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QUANTAM LIÉNARD II EQUATION AND JACOBI'S LAST MULTIPLIER

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Abstract. In this survey the role of Jacobi's last multiplier in mechanical systems with a position dependent mass is unveiled. In particular, we map the Liénard II equation $\ddot{x} + f(x)\dot{x}^2 + g(x) = 0$ to a position dependent mass system. The quantization of the Liénard II equation is then carried out using the point canonical transformation method together with the von Roos ordering technique. Finally we show how their eigenfunctions and eigenspectrum can be obtained in terms of associated Laguerre and exceptional Laguerre functions. By employing the exceptional Jacobi polynomials we construct three exactly solvable potentials giving rise to bound-state solutions of the Schrödinger equation.

Full text

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