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A STUDY ON ALMOST MATRIX SUMMABILITY OF FOURIER-JACOBI SERIES

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Abstract. In this paper, a quite new theorem on almost summability of Fourier-Jacobi series has been established. Our theorem extends and generalizes all previously known results of this line of work.

1 Introduction

The study of summability of Fourier-Jacobi series by Nörlund (N, p_n) summability method has been made by a number of researchers like Gupta [6], Choudhary [4], Thorpe [18], Pandey and Beohar [13], Beohar and Sharma [1], Pandey [12] and Tripathi, Tripathi and Yaday [19]. After a good amount of work in ordinary Nörlund summability of Jacobi series at the point $x = c$, Khare and Tripathi [8] discussed generalized Nörlund (N, p, q) summability of Jacobi series. The Nörlund (N, p_n) summability of Fourier-Jacobi series has also been studied by Prasad and Saxena [17]. Recently, the result of Prasad and saxena [17] has been generalized by Chandra [3] for generalized Nörlund (N, p, q) summability. Therefore, the purpose of present paper is to generalize the result of Chandra [3] to a more general class of almost matrix summability of Fourier-Jacobi series. Our important theorem extends and generalizes all previously known results of this line of work.
Here it is important to note that the almost matrix summability method includes as special cases the methods of almost $(C, 1)$, (C, δ), (N, p_n), $\left(\overline{N}, p_n\right)$ and (N, p, q) summability methods.

2 Definitions and Notations

Let $f(x)$ be a function defined in a closed interval $[-1, 1]$ such that the function

$$(1-x)^{\alpha}(1+x)^{\beta} f(x) \in L\,[-1,\,1]\,;\alpha > -1,\ \beta > -1.$$

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The Fourier-Jacobi series corresponding to the function $f(x)$ is given by

$$f(x) = \sum_{n=0}^{\infty} a_n P_n{}^{(\alpha,\beta)}(x) \tag{2.1}$$

where

$$a_n = \frac{(2n+\alpha+\beta+1)\,\Gamma(n+1)\,\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)} \cdot \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_n{}^{(\alpha,\beta)}(x) f(x)\, dx \tag{2.2}$$

and $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials defined by the generating function

$$2^{\alpha+\beta}\left(1-2xt+t^2\right)^{-1/2}\left[1-t+\left(1-2xt+t^2\right)^{1/2}\right]^{-\alpha}\left[1+t+\left(1-2xt+t^2\right)^{1/2}\right]^{-\beta} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\, t^n. \tag{2.3}$$

2.1 Definition.

Lorentz [9] has given the following definition:
A sequence is said to be almost convergent to a limit s if

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=m}^{n+m} s_k = s \text{ uniformly with respect to } m.$$

Let $T = (a_{n,k})$ be an infinite triangular matrix satisfying the Silverman-Toeplitz [20] condition of regularity i.e.

$$\sum_{k=0}^{n} a_{n,k} \to 1 \quad as \quad n \to \infty,$$

$$a_{n,k} = 0 \; for \; k > n$$

and

$$\sum_{k=0}^{n} |a_{n,k}| \le M, \text{ a finite constant.}$$

Now, let us introduce the concept of almost matrix summability with respect to an infinite regular matrix $T=(a_{n,k})$ such that the elements $a_{n,k}$ are non-negative and non-decreasing with k, and being

$$A_{n,\tau}=\sum_{k=0}^{\tau}a_{n,n-k},\ A_{n,n}=1\,\forall\, n. \tag{2.4}$$

Under these circumstances, if for a series $\sum\limits_{n=0}^{\infty}u_n$ with partial sums $\{\,s_n\}$ we denote

$$s_{n-k,m}=\frac{1}{n-k+1}\sum_{\nu=m}^{n-k+m}s_{\nu}, \tag{2.5}$$

we say that $\sum\limits_{n=0}^{\infty}u_n$ is almost matrix summable (T) to the sum s if

$$t_{n,m}=\sum_{k=0}^{n}a_{n,k}s_{k,m}=\sum_{k=0}^{n}a_{n,n-k}s_{n-k,m}\to s \tag{2.6}$$

uniformly with respect to m.
Seven important particular cases of matrix means are as follows:

1. $(C,1)$ mean when $a_{n,k}=\frac{1}{n+1}$, for $0\le k\le n$.
2. Harmonic mean when $a_{n,k}=\frac{1}{(n-k+1)\log n}$.
3. (C,δ) mean when $a_{n,k}=\dfrac{\binom{n-k+\delta+1}{\delta-1}}{\binom{n+\delta}{\delta}}$.
4. (H,p) mean when $a_{n,k}=\frac{1}{\log^{p-1}(n+1)}\prod\limits_{q=0}^{p-1}\log^q(k+1)$.
5. Nörlund mean [10] when $a_{n,k}=\frac{p_{n-k}}{P_n}$ where $P_n=\sum\limits_{k=0}^{n}p_k$.
6. Riesz $(\overline{N},p_n)$ mean when $a_{n,k}=\frac{p_k}{P_n}$.
7. Generalized Nörlund (N,p,q) mean [2] when $a_{n,k}=\frac{p_{n-k}\,q_k}{R_n}$ where $R_n=\sum\limits_{k=0}^{n}p_kq_{n-k}$.

We use the following notations:

$$F\left(\phi\right)=\left\{f\left(\cos\phi\right)-A\right\}\left(\sin\frac{\phi}{2}\right)^{2\alpha+1}\left(\cos\frac{\phi}{2}\right)^{2\beta+1}$$

where A being a fixed constant.

$$\psi(t) = \int_0^t |F(\phi)|\, d\phi.$$

$$M_n(\phi) = 2^{\alpha+\beta+1} \sum_{k=0}^{n-1} \frac{a_{n,n-k}}{n-k+1} \sum_{\nu=m}^{n-k+m} \lambda_\nu P_\nu^{(\alpha+1,\beta)}(\cos\phi).$$

$$\tau = \text{integral part of } \frac{1}{\phi} = \left[\frac{1}{\phi}\right].$$

$$A_{n,\tau} = \sum_{k=0}^{\tau} a_{n,n-k}.$$

$$\eta = \text{integral part of } \frac{1}{\delta} = \left[\frac{1}{\delta}\right].$$

3 Previous Results

Dealing with the Nörlund summability of Fourier-Jacobi series, Prasad and Saxena [17] have established the following:

3.1 Theorem 1.

If

$$F_1(t) = \int_0^t |F(\phi)|\, d\phi = O\left(\frac{\psi(t)\; t^{2\alpha+2}}{\theta(P_n)}\right) \text{ as } t \to 0 \tag{3.1}$$

where

$$F(\phi) = \{f(\cos\phi) - A\} \left(\sin\frac{\phi}{2}\right)^{2\alpha+1} \left(\cos\frac{\phi}{2}\right)^{2\beta+1},$$

$\psi(t)$ and $\theta(t)$ are non-negative, monotonic increasing functions of t such that

$$\psi(n)\log n = O(\theta(P_n)) \text{ as } n \to \infty,$$
$$n^{(2\alpha+1)/2} = o(P_n) \text{ as } n \to \infty$$

and

$$\sum_{k=2}^{n} \frac{P_k}{k^{(2\alpha+1)/2}\log k} = O\left(\frac{P_n}{n^{(2\alpha+1)/2}}\right) \text{ as } n \to \infty \tag{3.2}$$

then the Fourier-Jacobi series (2.1) is summable (N, p_n) *at the point* $x = +1$ *to sum* A*, provided that the condition*

$$-\frac{1}{2} \leq \alpha < \frac{1}{2}, \beta > -\frac{1}{2}$$

and the antipole condition

$$\int_{-1}^{b} (1+x)^{(2\beta-3)/4} \left| f(x) \right| dx \quad < \infty \tag{3.3}$$

are satisfied, where b *is fixed and* $\left(\overline{N}, p_n\right)$ *is regular Nörlund method defined by the real non-negative and non-increasing sequence* $\{p_n\}$ *such that* $P_n \to \infty$ *as* $n \to \infty$*.*

Generalizing Theorem 1, Chandra [3] has proved a following theorem:

3.2 Theorem 2.

Let (N, p, q) *be a generalized Nörlund method defined by a real non-negative, non-increasing sequence* $\{p_n\}$ *and a real non-negative, non-decreasing sequence* $\{q_n\}$*. Let* $\psi(t)$ *and* $\lambda(t)$ *non-negative, monotonic increasing functions of* t *such that*

$$\psi(n) \log n = O\left[\lambda(P_n)\right] \tag{3.4}$$

$$q_n P_n = O\left[(p * q) \log n\right] \tag{3.5}$$

and

$$\sum_{k=2}^{n} \frac{P_k}{k^{\frac{(2\alpha+1)}{2}} \log k} = O\left(\frac{(p*q)_n^{1-c}}{q_n n^{\frac{(2\alpha+1)}{2}}}\right) \tag{3.6}$$

as $n \to \infty$ *where* c *is a parameter with the restriction that* $0 \leq c \leq 1$*.*
In equation (3.5) and (3.6), the term $(p * q) = \sum_{k=0}^{n} p_{n-k} q_k$*.*
If

$$F_1(t) = \int_0^t \left| F(\phi) \right| d\phi = o\left(\frac{\psi(t)\ t^{2\alpha+2}}{\lambda(P_\tau)}\right) \tag{3.7}$$

as $t \to 0$*, where* $\tau = \left[\frac{1}{t}\right]$ *then the Fourier-Jacobi series (2.1) is summable* (N, p, q) *at the point* $x = +1$ *to the sum* A *provided that the condition*

$$-\frac{1}{2} \leq \alpha < \frac{1}{2},\ \beta > -\frac{1}{2}$$

and the antipole condition

$$\int_{-1}^{b} (1+x)^{(\beta-\alpha-1)/2} |f(x)| \, dx < \infty$$

are satisfied, where b is fixed.

4 Main Theorem

The object of present paper is to generalize Theorem 2 of Chandra [3] to more general class of almost matrix summability of Fourier-Jacobi series. In fact, we prove a following theorem:

4.1 Theorem.

Let $T = (a_{n,k})$ be an infinite regular triangular matrix such that $(a_{n,k})$ is non-negative and non-decreasing with k, $n^{(2\alpha-1)/2} A_{n,n} \to 0 \, as \, n \to \infty$.
For the range $-3/2 \le \alpha < 3/2$, $\beta > -1/2$ and for fixed b, the antipole condition

$$\int_{-1}^{b} (1+x)^{(2\beta-3)/4} |f(x)| \, dx < \infty \tag{4.1}$$

must be satisfied.
Now, if

$$\psi(t) = \int_{0}^{t} \left| \{f(\cos\phi) - A\} \left(\sin\frac{\phi}{2}\right)^{2\alpha+1} \left(\cos\frac{\phi}{2}\right)^{2\beta+1} \right| d\phi = O\left(\frac{t^{2\alpha+1}}{\xi\left(\frac{1}{t}\right)}\right) \, as \, t \to 0 \tag{4.2}$$

where $\xi(t)$ is positive, non-decreasing with t such that $\xi(n) \to \infty \, as \, n \to \infty$ and

$$\sum_{a}^{n} \frac{A_{n,k}}{k^{(2\alpha+1)/2} \, \xi(k)} = O\left(\frac{1}{n^{(2\alpha-1)/2}}\right) \tag{4.3}$$

where 'A' being a fixed positive integer then Fourier-Jacobi series (2.1) is almost matrix summable (T) at $x = 1$ to the sum 'A'.

5 Lemmas

The following lemmas are required for the proof of the theorem:

5.1 Lemma 1.

(Szegö [16]). If $\alpha > -1, \beta > -1$ then as $n \to \infty$,

$$P_n^{\alpha,\beta}(\cos\phi) = O(n^\alpha), 0 \le \phi \le \frac{1}{n} \tag{5.1}$$

$$= O\left(n^\beta\right), \pi - \frac{1}{n} \le \phi \le \pi \tag{5.2}$$

$$= \frac{1}{(n\pi)^{1/2}}\left(\sin\frac{\phi}{2}\right)^{-(2\alpha+1)/2}\left(\cos\frac{\phi}{2}\right)^{-(2\beta+1)/2}$$
$$\cdot \cos\left\{\left(\frac{2n+\alpha+\beta+1}{2}\right)\phi - (2\alpha+1)\frac{\pi}{4}\right\} + \frac{O(1)}{n\sin\phi}, \frac{1}{n} \le \phi \le \pi - \frac{1}{n} \tag{5.3}$$

5.2 Lemma 2.

(Gupta [6]). The antipole condition (4.3) implies that

$$\int_\delta^\pi \left(\cos\frac{\phi}{2}\right)^{(2\beta-1)/2} |f(\cos\phi) - A|\, d\phi < \infty \tag{5.4}$$

which further implies that

$$\int_0^{\frac{1}{n}} t^{(2\beta-1)/2} |f(-\cos t) - A|\, dt = O(1). \tag{5.5}$$

5.3 Lemma 3.

(Rhoades [15]). Let $\{u_n\}$ and $\{v_n\}$ be two real sequences and $\{u_n\}$ be non-negative. If $\{v_n\}$ is non-increasing then

$$\left|\sum_{k=1}^n u_k v_k\right| \le v_1 \max_{1\le r\le n}\left|\sum_{k=1}^r u_k\right|. \tag{5.6}$$

If $\{v_n\}$ is non-decreasing then

$$\left|\sum_{k=1}^n u_k v_k\right| \le 2v_n \max_{1\le r\le n}\left|\sum_{k=1}^r u_k\right|. \tag{5.7}$$

5.4 Lemma 4.

Under the conditin of the theorem on $(a_{n,k})$ for large n, uniformly in $0<\phi\leq\pi, 0\leq a\leq b\leq n$,

$$\left|\sum_{k=a}^{b}\frac{a_{n,n-k}}{n-k+1}\left[\sum_{\nu=m}^{n-k+m}\nu^{\alpha+\frac{1}{2}}\cos\left\{(\nu+\rho)\,\phi-r\right\}\right]\right|=O\left\{n^{\alpha-\frac{1}{2}}A_{n,\tau}\right\} \tag{5.8}$$

where $\rho=\frac{\alpha+\beta+2}{2}$, *and* $r=-\left(\alpha+\frac{3}{2}\right)\frac{\pi}{4}$.

Proof.

$$\left|\sum_{k=a}^{b}\frac{a_{n,n-k}}{n-k+1}\left[\sum_{\nu=m}^{n-k+m}\nu^{\alpha+\frac{1}{2}}\cos\left\{(\nu+\rho)\,\phi-r\right\}\right]\right|$$

$$=\left|\sum_{k=a}^{b}\frac{a_{n,n-k}}{n-k+1}\ \text{Real part of}\ \left[\sum_{\nu=m}^{n-k+m}\nu^{\alpha+\frac{1}{2}}\ e^{i\,\{(\nu+\rho)\phi-r\}}\right]\right|$$

$$\leq\left|\sum_{k=a}^{b}\frac{a_{n,n-k}}{n-k+1}\ \left[\sum_{\nu=m}^{n-k+m}\nu^{\alpha+\frac{1}{2}}\ e^{i\,\{\nu\,\phi\}}\right]\right|\left|e^{i\,(\rho\phi-r)}\right|$$

$$=\left|\sum_{k=a}^{b}\frac{a_{n,n-k}}{n-k+1}O\left\{\,(n-k)^{\alpha+\frac{1}{2}}e^{i\{(n-k)\phi\}}\right\}\right|$$

$$=\left|\sum_{k=a}^{b}\frac{a_{n,n-k}}{n-k+1}\left\{\,(n-k)^{\alpha+\frac{1}{2}}\ e^{-ik\phi}\right\}\right|\left|e^{in\phi}\right|$$

$$=O\left[\left|\sum_{k=a}^{\tau-1}\frac{a_{n,n-k}}{n-k+1}\ \left\{\,(n-k)^{\alpha+\frac{1}{2}}\ e^{-ik\phi}\right\}\right|+\left|\sum_{k=\tau}^{b}\frac{a_{n,n-k}}{n-k+1}\ \left\{\,(n-k)^{\alpha+\frac{1}{2}}\ e^{-ik\phi}\right\}\right|\right]$$

$$=O\left\{n^{\alpha-1/2}\right\}\left|\sum_{k=a}^{\tau-1}a_{n,n-k}\right|+O\left\{n^{\alpha-\frac{1}{2}}\right\}a_{n,n-\tau}\left|\sum_{\tau\leq k\leq r\leq b}^{r}e^{-ik\phi}\right|$$

by Abels Lemma and Lemma 3

$$=O\left\{n^{\alpha-\frac{1}{2}}\right\}\ A_{n,\tau}+O\left\{n^{\alpha-\frac{1}{2}}\right\}\ a_{n,n-\tau}\left|\frac{e^{-i\tau t}\left\{1-\left(e^{-it}\right)^{r-\tau+1}\right\}}{1-e^{-it}}\right|$$

$$=O\left\{n^{\alpha-\frac{1}{2}}\right\}A_{n,\tau}+O\left\{n^{\alpha-\frac{1}{2}}\right\}\ a_{n,n-\tau}\ \left|\frac{e^{-i\tau t}}{e^{-it/2}}\right|\left|\frac{1-e^{-it(r-\tau+1)}}{e^{it/2}-e^{-it/2}}\right|$$

$$=O\left\{n^{\alpha-\frac{1}{2}}\right\}A_{n,\tau}+O\left\{n^{\alpha-\frac{1}{2}}\right\}\ \frac{2a_{n,n-\tau}}{\sin\frac{t}{2}}$$

$$=O\left\{n^{\alpha-\frac{1}{2}}A_{n,\tau}\right\}+O\left\{n^{\alpha-\frac{1}{2}}\frac{a_{n,n-\tau}}{t}\right\}. \tag{5.9}$$

Now

$$\begin{aligned} A_{n,\tau} &= \sum_{k=0}^{\tau} a_{n,n-k} \\ &= a_{n,n} + a_{n,n-1} + \ldots + a_{n,n-\tau} \\ &\geq a_{n,n-\tau} + a_{n,n-\tau} + \ldots + a_{n,n-\tau} \\ &= (\tau+1)\, a_{n,n-\tau} \\ &\geq \frac{a_{n,n-\tau}}{\tau} \quad \text{since } \tau = \left[\frac{1}{t}\right] \end{aligned} \tag{5.10}$$

then

$$\frac{a_{n,n-\tau}}{\tau} = O\left(A_{n,\tau}\right). \tag{5.11}$$

By (5.9) and (5.11), we get

$$\left|\sum_{k=a}^{b} \frac{a_{n,n-k}}{n-k+1}\left[\sum_{\nu=m}^{n-k+m} \nu^{\alpha+\frac{1}{2}} \cos\left\{(\nu+\rho)\,\phi - r\right\}\right]\right| = O\left\{\,n^{\alpha-1/2} A_{n,\tau}\right\}.$$

5.5 Lemma 5

Under the hypothesis of the theorem

$$\sum_{k=0}^{n-1} \frac{a_{n,n-k}}{n-k+1} \sum_{\nu=m}^{n-k+m} \nu^{\alpha-\frac{1}{2}} = O\left(n^{\alpha-\frac{3}{2}}\right).$$

Proof.

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{a_{n,n-k}}{n-k+1} \sum_{\nu=m}^{n-k+m} \nu^{\alpha-\frac{1}{2}} &= \sum_{k=0}^{n-1} \frac{a_{n,n-k}}{n-k+1}\, O\left\{(n-k)^{\alpha-\frac{1}{2}}\right\} \\ &= O\left\{n^{\alpha-\frac{3}{2}}\right\} \sum_{k=0}^{n-1} a_{n,n-k} \quad \text{by Lemma 3} \\ &= O\left\{n^{\alpha-\frac{3}{2}}\right\} A_{n,n} \\ &= O\left\{n^{\alpha-\frac{3}{2}}\right\} \text{since } A_{n,n} = O\left(1\right) \end{aligned}$$

5.6 Lemma 6

$$M_n\left(\phi\right) = 2^{\alpha+\beta+1} \sum_{k=0}^{n-1} \left(\frac{a_{n,n-k}}{n-k+1}\right) \sum_{\nu=m}^{n-k+m} \lambda_\nu\, {P_\nu}^{(\alpha+1,\beta)} \left(\cos\phi\right)$$

where

$$\lambda_\nu = \frac{2^{-(\alpha+\beta+1)}\,\Gamma(\nu+\alpha+\beta+2)}{\Gamma(\alpha+1)\,\Gamma(\nu+\beta+1)} \approx \frac{2^{-(\alpha+\beta+1)}}{\Gamma(\alpha+1)}\nu^{\alpha+1}$$

then, for $\frac{1}{2} > \alpha \geq -\frac{1}{2}$, $\beta > -\frac{1}{2}$ *and if* $(a_{n,k})$ *satisfies the hypothesis of the theorem,*

$$M_n(\phi) = O\left(n^{2\alpha+1}\right) \ \textit{if}\ 0 \leq \phi \leq \frac{1}{n} \tag{5.12}$$

$$= O\left(n^{\alpha+\beta}\right) \ \textit{if}\ \pi - \frac{1}{n} \leq \phi \leq \pi \tag{5.13}$$

$$= O\left\{n^{\alpha-\frac{1}{2}} A_{n,\tau}\left(\sin\frac{\phi}{2}\right)^{-\alpha-\frac{3}{2}}\left(\cos\frac{\phi}{2}\right)^{-\beta-\frac{1}{2}}\right\}$$

$$+ O\left\{n^{\alpha-\frac{3}{2}}\left(\sin\frac{\phi}{2}\right)^{-\alpha-\frac{5}{2}}\left(\cos\frac{\phi}{2}\right)^{-\beta-\frac{3}{2}}\right\} \ \textit{if}\ \frac{1}{n} \leq \phi \leq \pi - \frac{1}{n} \tag{5.14}$$

Proof. For $0 \leq \phi \leq \frac{1}{n}$,

$$M_n(\phi) = O\left(2^{\alpha+\beta+1}\right)\left[\sum_{k=0}^{n-1}\frac{a_{n,n-k}}{n-k+1}\,O\left(2^{-(\alpha+\beta+1)}\right)\sum_{\nu=m}^{n-k+m}\nu^{2(\alpha+1)}\right] \text{ by (5.1)}$$

$$= O\left[\sum_{k=0}^{n-1}\frac{a_{n,n-k}}{n-k+1}\sum_{\nu=m}^{n-k+m}\nu^{2(\alpha+1)}\right]$$

$$= O\left[\sum_{k=0}^{n-1}\frac{a_{n,n-k}}{n-k+1}(n-k)^{2(\alpha+1)}\right]$$

$$= O\left(n^{2\alpha+1}\right)\left[\sum_{k=0}^{n-1}a_{n,n-k}\right] \quad \text{by Lemma 3}$$

$$= O\left(n^{2\alpha+1}\right)\left[\sum_{k=0}^{n}a_{n,n-k}\right]$$

$$= O\left(n^{2\alpha+1}\right)A_{n,n}$$

$$= O\left(n^{2\alpha+1}\right) \ \text{ since } A_{n,n} = O(1)$$

For $\pi - \frac{1}{n} \leq \phi \leq \pi$, using (5.2), we have

$$M_n(\phi) = \left[\sum_{k=0}^{n-1}\frac{a_{n,n-k}}{n-k+1}\sum_{\nu=m}^{n-k+m}O(\nu)^{\beta}O(\nu)^{(\alpha+1)}\right] \tag{5.15}$$

$$
\begin{aligned}
&= O\left|\sum_{k=0}^{n-1}\frac{a_{n,n-k}}{n-k+1}\sum_{\nu=m}^{n-k+m}O(\nu)^{(\alpha+\beta+1)}\right| \\
&= O\left|\sum_{k=0}^{n-1}\frac{a_{n,n-k}}{n-k+1}(n-k+m)^{(\alpha+\beta+1)}\right| \\
&= O(n+m)^{(\alpha+\beta+1)}\left|\sum_{k=0}^{n-1}\frac{a_{n,n-k}}{n-k+1}\right| \quad \text{by Lemma 3} \\
&= O(n)^{\alpha+\beta}O\left(1\right) \\
&= O\left(n^{\alpha+\beta}\right).
\end{aligned}
$$

If $\frac{1}{n}\leq\phi\leq\pi-\frac{1}{n}$, with notation as in Lemma 4, then we have

$$
\begin{aligned}
M_n\left(\phi\right) &= O\left(1\right)\sum_{k=0}^{n-1}\frac{a_{n,n-k}}{n-k+1} \\
&\cdot\left[\sum_{\nu=m}^{n-k+m}\left\{\nu^{\left(\alpha+\frac{1}{2}\right)}\left(\sin\frac{\phi}{2}\right)^{-\alpha-\frac{3}{2}}\left(\cos\frac{\phi}{2}\right)^{-\beta-\frac{1}{2}}cos\left(\nu\phi+\rho\phi-r\right)+\frac{O(1)}{\nu\sin\phi}\right\}\right] \quad \text{by(5.3)} \\
&= O\left(1\right)\sum_{k=0}^{n-1}\frac{a_{n,n-k}}{n-k+1}\left[\sum_{\nu=m}^{n-k+m}\left\{\nu^{\left(\alpha+\frac{1}{2}\right)}\left(\sin\frac{\phi}{2}\right)^{-\alpha-\frac{3}{2}}\left(\cos\frac{\phi}{2}\right)^{-\beta-\frac{1}{2}}\cos\left(\nu\phi+\rho\phi-r\right)\right\}\right] \\
&+ O\left(1\right)\sum_{k=0}^{n-1}\frac{a_{n,n-k}}{n-k+1}\left[\sum_{\nu=m}^{n-k+m}\left\{\nu^{\left(\alpha+\frac{1}{2}\right)}\left(\sin\frac{\phi}{2}\right)^{-\alpha-\frac{5}{2}}\left(\cos\frac{\phi}{2}\right)^{-\beta-\frac{3}{2}}\right\}\right] \\
&= O\left(1\right)\sum_{k=0}^{n-1}\frac{a_{n,n-k}}{n-k+1}\left[\sum_{\nu=m}^{n-k+m}\left\{\nu^{\left(\alpha+\frac{1}{2}\right)}\left(\sin\frac{\phi}{2}\right)^{-\alpha-\frac{3}{2}}\left(\cos\frac{\phi}{2}\right)^{-\beta-\frac{1}{2}}\cos\left(\nu\phi+\rho\phi-r\right)\right\}\right] \\
&+ O\left(1\right)\sum_{k=0}^{n-1}\frac{a_{n,n-k}}{n-k+1}\left[\sum_{\nu=m}^{n-k+m}\left\{\nu^{\left(\alpha-\frac{1}{2}\right)}\left(\sin\frac{\phi}{2}\right)^{\alpha-\frac{5}{2}}\left(\cos\frac{\phi}{2}\right)^{-\beta-\frac{3}{2}}\right\}\right] \quad \text{by Lemma 4} \\
&= O\left\{\left(\sin\frac{\phi}{2}\right)^{-\alpha-\frac{3}{2}}\left(\cos\frac{\phi}{2}\right)^{-\beta-\frac{1}{2}}\right\}\left[\sum_{k=0}^{n-1}\frac{a_{n,n-k}}{n-k+1}\sum_{\nu=m}^{n-k+m}\left\{\nu^{\left(\alpha+\frac{1}{2}\right)}\cos\left(\nu\phi+\rho\phi-r\right)\right\}\right] \\
&+ O\left\{\left(\sin\frac{\phi}{2}\right)^{-\alpha-\frac{5}{2}}\left(\cos\frac{\phi}{2}\right)^{-\beta-\frac{3}{2}}\right\}\left[\sum_{k=0}^{n-1}\frac{a_{n,n-k}}{n-k+1}\sum_{\nu=m}^{n-k+m}\left\{\nu^{\left(\alpha-\frac{1}{2}\right)}\right\}\right] \\
&= O\left\{\left(\sin\frac{\phi}{2}\right)^{-\alpha-\frac{3}{2}}\left(\cos\frac{\phi}{2}\right)^{-\beta-\frac{1}{2}}\right\}O\left\{n^{\alpha-\frac{1}{2}}A_{n,\tau}\right\} \\
&+ O\left\{\left(\sin\frac{\phi}{2}\right)^{-\alpha-\frac{5}{2}}\left(\cos\frac{\phi}{2}\right)^{-\beta-\frac{3}{2}}\right\}O\left\{n^{\alpha-\frac{3}{2}}\right\} \text{by Lemma (4) and (5)} \\
&= O\left(\left(n^{\alpha-1/2}A_{n,\tau}\right)\left(\sin\frac{\phi}{2}\right)^{-\alpha-\frac{3}{2}}\left(\cos\frac{\phi}{2}\right)^{-\beta-\frac{1}{2}}\right) \\
&+ O\left[\left(n^{\alpha-\frac{3}{2}}\right)\left(\sin\frac{\phi}{2}\right)^{-\alpha-\frac{5}{2}}\left(\cos\frac{\phi}{2}\right)^{-\beta-\frac{3}{2}}\right].
\end{aligned}
$$

6 Proof of the Theorem

Following Obrechkoff [11], the partial sum of the series (2.1) at the point $x = 1$ is given by

$$s_\nu(1) = 2^{\alpha+\beta+1}\lambda_\nu \int_0^\pi \left(\sin\frac{\phi}{2}\right)^{2\alpha+1}\left(\cos\frac{\phi}{2}\right)^{2\beta+1} f(\cos\phi)\, s'_\nu(1,\cos\phi)\, d\phi$$

where $s'_\nu(1,\cos\phi)$ denotes the ν^{th} partial sum of the series

$$\frac{\sum_{k=m}^{n-k+m} P_m^{(\alpha,\beta)}(1)\, P_m^{(\alpha,\beta)}(\cos\phi)}{g_m}$$

where

$$g_m = \frac{(2\nu+\alpha+\beta+1)\,\Gamma(\nu+1)\,\Gamma(\nu+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(\nu+\alpha+1)\,\Gamma(\nu+\beta+1)}.$$

Rao [14] has shown that

$$s'_\nu(1,\cos\phi) = \lambda_\nu P_\nu^{(\alpha+1,\beta)}(\cos\phi)$$

therefore

$$\begin{aligned} s_\nu(1) - A &= 2^{\alpha+\beta+1}\lambda_\nu \int_0^\pi \left(\sin\frac{\phi}{2}\right)^{2\alpha+1}\left(\cos\frac{\phi}{2}\right)^{2\beta+1} \{f(\cos\phi) - A\}\, P_\nu^{(\alpha+1,\beta)}(\cos\phi)\, d\phi \\ &= 2^{\alpha+\beta+1}\lambda_\nu \int_0^\pi F(\phi) P_\nu^{(\alpha+1,\beta)}(\cos\phi)\, d\phi \end{aligned}$$

where λ_ν is defined as in Lemma 6.
Then

$$\begin{aligned} s_{n-k,m}(1) - A &= \frac{1}{n-k+1}\sum_{\nu=m}^{n-k+m}\{S_\nu(1) - A\} \\ &= \frac{1}{n-k+1}\sum_{\nu=m}^{n-k+m}\left\{2^{\alpha+\beta+1}\lambda_\nu \int_0^\pi F(\phi)\, P_\nu^{(\alpha+1,\beta)}(\cos\phi)\, d\phi\right\}. \end{aligned}$$

Now

$$\sum_{k=0}^{n} a_{n,n-k}\{s_{n-k,m}(x) - A\} = \int_0^\pi F(\phi)\, M_n(\phi)\, d\phi.$$

Let us denote

$$I=\left[\int_{0}^{\frac{1}{n}}+\int_{\frac{1}{n}}^{\delta}+\int_{\delta}^{\pi-\frac{1}{n}}+\int_{\pi-\frac{1}{n}}^{\pi}\right]F\left(\phi\right)\,M_n\left(\phi\right)d\phi$$

where δ is a suitable constant such that $0<\eta<\pi$.
Now in order to prove the theorem, we have to show that

$$I=o\left(1\right)\ as\ n\to\infty$$

for which we need to prove

$$I_j=o\left(1\right)\ as\,n\to\infty\,\text{for}\,j=1,2,3,4. \tag{6.1}$$

Let us first consider,

$$\begin{aligned}I_1&=O\left[\int_{0}^{\frac{1}{n}}\left|F\left(\phi\right)\right|\,\left|M_n\left(\phi\right)\right|d\phi\right]\\&=O\left(n^{2\alpha+1}\right)\left[\int_{0}^{\frac{1}{n}}\left|F\left(\phi\right)\right|d\phi\right]\ \text{by (5.12)}\\&=O\left(n^{2\alpha+1}\right)\,O\left(\frac{1}{n^{2\alpha+1}\xi\left(n\right)}\right)\ \text{by(4.1)}\\&=O\left(\frac{1}{\xi\left(n\right)}\right)\\&=o\left(1\right)\ as\,n\to\infty\ \ \text{by hypothesis of the theorem.}\end{aligned}\tag{6.2}$$

In order to estimate I_2 we employ the asymptotic relation given in (5.14). Thus

$$\begin{aligned}I_2&=O\left[\int_{\frac{1}{n}}^{\delta}\left|F\left(\phi\right)\right|\,n^{\alpha-\frac{1}{2}}A_{n,\tau}\left(\sin\frac{\phi}{2}\right)^{-\alpha-\frac{3}{2}}d\phi\right]+O\left[\int_{\frac{1}{n}}^{\delta}\left|F\left(\phi\right)\right|\,n^{\alpha-\frac{3}{2}}\left(\sin\frac{\phi}{2}\right)^{-\alpha-\frac{5}{2}}d\phi\right]\\&=I_{2.1}+I_{2.2}\ \text{(say)}.\end{aligned}\tag{6.3}$$

Now,

$$I_{2.1} = O\left(n^{\frac{2\alpha-1}{2}}\right)\left[\int_{\frac{1}{n}}^{\delta} \frac{|F(\phi)|\, A_{n,\tau}}{\phi^{(2\alpha+3)/2}}\, d\phi\right]$$

$$= O\left(n^{\frac{2\alpha-1}{2}}\right)\left[o\left\{\left(\frac{\phi^{2\alpha+1}}{\xi\left(\frac{1}{\phi}\right)}\left(\frac{A_{n,\tau}}{\phi^{(2\alpha+3)/2}}\right)\right)\right\}_{\frac{1}{n}}^{\delta} + o\left\{\int_{\frac{1}{n}}^{\delta} \frac{\phi^{2\alpha+1}}{\xi\left(\frac{1}{\phi}\right)} \frac{d}{d\phi}\left(\frac{A_{n,\tau}}{\phi^{(2\alpha+3)/2}}\right) d\phi\right\}\right]$$

$$= o\left(\frac{A_{n,n}}{\xi(n)}\right) + o\left(n^{\frac{2\alpha-1}{2}} A_{n,\eta}\right) + o\left(n^{\frac{2\alpha-1}{2}}\right)\left\{\int_{\frac{1}{n}}^{\delta} \frac{\phi^{2\alpha+1}}{\xi\left(\frac{1}{\phi}\right)}\left(\frac{A_{n,\tau}}{\phi^{(2\alpha+5)/2}}\right) d\phi\right\}$$

$$= o(1) + o(1) + o\left(n^{\frac{2\alpha-1}{2}}\right)\int_{\frac{1}{\delta}}^{n}\left(\frac{A_{n,u}}{u^{(2\alpha+1)/2}\xi(u)}\right) du,$$

$\frac{1}{\phi} = u$,by the hypothesis of theorem and$\tau =$ integral part of $\frac{1}{\phi} = \left[\frac{1}{\phi}\right]$

$$= o(1) + o\left(n^{\frac{2\alpha-1}{2}}\right)\sum_{a}^{n} \frac{A_{n,k}}{k^{(2\alpha+1)/2}\,\xi(k)} \quad \text{where} \quad a = \left[\frac{1}{\delta}\right] + 1,\ n \geq \left[\frac{1}{t}\right]$$

Using (4.2),

$$I_{2.1} = o(1)\,. \tag{6.4}$$

Now consider,

$$I_{2.2} = O\left[\int_{\frac{1}{n}}^{\delta} |F(\phi)|\, n^{(2\alpha-3)/2}\left(\sin\left(\frac{\phi}{2}\right)\right)^{-(2\alpha+5)/2} d\phi\right]$$

$$= O\left(n^{(2\alpha-3)/2}\right)\left[\int_{\frac{1}{n}}^{\delta} \frac{F(\phi)}{(\phi)^{(2\alpha+5)/2}} d\phi\right]$$

$$= O\left(n^{(2\alpha-3)/2}\right)\left[\left\{\frac{1}{(\phi)^{(2\alpha+5)/2}}\, o\left(\frac{(\phi)^{(2\alpha+1)}}{\xi(1/\phi)}\right)\right\}_{\frac{1}{n}}^{\delta} + o\left\{\int_{\frac{1}{n}}^{\delta} \frac{(\phi)^{(2\alpha-5)/2}}{\xi(1/\phi)} d\phi\right\}\right]$$

$$= o\left(n^{(2\alpha-3)/2}\right) + o\left(n^{(2\alpha-3)/2}\right)\left(\frac{n^{-(2\alpha-3)/2}}{\xi(n)}\right) + o\left(n^{(2\alpha-3)/2}\right)\left\{\int_{\frac{1}{n}}^{\delta} \frac{(\phi)^{(2\alpha-5)/2}}{\xi(1/\phi)} d\phi\right\}$$

Using second mean value theorem for integral,

$$
\begin{aligned}
I_{2.2} &= o\left(n^{(2\alpha-3)/2}\right) + o\left(\frac{1}{\xi(n)}\right) + o\left(\frac{n^{(2\alpha-3)/2}}{\xi(n)}\right)\left\{\int_{\frac{1}{n}}^{\delta}(\phi)^{(2\alpha-5)/2}d\phi\right\} \\
&= o(1) + o\left(\frac{n^{(2\alpha-3)/2}}{\xi(n)}\right)\left\{\frac{(\phi)^{(2\alpha-3)/2}}{(2\alpha-3)/2}\right\}_{\frac{1}{n}}^{\delta}, \quad \frac{-3}{2} \le \alpha < \frac{3}{2} \\
&= o(1) + o\left(\frac{n^{(2\alpha-3)/2}}{\xi(n)}\right) + o\left(\frac{n^{(2\alpha-3)/2}}{\xi(n)}\right)\left(n^{-(2\alpha-3)/2}\right) \\
&= o(1) + o\left(\frac{n^{(2\alpha-3)/2}}{\xi(n)}\right) + o\left(\frac{1}{\xi(n)}\right) \\
&= o(1) + o(1) \\
&= o(1) \ \ as\ n \to \infty \ \text{ by the hypothesis of the theorem.}
\end{aligned} \tag{6.5}
$$

Now we consider,

$$
\begin{aligned}
I_3 &= O\left[\int_{\delta}^{\pi-\frac{1}{n}} \frac{|F(\phi)|\, A_{n,\tau}\, n^{(2\alpha-1)/2}}{\left(\sin\frac{\phi}{2}\right)^{(2\alpha+3)/2}\left(\cos\frac{\phi}{2}\right)^{(2\beta+1)/2}}d\phi\right] \\
&\quad + O\left(n^{(2\alpha-3)/2}\right)\left[\int_{\delta}^{\pi-\frac{1}{n}} \frac{|F(\phi)|}{\left(\sin\frac{\phi}{2}\right)^{(2\alpha+5)/2}\left(\cos\frac{\phi}{2}\right)^{(2\beta+3)/2}}d\phi\right] \\
&= O\left(n^{(2\alpha-1)/2}A_{n,\eta}\right)\left[\int_{\delta}^{\pi-\frac{1}{n}} |F(\cos\phi)-A|\left(\cos\frac{\phi}{2}\right)^{(2\beta-1)/2}\left(\cos\frac{\phi}{2}\right)d\phi\right] \\
&\quad + O\left(n^{(2\alpha-3)/2}\right)\left[\int_{\delta}^{\pi-\frac{1}{n}} |F(\cos\phi)-A|\left(\cos\frac{\phi}{2}\right)^{(2\beta-1)/2}d\phi\right] \\
&= O\left(n^{(2\alpha-1)/2}A_{n,\eta}\right) + O\left(n^{(2\alpha-3)/2}\right) \\
&= o(1) + o(1) \quad as\ n \to \infty \ \text{ by (5.4)} \\
&= o(1) \ \ as\ n \to \infty.
\end{aligned} \tag{6.6}
$$

Finally we consider,

$$
\begin{aligned}
I_4 &= O\left(n^{\alpha+\beta}\right)\left[\int_{\pi-\frac{1}{n}}^{\pi} |F(\phi)|\, d\phi\right] \text{ by (5.13)}\\
&= O\left(n^{\alpha+\beta}\right)\left[\int_{\pi-\frac{1}{n}}^{\pi} |f(\cos\phi)-A| \left(\sin\frac{\phi}{2}\right)^{(2\alpha+1)}\left(\cos\frac{\phi}{2}\right)^{(2\beta+1)} d\phi\right]\\
&= O\left(n^{\alpha+\beta}\right)\left[\int_{0}^{\frac{1}{n}} |f(-\cos t)-A| \left(\sin\frac{t}{2}\right)^{(2\beta+1)}\left(\cos\frac{t}{2}\right)^{(2\alpha+1)} dt\right] \text{ taking } \pi-\phi=t\\
&= O\left(n^{\alpha+\beta}\right)\left[\int_{0}^{\frac{1}{n}} |f(-\cos t)-A|\, (t)^{(2\beta+1)} dt\right]\\
&= O\left(n^{(2\alpha-3)/2}\right)\left[\int_{0}^{\frac{1}{n}} |f(-\cos t)-A|\, (t)^{(2\beta-1)/2} dt\right] \text{ since } \frac{-3}{2}\le\alpha<\frac{3}{2}\\
&= o(1) \text{ as } n\to\infty. \qquad (6.7)
\end{aligned}
$$

Combining from (6.1) to (6.7) yields,

$$I = o(1).$$

This completes the proof of the theorem.

7 Particular Cases

1. If $a_{n,k} = \frac{p_{n-k}}{P_n}$ and $\xi(x) = \frac{\log x}{x}$, the result of Gupta [6] becomes the particular case of our main theorem.
2. The result of Tripathi, Tripathi and Yadav [19] becomes the particular case of our main theorem if $a_{n,k}$ is defined as in particular case 1 and $\xi(x) = \frac{(P_{[x]})^c}{x}$, $0 < c < 1$.
3. If $a_{n,k} = \frac{p_{n-k}q_n}{R_n}$ where $R_n = \sum_{k=0}^{n} p_k q_{n-k}$ and $\xi(\nu)$ is as defined in particular case 1 then the result of Khare and Tripathi [8] becomes the particular case of our main theorem.

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