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COMMON FIXED POINT THEOREM FOR NONCOMPATIBLE MAPS IN PROBABILISTIC METRIC SPACE

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Abstract. In this paper, we prove a common fixed point theorem for noncompatible maps in probabilistic metric space using implicit relation. Our result does not require either the completeness of the space or continuity of the maps.

1 Introduction

K. Menger [6] introduced the notion of a probabilistic metric space in 1942 and since then the theory of probabilistic metric spaces has developed in many directions [12]. The idea of K. Menger was to use distribution functions instead of non-negative real numbers as values of the metric. The notion of a probabilistic metric space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. A probabilistic generalization of metric spaces appears to be interest in the investigation of physical quantities and physiological thresholds. It is also of fundamental importance in probabilistic functional analysis.

In 1986, Jungck [3] introduced the notion of compatible maps for a pair of self maps in metric space. Mishra [8] extended this notion of compatibility to probabilistic metric space. Several common fixed point theorems have been proved for compatible maps in probabilistic metric space. It is seen that most of the common fixed point theorems for contraction mappings invariably require a compatibility condition besides assuming continuity of at least one of the mappings. However, the study of common fixed points of noncompatible maps is also of great interest. Pant [9] initiated the study of common fixed points of noncompatible maps in metric spaces. In 2002, Aamri and El Moutawakil [1] defined a property (E.A) for self-maps which contained the class of noncompatible maps. Recently, Kubiaczyk and

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Sharma [4] studied the common fixed points of weakly compatible maps satisfying the property (E.A) in PM-spaces.

Mihet [7] and the present authors [5],[10] proved some fixed point theorems concerning probabilistic contractions satisfying an implicit relation.

The purpose of this paper is to obtain a common fixed point theorem for weakly compatible self-maps satisfying the property (E.A) using implicit relation in probabilistic metric space. Our result does not require either the completeness of the space or continuity of the maps.

For the sake of convenience, we first recall some definitions and notations.

2 Preliminaries

Definition 1. *[12] A mapping $F : R \to R^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in R} F(t) = 0$ and $\sup_{t\in R} F(t) = 1$.*

We shall denote by $\Im$ the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 2. *[12]A PM-space is an ordered pair (X,F), where X is a nonempty set of elements and F is a mapping from $X \times X$ to $\Im$, the collection of all distribution functions. The value of F at $(x, y) \in X \times X$ is represented by $F_{x,y}$. The functions $F_{x,y}$ are assumed to satisfy the following conditions:*

(i) $F_{x,y}(t) = 1$ for all $t > 0$ iff $x = y$;

(ii) $F_{x,y}(0) = 0$;

(iii) $F_{x,y}(t) = F_{y,x}(t)$;

(iv) if $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$ then $F_{x,z}(t + s) = 1$

for all $x, y, z \in X$ and $t, s > 0$.

Definition 3. *[12] A mapping $\triangle : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (briefly, t-norm) if the following conditions are satisfied:*

(i) $\triangle(a, 1) = a$ for all $a \in [0, 1]$;

(ii) $\triangle(a, b) = \triangle(b, a)$;

(iii) $\triangle(c, d) \geq \triangle(a, b)$ for $c \geq a$, $d \geq b$;

(iv) $\triangle(\triangle(a, b), c) = \triangle(a, \triangle(b, c))$;

for all $a, b, c, d \in [0, 1]$.

Definition 4. *[12] A Menger space is a triplet $(X, F, \triangle)$ where (X,F) is a PM-space and t-norm $\triangle$ is such that the inequality*

$$F_{x,z}(t + s) \geq \triangle(F_{x,y}(t), F_{y,z}(s))$$

holds for all $x, y, z \in X$ *and all* $t, s > 0$.

Every metric space (x,d) can be realized as a PM-space by taking $F : X \times X \to \Im$ *defined by* $F_{x,y}(t) = H(t - d(x, y))$ *for all x,y in X.*

Definition 5. *[8] Two self maps A and S of a PM-space* (X, F) *are said to be compatible if and only if* $F_{ASx_n,SAx_n}(t) \to 1$ *for all* $t > 0$*, whenever* $\{x_n\}$ *is a sequence in X such that* $Ax_n, Sx_n \to z$ *for some z in X.*

Definition 6. *[13] Self-maps A and S of a PM-space* (X, F) *are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if* $Az = Sz$ *some* $z \in X$*, then* $ASz = SAz$.

Remark 7. *Two compatible self-maps are weakly compatible, but the converse is not true (See [13], Ex. 1). Therefore the concept of weak compatibility is more general than that of compatibility.*

Definition 8. *[4] A pair of self-maps A and S on a PM-space* (X, F) *are said to satisfy the property (E.A), if there exists a sequence* $\{x_n\}$ *such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z, \text{ for some } z \in X.$$

Note that weakly compatible and property (E.A) are independent to each other (see [11], Ex. 2.2).

Remark 9. *From Definition 5, it is inferred that two self maps A and S on a PM-space* (X, F) *are noncompatible if and only if there exists at least one sequence* $\{x_n\}$ *in X such that* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ *for some* $z \in X$*, but for some* $t > 0$, $\lim_{n\to\infty} F_{ASx_n,SAx_n}(t)$ *is either less than 1 or nonexistent. Therefore, from Definition 8, it is easy to see that any noncompatible self-maps of* (X, F) *satisfy the property (E.A). But two mappings satisfying the property (E.A) need not be noncompatible (see [2], Ex. 1).*

3 Implicit Relation

The present authors [10] proved some fixed point theorems concerning probabilistic contractions satisfying the following implicit relation:

Let Φ be the class of all real continuous functions $\varphi : (R^+)^4 \to R$, non-decreasing in the first argument and satisfying the following conditions:

(R-1) $u, v \geq 0, \varphi(u, v, u, v) \geq 0$ or $\varphi(u, v, v, u) \geq 0$ implies that $u \geq v$.

(R-2) $\varphi(u, u, 1, 1) \geq 0$ for all $u \geq 1$.

Example 10. *[10] Define* $\varphi(t_1, t_2, t_3, t_4) = at_1 + bt_2 + ct_3 + dt_4$*, where* $a, b, c, d \in R$ *with* $a + b + c + d = 0$, $a > 0$, $a + c > 0$, $a + b > 0$ *and* $a + d > 0$. *Then* $\varphi \in \Phi$.

Example 11. *Define* $\varphi(t_1, t_2, t_3, t_4) = 14t_1 - 12t_2 + 6t_3 - 8t_4$. *Then* $\varphi \in \Phi$.

4 Result

Theorem 12. *Let $(X, F, \triangle)$ be a Menger space, where $\triangle$ is continuous t-norm. Further, let (A,S) and (B,T) be weakly compatible pairs of self-maps of X satisfying*

(1) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$;

(2) (A,S) or (B,T) satisfies the property (E.A);

(3) there exist $k \in (0, 1)$ and $\varphi \in \Phi$ such that

$$\varphi(F_{Ax,By}(kt), F_{Sx,Ty}(t), F_{Ax,Sx}(kt), F_{By,Ty}(t)) \geq 0;$$

for all $x, y \in X$, $t > 0$.

If the range of one of the maps A, B, S or T is a complete subspace of X, then A, B, S and T have a unique common fixed point.

Proof. If the pair (B,T) satisfies the property (E.A), then there exists a sequence $\{x_n\}$ in X such that $Bx_n \to z$ and $Tx_n \to z$, for some $z \in X$ as $n \to \infty$. Since $B(X) \subseteq S(X)$, there exist a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$. Hence, $Sy_n \to z$ as $n \to \infty$. Now we claim that $Ay_n \to z$ as $n \to \infty$. Suppose $Ay_n \to w(\neq z) \in X$, then by (3), we have

$\varphi(F_{Ay_n,Bx_n}(kt), F_{Sy_n,Tx_n}(t), F_{Ay_n,Sy_n}(kt), F_{Bx_n,Tx_n}(t)) \geq 0$,

that is,

$\varphi(F_{Ay_n,Bx_n}(kt), F_{Bx_n,Tx_n}(t), F_{Ay_n,Bx_n}(kt), F_{Bx_n,Tx_n}(t)) \geq 0$.

By (R-1) we have

$F_{Ay_n,Bx_n}(kt) \geq F_{Bx_n,Tx_n}(t)$.

Letting $n \to \infty$,

$F_{w,z}(kt) \geq F_{w,z}(t)$,

thus $w = z$. This shows that $Ay_n \to z$ as $n \to \infty$.

Suppose that $S(X)$ is a complete subspace of X. Then $z = Su$ for some $u \in X$. Subsequently, we have $Ay_n \to Su$, $Bx_n \to Su$, $Tx_n \to Su$ and $Sy_n \to Su$ as $n \to \infty$. By (3), we have

$\varphi(F_{Au,Bx_n}(kt), F_{Su,Tx_n}(t), F_{Au,Su}(kt), F_{Bx_n,Tx_n}(t)) \geq 0$.

Letting $n \to \infty$,

$\varphi(F_{Au,Su}(kt), 1, F_{Au,Su}(kt), 1) \geq 0$,

which implies, by (R-1) that is $Au = Su$.

The weak compatibility of A and S implies that $ASu = SAu$ and then $AAu = ASu = SAu = SSu$.

Now, since $A(X) \subseteq T(X)$, there exists a $v \in X$ such that $Au = Tv$. We show that $Tv = Bv$. By (3), we have

$\varphi(F_{Au,Bv}(kt), F_{Su,Tv}(t), F_{Au,Su}(kt), F_{Bv,Tv}(t)) \geq 0$,

that is, $\varphi(F_{Tv,Bv}(kt), 1, 1, F_{Bv,Tv}(t)) \geq 0$.

Thus, from (R-1), we have $Bv = Tv$. This implies $Au = Su = Tv = Bv$. The weak compatibility of B and T implies that $BTv = TBv$ and $TTv = TBv = BTv = BBv$.

Let us show that Au is the common fixed point of A, B, S and T. In view of (3), it follows

$\varphi(F_{AAu,Bv}(kt), F_{SAu,Tv}(t), F_{AAu,SAu}(kt), F_{Bv,Tv}(t)) \geq 0$,

that is, $\varphi(F_{AAu,Au}(kt), F_{AAu,Au}(t), 1, 1) \geq 0$.

Thus, from (R-2), we have $AAu = Au$.

Therefore, $Au = AAu = SAu$ and Au is a common fixed point of A and S. Similarly, we prove that Bv is a common fixed point of B and T. Since $Au = Bv$, we conclude that Au is a common fixed point of A, B, S and T. The proof is similar when $T(X)$ is assumed to be a complete subspace of X. The cases in which $A(X)$ or $B(X)$ is a complete subspace of X are similar to the cases in which $T(X)$ or $S(X)$ respectively, is complete since $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. If $Au = Bu = Su = Tu = u$ and $Av = Bv = Sv = Tv = v$, then (3) gives

$\varphi(F_{Au,Bv}(kt), F_{Su,Tv}(t), F_{Au,Su}(kt), F_{Bv,Tv}(t)) \geq 0$,

that is, $\varphi(F_{u,v}(kt), F_{u,v}(t), 1, 1) \geq 0$.

Thus, from (R-2), we have $u = v$. Therefore, the common fixed point of the involved maps is unique. □

The following example illustrates Theorem 12.

Example 13. *Let $X = [0, 15)$ and d be the usual metric on X. Let F be defined by*

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0. \ x, y \in X. \end{cases}$$

Then (X, F) is a noncomplete probabilistic metric space. Let A, B, S and T be self mappings on X defined as

$$A(X) = \begin{cases} 0, & \text{if } x = 0; \\ 3, & \text{if } 0 < x < 15. \end{cases}$$

$$B(X) = \begin{cases} 0, & \text{if } x = 0; \\ 7, & \text{if } 0 < x < 15. \end{cases}$$

$$S(X) = \begin{cases} 0, & \text{if } x = o; \\ 6, & \text{if } 0 < x \leq 10; \\ x - 7, & \text{if } 10 < x < 15. \end{cases}$$

$$T(X) = \begin{cases} 0, & \text{if } x = o; \\ 3, & \text{if } 0 < x \leq 10; \\ x - 3, & \text{if } 10 < x < 15. \end{cases}$$

Then A, B, S and T satisfy all the conditions of Theorem 12 with $k \in (0,1)$ and have a unique common fixed point at $x = 0$. Clearly, (A, S) and (B, T) are noncompatible if we suppose that $\{x_n\}$ is a sequence defined as $x_n = 10 + \frac{1}{n}, n \geq 1$ then we have $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = 7$. Hence, B and T satisfy the property (E.A). Also, $F_{ASx_n,SAx_n}(t) = \frac{t}{t+|3-6|} \neq 1$ and $F_{BTx_n,TBx_n}(t) = \frac{t}{t+|7-3|} \neq 1$. This shows that (A, S) and (B, T) are noncompatible pairs. As well as all the maps A, B, S and T are discontinuous at the common fixed point $x = 0$.

Conclusion. As two noncompatible maps of a Menger space $(X, F, \triangle)$ satisfy property (E.A.), therefore Theorem 12 also holds for noncompatible self maps.

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