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A SHORT SURVEY OF THE DEVELOPMENT OF FIXED POINT THEORY

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Abstract. In this survey paper, we collected the developmental history of fixed point theory. Some important results from beginning up to now are incorporated in this paper.

1 Introduction

The fixed point theorem states the existence of fixed points under suitable conditions. Recall that in case $f : X \to X$ is a function, then y is a fixed point of f if $fy = y$ is satisfied.
The famous Brouwer fixed point theorem was given in 1912 [8].

2 Brouwer fixed point theorem

The theorem states that if $f : B \to B$ is a continuous function and B is a ball in R^n, then f has a fixed point.

This theorem simply guarantees the existence of a solution, but gives no information about the uniqueness and determination of the solution.

For example, if $f : [0, 1] \to [0, 1]$ is given by $fx = x^2$, then $f0 = 0$ and $f1 = 1$, that is, f has 2 fixed points.

Several proofs of this theorem are given. Most of them are of topological in nature. A classical proof due to *Birkhoff and Kellog was given in 1922,* Similar classical proof was given in Linear Operators Volume 1, *Dunford and Schwartz 1958.*

Brouwer theorem gives no information about the location of fixed points. However, effective methods have been developed to approximate the fixed points. Such tools are useful in calculating zeros of functions.

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A polynomial equation $Px = 0$ can be written as $Fx - x = 0$ where $Fx - x = Px$. For example, consider $x^2 - 7x + 12 = 0$, where $Px = x^2 - 7x + 12$. We can write $Fx - x = Px = x^2 - 7x + 12$, so $x = (x^2 + 12)/7 = Fx$. Here F has two fixed points, $F3 = 3$ and $F4 = 4$.

The following books cover a good deal of fixed point theorems [1, 4, 7] and [32, 36].

This theorem is not true in infinite dimensional spaces. For example, if B is a unit ball in an infinite dimensional Hilbert space and $f : B \to B$ is a continuous function, then f need not have a fixed point. This was given by *Kakutani in 1941* [19].

The first fixed point theorem in an infinite dimensional Banach space was given by *Schauder in 1930*. The theorem is stated below:

Theorem 1. ***Schauder fixed point theorem***

If B is a compact, convex subset of a Banach space X and $f : B \to B$ is a continuous function, then f has a fixed point [34].

The Schauder fixed point theorem has applications in approximation theory, game theory and other scientific area like engineering, economics and optimization theory.

The compactness condition on B is a very strong one and most of the problems in analysis do not have compact setting. It is natural to prove the theorem by relaxing the condition of compactness. Schauder proved the following theorem [34].

Theorem 2. *If B is a closed bounded convex subset of a Banach space X and $f : B \to B$ is continuous map such that $f(B)$ is compact, then f has a fixed point.*

The above theorem was generalized to locally convex topological vector spaces by *Tychonoff in 1935* [37].

Theorem 3. *If B is a nonempty compact convex subset of a locally convex topological vector space X and $f : B \to B$ is a continuous map, then f has a fixed point.*

Further extension of Tychonoff's theorem was given by *Ky Fan* [12].

A very interesting useful result in fixed point theory is due to Banach known as the *Banach contraction principle* [5].

Theorem 4. *Recall that a map $f : X \to X$ is said to be a contraction map, if $d(fx, fy) \leq k\ d(x, y)$, where X is a metric space, $x, y \in X$ and $0 \leq k < 1$. Every contraction map is a continuous map, but a continuous map need not be a contraction map.*

For example,$fx = x$ is a continuous map but it is not a contraction map.

The method of successive approximation introduced by *Liouville in 1837* and systematically developed by *Picard in 1890* culminated in formulation by Banach known as the Banach contraction principle (BCP) is stated as below [5].

Theorem 5. ***(BCP)***
If X is a complete metric space and $f : X \to X$ is a contraction map, then f has a unique fixed point or $fx = x$ has a unique solution.

Proof. The proof of this theorem is constructive. Let $x_{n+1} = fx_n, n = 1, 2, \dots$. Then the sequence $\{x_n\}$ is a Cauchy sequence and converges to y in X. It is easy to show that $y = fy$, that is, y is a fixed point of f. Since f is a contraction map so y is a unique fixed point. □

The Banach contraction principle is important as a source of existence and uniqueness theorems in different branches of sciences. This theorem provides an illustration of the unifying power of functional analytic methods and usefulness of fixed point theory in analysis.

The important feature of the Banach contraction principle is that it gives the existence, uniqueness and the sequence of the successive approximation converges to a solution of the problem. The important aspect of the result is that existence, uniqueness and determination, all are given by Banach contraction principle.

Definition 6. *If $f : X \to X$ such that $d(fx, fy) \le d(x, y)$, for all $x, y \in X$, then f is said to be a nonexpansive map.*

A nonexpansive map need not have a fixed point in a complete metric space. For example, if $f : R \to R$ given by $fx = x + k$ where k is any number, then f has no fixed point.

A translation map has no fixed point. In case we have an identity map $I : R \to R$, then each point of I is a fixed point. The above examples illustrate that a nonexpansive map, unlike contraction map, need not have a fixed point and if it has a fixed point, then it may not be unique.

The famous fixed point theorem for nonexpansive maps was given by *Browder [9], Kirk [20] and Gohde [15]* independently in 1965.

Theorem 7. *If B is a closed bounded convex subset of a Hilbert space H and $f : B \to B$ is a nonexpansive map, then f has a fixed point.*

The following interesting question was proved by *Browder in 1967* [10].

If B is closed convex subset of a Banach space X and $f : B \to B$ is a nonexpansive map. If for each $r_i \in [0, 1)$ and any $y \in B$, we define $f_{r_i}x = r_i x + (1 - r_i)y$ for all $x \in B$, then $f_{r_i} : B \to B$, and each f_{r_i} is a contraction map with Lipschitz constant r_i. For r_i sufficiently close to 1, f_{r_i} is a contractive approximation of f.

By Banach contraction principle, each contraction map has a unique fixed point say $f_{r_i}x_{r_i} = x_{r_i}$ for each r_i, that is, $x_{r_i} = f_{r_i}x_{r_i} + (1 - r)y$. It is natural to ask if the sequence $\{x_{r_i}\}$ converges to a fixed point of f.

Since a nonexpansive map need not have a fixed point so in general the result is not affirmative. However, the following is due to *Browder* [10].

Theorem 8. *If C is a closed bounded convex subset of a Hilbert space H and $f : C \to C$ is a nonexpansive map. Define $f_r x = rfx + (1-r)y$ for some $y \in C$ and $0 < r < 1$. Let $x_r = f_r x_r$. Then, the sequence $\{x_r\}$ converges to a fixed point of f, closest to y. In case C is not bounded and f is not a selfmap, then a similar result is given in [35].*

In the study of fixed point theorems of nonexpansive mappings the following topics are of interest.
(i) The sequence of iterates $x_{n+1} = fx_n$ need not converge.
For example, if we consider $fx = -x$, for $x \in R$, then the sequence of iterates is an oscillatory sequence.
(ii) The nonexpansive map need not have a fixed point. Therefore for the study of nonexpansive map it is important to find that under what conditions the mapping is going to have a fixed point.

Here we give a brief development of the above areas. The method of successive approximation is useful in determining the solutions of equations. An early result dealing with the convergence of the sequence of iterates was given by *Krasnoselskii in 1955.* It is stated below [21].

Theorem 9. *If C is a closed bounded convex subset of a Banach space X and $f : C \to C$ a nonexpansive mapping with closure of $f(C)$ compact, then the sequence of iterates given by $(f_{\frac{1}{2}})^n x$ where $f_{\frac{1}{2}} x = \frac{1}{2} fx + \frac{1}{2} x$, converges to a fixed point of f. We note here that the fixed point of f and $f_{\frac{1}{2}}$ is the same. For example, if $fy = y$, then $f_{\frac{1}{2}} y = y$. The limit of the sequence $\{(f_{\frac{1}{2}})^n x\}$ converges to a fixed point of f.*

More generally, if C is a closed bounded convex subset of a Banach space X, then for $f : C \to C$, we consider $f_r x = rfx + (1-r)x$. In this case it is easy to see that $fy = y$ if and only if $f_r y = y$ and the sequence of iterates $(f_r^n)x$ converges to a fixed point of f.

Further extensions of iteration process due to *Mann* [22], *Ishikawa* [18], and *Rhoades* [30] are worth mentioning.

3 Results in Variational Inequality

Recently several interesting results for sequence of iterates are used to find the solutions of the Variational Inequality Problems (VIP). In most of the cases the basic tool has been the sequence of successive approximation used in the study of fixed point theory. A good deal of work has been associated with the nonexpansive maps. As the sequence of iterates for a nonexpansive map need not always converge therefore several researchers have tried to give techniques for convergence of the sequence of iterates.

The following result deals with the contraction maps in the study of variational inequality [24].

Theorem 10. *Let C be a nonempty closed convex subset of a Hilbert space H and $f : C \to H$ a continuous map such that $I - rf$ is a contraction map. Then the sequence of iterates*

$$u_{n+1} = Po(I - rf)u_n, u_0 \in C$$

converges to u where u satisfies the variational inequality $< fu, y - u >\geq 0$ for all $y \in C$.

Singh et. al. proved the following result for nonexpansive maps [36].

Theorem 11. *Let C be a closed convex subset of a Hilbert space H and $f : C \to H$ a continuous function such that $I - f$ is a nonexpansive map and let $(I - f)C$ be bounded. Then the sequence of iterates $u_{n+1} = Po(I - f)u_n, n = 1, 2, ..., u_1 \in C$ converges to u where u is a solution of the variational inequality $< fu, x - u >\geq 0$ for all $x \in C$, provided that $\lim_{n\to\infty} d(u_n, F) = 0$, where F is the set of fixed points of $Po(I - f) : C \to C$.*

The VIP is also closely associated with the best approximation problem so this technique can be applied to problems in approximation theory.

The following example is worth mentioning [11].

Theorem 12. *Let C_1 and C_2 be two closed convex sets in Hilbert space H and $g = P_1P_2$ of proximity maps. Convergence of $\{x_n\}$ to a fixed point of g is guaranteed if either*
(i) one set is compact or
(ii) one set is finite dimensional and the distance between the sets is attained.

The contraction, contractive and nonexpansive maps have been further extended to densifying, and 1- set contraction maps in 1969.

Several interesting results of fixed points were proved recently. A few results were proved separately for contraction maps and compact mappings (A continuous map with compact image is called a compact mapping). Both maps are densifying maps. Thus a fixed point theorem for densifying maps includes both for contraction and compact maps.

If $f : B \to R^n$, then f is said to be a nonself map. Most of the fixed point theorems have been given for self-maps.

In 1937 *Rothe* [31] gave a fixed point theorem for nonself maps (see also [4, 37]).

Theorem 13. ***Rothe***

If $f : B \to R^n$ is a continuous map, such that

$$f(\partial B) \subseteq B, \qquad (1)$$

then f has a fixed point.

The following condition for nonself map is called the *Altman's condition (1955).*

$$|fx - x|^2 \geq |fx|^2 - |x|^2.$$

There were a few results in fixed point theory dealing with combination for two maps- say one is contraction and the other one is compact.

Note that if we have f and g both continuous functions, then $f + g$ is also a continuous map and the fixed point theorem for continuous map is applicable for $f + g$. However, if f is a contraction map, then Banach contraction theorem is applied and if g is a compact map, then Schauder fixed point theorem is applicable. However, in such a case when f is contraction and g is a compact map, then for $f + g$ the fixed point theorem of densifying map is applicable.

We record a few definitions [4, 36]:

Definition 14. *Let C be a bounded subset of a metric space X. Define the measure of noncompactness $\alpha(C) = inf\{\epsilon > 0/C$ has a finite covering of subsets of diameter $\leq \epsilon\}$. The following properties of α are well known.*
Let A be a bounded subset of a metric space X. Then $\alpha(A) \leq \delta(A), \delta(A)$ is the diameter of A.
If $A \subseteq B$, then $\alpha(A) \leq \alpha(B)$, $\alpha($closure of $A) = \alpha(A)$
$\alpha(A \bigcup B) = max\{\alpha(A), \alpha(B)\}$
$\alpha(A) = 0$ *if and only if A is precompact.*

Definition 15. *A continuous mapping $f : X \to X$ is called a densifying map if for any bounded set A with $\alpha(A) > 0$, we have $\alpha f(A) < \alpha(A)$.*

In case $\alpha f(A) \leq \alpha(A)$, then f is said to be 1-set contraction. Note that a nonexpansive map is an example of 1-set contraction.

A contraction map is densifying and so is the compact mapping, that is, a function mapping closed sets to compact sets.

The following is a well known result [14, 25, 33].

Theorem 16. *Let $f : C \to C$ be a densifying map, where C is closed bounded and convex subset of a Banach space X. Then f has at least one fixed point in C.*

The contraction, contractive and nonexpansive maps have been further extended to densifying, and 1- set contraction maps in 1969. Several interesting results of fixed points were proved recently [27]. A few results were proved separately for contraction maps and compact mappings (A continuous map with compact image is called a compact mapping). Both maps are densifying maps. Thus a fixed point theorem for densifying maps includes both for contraction and compact maps.

In 1966, *Hartman and Stampacchia* [16] gave the following interesting result in variational inequalities.

Theorem 17. *If B is a unit ball in R^n and $f : B \to R^n$ a continuous function, then there is a $y \in B$ such that*

$$< fy, x - y > \geq 0. \tag{2}$$

for all $x \in B$.

Note: Let P be a metric projection onto B. Then $P(I - f)$ has a fixed point in B if and only if (2) has a solution.

The variational inequality theory is a very effective tool for handling problems in different branches of mathematics, engineering and theoretical physics. *Hartman and Stampacchia* [16] theorem yields Brouwer fixed point theorem as an easy corollary.

Let $g : B \to B$ be a continuous function, where B is a closed ball in R^n.
We have to show that g has a fixed point.
Let $f = I - g$. Then f is continuous and $f : B \to R^n$. Hence by using Hartman and Stampacchia theorem we get that there is a $y \in B$ such that $< fy, x - y > \geq 0$ for all $x \in B$.

Thus,$< (I - g)y, x - y > \geq 0$, that is $< y - gy, x - y > \geq 0$. Since $g : B \to B$, so by taking $x = gy$, we have $< y - gy, gy - y > \geq 0$. This is true only when $y = gy$. Hence g has a fixed point.

In 1969 the following result was given by Ky Fan commonly known as the best approximation theorem [13].

Theorem 18. *If C is a nonempty compact convex subset of a normed linear space X and $f : C \to X$ is a continuous function, then there is a $y \in C$ such that*

$$|fy - y| = inf|x - fy| \tag{3}$$

for all $x \in C$.
If P is a metric projection onto C, then Pof has a fixed point if and only if (3) holds. Recall that $d(x, C) = inf||x - y||$ for all $y \in C, x \neq C$.

The Ky Fan's theorem has been widely used in approximation theory, fixed point theory, variational inequalities, and other branches of mathematics.

Theorem 19. *If $f : B \to X$ is a continuous function and one of the following boundary conditions are satisfied, then f has a fixed point. Here B is a closed ball of radius r and center 0 (∂B stands for the boundary of the ball B).*
(i) $f(\partial B) \subseteq B$, (Rothe condition)
(ii) $|fx - x|^2 \geq |fx|^2 - |x|^2$, (Altman's condition)
(iii) If $fx = kx$ for $x \in \partial B$, then $k \leq 1$ (Leray Schauder condition)
(iv) If $f : B \to X$ and $fy \neq y$, then the line segment $[y, fy]$ has at least two elements of B. (Fan's condition).

In this survey we have restricted our presentation to single valued maps only. A vast literature is available for the fixed point theorems of multivalued maps. In 1941 Kakutani gave the following generalization of the Brouwer fixed point theorem to multivalued maps.

Theorem 20. *If F is a multivalued map on a closed bounded convex C subset of R^n, such that F is upper semicontinuous with nonempty closed convex values, then F has a fixed point.*

Recall that x is a fixed point of F if $x \in Fx$.

The fixed point theory of multivalued maps is useful in economics, game theory and minimax theory.

An important application of Kakutani fixed point theorem was made by Nash [23] in the proof of existence of an equilibrium for a finite game. Other applications of fixed point theorem of multivalued mapping are in mathematical programming, control theory and theory of differential equations.

Popa [28, 29] introduced implicit functions which are proving fruitful due to their unifying power besides admitting new contraction conditions. We also introduce an implicit function to prove our results [17]. The main theorem is listed below:

Theorem 21. *Let $\{S_1, S_2, ..., S_m\}, \{T_1, T_2, ..., T_n\}, \{I_1, I_2, ..., I_p\}$ and $\{J_1, J_2, ..., J_q\}$ be four families of self-mappings of a metric space (X, d) with $S = S_1S_2...S_m, T = T_1T_2...T_n, I = I_1I_2...I_p$ and $J = J_1J_2....J_q$ satisfying the following conditions:*
(a) $S(X) \subset J(X), T(X) \subset I(X)$,
(b) one of $S(X), T(X), I(X)$ and $J(X)$ is a complete subspace of X,
(c)$F(d(Sx, Ty), d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), d(Ix, Ty), d(Jy, Sx)) \leq 0$
for all $x, y \in X$ and $F \in \tau$. Then,
(d) (S, I) have a point of coincidence,
(e) (T, J) have a point of coincidence,
Moreover, if $S_iS_j = S_jS_i, I_kI_l = I_lI_k, T_rT_s = T_sT_r, J_tJ_u = J_uJ_t, S_iI_k = I_kS_i, I_kT_r = T_rI_k, T_rJ_t = J_tT_r, S_iJ_t = J_tS_i, S_iT_r = T_rS_i$ and $J_tI_k = I_kJ_t$ for all $i, j \in I_1 = \{1, 2, ..., m\}, k, l \in I_2 = \{1, 2, ..., p\}, r, s \in I_3 = \{1, 2, ..., n\}$ and $t, u \in I_4 = \{1, 2, ..., q\}$. Then (for all $i \in I_1, k \in I_2, r \in I_3$ and $t \in I_4$) S_i, I_k, T_r and J_t have a common fixed point.

Proof. For the proof and examples, see [17]. □

The most recent result for implicit functions is due to Javid Ali and M. Imdad [2]. They introduce an implicit function to prove their results because of their versatility of deducing several contraction conditions in one go. Some new forms of implicit relations are also introduced recently in [3] and [6].

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