Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 7 (2012), 39 – 47

ON COMMON FIXED POINT OF GENERALIZED CONTRACTIVE MAPPINGS IN METRIC SPACES

Mujahid Abbas and Hassen Aydi

Abstract. Existence of common fixed points is established for two self-mappings satisfying a generalized contractive condition. The presented results generalize several well known comparable results in the literature. We also study well-posedness of a common fixed point problem related to these mappings.

1 Introduction and preliminaries

Fixed point theory is one of the famous and traditional theories in mathematics and has a broad set of applications. In the existing literature on this theory, contractive conditions on the mappings play a vital role in proving the existence and uniqueness of a fixed point. Banach's contraction principle which gives an answer to the existence and uniqueness of a solution of an operator equation $Tx = x$, is the most widely used fixed point theorem in all of analysis. This principal is constructive in nature and is one of the most useful techniques in the study of nonlinear equations. There are many generalizations of the Banach's contraction mapping principle in the literature (see for example,[3], [4]). These generalizations were made either by using the contractive condition or by imposing some additional conditions on an ambient space. In 1968, Kannan [9] proved a fixed point theorem for a map satisfying a contractive condition that did not require continuity at each point. This paper was a genesis for a multitude of fixed point papers over the next two decades. In 1976 Jungck [6] extended and generalized the celebrated Banach contraction principle exploiting the idea of commuting maps. Sessa [14] coined the term weakly commuting maps. Jungck [7] generalized the notion of weak commutativity by introducing compatible maps and then weakly compatible maps [8]. Since then, many interesting coincidence and common fixed point theorems of compatible and weakly compatible maps under various contractive conditions and assuming the continuity of at least one of the mappings, have been obtained by a

2010 Mathematics Subject Classification: 54H25; 47H10.

Keywords: Coincidence point; Point of coincidence; Common fixed point; Contractive type mapping.

number of authors. Ćirić [5] studied necessary conditions to obtain a fixed point result of asymptotically regular mappings on complete metric spaces. The purpose of this paper is to present a common fixed point theorem for two mappings satisfying a generalized contractive condition. We also study the problem of well-posedness of common fixed point problem for the mappings considered in this paper.

2 Common fixed point theorem

A pair (f,T) of self-mappings on X is said to be weakly compatible if f and T commute at their coincidence point (i.e. $fTx = Tfx$ whenever $fx = Tx$). A point $y \in X$ is called point of coincidence of two self-mappings f and T on X if there exists a point $x \in X$ such that $y = Tx = fx$.

The following lemma is Proposition 1.4 of [1].

Lemma 1. *Let X be a non-empty set and the mappings T, $f : X \to X$ have a unique point of coincidence v in X. If the pair (f,T) is weakly compatible, then T and f have a unique common fixed point.*

Let (X, d) be a metric space, T and f be self-mappings on X, with $T(X) \subset f(X)$, and $x_0 \in X$. Choose a point x_1 in X such that $fx_1 = Tx_0$. This can be done since $T(X) \subset f(X)$. Continuing this process, having chosen $x_1, \dots, x_k$, we choose x_{k+1} in X such that

$$fx_{k+1} = Tx_k, \quad k = 0, 1, 2, \dots .$$

The sequence $\{fx_n\}$ is called a $T-$sequence with initial point x_0.

Definition 2. *Let T and f be self-mappings on a metric space X, with $T(X) \subset f(X)$, and $x_0 \in X$. A mapping T is said to be asymptotically $f-$regular at point x_0 if $d(fx_n, fx_{n+1}) \to 0$ as $n \to \infty$, where $\{fx_n\}$ is a $T-$sequence with initial point x_0.*

Let $F_i : [0,\infty) \to [0,\infty)$ be functions such that $F_i(0) = 0$ and F_i is continuous at 0 $(i = 1, 2)$. Our first result is the following:

Theorem 3. *Let (X, d) be a metric space. Let $T, f : X \to X$ be such that $T(X) \subset f(X)$. Assume that the following condition holds:*

$$\begin{aligned} d(Tx,Ty) &\leq a_1 F_1[\min\{d(fx,Tx), d(fy,Ty)\}] + a_2 F_2[d(fx,Tx)d(fy,Ty)] \\ &+ a_3 d(fx,fy) + a_4[d(fx,Tx) + d(fy,Ty)] + a_5[d(fx,Ty) + d(fy,Tx)] \quad (2.1) \end{aligned}$$

for all $x, y \in X$, where for $i = 1, ..., 5$, $a_i \geq 0$ such that for arbitrary fixed $k > 0$, $0 < \lambda_1 < 1$ and $0 < \lambda_2 < 1$, we have $a_4 + a_5 \leq \lambda_1$, $a_3 + 2a_5 \leq \lambda_2$ and $a_1, a_2 \leq k$. If $f(X)$ or $T(X)$ is a complete subspace of X and T is asymptotically $f-$regular at some point x_0 in X, then T and f have a point of coincidence.

Proof. Let x_0 be an arbitrary point in X and let $\{fx_n\}$ be a $T-$sequence with initial point x_0. Since T is asymptotically $f-$regular mapping at x_0, therefore $d(fx_n, fx_{n+1}) \to 0$ as $n \to \infty$. Now for $m > n$, we have

$$
\begin{aligned}
& d(fx_n, fx_m) \le d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{m+1}) + d(fx_{m+1}, fx_m) \\
= \quad & d(fx_n, fx_{n+1}) + d(Tx_n, Tx_m) + d(fx_{m+1}, fx_m) \\
\le \quad & d(fx_n, fx_{n+1}) + d(fx_{m+1}, fx_m) + a_1F_1[\min\{d(fx_n, Tx_n), d(fx_m, Tx_m)\}] \\
& +a_2F_2[d(fx_n, Tx_n)d(fx_m, Tx_m)] + a_3d(fx_n, fx_m) + a_4[d(fx_n, Tx_n) + \\
& d(fx_m, Tx_m)] + a_5[d(fx_n, Tx_m) + d(fx_m, Tx_n)] \\
= \quad & d(fx_n, fx_{n+1}) + d(fx_{m+1}, fx_m) + a_1F_1[\min\{d(fx_n, fx_{n+1}), d(fx_m, fx_{m+1})\}] \\
& +a_2F_2[d(fx_n, fx_{n+1})d(fx_m, fx_{m+1})] + a_3d(fx_n, fx_m) + a_4[d(fx_n, fx_{n+1}) + \\
& d(fx_m, fx_{m+1})] + a_5[d(fx_n, fx_{m+1}) + d(fx_m, fx_{n+1})] \\
\le \quad & d(fx_n, fx_{n+1}) + d(fx_{m+1}, fx_m) + a_1F_1[\min\{d(fx_n, fx_{n+1}), d(fx_m, fx_{m+1})\}] \\
& +a_2F_2[d(fx_n, fx_{n+1})d(fx_m, fx_{m+1})] \\
& +a_3d(fx_n, fx_m) + a_4[d(fx_n, fx_{n+1}) + d(fx_m, fx_{m+1})] \\
& +a_5[d(fx_n, fx_m) + d(fx_m, fx_{m+1}) + d(fx_m, fx_n) + d(fx_n, fx_{n+1})] \\
= \quad & (1 + a_4 + a_5)[d(fx_n, fx_{n+1}) + d(fx_{m+1}, fx_m)] + a_1F_1[\min\{d(fx_n, fx_{n+1}), d(fx_m, fx_{m+1})\}] \\
& +a_2F_2[d(fx_n, fx_{n+1})d(fx_m, fx_{m+1})] + (a_3 + 2a_5)d(fx_n, fx_m) \\
\le \quad & (1 + \lambda_1)[d(fx_n, fx_{n+1}) + d(fx_{m+1}, fx_m)] + kF_1[\min\{d(fx_n, fx_{n+1}), d(fx_m, fx_{m+1})\}] \\
& +kF_2[d(x_n, fx_{n+1})d(fx_m, fx_{m+1})] + \lambda_2d(fx_n, fx_m).
\end{aligned}
$$

Thus we obtain that

$$
\begin{aligned}
& (1 - \lambda_2)d(fx_n, fx_m) \\
\le \quad & (1 + \lambda_1)[d(fx_n, fx_{n+1}) + d(fx_{m+1}, fx_m)] + kF_1[\min\{d(fx_n, fx_{n+1}), d(fx_m, fx_{m+1})\}] \\
& +kF_2[d(fx_n, fx_{n+1})d(fx_m, fx_{m+1})].
\end{aligned}
$$

Since T is asymptotically $f-$regular and F_1 and F_2 are continuous at zero, then the right-hand side of the above inequality tends to zero, as $m, n \to \infty$. Thus,

$$\lim_{m,n\to\infty} d(fx_n, fx_m) = 0.$$

It follows that $\{fx_n\}$ is a Cauchy sequence in X. If $f(X)$ is a complete subspace of X, there exist $u, p \in X$ such that $fx_n \to p = fu$ (this holds also if $T(X)$ is complete with $p \in T(X)$). We claim that u is a coincidence point of f and T. If not, then

$d(fu, Tu) > 0$. From (2.1), we obtain

$$\begin{aligned}
& d(fu,Tu) = d(p,Tu) \leq d(p,fx_{n+1}) + d(fx_{n+1},Tu) \\
= \;& d(p,fx_{n+1}) + d(Tx_n,Tu) \\
\leq \;& d(p,fx_{n+1}) + a_1F_1[\min\{d(fx_n,Tx_n),d(fu,Tu)\}] + a_2F_2[d(fx_n,Tx_n)d(fu,Tu)] \\
& +a_3d(fx_n,fu) + a_4[d(fx_n,Tx_n) + d(fu,Tu)] + a_5[d(fx_n,Tu) + d(fu,Tx_n)] \\
\leq \;& d(p,fx_{n+1}) + a_1F_1[\min\{d(fx_n,fx_{n+1}),d(p,Tu)\}] + a_2F_2[d(fx_n,fx_{n+1})d(p,Tu)] \\
& +a_3d(fx_n,p) + a_4[d(fx_n,fx_{n+1}) + d(p,Tu)] \\
& +a_5[d(fx_n,p) + d(p,Tu) + d(p,fx_{n+1})] \\
= \;& (1+a_5)d(p,fx_{n+1}) + a_1F_1[\min\{d(fx_n,fx_{n+1}),d(p,Tu)\}] + a_2F_2[d(fx_n,fx_{n+1})d(p,Tu)] \\
& +(a_3+a_5)d(fx_n,p) + (a_4+a_5)d(p,Tu) + a_5[d(p,fx_{n+1})] \\
\leq \;& (1+a_5)d(p,fx_{n+1}) + a_1F_1[\min\{d(fx_n,fx_{n+1}),d(p,Tu)\}] \\
& +a_2F_2[d(fx_n,fx_{n+1})d(p,Tu)] + (a_3+a_5)d(fx_n,p) \\
& +(a_4+a_5)d(p,Tu) + a_5[d(p,fx_{n+1})],
\end{aligned}$$

which on taking limit as $n \to \infty$ gives that,

$$d(p,Tu) \leq (a_4+a_5)d(p,Tu),$$

a contradiction, and so $p = fu = Tu$ is a point of coincidence of f and T. □

Lemma 4. *Let $(X,\, d)$ be a metric space. Let $T, f : X \to X$ be such that $T(X) \subset f(X)$. Assume that the following condition holds:*

$$\begin{aligned}
& d(Tx,Ty) \leq a_1F_1[\min\{d(fx,Tx),d(fy,Ty)\}] + a_2F_2[d(fx,Tx)d(fy,Ty)] \\
& +a_3d(fx,fy) + a_4[d(fx,Tx) + d(fy,Ty)] + a_5[d(fx,Ty) + d(fy,Tx)]
\end{aligned}$$

for all $x, y \in X$, where for $i = 1, ..., 5$, $a_i \geq 0$ such that for arbitrary fixed $k > 0$, $0 < \lambda_1 < 1$ and $0 < \lambda_2 < 1$, we have $a_4 + a_5 \leq \lambda_1$, $a_3 + 2a_5 \leq \lambda_2$ and $a_1, a_2 \leq k$. Then, T and f have at most a unique point of coincidence.

Proof. Assume that there exist points p, p^* in X such that $p = fu = Tu$ and $p^* = fu^* = Tu^*$, for some u, u^* in X. From

$$\begin{aligned}
d(p,p^*) \;&=\; d(Tu,Tu^*) \\
&\leq\; a_1F_1[\min\{d(fu,Tu),d(fu^*,Tu^*)\}] + a_2F_2[d(fu,Tu)d(fu^*,Tu^*)] \\
&\quad +a_3d(fu,fu^*) + a_4[d(fu,Tu) + d(fu^*,Tu^*)] + a_5[d(fu,Tu^*) + d(fu^*,Tu)] \\
&=\; a_3d(p,p^*) + a_5[d(p,p^*) + d(p^*,p)] \\
&=\; (a_3+2a_5)d(p,p^*),
\end{aligned}$$

we deduce that $p = p^*$. □

From Theorem 3 and Lemma 4, we obtain the following theorem.

Theorem 5. *Let* $(X,\, d)$ *be a metric space. Let* $T, f : X \to X$ *be such that* $T(X) \subset f(X)$*. Assume that* T *and* f *satisfy condition (2.1) for all* $x, y \in X$*. If* $f(X)$ *or* $T(X)$ *is a complete subspace of* X *and the pair* (T, f) *is weakly compatible, then* T *and* f *have a unique common fixed point provided that* T *is asymptotically* $f-$*regular at some point* x_0 *in* X*.*

Proof. By Theorem 3 and Lemma 4, T and f have a unique point of coincidence. Since the pair (T, f) is weakly compatible, by Lemma 1, T and f have a unique common fixed point. □

Taking $a_1 = a_2 = 0$ in the inequality (2.1), we have the following corollary.

Corollary 6. *Let* $(X,\, d)$ *be a metric space. Let* $T, f : X \to X$ *be such that* $T(X) \subset f(X)$*. Assume that the following condition holds:*

$$d(Tx, Ty) \le a_3 d(fx, fy) + a_4[d(fx, Tx) + d(fy, Ty)] + a_5[d(fx, Ty) + d(fy, Tx)]$$

for all $x, y \in X$*, where for* $i = 3, ..., 5$*,* $a_i \ge 0$ *such that for arbitrary fixed* $0 < \lambda_1 < 1$ *and* $0 < \lambda_2 < 1$*, we have* $a_4 + a_5 \le \lambda_1$ *and* $a_3 + 2a_5 \le \lambda_2$*. If* $f(X)$ *or* $T(X)$ *is a complete subspace of* X *and* T *is asymptotically* $f-$*regular at some point* x_0 *in* X*, then* T *and* f *have a point of coincidence.*

Remark 7. *Theorem 3 and Lemma 4 remain true if we replace the real numbers* a_i *by real functions* $a_i(x, y)$ *for* $x, y \in X$ *and* $i = 1, ..., 5$*.*

As a consequence of Theorem 3, Lemma 4 and 5, we obtain the following result of Ćirić [5] as a corollary.

Corollary 8. *Let* $(X,\, d)$ *be a complete metric space. Let* $T : X \to X$ *be such that the following condition holds:*

$$\begin{aligned} d(Tx, Ty) &\le a_1 F_1[\min\{d(x, Tx), d(y, Ty)\}] + a_2 F_2[d(x, Tx)d(y, Ty)] \\ &+ a_3 d(x, y) + a_4[d(x, Tx) + d(y, Ty)] + a_5[d(x, Ty) + d(y, Tx)] \end{aligned} \tag{2.2}$$

where for $i = 1, ..., 5$*,* $a_i \ge 0$ *such that for arbitrary fixed* $k > 0$*,* $0 < \lambda_1 < 1$ *and* $0 < \lambda_2 < 1$*, we have* $a_4 + a_5 \le \lambda_1$*,* $a_3 + 2a_5 \le \lambda_2$ *and* $a_1, a_2 \le k$*. If* T *is asymptotically regular at some point* x_0 *in* X*, then* T *has a (unique) fixed point.*

Remark 9. *The fact that* T *is asymptotically regular at some* $x_0 \in X$ *corresponds to* T *is asymptotically* I_X-*regular at some* $x_0 \in X$*. Our results extend Theorem 1 in 8 and in turn extend and generalize results of Sharma and Yuel [13] and Babu, Sandhya and Kameswari [2] (of course when the constants* $(a_i)_{i=1,...,5}$ *are taken real functions* $a_i(x, y)$ *).*

We give an example to support our results.

Example 10. *Let $X = [0, +\infty)$ be endowed with the usual metric. Define $f, T : X \to X$ by*

$$Tx = 2x \text{ and } fx = 3x.$$

Let $x_0 = 1$ and the sequence $\{x_n\}_{n\geq 1}$ be given by $x_n = (\frac{2}{3})^n$. Note that $\{fx_n\}$ is a $T-$sequence with initial point x_0. Since $d(fx_n, fx_{n+1}) \to 0$ as $n \to \infty$,the mapping T is asymptotically $f-$regular at the point x_0. Also, $T(X) \subset f(X)$, $T(X)$ is a complete subset of X, the pair (f, T) is weakly compatible and the inequality (2.1) holds for all $x, y \in X$ with

$$F_1 = F_2 = 1, \quad a_1 = a_2 = a_4 = a_5 = 0 \quad \text{and} \quad a_3 = \frac{3}{4}.$$

Thus $f, T : X \to X$ satisfy all conditions of Theorem 5. Moreover, $u = 0$ is the common fixed point of f and T.

Now, we have the following result on the continuity on the set of common fixed points. Let $CF(T, f)$ denote the set of all common fixed points of T and f.

Theorem 11. *Let (X, d) be a metric space. Assume that $T, f : X \to X$ satisfy condition (2.1) for all $x, y \in X$. If $CF(T, f) \neq \emptyset$, then T is continuous at $p \in CF(T, f)$ whenever f is continuous at p.*

Proof. Fix $p \in CF(T, f)$. Let (z_n) be any sequence in X converging to p. Then by taking $y := z_n$ and $x := p$ in (2.1), we get

$$\begin{aligned} d(Tp, Tz_n) \leq\ & a_1F_1[\min\{d(fp, Tp), d(fz_z, Tz_n)\}] + a_2F_2[d(fp, Tp)d(fz_n, Tz_n)] \\ & + a_3d(fp, fz_n) + a_4[d(fp, Tp) + d(fz_n, Tz_n)] + a_5[d(fp, Tz_n) + d(fz_n, Tp)] \end{aligned}$$

which, in view of $Tp = fp$, we obtain

$$\begin{aligned} d(Tp, Tz_n) &\leq a_3d(fp, fz_n) + a_4[d(fz_n, Tz_n)] + a_5[d(fp, Tz_n) + d(fz_n, Tp)] \\ &\leq a_3d(fp, fz_n) + a_4[d(fz_n, fp)] + a_4[d(Tp, Tz_n)] + a_5[d(Tp, Tz_n) + d(fz_n, fp)]. \end{aligned}$$

Now, by letting $n \to \infty$ we get

$$\limsup_{n\to\infty} d(Tp, Tz_n) \leq (a_4 + a_5) \limsup_{n\to\infty} d(Tp, Tz_n),$$

whenever f is continuous at p. The last inequality is true only if $\limsup_{n\to\infty} d(Tp, Tz_n) = 0$. We get that $Tz_n \to Tp$ as $n \to \infty$. $\square$

The following example shows that self-maps f and T of a complete metric space X may not have a common fixed point in X. Here pair (T, f) satisfies the inequality (2.1), and both f and T are continuous on X. Note that (T, f) is not weakly compatible.

Example 12. *Let $X = \mathbb{R}$ endowed with the usual metric. We define mappings $f, T : X \to X$ by $Tx = \frac{x+1}{4}$ and $fx = \frac{x}{2}$ for $x \in X$. Taking $a_1 = a_2 = a_4 = a_5 = 0$ and $a_3 = \frac{1}{2}$, the inequality (2.1) holds for any $x, y \in X$. But, f and T have no common fixed points, that is, $CF(T, f) = \emptyset$. Here, the unique coincidence point is $u = 1$ and $fT1 \neq Tf1$, so the pair (T, f) is not weakly compatible.*

3 Well-Posedness

The notion of well-posedness of a fixed point has evoked much interest to several mathematicians. Recently, Karapinar [10] studied well-posed problem for a cyclic weak $\phi-$contraction mapping on a complete metric space (see also, [11, 12]).

Definition 13. *A common fixed point problem of self-maps f and T on X, $CFP(f, T, X)$, is called well-posed if $CF(f, T)$ (the set of common fixed points of f and T) is singleton and for any sequence $\{x_n\}$ in X with $x^* \in CF(S, T)$ and $\lim_{n\to\infty} d(fx_n, x_n) = \lim_{n\to\infty} (Tx_n, x_n) = 0$ implies $x^* = \lim_{n\to\infty} x_n$.*

Theorem 14. *Suppose that T and f be self-maps on X as in Theorem 3 and lemma 4. Then, the common fixed point problem of f and T is well-posed.*

Proof. From Theorem 3 and lemma 4, the mappings f and T have a unique common fixed point, say $u \in X$. Let $\{x_n\}$ be a sequence in X and $\lim_{n\to\infty} d(fx_n, x_n) = \lim_{n\to\infty} (Tx_n, x_n) = 0$. With loss of generality, we may suppose that $u \neq x_n$ for every non-negative integer n. Then, having in mind $fu = Tu = u$ and from the triangle inequality and (2.1), we have

$$
\begin{aligned}
& d(u, x_n) = d(Tu, x_n) \leq d(Tx_n, Tu) + d(Tx_n, x_n) \\
\leq \; & d(Tx_n, x_n) + a_1 F_1[\min\{d(fx_n, Tx_n), d(fu, Tu)\}] + a_2 F_2[d(fx_n, Tx_n) d(fu, Tu)] \\
& + a_3 d(fx_n, fu) + a_4[d(fx_n, Tx_n) + d(fu, Tu)] + a_5[d(fx_n, Tu) + d(fu, Tx_n)] \\
& d(Tx_n, x_n) + a_3 d(fx_n, u) + a_4[d(fx_n, Tx_n)] + a_5[d(fx_n, u) + d(u, Tx_n)] \\
\leq \; & d(Tx_n, x_n) + a_3 d(fx_n, x_n) + a_3 d(x_n, u) + a_4[d(fx_n, x_n)] \\
& + a_4[d(x_n, Tx_n)] + a_5[d(fx_n, x_n) + d(x_n, u) + d(u, x_n) + d(x_n, Tx_n)].
\end{aligned}
$$

Letting $n \to \infty$, we get that

$$
\limsup_{n\to\infty} d(u, x_n) \leq (a_3 + 2a_5) \limsup_{n\to\infty} d(u, x_n),
$$

which holds unless, $\limsup_{n\to\infty} d(u, x_n) = 0$. We deduce, $x_n \to u$ as $n \to \infty$. This completes the proof of Theorem 14. □

References

[1] M. Abbas, G. Jungck, *Common fixed results for noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl. **341**(2008), 416–420. MR2394094(2008m:47071). Zbl 0960.43003.

[2] G. V. R. Babu, M. L. Sandhya, M. V. R. Kameswari, *A note on a fixed point theorem of Berinde on weak contractions*, Carpathian J. Math. **24**(2008), 8-12. MR2410201(2009e:54086). Zbl 1199.54205.

[3] Lj. B. Ćirić, *Generalized contractions and fixed point theorems*, Publ. Inst. Math. (Beograd), **12**(1971), 19–26. MR0309092(46#8203). Zbl 0234.54029.

[4] Lj. B. Ćirić, *A generalization of Banach's contraction principle,* Proc. Amer. Math. Soc. **45**(1974), 267–273. MR0356011 (50#8484). Zbl 0291.54056.

[5] Lj.B. Ćirić, *Fixed point of asymptotically regular mappings*, Mathematical Communications, **10**(2005), 111-114. MR2199099. Zbl 1089.54515.

[6] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly , **73**(1976), 261–263. MR0400196(53#4031).

[7] G. Jungck, *Compatible mappings and common fixed points,* Int. J. Math. Math. Sci. **9**(1986), 771-779. MR1255221.

[8] G. Jungck, *Common fixed points for noncontinuous nonself maps on nonmetric spaces*, Far East J. Math. Sci. **4**(1996), 199–215. MR1426938. Zbl 0928.54043 .

[9] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc. **10**(1968), 71–76. MR0300264(45#9310). Zbl 0257.54044.

[10] E. Karapınar, *Fixed point theory for cyclic weak ϕ- contraction*, Appl. Math. Lett. **24**(2011), 822–825. Zbl pre05874611 .

[11] M. Păcurar, I. A. Rus, *Fixed point theory for cyclic φ- contractions*, Nonlinear Anal. **72**(2010), 1181–1187. Zbl 1191.54042.

[12] S. Reich, A. J. Zaslawski, *Well posedness of fixed point problems*, Far East J. Math. Sci. Special Volume, Part III (2001) 393-401. MR1888108(2003d:54058).

[13] P. L. Sharma and A. K. Yuel, *Fixed point theorems under asymptotic regularity at a point*, Math. Sem. Notes, **35** (1982), 181–190. MR0834895(87f:54061).

[14] S. Sessa, *On a weak commutativity condition of mappings in fixed point consideration*, Publ. Inst. Math. **32** (1982), 149–153. MR0710984(85f:54107). Zbl 0832.54034 .

****

M. Abbas
Department of Mathematics,
Lahore University of Management Sciences,
54792-Lahore, Pakistan.
e-mail: mujahid@lums.edu.pk

H. Aydi
Université de Sousse,
Institut Supérieur d'Informatique et des
Technologies de Communication de Hammam Sousse,
Route GP1, Hammam Sousse-4011, Tunisie.
e-mail: hassen.aydi@isima.rnu.tn