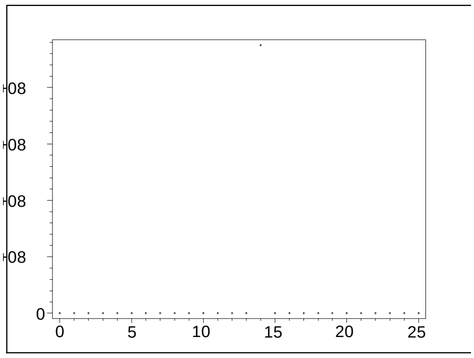
Picture 5



Picture 6



Picture 7



Picture 8

4 Conclusions. Open problems

From the above comparison, some concluding remarks issue:

1. Although the deformation efficiencies relations have a simpler form for the periodic flow, it has *also a non-linear behavior*. This is the most important feature in this context.

2. The analysis for the periodic flow needs more – *irrational*- values for the versors, comparing to that of non periodic case, where there have to be tested a lot of *parameters values*. In the 3D (more complicated) model, the non linearity issue from the parameters behavior. We skip here the details related to all the parameters involved in the 3D model [1] and [5].

3. Analyzing more statistical cases will complete the analysis of the deformation efficiency. It would be useful to search rare events for this model. In [1], the rare events were defined as the events of breaking up of filaments of the material exposed to the experiment [5]. The conclusion was that the turbulent mixing for a 3D flow *can be associated to a far from equilibrium phenomena*. The problem is if the same thing can happen with periodic flows. This is a proximal aim.

4. These conclusions are preliminary, as we have not exhausted all irrational cases for the length and surface versors, all situations for the time period T_{ext}.

It is important to note that e_λ and e_η can be approached *both* as differential equations and as functions of some parameters, with some properties. In this context a detailed analysis for *special functions* $\omega(r)$ (of the rotational component of TW model) would give new analytical information. A detailed analysis for both deformations efficiencies will come soon.

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