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OPTIMAL ALTERNATIVE TO THE AKIMA'S METHOD OF SMOOTH INTERPOLATION APPLIED IN DIABETOLOGY

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Abstract. It is presented a new method of cubic piecewise smooth interpolation applied to experimental data obtained by glycemic profile for diabetics. This method is applied to create a soft useful in clinical diabetology. The method give an alternative to the Akima's procedure of the derivatives computation on the knots from [1] and have an optimal property.

1 Introduction

We construct here a method of cubic piecewise smooth interpolation which combine the calculus of the derivatives on the interior knots, given by the Akima's procedure (see [1]), with the cubic piecewise smooth interpolation formula from [6] and a new procedure to compute the derivatives on the first two and last three knots. Such new procedure is proposed here. The derivatives on the first two knots and on the last three knots are calculated such that to be minimized the quadratic oscillation in average (defined below) of the smooth interpolation function. The quadratic oscillation in average (defined in [2]) was introduced to measure the geometrical distance between the graphs of the interpolation function and of the polygonal line joining the interpolation points. In the interior knots where we can use the Akima's method (see [1]), we prefer this method because give a natural computation of the derivatives.
This numerical method is implemented here to obtain a proper soft which is tested and used on diabetology measurements. For patients which present blood-glucose

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homeostasis, the numerical method can be combined with the mathematical model obtained in [3] the aim to determine the critical hypoglycemia characteristic for such patients.

2 The cubic piecewise smooth interpolation.

In the plane tOx consider the points (t_i, x_i), $i = \overline{0,n}$ where t_i, represent moments of time and x_i represent measured values of a parameter (which varies in time). Let the vector $x = (x_0, ..., x_n)$, $h_i = t_i - t_{i-1}$, $\quad i = \overline{1,n}$ and the slopes,

$$m_i = \frac{x_{i+1} - x_i}{t_{i+1} - t_i}, \quad i = \overline{0, n-1}. \tag{1}$$

We use these slopes to compute x_i', , $i = \overline{2, n-3}$, applying the Akima's procedure from [1],

$$x_i' = \frac{|m_{i+2} - m_{i+1}| \cdot m_{i-1} + |m_{i-1} - m_{i-2}| \cdot m_{i+1}}{|m_{i+2} - m_{i+1}| + |m_{i-1} - m_{i-2}|}, \quad i = \overline{2, n-3} \tag{2}$$

By (2) we see that m_i, $\quad i = \overline{0, n-1}$ is not enough to compute

$$x_0', \quad x_1', \quad x_{n-2}', \quad x_{n-1}', \quad x_n'$$

too. Therefore H. Akima propose in [1] an artificial computation of

$$m_{-2}, \quad m_{-1}, \quad m_m, \quad m_{n+1}, \quad m_{n+2}$$

and then the treatment of the end points is a weakness of the method (as it is mentioned in [1] and [5]). Moreover in [1] is not given the error estimation.

To interpolate the data (t_i, x_i), $i = \overline{1,n}$, we define $F : [t_0, t_n] \longrightarrow \mathbb{R}$ by his restrictions F_i, $i = \overline{1,n}$, to the intervals $[t_{i-1}, t_i]$, $i = \overline{1,n}$. The functions F_i are cubic polynomials and have expression as in [6] and [5]:

$$F_i(t) = \frac{(t_i - t)^2 (t - t_{i-1})}{h_i^2} \cdot x_{i-1}' - \frac{(t - t_{i-1})^2 (t_i - t)}{h_i^2} \cdot x_i' +$$

$$+ \frac{(t_i - t)^2 \left[2(t - t_{i-1}) + h_i\right]}{h_i^3} \cdot x_{i-1} + \frac{(t - t_{i-1})^2 \left[2(t_i - t) + h_i\right]}{h_i^3} \cdot x_i, \quad t \in [t_{i-1}, t_i], \tag{3}$$

that is, with other notations,

$$F_i(t) = A_i(t) \cdot x_{i-1}' + B_i(t) \cdot x_i' + C_i(t) \cdot x_{i-1} + E_i(t) \cdot x_i. \tag{4}$$

Definition 1. *([2]) Corresponding to the points* (t_i, x_i) , $i = \overline{0,n}$, *let the polygonal line* $D(x) : [t_0, t_n] \longrightarrow \mathbb{R}$ *having the restrictions to the intervals* $[t_{i-1}, t_i]$, $i = \overline{1,n}$, D_i *given by*

$$D_i(t) = x_{i-1} + \frac{x_i - x_{i-1}}{t_i - t_{i-1}} \cdot (t - t_{i-1}).$$

The quadratic oscillation in average of F *is*

$$\rho(F,x) = \sqrt{\sum_{i=1}^{n} \cdot \int_{t_{i-1}}^{t_i} [F_i(t) - D_i(t)]^2 dt}.$$

We will obtain x'_0, x'_1, x'_{n-2}, x'_{n-1}, and x'_n such that the quadratic oscillation in average of F to be minimal. In this aim we will consider the residual

$$R\left(x'_0, x'_1, x'_{n-2}, x'_{n-1}, x'_n\right) = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} [F_i(t) - D_i(t)]^2 dt. \tag{5}$$

3 Optimal property and the error estimation

Theorem 2. *There exist an unique point* $\left(\overline{x'_0}, \overline{x'_1}, \overline{x'_{n-2}}, \overline{x'_{n-1}}, \overline{x'_n}\right) \in \mathbb{R}^5$ *which minimize the quadratic oscillation in average* $\rho(F,x)$.

Proof. We will minimize the residual

$$R\left(x'_0, x'_1, x'_{n-2}, x'_{n-1}, x'_n\right) = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} [F_i(t) - D_i(t)]^2 dt.$$

applying the least squares method. Therefore, we solve the systems given by the conditions

$$\frac{\partial R}{\partial x'_0} = 0, \qquad \frac{\partial R}{\partial x'_1} = 0 \tag{6}$$

$$\frac{\partial R}{\partial x'_{n-2}} = 0, \qquad \frac{\partial R}{\partial x'_{n-1}} = 0, \qquad \frac{\partial R}{\partial x'_{n-1}} = 0. \tag{7}$$

The system (6), according to (4), have the form :

$$\begin{cases} \left(\int\limits_{t_0}^{t_1} A_1^2(t)\,dt\right)\cdot x_0' + \left(\int\limits_{t_0}^{t_1} A_1(t)\cdot B_1(t)\,dt\right)\cdot x_1' = \int\limits_{t_0}^{t_1} A_1(t)\cdot D_1(t)\,dt - \\ \quad - \left(\int\limits_{t_0}^{t_1} A_1(t)\cdot C_1(t)\,dt\right)\cdot x_0 - \left(\int\limits_{t_0}^{t_1} A_1(t)\cdot E_1(t)\,dt\right)\cdot x_1 \\ \left(\int\limits_{t_0}^{t_1} A_1(t)\cdot B_1(t)\,dt\right)\cdot x_0' + \left(\int\limits_{t_0}^{t_1} B_1^2(t)\,dt + \int\limits_{t_1}^{t_2} A_2^2(t)\,dt\right)\cdot x_1' = \int\limits_{t_0}^{t_1} B_1(t)\cdot D_1(t)\,dt - \\ - \left(\int\limits_{t_1}^{t_2} A_2(t)\cdot B_2(t)\,dt\right)\cdot x_2' - \left(\int\limits_{t_0}^{t_1} B_1(t)\cdot C_1(t)\,dt\right)\cdot x_0 - \left(\int\limits_{t_0}^{t_1} B_1(t)\cdot E_1(t)\,dt\right)\cdot x_1 + \\ \quad + \int\limits_{t_1}^{t_2} A_2(t)\cdot D_2(t)\,dt - \left(\int\limits_{t_1}^{t_2} A_2(t)\cdot C_2(t)\,dt\right)\cdot x_1 - \left(\int\limits_{t_1}^{t_2} A_2(t)\cdot E_2(t)\,dt\right)\cdot x_2. \end{cases} \tag{8}$$

Since

$$\frac{\partial^2 R}{\partial x_0'^2} = 2\left(\int\limits_{t_0}^{t_1} A_1^2(t)\,dt\right) > 0$$

and the determinant of the Hesse matrix

$$\begin{pmatrix} \frac{\partial^2 R}{\partial x_0'^2} & \frac{\partial^2 R}{\partial x_0' \partial x_1'} \\ \frac{\partial^2 R}{\partial x_0' \partial x_1'} & \frac{\partial^2 R}{\partial x_1'^2} \end{pmatrix}$$

is

$$\delta_1 = 4\left(\int\limits_{t_0}^{t_1} A_1^2(t)\,dt\right)\cdot\left(\int\limits_{t_0}^{t_1} B_1^2(t)\,dt\right) - 4\left(\int\limits_{t_0}^{t_1} A_1(t)\cdot B_1(t)\,dt\right)^2 +$$

$$+4\left(\int\limits_{t_0}^{t_1} A_1^2(t)\,dt\right)\cdot\left(\int\limits_{t_1}^{t_2} A_2^2(t)\,dt\right) > 0$$

we infer that the system (8) have unique solution $\left(\overline{x_0'}, \overline{x_1'}\right)$. The system (7) according to (4), is :

$$
\begin{cases}
\left(\int_{t_{n-3}}^{t_{n-2}} B_{n-2}^2(t)\,dt + \int_{t_{n-2}}^{t_{n-1}} A_{n-1}^2(t)\,dt\right)\cdot x'_{n-2} + \left(\int_{t_{n-2}}^{t_{n-1}} A_{n-1}(t)\cdot B_{n-1}(t)\,dt\right)\cdot x'_{n-1} \\
\quad = -\int_{t_{n-3}}^{t_{n-2}} B_{n-2}(t)\,[A_{n-2}(t)\cdot x'_{n-3} + C_{n-2}(t)\cdot x_{n-3} + E_{n-2}(t)\cdot x_{n-2} - \\
\quad -D_{n-2}(t)]dt - \int_{t_{n-2}}^{t_{n-1}} A_{n-1}(t)\,[C_{n-1}(t)\cdot x_{n-2} + E_{n-1}(t)\cdot x_{n-1} - D_{n-1}(t)]dt \\
\left(\int_{t_{n-2}}^{t_{n-1}} A_{n-1}(t)\cdot B_{n-1}(t)\,dt\right)\cdot x'_{n-2} + \left(\int_{t_{n-2}}^{t_{n-1}} B_{n-1}^2(t)\,dt + \int_{t_{n-1}}^{t_n} A_n^2(t)\,dt\right)\cdot x'_{n-1} + \\
+\left(\int_{t_{n-1}}^{t_n} A_n(t)\,B_n(t)\,dt\right)\cdot x'_n = -\int_{t_{n-2}}^{t_{n-1}} B_{n-1}(t)\,[C_{n-1}(t)\cdot x_{n-2} + E_{n-1}(t)\cdot x_{n-1} - \\
\quad -D_{n-1}(t)]dt - \int_{t_{n-1}}^{t_n} A_n(t)\cdot[C_n(t)\cdot x_{n-1} + E_n(t)\cdot x_n - D_n(t)]dt \\
\left(\int_{t_{n-1}}^{t_n} A_n(t)\cdot B_n(t)\,dt\right)\cdot x'_{n-1} + \left(\int_{t_{n-1}}^{t_n} B_n^2(t)\,dt\right)\cdot x'_n = \\
\quad = -\int_{t_{n-1}}^{t_n} B_n(t)\cdot[C_n(t)\cdot x_{n-1} + E_n(t)\cdot x_n - D_n(t)]dt
\end{cases}
\tag{9}
$$

Since

$$
\frac{\partial^2 R}{\partial x'^2_{n-2}} = 2\left(\int_{t_{n-3}}^{t_{n-2}} B_{n-2}^2(t)\,dt + \int_{t_{n-2}}^{t_{n-1}} A_{n-1}^2(t)\,dt\right) > 0
$$

and the determinant of the Hesse matrix

$$
\begin{pmatrix}
\frac{\partial^2 R}{\partial x'^2_{n-2}} & \frac{\partial^2 R}{\partial x'_{n-2}\partial x'_{n-1}} \\
\frac{\partial^2 R}{\partial x'_{n-1}\partial x'_{n-2}} & \frac{\partial^2 R}{\partial x'^2_{n-1}}
\end{pmatrix}
$$

is

$$
\delta_2 = 4\left(\int_{t_{n-2}}^{t_{n-1}} A_{n-1}^2(t)\,dt\right)\left(\int_{t_{n-2}}^{t_{n-1}} B_{n-1}^2(t)\,dt\right) - 4\left(\int_{t_{n-2}}^{t_{n-1}} A_{n-1}(t)\cdot B_{n-1}(t)\,dt\right)^2 +
$$

$$+4\left(\int_{t_{n-3}}^{t_{n-2}} B_{n-2}^2(t)\,dt\right)\cdot\left(\int_{t_{n-2}}^{t_{n-1}} B_{n-1}^2(t)\,dt+\int_{t_{n-1}}^{t_n} A_n^2(t)\,dt\right)+$$

$$+4\left(\int_{t_{n-2}}^{t_{n-1}} A_{n-1}^2(t)\,dt\right)\cdot\left(\int_{t_{n-1}}^{t_n} A_n^2(t)\,dt\right)>0,$$

and in addition for the determinant of the Hesse matrix

$$\begin{pmatrix} \frac{\partial^2 R}{\partial x'^2_{n-2}} & \frac{\partial^2 R}{\partial x'_{n-2}\partial x'_{n-1}} & \frac{\partial^2 R}{\partial x'_{n-2}\partial x'_n} \\ \frac{\partial^2 R}{\partial x'_{n-1}\partial x'_{n-2}} & \frac{\partial^2 R}{\partial x'^2_{n-1}} & \frac{\partial^2 R}{\partial x'_{n-1}\partial x'_n} \\ \frac{\partial^2 R}{\partial x'_{n-2}\partial x'_n} & \frac{\partial^2 R}{\partial x'_{n-1}\partial x'_n} & \frac{\partial^2 R}{\partial x'^2_n} \end{pmatrix}$$

we have

$$\Delta=8\left(\int_{t_{n-2}}^{t_{n-1}} A_{n-1}^2(t)\,dt+\int_{t_{n-3}}^{t_{n-2}} B_{n-2}^2(t)\,dt\right)\cdot\left[\left(\int_{t_{n-1}}^{t_n} A_n^2(t)\,dt\right)\cdot\left(\int_{t_{n-1}}^{t_n} B_n^2(t)\,dt\right)-\right.$$

$$\left.-\left(\int_{t_{n-1}}^{t_n} A_n(t)\,B_n(t)\,dt\right)^2\right]+8\left(\int_{t_{n-1}}^{t_n} B_n^2(t)\,dt\right)\cdot\left[\left(\int_{t_{n-2}}^{t_{n-1}} A_{n-1}^2(t)\,dt\right)\cdot\right.$$

$$\left.\cdot\left(\int_{t_{n-2}}^{t_{n-1}} B_{n-1}^2(t)\,dt\right)-\left(\int_{t_{n-2}}^{t_{n-1}} A_{n-1}(t)\cdot B_{n-1}(t)\,dt\right)^2\right]+$$

$$+8\left(\int_{t_{n-1}}^{t_n} B_n^2(t)\,dt\right)\cdot\left(\int_{t_{n-2}}^{t_{n-1}} B_{n-1}^2(t)\,dt\right)\cdot\left(\int_{t_{n-3}}^{t_{n-2}} B_{n-2}^2(t)\,dt\right)>0,$$

according to the Cauchy-Buniakovski-Schwarz's inequality, follows that the system (9) have unique solution $\left(\overline{x'_{n-2}},\overline{x'_{n-1}},\overline{x'_n}\right)$. Consequently, there exist an unique point $\left(\overline{x'_0},\overline{x'_1},\overline{x'_{n-2},x'_{n-1}},\overline{x'_n}\right)\in\mathbb{R}^5$ for which the residual $R\left(\overline{x'_0},\overline{x'_1},\overline{x'_{n-2},x'_{n-1}},\overline{x'_n}\right)$ is minimal. Because

$$\rho(F,x)=\sqrt{R\left(x'_0,x'_1,x'_{n-2},x'_{n-1},x'_n\right)},$$

we infer that for this point, the quadratic oscillation in average, $\rho(F,x)$ is minimal. □

From (3) follows that $F \in C^1[t_0,t_n], F \notin C^2[t_0,t_n]$, but F'' is piecewise continuous. Then, F'' is bounded.

Let $F\left(\overline{x_0'},\overline{x_1'},\overline{x_{n-2}'},\ \overline{x_{n-1}'},\overline{x_n'}\right)$ the smooth interpolation function defined by his restrictions in (3) and obtained in the above theorem. For this function we have the error estimation obtained below.

Theorem 3. *If x_i, $i=\overline{0,n}$ are values of a function $f:[t_0,t_n]\longrightarrow\mathbb{R}$, that is $f(t_i)=x_i$, $\forall i=\overline{0,n}$ and if $f\in C^1[t_0,t_n]$ with Lipschitzian first derivative, then the function $F\in C^1[t_0,t_n]$ given in (3) realize the piecewise smooth interpolation of the function f on the knots t_i, $i=\overline{0,n}$ and the following error estimation holds :*

$$\|f-F\|_c \leqslant (L'+M)\cdot\max\{h_i^2 : i=\overline{1,n}\}, \tag{10}$$

where L' is the Lipschitz constant of f' and

$$M=\max\{\max(|F_i''(t_{i-1})|,|F_i''(t_i)|) : i=\overline{1,n}\}.$$

Proof. Consider $\varphi=f-F$. Since $f(t_i)=F(t_i)=x_i$, $\forall i=\overline{0,n}$, we infer that $\varphi(t_i)=0$, $\forall i=\overline{0,n}$. Therefore, for any $i=\overline{1,n}$ there exist $\xi_i\in(t_{i-1},t_i)$ such that $\varphi'(\xi_i)=0$.

For any $t\in[t_0,t_n]$ there exist $j\in\{1,...,n\}$ such that $t\in[t_{j-1},t_j]$. We have,

$$|f(t)-F(t)|=\left|\int_{t_{j-1}}^{t}[f'(s)-F'(s)]ds\right|\leq$$

$$\leq\int_{t_{j-1}}^{t}(|f'(s)-f'(\xi_j)|+|f'(\xi_j)-F'(\xi_j)|+|F'(\xi_j)-F'(s)|)ds\leq$$

$$\leq\int_{t_{j-1}}^{t}(L'\cdot|s-\xi_j|+\|F_j''\|_C\cdot|s-\xi_j|)ds\leq(L'+\|F_j''\|_C)\cdot h_j^2.$$

Since F_i'' is first order polynomial $\forall i=\overline{1,n}$ we get

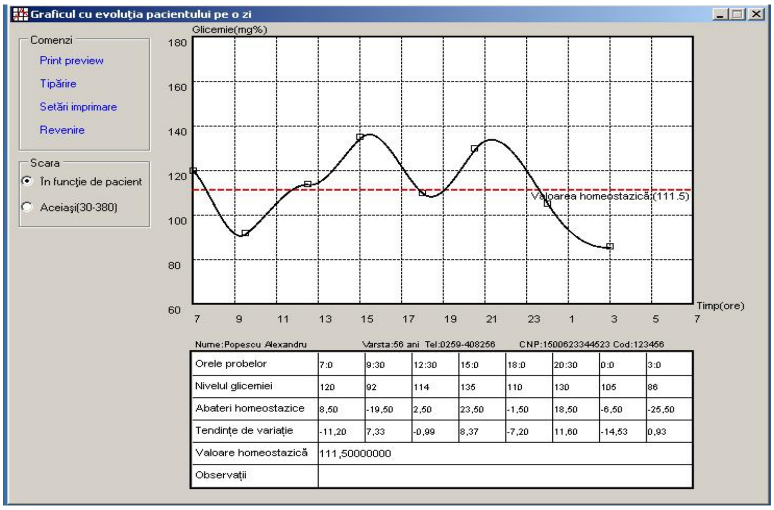
$$\|F_i''\|_C=\max(|F_i''(t_{i-1})|,|F_i''(t_i)|),$$

and obtain the estimation (10). We conclude that similar error estimation holds for the smooth interpolation function $F\left(\overline{x_0'},\overline{x_1'},\overline{x_{n-2}'},\ \overline{x_{n-1}'},\overline{x_n'}\right)$. □

4 Application

From the above section follows that the presented method optimally improves on the first two knots and on the last three knots, the Akima's method from [1]. Moreover, gives the error estimation of order $O(h^2)$. This numerical method is used to obtain a soft applicable in diabetology at the fitting of glycemic profile experimental data. The soft was created in C# and was tested on data harvested in October 2006 for five patients.

As example, for the patient no. 4 blood- glucose levels (in mg/dl) was measured at the hours 7:00,9:30,12:30,15:00,18:00,20:30,0:00,3:00 and the values obtained were 120,92,114,135,110,130,105,86 (in mg/dl). Variation trends on those moments were : -11.20, 7.33, -0.99, 8.37, -7.20, 11.60, -14.53, 0.93. The graphic result of this patient is:



Graphic for patient no. 4

The presented method is sufficiently accurate in the aim to approximate the daily evolution of the blood glucose levels (continuously given by the Holder program) and cheaper than the Holder method which use a consumable enzyme. Therefore, represent a more economic method.

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