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POSITIVE BLOCK MATRICES ON HILBERT AND KREIN C^*-MODULES

M. Dehghani, S. M. S. Modarres and M. S. Moslehian

Abstract. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert C^*-modules. In this paper we give some necessary and sufficient conditions for the positivity of a block matrix on the Hilbert C^*-module $\mathscr{H}_1 \oplus \mathscr{H}_2$. If $(\mathscr{H}_1, J_1)$ and $(\mathscr{H}_2, J_2)$ are two Krein C^*-modules, we study the $\tilde{\mathbf{J}}$-positivity of 2×2 block matrix

$$\begin{pmatrix} A & X \\ X^\sharp & B \end{pmatrix}$$

on the Krein C^*-module $(\mathscr{H}_1 \oplus \mathscr{H}_2, \tilde{\mathbf{J}} = J_1 \oplus J_2)$, where $X^\sharp = J_2 X^* J_1$ is the (J_2, J_1)-adjoint of the operator X. We prove that if A is J_1-selfadjoint and B is J_2-selfadjoint and A is invertible, then the operator $\begin{pmatrix} A & X \\ X^\sharp & B \end{pmatrix}$ is $\tilde{\mathbf{J}}$-positive if and only if $A \geq^{J_1} 0$, $B \geq^{J_2} 0$ and $X^\sharp A^{-1} X \leq^{J_2} B$. We also present more equivalent conditions for the $\tilde{\mathbf{J}}$-positivity of this operator.

Full text

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Mohammad Sal Moslehian
Department of Pure Mathematics,
Center of Excellence in Analysis on
Algebraic Structures (CEAAS),
Ferdowsi University of Mashhad,
P. O. Box 1159, Mashhad 91775, Iran.
e-mail: moslehian@um.ac.ir,
moslehian@member.ams.org

Mahdi Dehghani
Department of Pure and Applied Mathematics,
Yazd University,
Yazd P.O. Box 89195-741, Iran.
Department of Pure Mathematics,
Center of Excellence in Analysis on
Algebraic Structures (CEAAS),
Ferdowsi University of Mashhad,
P. O. Box 1159, Mashhad 91775, Iran.
e-mail: e.g.mahdi@gmail.com

Seyed Mohammad Sadegh Modarres
Department of Pure and Applied Mathematics,
Yazd University,
Yazd P. O. Box 89195-741, Iran.
e-mail: smodares@yazduni.ac.ir

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