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THE T(1) THEOREM REVISITED

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Respectfully dedicated to the memory of Jean-Lin Journé (1957-2016)

Abstract. The main purpose of this article is to present a proof of the T(1) Theorem that uses a continuous version of the Cotlar-Knapp-Stein lemma, due to A. P. Calderón and R. Vaillancourt.

1 Introduction

More than thirty years have passed since the publication of David's and Journé's celebrated T(1) Theorem [38], a central result in Harmonic Analysis, characterizing the L^2 continuity of operators in a certain class. In that time, numerous new proofs, extensions and generalizations, as well as excellent expositions at different levels, have appeared. We cite, as examples, [33], [39], [41], [26], [45], [46], [85], [89], [90], [62], [80], [81], [82], [83], [25], [40], [86], [44], [77]. For a more comprehensive list, we refer to the bibliography in [25], [77] and [44].

The original proof of David and Journé first reduces the operator to "zero initial conditions", thus, introducing some useful cancellation properties. This is done using a suitable realization of the so called paraproducts (see, for instance, [25], p. 40; [82]), defined by Bony [8]. Then, it uses an "almost orthogonality" principle to prove the L^2 continuity of the reduced operator. We point out that shortly after David's and Journé's proof was announced, Coifman and Meyer came up with a one-step proof [33] that avoids the "almost orthogonality" argument.

Going back to the original proof, the "almost orthogonality" principle used is the remarkable Cotlar's lemma [35], in a version known as Cotlar-Knapp-Stein lemma (see, for instance, [58]; [77], p. 280).

However, earlier results by Calderón and Vaillancourt on the boundedness of pseudo-differential operators [20], had relied on their "continuous" version, that is to say, a version involving integrals instead of sums, of the Cotlar-Knapp-Stein

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lemma. Thus, it seemed quite natural to write a proof of the T(1) Theorem that uses the Calderón-Vaillancourt version [20] of the lemma. This is the main purpose of this article. Although the Calderón-Vaillancourt lemma is cited in the literature (see, for instance, [77], p. 318, further results 5.5 (ii); [44], p. 283, Exercise 4.5.8; p. 324, historical notes), we do not know of any source that presents or suggests such a proof.

Our work is organized as follows: It commences with a section devoted to standard kernels and to the operators associated with them. These notions were introduced by Coifman and Meyer [30], as a natural formulation encompassing the singular integrals defined and studied by Calderón and Zygmund (see, for instance, [21], [22], [23], [15]), as well as other important operators related to pseudo-differential operators, commutators and Cauchy integrals. We discuss, in detail, the non-trivial relationship between an operator and its kernel, bringing in results and observations from various sources that we dutifully acknowledge. In the next section we motivate and present in some detail several operators of great significance, that fall into the framework outlined in the previous section. We do not aim for completeness, we just want to have at hand a sufficiently rich collection of examples to allow us, later, to discuss the applicability of the theorem. Next, we state the T(1) Theorem, taking the time to analyze the meaning and independence of its hypotheses. The proof of the theorem, including a few preliminary definitions and results, is the subject of the next two sections. In the first one, we explain a reduction step that follows the original presentation by David and Journé. We include the detailed proof of all the statements made. It is in the second section where we use the Calderón-Vaillancourt version of the Cotlar-Knapp-Stein lemma, to prove the L^2 continuity of the reduced operator. In the section that follows, we examine the applicability of the T(1) Theorem to the particular operators introduced earlier. Lastly, let us point out that there are various definitions of standard kernel. For this reason, we think that it is of interest to make precise the relationship between a few of the most common formulations. Thus, we dedicate a last section to this endeavour. The article ends with an extensive list of references.

There are many excellent accounts of the background information leading to the concepts and results used in this article. For instance, we cite [75], [78], [40], [25], [43], [44], [86] and [77], and some of the references therein.

We use the standard notation in the subject and we work with functions and distributions that might take real or complex values. Unless otherwise indicated, the underlying space will be $\mathbb{R}^n$. Typically, we will denote C a positive constant, only depending on specific parameters and possibly varying at different occurrences.

2 Standard kernels and the operators associated with them

Given a distribution $k \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$, we can define, uniquely, an operator $T : \mathcal{D} \to \mathcal{D}'$, linear and continuous, as

$$(T\varphi, \psi) = (k, \varphi \otimes \psi), \tag{2.1}$$

where $\varphi, \psi \in \mathcal{D}$ and the duality is understood as $(\mathcal{D}', \mathcal{D})$ on the left and$(\mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n), \mathcal{D}(\mathbb{R}^n \times \mathbb{R}^n))$ on the right. Conversely, a famous theorem due to L. Schwartz [73], asserts that given a linear and continuous operator $T : \mathcal{D} \to \mathcal{D}'$, there exists a unique distribution $k \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$, called the distribution kernel of T, so that the representation (2.1) holds.

The T(1) Theorem characterizes the continuity on the space L^2, of operators T for which their distribution kernel k satisfies certain size and smoothness conditions. Definition 1 below provides one of the classical formulations of such conditions. In the last section, we will analyze how it is related to other versions.

In what follows we will indicate as Δ the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$, that is to say, $\Delta = \{(x, x)\,; x \in \mathbb{R}^n\}$.

Definition 1. *(Standard kernel) Given $k \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$, we say that k is a standard kernel if its restriction to $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$ is a continuous function satisfying, for some $C > 0$ and for a fixed $0 < \delta \leq 1$, the following conditions:*

1. $|k(x, y)| \leq \frac{C}{|x-y|^n}$, *when* $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$,
2. $|k(x, y) - k(z, y)| \leq C\frac{|x-z|^\delta}{|y-z|^{n+\delta}}$, *when* $|y - z| \geq c\,|x - z|$ *for some* $c > 1$,
3. $|k(y, x) - k(y, z)| \leq C\frac{|x-z|^\delta}{|y-z|^{n+\delta}}$, *when* $|y - z| \geq c\,|x - z|$ *for some* $c > 1$.

Remark 2. *Although in conditions 2) and 3) of Definition 1 we only ask for c to be larger than one, we will see immediately that, in some situations, c needs to be at least* 2.

Lemma 3. *If k satisfies 2) and 3) in Definition 1, for some $C > 0$, $c \geq 2$, and a given $0 < \delta \leq 1$, then it will also satisfy the same conditions for any δ' with $0 < \delta' < \delta$ and the same constants C and c.*

The kernel k satisfies 2) with $\delta = 1$, when $|y - z| \geq c\,|x - z|$ for some $c > 2$ and an appropriate $C > 0$, if it satisfies the condition

$$|\nabla_x k(x, y)| \leq \frac{C}{|x-y|^{n+1}},$$

which we will indicate as 2'), for some $C > 0$.

Finally, the kernel k satisfies 3) with $\delta = 1$, when $|y-z| \geq c\,|x-z|$ for some $c > 2$ and an appropriate $C > 0$, if it satisfies the condition

$$|\nabla_x k\,(y,x)| \leq \frac{C}{|x-y|^{n+1}},$$

indicated 3'), for some $C > 0$.

Proof. The first assertion follows from the estimate

$$\frac{|x-z|^{\delta}}{|y-z|^{n+\delta}} = \frac{|x-z|^{\delta-\delta'}}{|y-z|^{\delta-\delta'}} \frac{|x-z|^{\delta'}}{|y-z|^{n+\delta'}} \leq \left(\frac{1}{2}\right)^{\delta-\delta'} \frac{|x-z|^{\delta'}}{|y-z|^{n+\delta'}} \leq \frac{|x-z|^{\delta'}}{|y-z|^{n+\delta'}}.$$

As for the second assertion, if condition 2') holds, then, for $|x-z| \leq \frac{|y-z|}{c}$ we can write

$$\begin{aligned} |k\,(x,y) - k\,(z,y)| &\leq |(\nabla_x k)\,(x + t\,(z-x)\,,y)|\,|x-z| \\ &\leq C\frac{|x-z|}{|x-y+t\,(z-x)|^{n+1}} \leq C\frac{|x-z|}{(|x-y| - |x-z|)^{n+1}} \\ &\leq C\frac{|x-z|}{(|y-z| - 2\,|x-z|)^{n+1}} \\ &\leq C\left(\frac{c-2}{c}\right)^{n+1} \frac{|x-z|}{|y-z|^{n+1}}, \end{aligned}$$

with a similar estimate for $|k\,(y,x) - k\,(y,z)|$, if 3') holds.

This completes the proof of the lemma. □

Remark 4. *Given a Lipschitz function $\varphi : \mathbb{R}^n \to \mathbb{R}$ for $n \geq 2$, if $\Omega \subseteq \mathbb{R}^{n+1}$ is the open set defined by $t > \varphi\,(x)$, the kernel of the double layer potential is defined, on $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$, as*

$$k_{\varphi}\,(x,y) = \frac{\varphi\,(x) - \varphi\,(y) - (x-y)\cdot(\nabla\varphi)\,(y)}{\left(|x-y|^2 + (\varphi\,(x) - \varphi\,(y))^2\right)^{(n+1)/2}}.$$

As Meyer observed in [63], the kernel $k_{\varphi}\,(x,y)$ satisfies 1) and 2) in Definition 1, but not 3). Likewise, the kernel $k^t_{\varphi}\,(x,y) = k_{\varphi}\,(y,x)$, satisfies 1) and 3), but not 2). In Example 18 of the section that follows, we consider in some detail the $n = 2$ version of this example.

Definition 5. *(Operator associated with a kernel) Given a linear and continuous operator $T : \mathcal{D} \to \mathcal{D}'$, we say that T is associated with a kernel if the distribution kernel k of T coincides with a continuous function, also denoted k, on $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$, and the following pointwise representation holds:*

$$T(\varphi)(x) = \int_{\mathbb{R}^n} k(x, y)\, \varphi(y)\, dy,$$

for $\varphi \in \mathcal{D}$ and $x \in \mathbb{R}^n \backslash supp(\varphi)$.

Definition 6. *(Operator associated with a standard kernel) Given a linear and continuous operator $T : \mathcal{D} \to \mathcal{D}'$, we say that T is associated with a standard kernel if the following two conditions hold:*

i) *The distribution kernel of T coincides, away from the diagonal, with a function that is a standard kernel in the sense of Definition 1.*

ii) *The following pointwise representation holds:*

$$T(\varphi)(x) = \int_{\mathbb{R}^n} k(x, y)\, \varphi(y)\, dy, \tag{2.2}$$

for $\varphi \in \mathcal{D}$ and $x \in \mathbb{R}^n \backslash supp(\varphi)$.

Remark 7. *Definition 1 and Definition 6 say that the restriction to the complement of the diagonal of the distribution kernel of T, is a function with prescribed singularities as $|x - y| \to 0$, that are at the edge of integrability. For more on these definitions, we refer to ([30], p. 79).*

Remark 8. *As we said before, there is a bijection between the class of linear and continuous operators $T : \mathcal{D} \to \mathcal{D}'$ and $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$. Moreover, if T is associated with a standard kernel k, then k is, pointwise, uniquely determined in $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$. In fact, for $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$ fixed, let $\{\varphi_j\}_{j \geq 1}$ be so that, for all $j \geq 1$, $\varphi_j \in \mathcal{D}$, $x_0 \notin supp(\varphi_j)$and $\varphi_j \to \delta_{y_0}$ in $\mathcal{D}'$ as $j \to \infty$. Here, δ_{y_0} denotes the Dirac distribution supported on $\{y_0\}$. Then,*

$$T(\varphi_j)(x_0) = \int_{\mathbb{R}^n} k(x_0, y)\, \varphi_j(y)\, dy = (\varphi_j, k(x_0, y)) \underset{j \to \infty}{\to} k(x_0, y_0).$$

That is to say, there is $\lim_{j \to \infty} T(\varphi_j)(x_0)$ in $\mathbb{C}$ and this limit gives the value $k(x_0, y_0)$, independently of the approximation $\{\varphi_j\}_{j \geq 1}$, provided that this approximation satisfies the stated conditions. On the other hand, neither the operator T nor its distribution kernel k are uniquely determined by the restriction of k to $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$. In fact, let us consider I, the identity operator, associated with the Dirac-like distribution kernel $\delta(x - y)$, and the zero operator, associated with the identically *zero distribution kernel. In both cases, the restriction of the kernel to* $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$ *is the identically zero function, proving our assertion.*

Likewise, given a continuous function $k : \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \to \mathbb{C}$ *satisfying Definition 1, not always exists a distribution kernel that coincides with* k *on* $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$*. For instance, that is the case when* $k(x,y) = |x-y|^{-n}$ *([44], p. 212, Example 4.1.3). Let us remark that the function* $|x|^{-n}$ *defined on* $\mathbb{R}^n \setminus \{0\}$*, satisfies the estimate*

$$\left| \nabla_x \frac{1}{|x|^n} \right| \leq \frac{C}{|x|^{n+1}},$$

as can be seen by a straightforward application of the Mean Value Theorem. So, $k(x,y) = |x-y|^{-n}$ *does satisfy Definition 1 with* $\delta = 1$*, when* $|y-z| \geq c\,|x-z|$ *for some* $c > 2$ *and an appropriate* $C > 0$*, as shown in Lemma 3.*

If the function k *satisfying Definition 1 is antisymmetric, that is to say if* $k(y,x) = -k(x,y)$*, it extends to a distribution kernel. We will prove this assertion in Section 7.*

For more on the relationship between kernels and operators, see ([30], Chapter IV).

Remark 9. *We will assume, from now on, that* $c = 2$ *in Definition 1 (see [4]). This is how conditions 2) and 3) in Definition 1 are typically stated. For more on this, we refer to the comparison of various pointwise as well as integral conditions on the kernel, discussed in the last section.*

Definition 10. *Given a linear and continuous operator* $T : \mathcal{D} \to \mathcal{D}'$*, we can define another operator,* $T^t : \mathcal{D} \to \mathcal{D}'$*, called the transpose of* T*, as*

$$\left(T^t \varphi, \psi\right) = (T\psi, \varphi),$$

or, in terms of their distribution kernels,

$$(k, \psi \otimes \varphi) = \left(k^t, \varphi \otimes \psi\right),$$

where k^t *is called the transpose kernel of* K*.*

If k is given by a function $k(x,y)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$, then the transpose distribution kernel k^t as given by Definition 10, coincides with the function $k(y,x)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$.

Definition 11. *(CZO) An operator* T *associated with a standard kernel* k *is called a Calderón-Zygmund operator, in brief CZO, if* T *extends to a continuous operator from* L^2 *into itself, with norm denoted* $\|T\|_{L(L^2)}$*.*

Remark 12. *The notions of standard kernel, operator associated to a standard kernel and Calderón-Zygmund operator, were introduced by Coifman and Meyer [30], to bring together many particular situations of great interest. They originated with the theory of singular integrals due to Calderón and Zygmund.*

Remark 13. *The partial derivative operator* ∂_{x_j}*, associated with the Dirac-like distribution kernel* $-\partial_{y_j}(\delta(x-y))$*, is not a CZO. In fact, if it were, setting* $\varphi_R(x) = \varphi(Rx)$ *for* $\varphi \in \mathcal{D}$*, we would have the estimate*

$$\left\|\partial_{x_j}(\varphi_R)\right\|_{L^2} \leq C\,\|\varphi_R\|_{L^2},$$

for some $C > 0$ *and all* $R > 0$*, or*

$$R\left\|\partial_{x_j}(\varphi)\right\|_{L^2} \leq C\,\|\varphi\|_{L^2},$$

which is not possible.

Every pseudo-differential operator with symbol in the Hörmander class $S^0_{1,\delta}$ *for* $0 \leq \delta < 1$*, is a CZO (see, for instance, [1]). However, although the pseudo-differential operators with symbol in the Hörmander class* $S^0_{1,1}$ *are associated with standard kernels (see, for instance, [61]; [1]; [77]), p. 271, Proposition 1), they are not always CZOs [24]. In Section 4 and Section 7, we consider these assertions in detail, while in the next section, we take a closer look at pseudo-differential operators in various forms.*

As observed in Remark 4, the kernel of the double layer potential is not standard in the sense of Definition 1. However, it extends to an operator that is continuous on L^2 *[63].*

Given two CZOs T_1 *and* T_2 *such that their kernels are equal on* $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$*, the difference* $T_1 - T_2$ *is the operator given on* L^2 *as multiplication by an* L^∞ *function ([44], p. 218, Proposition 4.1.11 (4)). This assertion fully extends the example* $T_1 = I$*,* $T_2 = 0$ *discussed in Remark 8.*

If T *is a CZO, then* T^t *is the transpose in the sense of the real inner product structure of* L^2*. Although we will not insist on it, we could also consider the adjoint operator* T^* *if we use, instead, the complex inner product structure of* L^2*. In this case,* T *will be associated with the standard kernel* $\overline{k(y,x)}$*.*

3 Examples

We now present, and to certain degree motivate, several operators that fall within the framework of the previous section. They will prove to be of great significance when we discuss the applicability of the T(1) Theorem, in Section 7. Each of the operators we consider will have a specific natural domain, but, for simplicity, we will work formally or, when appropriate, we will use $\mathcal{D}$ as domain. Unless a reference is given, we refer to the sources cited towards the end of the introduction, for a comprehensive account.

Example 14. *If* $f(z) = u(z) + iv(z)$ *is holomorphic on the upper half plane* $\operatorname{Im}(z) \geq 0$ *and* $zf(z)$ *is bounded, the restrictions to the real axis of the functions* u *and* v *are* *related by the formula*

$$v(x) = pv\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(y)}{x-y}dy, \tag{3.1}$$

where, as usual,

$$pv\int_{-\infty}^{\infty} = \lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon}.$$

Now, given $u \in \mathcal{D}(\mathbb{R})$, the correspondence given by (3.1) is the Hilbert transform $H(u)$ of u. It is associated with the standard, antisymmetric and translation invariant kernel, $k(x,y) = \frac{1}{\pi}\frac{1}{x-y}$. As observed in Remark 8, this kernel has an extension to a distribution kernel, which is the distribution $pv\frac{1}{\pi x}$ acting on test functions by convolution. Let us point out that given $f \in \mathcal{D}(\mathbb{R})$, we can write

$$H(f)(x) = i\int_{-\infty}^{\infty} e^{-2\pi i x\xi} sgn(\xi)\widehat{f}(\xi)\,d\xi,$$

where $sgn(\xi) = 1$ when $\xi \geq 0$ and -1 when $\xi < 0$. Thus, H is a CZO.

In spite of its strong complex analysis flavor, the operator H has very interesting applications in real analysis. For instance, if we fix again $f \in \mathcal{D}$, let us consider, for $N > 0$,

$$S_N(f)(x) = \int_{-N}^{N} e^{-2\pi i x\xi}\widehat{f}(\xi)\,d\xi, \tag{3.2}$$

where we define, as before,

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{2\pi i y\xi} f(y)\,dy. \tag{3.3}$$

Using (3.3) in (3.2), an easy calculation shows that

$$S_N(f)(x) = \frac{e^{2\pi i N x}}{2i}H\left(e^{-2\pi i N y}f\right)(x) - \frac{e^{-2\pi i N x}}{2i}H\left(e^{2\pi i N y}f\right)(x).$$

That is to say, estimates for the operator S_N, will follow from continuity results for the Hilbert transform.

Example 15. *The family $\{R_j\}_{1\leq j\leq n}$ of Riesz transforms, defined, for each j, as convolution with the distribution*

$$C_n pv\frac{x_j}{|x-y|^{n+1}},$$

or, on the Fourier transform side, as

$$\widehat{R_j(f)}(\xi) = i\frac{\xi_j}{|\xi|}\widehat{f}(\xi), \tag{3.4}$$

for, say, $f \in \mathcal{D}$*, is the counterpart of the Hilbert transform, for* $n > 1$*. Each operator* R_j *is a CZO with* L^2 *norm* ≤ 1*. The Riesz transforms have many important applications. As a small illustration, let us observe that, from (3.4), we can deduce the identity*

$$\partial^2_{x_j x_l}(f) = R_j R_l \Delta(f),$$

where Δ *denotes the Laplace operator. Thus,*

$$\left\| \partial^2_{x_j x_l}(f) \right\|_{L^2} \leq \|\Delta(f)\|_{L^2},$$

showing that the Laplace operator controls, in the L^2 *norm, each second order derivative.*

In the previous section, we mentioned briefly (see Remark 13) the notion of pseudo-differential operator with symbol in the Hörmander class $S^0_{1,\delta}$. The next example shows, in particular, how pseudo-differential operators originated.

Example 16. *Given a linear partial differential operator* L*, formally written as*

$$L = \sum_{|\alpha| \leq m} a_\alpha(x)\, \partial^\alpha, \tag{3.5}$$

we can use the Fourier transform to give an integral representation for L*. Namely, if, say,* $f \in \mathcal{D}$*,*

$$L(f)(x) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} \left[\sum_{|\alpha| \leq m} a_\alpha(x)\,(-2\pi i x \xi)^\alpha \right] \widehat{f}(\xi)\, d\xi. \tag{3.6}$$

An operator L *is called pseudo-differential if it is written as in (3.6), with the function in brackets replaced by a general function* $a(x,\xi)$*, called symbol of the operator, that will satisfy various types of conditions. The general idea is that the function* $a(x,\xi)$*, although no longer a polynomial in* ξ*, should satisfy estimates close to those expected of* $\sum_{|\alpha|\leq m} a_\alpha(x)(-2\pi i x\xi)^\alpha$*. The aim is to have an algebra, perhaps self-adjoint, containing approximate inverses, or parametrices, for certain linear differential operators. However, if a given class of pseudo-differential operators is to be closed under composition, it should be possible, as well, to compose freely the differential operators in the class. As a consequence, the coefficients in (3.5) have to be in* C^∞*, unless a roundabout technique is used. This technique is due to A. P. Calderón, who used it to prove very general uniqueness results for certain partial differential problems (see [10] and [12]). In what follows, we outline the technique's main features. This is not an unnecessary digression, since it will provide motivation for other examples of CZOs.*

Let $\varphi \in C^{\infty}$ *be so* $\varphi(\xi) = |\xi|$ *for* $|\xi| \geq 1$ *and* $\varphi(\xi) > 0$ *everywhere. Let us call* $p(x, \xi)$ *the variable coefficient polynomial that appears in (3.5) and let us write*

$$p(x, \xi) = [q(x, \xi) + r(x, \xi)] \varphi(\xi)^{m},$$

where

$$q(x, \xi) = \sum_{|\alpha| = m} a_{\alpha}(x) (-2\pi i x \xi)^{\alpha} \varphi(\xi)^{-m},$$
$$r(x, \xi) = \sum_{|\alpha| < m} a_{\alpha}(x) (-2\pi i x \xi)^{\alpha} \varphi(\xi)^{-m}.$$

Let

$$K(f)(x) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} q(x, \xi) \widehat{f}(\xi) d\xi,$$
$$R(f)(x) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} r(x, \xi) \widehat{f}(\xi) d\xi.$$

So,

$$L(f) = (K + R) \Lambda^{m}(f),$$

where

$$\widehat{\Lambda(f)} = \varphi \widehat{f}.$$

The function $q(x, \xi)$ *is bounded and coincides, for* $|\xi| \geq 1$*, with a homogeneous function in* ξ *of degree zero. On the other hand,* R *and* $R\partial_{x_j}$ *for* $1 \leq j \leq n$*, can be extended to continuous operators on* L^p*, for* $1 < p < \infty$*, if the coefficients* $a_{\alpha}(x)$ *are bounded functions. Finally, it is well known how the operator* Λ *acts on many functional spaces.*

Adding all up, we propose the following definition for the operators in our class:

Let

$$L(f)(x) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} q(x, \xi) \widehat{f}(\xi) d\xi + R(f)(x), \tag{3.7}$$

where

i) *The function* $q(x, \xi)$ *is bounded and coincides, for* $|\xi| \geq 1$*, with a homogeneous function in* ξ *of degree zero.*

ii) *The operators* R *and* $R\partial_{x_j}$*, for* $1 \leq j \leq n$*, extend to continuous operators on* L^p*, for* $1 < p < \infty$*.*

So far, we have not said how much regularity we will impose on $q(x, \xi)$*, as a function of* x*. Certainly,* q *could not be better, as a function of* x*, than the coefficients of the differential operators we want to have in the class. Furthermore, it seems* *not advisable to consider differential operators whose coefficients, for $|\alpha| = m$, are not Lipschitz. Indeed, the first order operator $\partial_{x_1} + b(x_2)\partial_{x_2}$, where b is a Hölder function with positive exponent less than 1, is not locally solvable at the origin in $\mathbb{R}^2$, because the associated vector field does not have unique trajectories (see, for instance, [56]).*

iii) *Thus, we will assume that the function $q(x,\xi)$ in (3.7) is bounded, belongs to C^∞ as a function of ξ for each x fixed, is Lipschitz as a function of x uniformly on ξ, and coincides, for $|\xi| \geq 1$, with a homogeneous function in ξ of degree zero.*

Under these hypothesis, the first term in (3.7) is a CZO, associated to the standard kernel k (see Definition 1) given, formally, by the integral

$$k(x,y) = \int_{\mathbb{R}^n} e^{-2\pi i(x-y)\cdot\xi} q(x,\xi)\, d\xi,$$

which exists in the sense of an oscillatory integral. The class of such operators L becomes a non self-adjoint algebra, which has been very useful in proving existence and uniqueness of, and a priori estimates for, solutions of particular linear differential problems. Besides [10] and [12], we cite here [16], [11], [13] and [23].

Let us mention that in proving that the class of operators L is closed under composition, it suffices to use, instead of q, the homogeneous function with which it coincides for $|\xi| \geq 1$. In doing so, we introduce an error term that is a very well behaved operator. Furthermore, it is possible to exploit the very desirable properties the Fourier transform has on homogeneous functions.

Example 17. *To motivate another example of a CZO, we will look at two very simple operators in the class introduced in the previous discussion. For $n = 1$, we consider the operator M_a of multiplication by a bounded Lipschitz function a and the Hilbert transform H. It should be clear that the composition $M_a H$ belongs to the class. Thus, in order to prove that HM_a is also in the class, it would be enough to show it for the commutator*

$$[H, M_a] = HM_a - M_a H.$$

We can write

$$\begin{aligned}
[H, M_a]\frac{df}{dx} &= (HM_a - M_a H)\frac{df}{dx} \\
&= H\left(M_a \frac{df}{dx} - \frac{d}{dx} M_a(f)\right) + H\frac{d}{dx} M_a(f) - M_a H \frac{df}{dx} \\
&= H\left[M_a, \frac{d}{dx}\right](f) + \left[H\frac{d}{dx}, M_a\right](f). \qquad (3.8)
\end{aligned}$$

For our purpose, we focus on the second term in (3.8). It can be written, formally, as

$$\left[H\frac{d}{dx}, M_a\right](f)(x) = \int_{-\infty}^{\infty} \frac{a(x)-a(y)}{x-y}\frac{f(y)}{x-y}dy. \tag{3.9}$$

For reasons that will become clear soon, this operator is called the first commutator. It is associated with an antisymmetric standard kernel and it reduces to H when $a(x) = x$.

Example 18. *Let $D \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary ∂D. Given a function f defined on ∂D, the function F formally defined as*

$$F(x) = -\frac{1}{2\pi}\int_{\partial D} f(\overline{y})\,\partial_{n_{\overline{y}}}\log\frac{1}{|x-\overline{y}|}d\sigma_{\overline{y}}$$

for $x \in D$, is harmonic. Moreover, for $\overline{u} \in \partial D$ fixed, we can take formally,

$$-2\pi\lim_{x\to\overline{u}} F(x) = \frac{1}{2}f(\overline{u}) + pv\int_{\partial D} f(\overline{y})\,\partial_{n_{\overline{y}}}\log\frac{1}{|\overline{u}-\overline{y}|}d\sigma_{\overline{y}}.$$

Thus, F will be, at least formally, a solution of the Dirichlet problem associated with the Laplace operator, if given a function g defined on ∂D, it is possible to find f so that, for $\overline{u} \in \partial D$,

$$-2\pi g(\overline{u}) = \frac{1}{2}f(\overline{u}) + \int_{\partial D} f(\overline{y})\,\partial_{n_{\overline{y}}}\log\frac{1}{|\overline{u}-\overline{y}|}d\sigma_{\overline{y}}. \tag{3.10}$$

Again, the integral in (3.10) has to be suitably interpreted.

If $\operatorname{supp}(f)$ is sufficiently small, we can write, by hypothesis, $\overline{y} = (y, a(y))$ and $\overline{u} = (u, a(u))$ for $\overline{y}$ and $\overline{u}$ in the support of f, where a is a Lipschitz function. The Rademacher's theorem ([47], p. 18), states that the function a is differentiable almost everywhere. Moreover, as a consequence of the Lipschitz condition, a' is in $L^\infty(\mathbb{R})$. So, if we recall that $\partial_{n_{\overline{y}}} = n_{\overline{y}} \cdot \nabla_{\overline{y}}$ and we use the local parametrization of ∂D, the integral on the right hand side of (3.10) can be written, up to a multiplicative constant, as ([25], p. 10)

$$\int_{-\infty}^{\infty} f(y, a(y))\frac{a(u)-a(y)-(u-y)a'(y)}{(u-y)^2+(a(u)-a(y))^2}dy,$$

or

$$\int_{-\infty}^{\infty} \frac{f(y,a(y))}{u-y}\frac{\frac{a(u)-a(y)}{u-y}}{1+\left(\frac{a(u)-a(y)}{u-y}\right)^2}dy$$
$$-\,pv\int_{-\infty}^{\infty}\frac{f(y,a(y))}{u-y}\frac{a'(y)}{1+\left(\frac{a(u)-a(y)}{u-y}\right)^2}dy.$$

When studying $L^2(\mathbb{R})$ continuity, the factor $a'(y)$ is irrelevant, so we can write both terms as

$$\int_{-\infty}^{\infty} G\left(\frac{a(u)-a(y)}{u-y}\right)\frac{h(y)}{u-y}dy, \tag{3.11}$$

for suitable functions G.

If we replace G, formally, by its Taylor expansion, (3.11) will be a sum of integrals of the form

$$C_m\int_{-\infty}^{\infty}\left(\frac{a(u)-a(y)}{u-y}\right)^m\frac{h(y)}{u-y}dy. \tag{3.12}$$

For $m=1$, (3.12) is, except for a constant factor, the first commutator (3.9). In general, (3.12) differs only in a constant factor from the m-th commutator

$$[M_a,[M_a,[...[M_a,H\left(\frac{d}{dx}\right)^m]...]]].$$

Similar representations can be obtained when considering other boundary value problems for the Laplace operator, on bounded domains in $\mathbb{R}^n$. More than the type of problem, what is crucial for this approach is to consider a domain with a boundary regular enough, so the resulting integrals can be properly interpreted and will enjoy certain continuity properties. We will go back to this matter in Section 7.

The commutator operators (3.9) and (3.12), due to Calderón, are CZOs (see [14] and [18]), when a is a Lipschitz function, possibly unbounded. From our discussion in Example 16 and Example 18, it should be clear how these operators arise in connection with the theory of linear partial differential equations.

Example 19. *Let Γ be a rectifiable curve in the complex plane, given as the graph $z(t)=t+ia(t)$, of a Lipschitz function $a:\mathbb{R}\to\mathbb{R}$. Given a suitable f defined on Γ, the Cauchy integral*

$$\frac{1}{2\pi i}\int_\Gamma\frac{f(w)}{z-w}dw,$$

defines a holomorphic function $F(z)$ in the complement of Γ. Moreover, the non tangential limit of $F(z)$, as z approaches Γ, exists and is given by [12]

$$\pm\frac{1}{2}f(t,a(t))+\int_{-\infty}^{\infty}f(s,a(s))\frac{1+ia'(s)}{(s-t)+i(a(s)-a(t))}ds, \tag{3.13}$$

where the sign depends on whether z approaches Γ from above or from below and the integral in (3.13), once again, needs to be interpreted in an appropriate way. Except for the almost everywhere bounded factor $1+ia'(s)$, the integral above coincides with the Hilbert transform when Γ is the real axis, and it is essentially a particular case *of (3.11). Thus, it is natural to expect, as it is the case, that commutator operators similar to (3.12) will be relevant when studying the* $L^2(\mathbb{R})$ *continuity.*

It was first proved by Calderón ([18] and [19]) that there exists $\varepsilon > 0$ *small, so that the operator defined by the integral in (3.13) is a CZO, when* $\|a'\|_{L^\infty(\mathbb{R})} < \varepsilon$*. At the time, it was not known whether* ε *had a lower bound. The restriction on* $\|a'\|_{L^\infty(\mathbb{R})}$ *was lifted by Coifman, McIntosh and Meyer [28]. David showed in [36] how to derive the* $L^2(\mathbb{R})$ *continuity of the operator for* $\|a'\|_{L^\infty(\mathbb{R})}$ *large, from the small norm case. We will take another look at these ideas in Section 7.*

The book by Muskhelishvili [69] has a very interesting historical account of the Cauchy integral.

Example 20. *We conclude this section by taking a second look at pseudo-differential operators. To be sure, there are many different versions of what a pseudo-differential operator could be, depending on the conditions imposed on the function that replaces the bracket in (3.6). For instance, we refer to [17], [48], [59], [87], [88], [60], [74], [49], [50], [52], [71], [84], [7], [30] and [77].*

We will consider here the Hörmander class $L^m_{\rho,\delta}$*,* $m \in \mathbb{R}$*,* $0 \le \rho, \delta \le 1$*, introduced in [50], [60] and [88], of operators with symbol in* $S^m_{\rho,\delta}$*. The case* $m = 0$*,* $\rho = 1$*,* $0 \le \delta < 1$*, has already been mentioned in the previous section (see Remark 13).*

An operator L *is in* $L^m_{\rho,\delta}$ *when it can be written as*

$$L(f)(x) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} p(x,\xi)\,\widehat{f}(\xi)\,d\xi \tag{3.14}$$

for $f \in \mathcal{D}$*, where* p *belongs to* $S^m_{\rho,\delta}$*. This means that* $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ *and*

$$\left|\partial^\alpha_x \partial^\beta_\xi p(x,\xi)\right| \le C_{\alpha\beta}(1+|\xi|)^{m-\rho|\beta|+\delta|\alpha|},$$

for all multi indices α*,* β*. The function* p *is called the symbol of the operator because* p *is uniquely determined by* L *(see, for instance, [77], p. 261, further results 7.1)*

When $\rho > 0$ *and* $\delta < 1$*, the operator* L *is pseudo local, which is equivalent to saying that its distribution kernel* k *coincides, away from the diagonal, with a* C^∞ *function. Furthermore, each derivative* $\partial^\alpha_x \partial^\beta_y k(x,y)$ *decays rapidly as* $|x-y| \to \infty$*. Very precise estimates can be obtained for* $|x-y| < 1$ *([2], [3]). It was already known ([30], p. 87, Proposition 1), that operators in the class* $L^0_{1,0}$ *are associated with distribution kernels* k *that coincide with standard kernels away from the diagonal.*

Chapters VI and VII in [77] have a very insightful discussion of the class $L^m_{\rho,\delta}$ *for different values of the parameters, from which arise the so called classical operators as well as the exotic operators [30].*

The difficulties we have encountered when trying to make sense of certain integrals, are due to the minimal smoothness conditions assumed ([25], Chapter I). In the terminology of [30], operators such as the m-th commutator and the Cauchy integral on Lipschitz curves, lie beyond (au delà) the classical pseudo-differential operators with symbols in $S^0_{1,0}$.

Although our main interest has been on the L^2 continuity of the operators considered in this section, they actually enjoy various continuity properties in different functional spaces ([18], [50], [29], [84], [42], [7], [30], [19], [66], [31], [32], [27], [57], [51], [55], [70], [67]). The monograph [86] has an in depth analysis of these matters.

4 The statement of the T(1) Theorem

As we will see shortly, the space of functions with bounded mean oscillation, BMO, plays a crucial role in the formulation of the T(1) Theorem. So, we begin by recalling its definition.

Definition 21. *Given a locally integrable function f, we say that f belongs to BMO if*

$$\sup_Q \frac{1}{|Q|}\int_Q |f - f_Q|\,dx < \infty, \tag{4.1}$$

where Q denotes any cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, $|Q|$ is the Lebesgue measure of Q and f_Q is the average of f over Q. The left hand side of (4.1) is zero when f is a constant function. The quantity appearing in (4.1) gives the seminorm of f in BMO, denoted $\|f\|_{BMO}$. Equivalently, we can impose the condition

$$\sup_Q \left(\inf_{c\in\mathbb{R}} \frac{1}{|Q|}\int_Q |f - c|\,dx\right) < \infty. \tag{4.2}$$

Remark 22. *Occasionally, we will find convenient to use in (4.1), balls instead of cubes. We will do so without any further comment.*

It is clear that $L^\infty \subseteq BMO$. The standard example of an unbounded function in BMO is $\log|x|$. Another example, less often encountered in the literature, is $\log|P(x)|$, where $P(x)$ is any non zero polynomial ([76], p. 332, Theorem 6; [77], p. 177, further results 6.1; p. 219, further results 6.5). For more examples, see ([77], p. 178, further results 6.3; p. 218, further results 6.2). We will go back to some of these examples in Section 7. An examination of any of the examples would tell us that functions in BMO do not seem to grow wildly at infinity. That this is the case for any function in BMO, is a consequence of the following property (see, for instance, [77], p. 141, 1.1.4)

If $f \in BMO$, then

$$\int_{\mathbb{R}^n} |f(x)|\,(1+|x|)^{-n-1}\,dx < \infty. \tag{4.3}$$

However, this property does not characterize BMO. For instance, the function $sgn(x)\ln|x|$ does not belong to BMO ([25], p. 35). That is to say, (4.1) is not just *a size condition. For a characterization of BMO involving (4.3), see ([25], p. 39, Theorem 12).*

A second ingredient in the statement of the T(1) Theorem, in need of clarification, will be the so called weak boundedness property, in short WBP, which we now define.

Definition 23. *Let $T : \mathcal{D} \to \mathcal{D}'$ be a linear and continuous operator. Then, we say that T has the WBP, if for each bounded subset $\mathcal{B}$ of $\mathcal{D}$, there exists $C > 0$ such that*

$$\left|\left(T\left(\varphi_t^{x_0}\right), \psi_t^{x_0}\right)_{\mathcal{D}',\mathcal{D}}\right| \leq \frac{C}{t^n}, \tag{4.4}$$

for every $\varphi, \psi \in \mathcal{B}$, $x_0 \in \mathbb{R}^n$ and $t > 0$, where

$$f_t^{x_0}(x) = \frac{1}{t^n} f\left(\frac{x - x_0}{t}\right).$$

Remark 24. *The transpose operator T^t has the WBP if T does. Also, any linear and continuous operator on L^2 has it. On the other hand, the operator ∂_{x_j} does not satisfy the property. The verification of these claims is straightforward and it will be omitted. The WBP is truly weaker that the full L^2 continuity. Indeed, any pseudo-differential operator with symbol in the Hörmander class $S^0_{1,1}$ has the WBP, although it might not be bounded on L^2. In fact, every pseudo-differential operator L of the form (3.14) will have the WBP if we just assume that the symbol p is a bounded function. Let us outline the proof of this claim.*

Given a bounded subset $\mathcal{B}$ of $\mathcal{D}$ we fix, as in Definition 23, $\varphi, \psi \in \mathcal{B}$, $x_0 \in \mathbb{R}^n$ and $t > 0$. Then,

$$\left(L\left(\varphi_t^{x_0}\right), \psi_t^{x_0}\right)_{\mathcal{D}',\mathcal{D}} = \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{-2\pi i (x - x_0)\cdot\xi} p(x, \xi)\, \widehat{\varphi}(t\xi)\, \psi_t(x - x_0)\, d\xi dx.$$

With the change of variables $t\xi = \eta$ and $\frac{x - x_0}{t} = z$, the integral above reduces to

$$\frac{1}{t^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{-2\pi i z\cdot\eta} p\left(tz + x_0, \frac{\eta}{t}\right) \widehat{\varphi}(\eta)\, \psi(z)\, d\eta dz.$$

So,

$$\left|\left(L\left(\varphi_t^{x_0}\right), \psi_t^{x_0}\right)_{\mathcal{D}',\mathcal{D}}\right| \leq \frac{1}{t^n} \|p\|_{L^\infty} \|\widehat{\varphi}\|_{L^1} \|\psi\|_{L^1},$$

from which (4.4) follows, for a constant $C > 0$ that depends on $\mathcal{B}$.

To see that the WBP is truly weaker that the full L^2 continuity, we look at the pseudo-differential operator in $L^0_{1,1}$ defined by Ching [24] and already mentioned in Remark 13. Although the operators in $L^0_{1,1}$ are all associated with standard kernels and satisfy the WBP, Ching constructed a type of operator in the class that is not always bounded on L^2. Indeed, he considered the symbol

$$p(x, \xi) = \sum_{k \geq 0} a_k e^{2\pi i 5^k 3 e_1 \cdot x} \psi\left(\frac{\xi}{5^k}\right), \tag{4.5}$$

where $\{a_k\}_{k\geq 0}$ *is a bounded sequence,* $e_1 = (1, 0, ..., 0)$, $\psi \in \mathcal{D}$, *supp* $(\psi) \subseteq \{1 \leq |\xi| \leq 5\}$ *and* $\psi(\xi) = 1$ *when* $2 \leq |\xi| \leq 4$. *The symbol* p *is in* $S^0_{1,1}$, *so the operator* L *with symbol* p *has the WBP. However, Ching proved that if* L *is continuous on* L^2, *the sequence* $\{a_k\}_{k\geq 0}$ *has to be square summable. We will take a final look at Ching's counterexample in Section 7.*

We continue our discussion, prior to stating the T(1) Theorem, by proving the following important result:

Proposition 25. *Let* $T : \mathcal{D} \to \mathcal{D}'$ *be a linear and continuous operator, associated with a standard kernel* k. *Then, the action of* T *on* $f \in C^\infty \cap L^\infty$ *can be defined as a linear and continuous functional on the closed subspace of* $\mathcal{D}$,

$$\mathcal{D}_0 = \left\{ \varphi \in \mathcal{D} : \int_{\mathbb{R}^n} \varphi(x)\, dx = 0 \right\}.$$

Proof. We fix $f \in C^\infty \bigcap L^\infty$ and we fix a bounded subset $\mathcal{B}$ of $\mathcal{D}_0$. We can assume that *supp* $(g) \subseteq B(z, r)$, an open ball centered at z with radius r, for every $g \in \mathcal{B}$. Next, we pick a function $\theta \in \mathcal{D}$ so that, say, $\theta(x) = 1$ for $|x - z| < 3r$ and $\theta(x) = 0$ for $|x - z| \geq 4r$.

Then, the action of $T(\theta f)$ on $g \in \mathcal{D}_0$ is well defined in the duality $(\mathcal{D}', \mathcal{D})$ and can also be expressed as $\left(T^t(g), \theta f\right)$ by reflexivity.

On the other hand, since $\int_{\mathbb{R}^n} g(x)\, dx = 0$, we can write, for $x \in \mathbb{R}^n \setminus$ *supp* (g),

$$T^t(g)(x) = \int_{\mathbb{R}^n} k(y, x)\, g(y)\, dy = \int_{|y-z|<r} (k(y, x) - k(z, x))\, g(y)\, dy,$$

so, the iterated integral

$$\int_{|x-z|\geq 3r} |1 - \theta(x)|\, |f(x)| \left(\int_{|y-z|<r} |k(y, x) - k(z, x)|\, |g(y)|\, dy \right) dx \tag{4.6}$$

exists. Let us observe that $x \in$ *supp* $((1 - \theta) f)$ and $y \in$ *supp* (g), imply that

$$|x - z| \geq 3r > 3|y - z| > 2|y - z|.$$

Thus, (4.6) can be estimated by

$$C \|f\|_{L^\infty} \left(\int_{\mathbb{R}^n} |g(y)|\, dy \right) \left(\int_{|x-z|\geq 2|y-z|} |k(y, x) - k(z, x)|\, dx \right) \tag{4.7}$$

$$\leq C \|f\|_{L^\infty} \|g\|_{L^\infty} \int_{|x-z|\geq 2|y-z|} |k(y, x) - k(z, x)|\, dx, \tag{4.8}$$

where we have used that there is a compact set $K \subset \mathbb{R}^n$, so that *supp* $(g) \subseteq K$ for all $g \in \mathcal{B}$ ([6], p. 41).

Since the kernel k satisfies condition 2) in Definition 1, the integral in (4.8) is bounded by a fixed constant. In fact,

$$\begin{aligned}\int_{|x-z|\geq 2|y-z|} |k(y,x)-k(z,x)|\,dx &\leq C\int_{|x-z|\geq 2|y-z|}\frac{|y-z|^{\delta}}{|x-z|^{n+\delta}}dx \\ &\leq C\,|y-z|^{\delta}\int_{2|y-z|}^{+\infty} r^{n-1-n-\delta}dr \\ &= C, \end{aligned} \tag{4.9}$$

where the constant C only depends on fixed parameters, such as n and δ.

Finally, we can bound (4.6) with

$$C\,\|f\|_{L^{\infty}}\,\|g\|_{L^{\infty}}\,,$$

where the constant C ultimately depends on $\mathcal{B}$, among other fixed parameters.

So, for θ fixed, we can define the action of $T(f)$ on g, as

$$\left(T^{t}(g),\theta f\right)+\int_{\mathbb{R}^n}(1-\theta(x))\,f(x)\left(\int_{\mathbb{R}^n}k(y,x)\,g(y)\,dy\right)dx, \tag{4.10}$$

which gives $T(f)$ as a linear and continuous functional on $\mathcal{D}_0$. Let us recall, once again, that $(,)$ denotes the $(\mathcal{D}',\mathcal{D})$ duality.

Definition (4.10) does not depend on θ, provided that θ satisfies the aforementioned conditions. In fact, if θ_1 and θ_2 are two such functions,

$$\begin{aligned}&\int_{\mathbb{R}^n}(1-\theta_1(x))\,f(x)\left(\int_{\mathbb{R}^n}k(y,x)\,g(y)\,dy\right)dx \\ &-\int_{\mathbb{R}^n}(1-\theta_2(x))\,f(x)\left(\int_{\mathbb{R}^n}k(y,x)\,g(y)\,dy\right)dx \\ &=\int_{\mathbb{R}^n}(\theta_2(x)-\theta_1(x))\,f(x)\left(\int_{\mathbb{R}^n}k(y,x)\,g(y)\,dy\right)dx \\ &=-\left(T^{t}(g),(\theta_1-\theta_2)\,f\right).\end{aligned}$$

Let us observe that if f happens to be in $\mathcal{D}$, both θf and $(1-\theta)\,f$ are in $\mathcal{D}$ as well, and we can write

$$\begin{aligned}(T(f),g)&=(T(\theta f),g)+(T((1-\theta)\,f),g)_{\mathcal{D}',\mathcal{D}} \\ &=\left(T^{t}(g),\theta f\right) \\ &+\int_{\mathbb{R}^n}(1-\theta(x))\,f(x)\left(\int_{\mathbb{R}^n}k(y,x)\,g(y)\,dy\right)dx.\end{aligned}$$

So, $T(f)$, defined as a distribution in $\mathcal{D}'$, coincides with the definition given by (4.10).

Finally, we point out that in justifying (4.10) it suffices to assume that the integral in (4.8) is bounded by a fixed constant, instead of using the pointwise estimate from condition 2) in Definition 1. For another instance where a similar condition appears, see Remark 31 below. We will revisit this idea in the last section.

The proof of the proposition is now complete. □

Remark 26. *Given $f \in C^\infty \bigcap L^\infty$, the action of $T(f)$ on $g \in \mathcal{D}_0$ can be defined, equivalently, as*

$$(T(f), g) = \lim_{j\to\infty} (T(\theta_j f), g), \tag{4.11}$$

where $\theta \in \mathcal{D}$, $\theta(x) = 1$ for $|x| < 1$, $\theta(x) = 0$ for $|x| > 2$, and $\theta_j(x) = \theta\left(\frac{x}{j}\right)$. This assertion follows from (4.10), observing that for such a function θ_j, the second term in (4.10) will converge to zero as $j \to \infty$.

Remark 27. *Proposition 25 and Remark 26 apply as well to the transpose operator T^t, associated with the standard kernel $k^t(x,y) = k(y,x)$.*

Remark 28. *For future reference, we point out that $\mathcal{D}_0$ is dense (see [38], p. 372) in the Hardy space H^1, the topological predual of BMO (see [77], Chapter IV, 1.2).*

We are now ready to state the T(1) Theorem.

Theorem 29. *(T(1) Theorem) Let T be an operator associated with a standard kernel. Then, T is a CZO if and only if the following three conditions hold:*

1. *T has the WBP,*
2. *$T(1) \in BMO$,*
3. *$T^t(1) \in BMO$,*

where 1 denotes the function in $C^\infty \bigcap L^\infty$ that is identically equal to one.

Remark 30. *First of all, Proposition 25, tells us how to define $T(1)$ and $T^t(1)$. Moreover, the statement of Theorem 29 is invariant by transposition, which makes sense, since the definition of standard kernel is transpose invariant and the class of CZOs is closed under transposition. Furthermore, conditions 1), 2) and 3) in Theorem 29 are independent. For example, the operator M of multiplication by the BMO function $\log|x|$ is associated with the identically zero kernel and it satisfies $M(1)$ and $M^t(1) \in BMO$, since $M(1) = M^t(1) = \log|x|$. However, M does not have the WBP property. In fact, for any $\varphi = \varphi(r) \in \mathcal{D}$ radial and $t > 0$,*

$$(M(\varphi_t), \varphi_t)_{\mathcal{D}',\mathcal{D}} = \frac{C_n}{t^{2n}} \int_0^{+\infty} \ln r\, \varphi^2\left(\frac{r}{t}\right) r^{n-1} dr.$$

With the change of variable $r \to s = \frac{r}{t}$, *we can write*

$$\begin{aligned}&(M(\varphi_t), \varphi_t)_{\mathcal{D}', \mathcal{D}} \\ &= \frac{C_n}{t^n}\left[\int_0^{+\infty} \ln r\, \varphi^2(s)\, s^{n-1} ds + \ln t \int_0^{+\infty} \varphi^2(s)\, s^{n-1} ds\right],\end{aligned}$$

where the second term within the brackets is not bounded on t.

As we observed in Remark 24, any pseudo-differential operator L *with symbol* $p(x,\xi)$ *in* $S^0_{1,1}$ *has the WBP. It follows from (4.11) that* $L(1) = p(x,0)$, *so* $L(1) \in L^\infty \subset BMO$. *However, the condition* $L^t(1) \in BMO$ *cannot always be true, otherwise, by Theorem 29, operators in* $L^0_{1,1}$ *would always be bounded on* L^2. *The example due to C.-H. Ching [24], briefly considered in Remark 24, shows that this is not the case. Actually, this very interesting example, can be used to demonstrate that conditions 1) and 3) do not imply 2) and that condition 1) does not imply either 2) or 3), etc. We will revisit these claims in Section 7.*

Remark 31. *Let us see that CZOs satisfy the conditions in Theorem 29. We already know that CZOs have the WBP. We also know that a CZO* T *will satisfy condition 2) if and only if it satisfies condition 3). To see that* T *satisfies condition 2), we will actually show that* T *can be defined on* L^∞, *in a manner compatible with Proposition 25, and that this definition yields a continuous operator from* L^∞ *to* BMO. *We follow Theorem 24 in ([30], p. 117).*

Suppose that T *is a CZO. Given* $f \in L^\infty$ *and given a ball* $B(z,r)$, *we write* $f = f\varphi + f(1-\varphi)$, *where* $\varphi \in \mathcal{D}$, φ *is identically one on* $B(z,2r)$ *and* $supp(\varphi) \subseteq B(z,3r)$. *Let us observe that* $f\varphi \in L^2$, *so* $T(f\varphi)$ *is defined, almost everywhere, as an* L^2 *function. If* k *denotes a standard kernel for the operator,* $|k(x,y) - k(z,y)|$ *is integrable for* $|y-z| \geq 2r$, *as a function of* y, *when* $x \in B(z,r)$ *and the integral is estimated by a constant. Thus, the product* $(k(x,y) - k(z,y)) f(y)(1-\varphi(y))$ *is well defined, almost everywhere, and we can write*

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} (k(x,y) - k(z,y)) f(y)(1-\varphi(y))\, dy\right| \\ &\leq C\|f\|_{L^\infty} \int_{|y-z|\geq 2r} |k(x,y) - k(z,y)|\, dy. \qquad (4.12)\end{aligned}$$

We claim that the integral in (4.12) satisfies the estimate

$$\sup_{x\in B(z,r)} \int_{|y-z|\geq 2r} |k(x,y) - k(z,y)|\, dy \leq C,$$

where the constant C *only depends on fixed parameters. This is proved similarly to the estimate of the integral in (4.8). In fact, using condition 2) in Definition 1, we* *have, for $x \in B(z,r)$ fixed,*

$$\int_{|y-z|\geq 2r} |k(x,y)-k(z,y)|\,dy \leq C\int_{|y-z|\geq 2r} \frac{|x-z|^{\delta}}{|y-z|^{n+\delta}}dx$$
$$= Cr^{\delta}\int_{2r}^{+\infty} r^{n-1-n-\delta}dr = C.$$

So, we have the estimates

$$\frac{1}{|B|}\int_B |T(f\varphi)(x)|\,dx \leq \left(\frac{1}{|B|}\int_B |T(f\varphi)(x)|^2\,dx\right)^{1/2}$$
$$\leq C\,\|f\|_{L^\infty}$$

and

$$\frac{1}{|B|}\int_B \left|\int_{\mathbb{R}^n} (k(x,y)-k(z,y))\,f(y)\,(1-\varphi(y))\,dy\right| dx$$
$$\leq C\,\|f\|_{L^\infty}\,.$$

If we define $T(f)(x)$, modulo additive constants and almost everywhere, as

$$T(f\varphi)(x)+\int_{\mathbb{R}^n} (k(x,y)-k(z,y))\,f(y)\,(1-\varphi(y))\,dy,$$

we can see, as in Proposition 25, that this definition does not depend on φ satisfying suitable properties. Furthermore, the estimates above show that T is indeed continuous from L^∞ to BMO. If $f \in C^\infty \cap L^\infty$, the definition of $T(f)$ we just discussed, agrees with the definition given in Proposition 25.

Finally, let us observe that the integral in (4.12) being bounded by a fixed constant for $x \in B(z,r)$ and every z and r, is another instance of (4.9).

When T is a pseudo-differential operator in the class $L^0_{1,\delta}$, $0 \leq \delta < 1$, its kernel decays rapidly at infinity (see Example 20), so there are no difficulties associated with the definition of T on L^∞.

We observed in Remark 4 that the kernel of the double layer potential T associated with an open set with a Lipschitz boundary is not a standard kernel. In the terminology of the T(1) Theorem, the kernel is not smooth enough in the y variable to define $T^t(1)$. However, T is bounded on L^2 [63].

From all we have said, it should be clear that when T is a CZO, conditions 1), 2) and 3) in Theorem 29 hold. The next two sections will be dedicated to prove the converse, starting with the ...

5 Reduction to the case $T(1) = T^t(1) = 0$

This is a crucial step in the original proof of the T(1) Theorem, and it is based on the following lemma:

Lemma 32. *Given $a \in BMO$, there exists a CZO, L, such that $L(1) = a$ and $L^t(1) = 0$.*

There are several ways of building such an operator L (see, for instance, [38]; [44], p. 212, Section 8.4). All of them essentially consist of redefining the pointwise product by a BMO function. As we saw in Remark 30, this multiplication operator does not even have, in general, the WBP. The new operation, called generically a paraproduct ([30], Appendix I; [25], Chapter III (3) and (5)), gives a bilinear action from $L^2 \times BMO$ into L^2. Incidentally, paraproducts are related to a very interesting subclass of $L^m_{1,1}$, the paradifferential operators ([8], [61], [79], [53] and [54]).

The proof of the lemma, given below, is a detailed account of the paraproduct construction presented in [38].

Remark 33. *Assuming that we have proved Lemma 32 and that T is a fixed operator as in Theorem 29, let L, M be CZOs satisfying*

$$L(1) = T(1)\,,\ L^t(1) = 0,$$
$$M(1) = T^t(1)\,,\ M^t(1) = 0.$$

Then, the operator S defined as

$$S = T - L - M^t,$$

will satisfy

$$S(1) = S^t(1) = 0.$$

The proof of the lemma uses the notion of Carleson measure, which was introduced by L. Carleson, in order to solve the following problem: Given the Poisson kernel $P(x) = c_n \left(1 + |x|^2\right)^{-(n+1)/2}$ for the upper-half space $\mathbb{R}^{n+1}_+$, characterize the measures μ on $\mathbb{R}^{n+1}_+$ for which

$$\int_{\mathbb{R}^{n+1}_+} |(P_t * f)(x)|^p \, d\mu(x,t) \leq C_p \|f\|^p_{L^p}\,, \tag{5.1}$$

for $1 < p < \infty$, where

$$P_t(x) = \frac{1}{t^n} P\left(\frac{x}{t}\right). \tag{5.2}$$

The solution to this problem is as follows: A measure μ on $\mathbb{R}^{n+1}_+$ satisfies (5.1) if and only if there is $C > 0$ such that for every cube $Q \subset \mathbb{R}^n$ the condition

$$\mu\left(\overline{Q}\right) \leq C\,|Q| \tag{5.3}$$

holds, where $\overline{Q}$ denotes the cube in $\mathbb{R}^{n+1}_{+}$ with base Q and $|Q|$ is the Lebesgue measure of Q. A measure μ on $\mathbb{R}^{n+1}_{+}$ satisfying (5.3) is called Carleson measure. The main example of a Carleson measure is provided by the following result:

Proposition 34. *([77], p. 159) Given $\psi \in \mathcal{S}$ with $\int_{\mathbb{R}^n} \psi(x)\,dx = 0$ and given $a \in BMO$, the measure $|(a * \psi_t)(x)|^2 \frac{dxdt}{t}$ is a Carleson measure, where the scaling ψ_t of ψ is defined as in (5.2). Moreover,*

$$\left(\int_{\mathbb{R}^{n+1}_{+}} |(P_t * f)(x)|^2 \, |(\psi_t * a)(x)|^2 \frac{dxdt}{t} \right)^{1/2} \leq C \|a\|_{BMO} \|f\|_{L^2} . \tag{5.4}$$

Remark 35. *Proposition 34 has the following extension: Let $\varphi \in L^1$ be bounded by a radial function $\Phi(|x|)$ where Φ is decreasing and integrable, and let μ be any Carleson measure. Then,*

$$\left(\int_{\mathbb{R}^{n+1}_{+}} |(\varphi_t * f)(x)|^2 \, d\mu \right)^{1/2} \leq C \|f\|_{L^2} ,$$

for some $C > 0$ depending on Φ and μ.

In particular, if $\mu = |(\psi_t * a)(x)|^2 \frac{dxdt}{t}$ as in Proposition 34,

$$\left(\int_{\mathbb{R}^{n+1}_{+}} |(\varphi_t * f)(x)|^2 \, |(\psi_t * a)(x)|^2 \frac{dxdt}{t} \right)^{1/2} \leq C \|a\|_{BMO} \|f\|_{L^2} ,$$

with C depending on Φ.

Let us point out that, in the proof of Lemma 32, P_t will be the operator defined on $\mathcal{D}$ as the convolution with the function φ_t. Since we will no longer refer to the Poisson kernel, this notation should not cause any confusion.

We now prove Lemma 32.

Proof. We give here a detailed account of the construction presented in [38]. We fix a radial function as in Remark 35, such that $\varphi \in \mathcal{D}$, $supp(\varphi) \subseteq \{x : |x| \leq 1\}$ and $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$. Let $\varphi_t(x) = \frac{1}{t^n}\varphi\left(\frac{x}{t}\right)$ and let P_t be the operator acting on $\mathcal{D}$ as the convolution with φ_t. Likewise, let Q_t the operator acting on $\mathcal{D}$ as the convolution with ψ_t, where

$$\psi(x) = \sum_{j=1}^{n} \partial_{x_j}(x_j \varphi(x)) .$$

Let us observe that still ψ is a radial function in $\mathcal{D}$ and $supp(\psi) \subseteq \{x : |x| \leq 1\}$, but $\int_{\mathbb{R}^n} \psi(x)\,dx = 0$. Moreover, in the sense of convolution operators defined on $\mathcal{D}$,

$$Q_t = -t \frac{d}{dt} P_t,$$

for $t > 0$. Indeed,

$$\begin{aligned}
-t\frac{d}{dt}\left(\varphi_t\left(x\right)\right) &= -t\frac{d}{dt}\left(\frac{1}{t^n}\varphi\left(\frac{x}{t}\right)\right) = nt\frac{1}{t^{n+1}}\varphi\left(\frac{x}{t}\right) \\
&- \frac{1}{t^{n-1}}\frac{d}{dt}\left(\varphi\left(\frac{x}{t}\right)\right) \\
&= \frac{1}{t^n}\sum_{j=1}^{n}\varphi\left(\frac{x}{t}\right) - \frac{1}{t^{n-1}}\sum_{j=1}^{n}\left(-\frac{x_j}{t^2}\right)\left(\partial_{x_j}\left(\varphi\right)\right)\left(\frac{x}{t}\right) \\
&= \frac{1}{t^n}\sum_{j=1}^{n}\left(\varphi\left(\frac{x}{t}\right) + \left(\frac{x_j}{t}\right)\left(\partial_{x_j}\left(\varphi\right)\right)\left(\frac{x}{t}\right)\right) \\
&= \left(\sum_{j=1}^{n}\partial_{x_j}\left(x_j\varphi\left(x\right)\right)\right)_t = \psi_t\left(x\right).
\end{aligned}$$

Our goal is to prove that $\int_0^\infty Q_t\left(Q_t\left(a\right)\right)P_t\frac{dt}{t}$ can be defined, in a weak sense, as a CZO, satisfying, up to a normalizing factor, the conditions stated in the lemma. More specifically, if

$$\left(L_m\left(f\right), g\right) = \int_{\frac{1}{m}}^{m}\left(\left(Q_t\left(Q_t\left(a\right)\right)P_t\right)\left(f\right), g\right)\frac{dt}{t} \tag{5.5}$$

for $m \geq 1$, we will show that L_m is a CZO uniformly with respect to m and that for $f, g \in \mathcal{D}$ there exists

$$\lim_{m\to\infty}\left(L_m\left(f\right), g\right) = \left(L\left(f\right), g\right), \tag{5.6}$$

where L is a CZO satisfying, with an appropriate normalization, $L\left(1\right) = a$ and $L^t\left(1\right) = 0$.

Let us recall that $(,)$ still signifies the $(\mathcal{D}', \mathcal{D})$ duality. Any other duality will be specifically noted.

We begin by proving that L_m is associated with a standard kernel. For $t > 0$ fixed, we write

$$\begin{aligned}
\left(Q_t\left(Q_t\left(a\right)\right)P_t\left(f\right), g\right) &= \left(\left(Q_t\left(a\right)\right)P_t\left(f\right), Q_t\left(g\right)\right) \\
&= \int_{\mathbb{R}^{3n}}\left(Q_t\left(a\right)\right)\left(u\right)\varphi_t(u-y)\psi_t\left(u-x\right)f\left(y\right)g\left(x\right)dxdydu.
\end{aligned}$$

So, $L_m : \mathcal{D} \to \mathcal{D}'$ is an integral operator with kernel $\mathcal{L}_m$ defined as

$$\mathcal{L}_m\left(x, y\right) = \int_{\frac{1}{m}}^{m}\int_{\mathbb{R}^n}\left(Q_t\left(a\right)\right)\left(u\right)\varphi_t(u-y)\psi_t\left(u-x\right)du\frac{dt}{t}.$$

Taking into account that $\int_{\mathbb{R}^n}\psi\left(x\right)dx = 0$, we have

$$\left|\left(Q_t\left(a\right)\right)\left(u\right)\right| = \left|\frac{1}{t^n}\int_{B(u,t)}\psi\left(\frac{u-v}{t}\right)\left(a\left(v\right) - a_{B(u,t)}\right)dv\right|,$$

where $a_{B(u,t)}$ is the average of a on the ball $B(u,t)$. Thus,

$$|(Q_t(a))(u)| \leq C \|a\|_{BMO}, \tag{5.7}$$

for some $C > 0$ independent of t. We observe now that $\int_{\mathbb{R}^n} (Q_t(a))(u)\, \varphi_t(u-y)\psi_t(u-x)\, du$ is supported on $\left\{(x,y) \in \mathbb{R}^{2n} : |x-y| \leq 2t\right\}$, where we can write

$$1 \leq 1 + \frac{|x-y|}{t} \leq 3.$$

Thus, for any $N \geq 1$ and an appropriate constant $C > 0$, again not depending of t,

$$\begin{aligned} &\left|\int_{\mathbb{R}^n} (Q_t(a))(u)\, \varphi_t(u-y)\psi_t(u-x)\, du\right| \\ &\leq \frac{C}{t^n} \|a\|_{BMO} \frac{1}{\left(1+\frac{|x-y|}{t}\right)^N}. \end{aligned} \tag{5.8}$$

If we choose, for instance, $N = n+1$,

$$\begin{aligned} |\mathcal{L}_m(x,y)| &\leq C \|a\|_{BMO} \int_0^\infty \frac{1}{t^n} \frac{1}{\left(1+\frac{|x-y|}{t}\right)^{n+1}} \frac{dt}{t} \\ &= C \|a\|_{BMO} \frac{1}{|x-y|^n}. \end{aligned} \tag{5.9}$$

Similar calculations with N at least equal to $n+2$, will show that

$$|\nabla_{x,y}\mathcal{L}_m(x,y)| \leq C \|a\|_{BMO} \frac{1}{|x-y|^{n+1}}, \tag{5.10}$$

for some $C > 0$ not depending on m. So, according to Lemma 3, the function $\mathcal{L}_m(x,y)$ is a standard kernel in the sense of Definition 1, uniformly on m.

We will now prove that the sequence of kernels $\{\mathcal{L}_m\}_{m\geq 1}$, as well as the sequence $\{\nabla_{x,y}\mathcal{L}_m\}_{m\geq 1}$, converges uniformly on each compact subset of $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$, to a function $\mathcal{L}$ that, according to (5.9) and (5.10), will then be a standard kernel, with $\delta = 1$ and constant C proportional to $\|a\|_{BMO}$. In fact, according to (5.8), we can write

$$\begin{aligned} |\mathcal{L}_{m+p}(x,y) - \mathcal{L}_m(x,y)| &\leq C \|a\|_{BMO} \left(\int_{\frac{1}{m+p}}^{\frac{1}{m}} + \int_m^{m+p}\right) \frac{1}{t^n} \frac{1}{\left(1+\frac{|x-y|}{t}\right)^{n+1}} \frac{dt}{t} \\ &\leq C \|a\|_{BMO} \left(\int_0^{\frac{1}{m}} + \int_m^\infty\right) \frac{1}{t^n} \frac{1}{\left(1+\frac{|x-y|}{t}\right)^{n+1}} \frac{dt}{t}. \end{aligned} \tag{5.11}$$

Thus, there exists

$$\lim_{m\to\infty} \mathcal{L}_{m+p}(x,y) - \mathcal{L}_m(x,y) = 0,$$

for every $p \geq 1$, uniformly on each compact subset of $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$. A similar reasoning will prove that the sequence $\{\nabla_{x,y}\mathcal{L}_m\}_{m\geq 1}$ is Cauchy, uniformly on each compact subset of $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$.

We claim that the operator L_m is bounded on L^2, with norm bounded independently of m. In fact, given $f, g \in \mathcal{D}$,

$$\begin{aligned}
|(L_m f, g)|^2 &= \left| \int_{\frac{1}{m}}^{m} ((Q_t(a)) P_t(f), Q_t(g)) \frac{dt}{t} \right|^2 \\
&= \left| \int_{\frac{1}{m}}^{m} \int_{\mathbb{R}^n} (Q_t(a))(x) (P_t(f))(x) (Q_t(g))(x) \frac{dxdt}{t} \right|^2 \\
&\leq \int_0^\infty \int_{\mathbb{R}^n} |(Q_t(a))(x) (P_t(f))(x)|^2 \frac{dxdt}{t} \qquad (5.12)\\
&\times \int_0^\infty \int_{\mathbb{R}^n} |(Q_t(g))(x)|^2 \frac{dxdt}{t}. \qquad (5.13)
\end{aligned}$$

Now, for $t > 0$ fixed,

$$\int_{\mathbb{R}^n} |(Q_t(g))(x)|^2 dx = \int_{\mathbb{R}^n} \left|\widehat{\psi}(t\xi)\right|^2 |\widehat{g}(\xi)|^2 d\xi.$$

Since $\widehat{\psi}(0) = 0$, we have the estimate

$$\left|\widehat{\psi}(\xi)\right| \leq \begin{cases} C|\xi| & \text{for } |\xi| \leq 1 \\ C|\xi|^{-1} & \text{for } |\xi| \geq 1 \end{cases}.$$

So, (5.13) can be bounded by

$$\begin{aligned}
&C \int_{\mathbb{R}^n} \left(\int_0^{\frac{1}{|\xi|}} |\xi|^2 t^2 \frac{dt}{t} + \int_{\frac{1}{|\xi|}}^\infty \frac{1}{|\xi|^2 t^2} \frac{dt}{t} \right) |\widehat{g}(\xi)|^2 d\xi \\
&= C \|g\|_{L^2}^2 .
\end{aligned}$$

As for (5.12), Proposition 34 shows that $|(Q_t(a))(x)|^2 \frac{dxdt}{t}$ is a Carleson measure, so it satisfies (5.3). More specifically, there exists $C > 0$ such that for every cube $B \subset \mathbb{R}^n$, if $\overline{B}$ denotes the cube in $\mathbb{R}^{n+1}_+$ with base B,

$$\int_{\overline{B}} |(Q_t(a))(x)|^2 \frac{dxdt}{t} \leq C \|a\|_{BMO}^2 |B| .$$

According to Remark 35,

$$\int_0^\infty \int_{\mathbb{R}^n} |(P_t(f))(x)|^2 |(Q_t(a))(x)|^2 \frac{dxdt}{t} \leq C \|a\|_{BMO}^2 \|f\|_{L^2}^2 .$$

So,

$$|(L_m f, g)| \leq C \|a\|_{BMO} \|f\|_{L^2} \|g\|_{L^2} . \tag{5.14}$$

Next, we claim that the sequence $\{L_m\}_{m\geq 1}$ of operators from $\mathcal{D}$ to $\mathcal{D}'$, converges. Given $f, g \in \mathcal{D}$,

$$\begin{aligned} &|(L_{m+p}(f) - L_m(f), g)|^2 \\ &= \left| \left(\int_{\frac{1}{m+p}}^{\frac{1}{m}} + \int_m^{m+p} \right) \int_{\mathbb{R}^n} (Q_t(a))(x) (P_t(f))(x) (Q_t(g))(x) \frac{dxdt}{t} \right|^2 \\ &\leq \int_0^{\frac{1}{m}} \int_{\mathbb{R}^n} |(Q_t(a))(x) (P_t(f))(x)|^2 \frac{dxdt}{t} \\ &+ \int_m^{\infty} \int_{\mathbb{R}^n} |(Q_t(g))(x)|^2 \frac{dxdt}{t}. \end{aligned}$$

Then, each term converges to zero, as $m \to \infty$. According to (5.14), if L is the operator limit,

$$|(L(f), g)| \leq C \|a\|_{BMO} \|f\|_{L^2} \|g\|_{L^2} .$$

According to Definition 5, given $f, g \in \mathcal{D}$ with disjoint supports,

$$(L_m(f), g) = \int_{\mathbb{R}^{2n}} \mathcal{L}_m(x, y) f(y) g(x) \, dxdy.$$

So, taking $\lim_{m\to\infty}$, we get

$$(L(f), g) = \int_{\mathbb{R}^{2n}} \mathcal{L}(x, y) f(y) g(x) \, dxdy.$$

Thus, L is a CZO and $\|L\|_{L(L^2)} \leq C \|a\|_{BMO}$.

We are left to prove that, with an appropriate normalization, $L(1) = a$ and $L^t(1) = 0$. From (4.10), if $g \in \mathcal{D}_0$,

$$\begin{aligned} &\left(L_m^t(1), g\right) \\ &= (L_m(g), \theta) + \int_{\mathbb{R}^n} (1 - \theta(x)) \left(\int_{\mathbb{R}^n} \mathcal{L}_m(x, y) g(y) \, dy \right) dx. \end{aligned}$$

Since the sequence $\{L_m\}_{m\geq 1}$ of operators from $\mathcal{D}$ to $\mathcal{D}'$ converges to L, there is

$$\lim_{m\to\infty} (L_m(g), \theta) = (L(g), \theta).$$

According to (5.11), we can say that the sequence of kernels $\{\mathcal{L}_m\}_{m\geq 1}$ converges uniformly to $\mathcal{L}$ on $supp(1-\theta) \times supp(g) \subset \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$. So, there is

$$\begin{aligned} &\lim_{m\to\infty} \int_{\mathbb{R}^n} (1 - \theta(x)) \left(\int_{\mathbb{R}^n} \mathcal{L}_m(x, y) g(y) \, dy \right) dx \\ &= \int_{\mathbb{R}^n} (1 - \theta(x)) \left(\int_{\mathbb{R}^n} \mathcal{L}(x, y) g(y) \, dy \right) dx. \end{aligned}$$

That is to say,

$$\left(L^t(1), g\right) = \lim_{m\to\infty} \left(L_m^t(1), g\right). \tag{5.15}$$

Next, we observe that, in the weak sense,

$$L_m^t = \int_{\frac{1}{m}}^m P_t(Q_t(a))\, Q_t \frac{dt}{t}.$$

In fact, according to (5.5), for $f, g \in \mathcal{D}$,

$$\begin{aligned}(L_m(f), g) = \left(f, L_m^t(g)\right) &= \int_{\frac{1}{m}}^m \left(f, (Q_t(Q_t(a))\, P_t)^t(g)\right) \frac{dt}{t} \\ &= \int_{\frac{1}{m}}^m (f, P_t(Q_t(a)\, Q_t)(g)) \frac{dt}{t},\end{aligned}$$

where we have used the continuity of the transposition operation and the fact that P_t and Q_t are defined by convolution with the radial functions φ_t and ψ_t, respectively. Let us point out that the upper index t always indicates the transpose,

According to Remark 26, is

$$\left(L_m^t(1), g\right) = \lim_{j\to\infty} \left(L_m^t(\theta_j), g\right).$$

Since

$$\left(L_m^t(\theta_j), g\right) = \int_{\frac{1}{m}}^m (P_t(Q_t(a))\, Q_t(\theta_j), g) \frac{dt}{t}$$

and

$$\lim_{j\to\infty} Q_t(\theta_j) = Q_t(1) = 0,$$

we conclude that $L^t(1) = 0$.

Let us see that a suitable normalization will give us $L(1) = a$. In fact, similarly to (5.15), we can prove that there is

$$\lim_{m\to\infty} (L_m(1), g) = (L(1), g).$$

Now, for $m \geq 1$ fixed, using that $P_t(1) = 1$, we have

$$\begin{aligned}(L_m(1), g) &= \int_{\frac{1}{m}}^m (Q_t(Q_t(a))\, P_t(1), g) \frac{dt}{t} \\ &= \int_{\frac{1}{m}}^m \left(Q_t(a), Q_t^t(g)\right) \frac{dt}{t},\end{aligned} \tag{5.16}$$

where the upper index t, as always, denotes the transpose and should not be confused with the integration variable.

From (5.7),

$$\|Q_t(a)\|_{L^\infty} \leq C \|a\|_{BMO}.$$

Moreover, we claim that

$$\left\|Q_t^t(g)\right\|_{L^1} \leq \begin{cases} Ct & \text{for } 0 < t \leq 1, \\ \frac{C}{t} & \text{for } t \geq 1. \end{cases}$$

In fact, let K be a compact subset of $\mathbb{R}^n$ containing the support of $\psi_1 * g$. Then, if $0 < t \leq 1$, we can write, using that $\int_{\mathbb{R}^n} \psi(x)\, dx = 0$,

$$\begin{aligned}\int_{\mathbb{R}^n} \left|Q_t^t(g)(x)\right| dx &\leq \int_K \left(\int_{|z|\leq t} |\psi_t(z)|\, |g(x+z) - g(x)|\, dz\right) dx \\ &\leq \|\nabla g\|_{L^\infty} \int_K \left(\int_{|z|\leq t} |\psi_t(z)|\, |z|\, dz\right) dx \\ &\leq Ct.\end{aligned}$$

Now, if χ indicates the characteristic function of the ball $B(0,1)$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \left|Q_t^t(g)(x)\right| dx &\leq \int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} |\psi_t(y-x) - \psi_t(x)|\, |g(y)|\, \chi\left(\frac{y-x}{t}\right) dy\right| dx \\ &\leq \frac{1}{t^{n+1}} \int_{\mathbb{R}^n} \\ &\left(\int_{\mathbb{R}^n} \chi\left(\frac{y-x}{t}\right) \left|(\nabla\psi)\left((1-\lambda)\frac{x}{t} - \lambda\frac{y-x}{t}\right)\cdot y\right| |g(y)|\, dy\right) dx \\ &\leq \frac{1}{t^{n+1}} \|\nabla\psi\|_{L^\infty} \int_{\mathbb{R}^n} |y|\, |g(y)| \left(\int_{\mathbb{R}^n} \chi\left(\frac{y-x}{t}\right) dx\right) dy \\ &= \frac{C}{t}.\end{aligned}$$

All in all, we have proved that (5.16) converges to $\int_0^\infty \left(Q_t(a), Q_t^t(g)\right)_{L^\infty, L^1} \frac{dt}{t}$ as $m \to \infty$. So, we arrive at the identity

$$(L(1), g) = \int_0^\infty \left(Q_t(a), Q_t^t(g)\right)_{L^\infty, L^1} \frac{dt}{t},$$

or

$$\begin{aligned}(L(1), g) &= \int_0^\infty \left(Q_t^2(a), g\right)_{L^\infty, L^1} \frac{dt}{t} \\ &= \left(\int_0^\infty Q_t^2(a) \frac{dt}{t}, g\right)_{L^\infty, L^1}. \qquad (5.17)\end{aligned}$$

If we could say that $\int_0^\infty Q_t^2(a)\frac{dt}{t}=a$, we would be done. So, all that is left is to find an appropriate normalization factor. Via Fourier transform, we observe that

$$\int_0^\infty \left(\widehat{\psi}(t\xi)\right)^2\frac{dt}{t}=\int_0^\infty (F(t|\xi|))^2\frac{dt}{t}=\int_0^\infty (F(s))^2\frac{ds}{s},$$

which is a positive number for $\xi\neq 0$, since ψ radial implies that $\widehat{\psi}$ is radial also. On the other hand, from (5.17),

$$\begin{aligned}\int_{\mathbb{R}^n}(\psi_t*\psi_t*a)(x)\,g(x)\,dx&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}a(y)(\psi_t*\psi_t)(x-y)\,g(x)\,dydx\\&=\left(\widehat{a},\left(\widehat{\psi_t}\right)^2\widehat{g}\right)=\left(\widehat{a},\left(\widehat{\psi}(t\cdot)\right)^2\widehat{g}\right),\end{aligned}\tag{5.18}$$

where the parentheses in (5.18) indicate, for each $t>0$, the duality between S', the space of tempered distributions and S, the Schwartz space. Indeed, using (4.3) in Remark 22, we can see that $a\in S'$, or equivalently, $\widehat{a}\in S'$.

If we set

$$\left(\int_0^\infty (F(s))^2\frac{ds}{s}\right)^{-1}=A,$$

and integrate both sides of the equality

$$\int_{\mathbb{R}^n}(\psi_t*\psi_t*a)(x)\,g(x)\,dx=\left(\widehat{a},\left(\widehat{\psi}(t\cdot)\right)^2\widehat{g}\right)$$

with respect to t, the function $A\psi$ will satisfy

$$\left(\int_0^\infty Q_t^2(a)\frac{dt}{t},g\right)=(a,g),$$

for all $g\in\mathcal{D}_0$, or

$$L(1)=a,$$

according to Remark 28.

This completes the proof of the lemma. □

Remark 36. *The proof of $L(1)=a$, borrows from ([25], pp. 17 and 43).*

6 A continuous proof of the T(1) Theorem for the modified operator

Here is the version of the T(1) Theorem that we are left to prove:

Theorem 37. *Let T be an operator associated with a standard kernel. Assume that T satisfies the following three conditions:*

1. *T has the WBP,*

2. $T(1) = 0$,

3. $T^t(1) = 0$.

Then, T is a CZO.

In the original proof of this theorem [38], the main tool is the Cottlar-Knapp-Stein lemma. Instead, we will use here the continuous version of the lemma, due to A. P. Calderón and R. Vaillancourt [20], that we state now:

Lemma 38. *([20]) Let $(\mathcal{T}, \Sigma, \mu)$ be a measure space and let $\{A_t\}_{t\in\mathcal{T}}$ be a family of operators in $L(H)$, the space of linear and continuous operators from a Hilbert space H into itself. Suppose that the map*

$$\mathcal{T} \longmapsto L(H)$$
$$t \mapsto A_t$$

is μ weakly-measurable, that is, for $f, g \in H$, the map $\langle A_t f, g\rangle_H$ is μ measurable. Moreover, suppose that the following conditions hold:

a) *There exist $\mu \times \mu$ measurable functions $h_1(t,s)$ and $h_2(t,s)$, defined on $\mathcal{T} \times \mathcal{T}$, satisfying the estimates*

$$\left\|A_t^t A_s\right\|_{L(H)} \le h_1^2(t,s), \tag{6.1}$$
$$\left\|A_t A_s^t\right\|_{L(H)} \le h_2^2(t,s). \tag{6.2}$$

where, as always, the upper index denotes the transpose.

b) *The integral $\int_{\mathcal{T}} h_1(t,r)\, h_2(r,s)\, d\mu(r)$ is a $\mu \times \mu$ measurable function $h(t,s)$ and $\int_{\mathcal{T}} h(t,s)\, f(s)\, d\mu(s)$ defines a linear and continuous operator B from $L^2(\mathcal{T}, \mu)$ into itself. Furthermore, for some $C > 0$,*

$$\|B\|_{L(L^2(\mathcal{T},\mu))} \le C^2.$$

Then, for each $J \in \Sigma$ with finite μ measure, the operator valued integral $\int_J A_t d\mu(t)$ belongs to $L(H)$ and

$$\left\|\int_J A_t d\mu(t)\right\|_{L(H)} \le C. \tag{6.3}$$

With this powerful tool in hand, we are ready to prove Theorem 37.

Proof. First of all, according to the first part of Lemma 3, we can assume that the kernel of the operator T satisfies Definition 1 with $\delta < 1$. The purpose of this assumption is to simplify some of the estimates needed later on.

Let $\varphi \in \mathcal{D}$ be a radial function such that $supp\,(\varphi) \subseteq \{|x| \le 2\}$, $\varphi(x) = 1$ when $|x| \le 1$, $0 \le \varphi(x) \le 1$ for all $x \in \mathbb{R}^n$ and $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$. As before, let $\varphi_t(x) = \frac{1}{t^n}\varphi\left(\frac{x}{t}\right)$ and let P_t be the operator acting on $\mathcal{D}$ as the convolution with φ_t.

Given $m \ge 1$, $f, g \in \mathcal{D}$, we define the operator

$$(T_m(f), g) = \int_{\frac{1}{m}}^{m} \left(\frac{d}{dt}(P_t T P_t)(f), g\right) dt \tag{6.4}$$

$$= ((P_m T P_m)(f), g) - \left(\left(P_{1/m} T P_{1/m}\right)(f), g\right). \tag{6.5}$$

We recall that, unless otherwise noted, $(,)$ still indicates the $(\mathcal{D}', \mathcal{D})$ duality.

Let us consider each term in (6.5) separately. Since $\left\{P_{1/m}\right\}_{m\ge1}$ is an approximation of the identity and T is continuous from $\mathcal{D}$ into $\mathcal{D}'$, there is

$$\lim_{m\to\infty}\left(\left(P_{1/m} T P_{1/m}\right)(f), g\right) = \lim_{m\to\infty}\left(T P_{1/m}(f), P_{1/m} g\right) = (T(f), g).$$

Next, we use the WBP to show that $(P_m T P_m(f), g) = (T P_m(f), P_m(g))$ converges to zero as $m \to \infty$. In fact,

$$(T(P_m(f)), P_m(g)) = \left(T_v\left(\int_{\mathbb{R}^n} \varphi_m(v-y) f(y)\,dy\right), \int_{\mathbb{R}^n} \varphi_m(u-x) g(x)\,dx\right)$$
$$= \int_{\mathbb{R}^{2n}} (T_v(\varphi_m(v-y)), \varphi_m(u-x)) f(y) g(x)\,dydx,$$

where T_v indicates that the operator T is acting on the variable v. For x, y fixed, we can write

$$(T_v(\varphi_m(v-y)), \varphi_m(u-x)) = \left(T_v\left(\frac{1}{m^n}\varphi\left(\frac{v-y}{m}\right)\right), \frac{1}{m^n}\varphi\left(\frac{y-x}{m} + \frac{u-y}{m}\right)\right).$$

We observe that, when $x \in supp\,(g)$, $y \in supp\,(f)$, and $m \ge 1$, the family of functions $\left\{\varphi\left(\frac{y-x}{m} + \cdot\right)\right\}_{x,y,m}$ is a bounded subset of $\mathcal{D}$. Thus, since the operator T has the WBP,

$$|(T_v(\varphi_m(v-y)), \varphi_m(u-x))| \le \frac{C}{m^n},$$

uniformly with respect to $x \in supp\,(g)$ and $y \in supp\,(f)$. That is to say, there exists

$$\lim_{m\to\infty}(T P_m(f), P_m(g)) = 0.$$

Let us take a closer look at the integral in (6.4). In the weak sense,

$$\frac{d}{dt}(P_t T P_t) = \left(\frac{d}{dt}\varphi_t * \cdot\right) T P_t + P_t T\left(\frac{d}{dt}\varphi_t * \cdot\right).$$

As it was indicated in the proof of Lemma 32,

$$\frac{d\varphi_t}{dt}(x) = -\frac{1}{t}\left(\sum_{j=1}^{n} \partial_{x_j}(x_j\varphi)\right)_t = -\frac{1}{t}\psi_t(x),$$

where $\int_{\mathbb{R}^n} \psi(x)\,dx = 0$.

So, if Q_t is the operator of convolution with ψ_t, we have proved that, in the weak sense,

$$T = \int_0^\infty (Q_t T P_t + P_t T Q_t)\frac{dt}{t}. \tag{6.6}$$

We will show that the two families of operators, $\{Q_t T P_t\}_{t>0}$ and $\{P_t T Q_t\}_{t>0}$, satisfy the hypotheses of Lemma 38, with $H = L^2$, $\mathcal{T} = (0,\infty)$, and $d\mu = \frac{dt}{t}$. Actually, we will only consider $\{Q_t T P_t\}_{t>0}$, since $P_t T Q_t = \left(Q_t T^t P_t\right)^t$ and both operators, T and T^t, satisfy the same conditions. Once again, the upper index t indicates, as always, the transpose.

For $t > 0$ fixed, $Q_t T P_t$ is certainly a linear and continuous operator from $\mathcal{D}$ into $\mathcal{D}'$. Moreover, it is an integral operator whose kernel $\mathcal{K}(x,y,t)$ is the $C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+)$ function given by

$$\mathcal{K}(x,y,t) = (T_v(\varphi_t(v-y)), \psi_t(u-x)). \tag{6.7}$$

We claim that $\mathcal{K}(x,y,t)$ satisfies the following Poisson like conditions:

i)

$$|\mathcal{K}(x,y,t)| \le C\frac{t^\delta}{(|x-y|+t)^{n+\delta}},$$

ii)

$$|\mathcal{K}(x,y,t) - \mathcal{K}(z,y,t)| + |\mathcal{K}(y,x,t) - \mathcal{K}(y,z,t)| \le C\frac{|x-z|}{t}\frac{t^\delta}{(|y-z|+t)^{n+\delta}},$$

if $2|x-z| \le |y-z| + t$, where $0 < \delta \le 1$ is the same parameter as in Definition 1.

as well as the cancellation conditions

iii)

$$\int_{\mathbb{R}^n} \mathcal{K}(x,y,t)\,dx = \int_{\mathbb{R}^n} \mathcal{K}(x,y,t)\,dy = 0.$$

To prove i) we will consider two cases, $|x-y| \leq at$ and $|x-y| \geq at$, for some $a>0$ to be chosen. We begin with the case $|x-y| \leq at$. As before, we can write

$$\psi_t(u-x) = \frac{1}{t^n}\psi\left(w + \frac{u-y}{t}\right),$$

where $w = \frac{y-x}{t}$. Thus, the family $\{\varphi, \psi(w+\cdot)\}_{|w|\leq a}$ is a bounded subset of $\mathcal{D}$. Since T has the WBP, we conclude from (6.7) that

$$|\mathcal{K}(x,y,t)| \leq \frac{C}{t^n} \leq C\frac{t^\delta}{(|x-y|+t)^{n+\delta}}.$$

Now let us assume that $|x-y| \geq at$. Since $supp\,(\varphi_t(v-y)) \subseteq \{|v-y| \leq 2t\}$, and $supp\,(\psi_t(u-x)) \subseteq \{|u-x| \leq 2t\}$, they will be disjoint if $a \geq 5$. Thus, we can write

$$\mathcal{K}(x,y,t) = \int_{\mathbb{R}^{2n}} k(u,v)\,\varphi_t(v-y)\,\psi_t(u-x)\,dvdu$$

where $k(x,y)$ is the standard kernel of T, uniquely determined by T according to Remark 8.

Since $\int_{\mathbb{R}^n}\psi(x)\,dx = 0$, we can also write

$$\mathcal{K}(x,y,t) = \int_{\mathbb{R}^{2n}} (k(u,v) - k(x,v))\,\varphi_t(v-y)\,\psi_t(u-x)\,dvdu.$$

Our assumptions imply that

$$|x-v| \geq |x-y| - |y-v| \geq (a-2)\,t.$$

So, $(a-2)\,t \geq 2\,|x-u|$ if $a \geq 6$. We settle on $a=6$. Hence, $|x-v| \geq 2\,|x-u|$. Thus, by condition 2) in Definition.1,

$$|k(u,v) - k(x,v)| \leq C\frac{|u-x|^\delta}{|v-x|^{n+\delta}}.$$

Moreover,

$$|v-x| \geq |x-y| - |v-y| \geq |x-y| - 2t \geq \frac{2}{3}\,|x-y|\,.$$

Then,

$$|k(u,v) - k(x,v)| \leq C\frac{|x-u|^\delta}{|x-y|^{n+\delta}},$$

from which we can write,

$$|\mathcal{K}(x,y,t)| \leq \frac{C}{|x-y|^{n+\delta}}\int_{\mathbb{R}^{2n}} |u-x|^\delta\,\varphi_t(v-y)\,|\psi_t(u-x)|\,dvdu. \tag{6.8}$$

Since $|u-x| \leq 2t$ on $supp\,(\psi_t\,(u-x))$ and we are assuming $|x-y| \geq 6t$, we have

$$\begin{aligned} |x-y| &= \frac{1}{2}|x-y| + \frac{1}{2}|x-y| \\ &\geq \frac{1}{2}|x-y| + 3t \geq \frac{1}{2}(|x-y|+t)\,. \end{aligned}$$

Substituting in (6.8) we get i). Let us prove ii).

We write

$$\begin{aligned} \mathcal{K}(x,y,t) - \mathcal{K}(z,y,t) &= \int_0^1 (\nabla_x \mathcal{K})(z+s(x-z),y,t)\cdot(x-z)\,ds \\ &= \int_0^1 (T_v(\varphi_t(v-y)), -(\nabla_x(\psi_t))(u-z-s(x-z)))\cdot(x-z)\,ds, \end{aligned}$$

and we observe that the proof of estimate i) still holds if we replace in (6.7) the function ψ with any other function $\widetilde{\psi}$ in D_0. So, we apply estimate i) to the kernel $\left(T_v(\varphi_t(v-y)), -\frac{1}{t}\left(\partial_{x_j}\psi\right)_t(u-z-s(x-z))\right)$ for each $1 \leq j \leq n$, with $\widetilde{\psi} = \partial_{x_j}\psi$, to get

$$|\mathcal{K}(x,y,t) - \mathcal{K}(z,y,t)| \leq C\frac{|x-z|}{t}\int_0^1 \frac{t^{\delta}}{(|z+s(x-z)-y|+t)^{n+\delta}}ds.$$

Since we are assuming that $2|x-z| < |y-z| + t$, we have

$$\begin{aligned} |z+s(x-z)-y| + t &\geq |y-z| - |x-z| + t \\ &\geq \frac{1}{2}(|y-z|+t)\,, \end{aligned}$$

giving us the first half of ii). The proof of the other half follows the same idea, so we will omit it.

Finally, we prove iii).

Since the operator T is linear and continuous from $\mathcal{D}$ into $\mathcal{D}'$ and $\int_{\mathbb{R}^n}\psi(x)\,dx = 0$, we get

$$\begin{aligned} \int_{\mathbb{R}^n}\mathcal{K}(x,y,t)\,dx &= \int_{\mathbb{R}^n}(T_v(\varphi_t(v-y)), \psi_t(u-x))\,dx \\ &= \left(T_v(\varphi_t(v-y)), \int_{\mathbb{R}^n}\psi_t(u-x)\,dx\right) = 0 \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^n}\mathcal{K}(x,y,t)\,dy &= \int_{\mathbb{R}^n}(T_v(\varphi_t(v-y)), \psi_t(u-x))\,dy = \left(T_v\left(\int_{\mathbb{R}^n}\varphi_t(v-y)\,dy\right), \psi_t(u-x)\right) \\ &= (T(1), \psi_t(u-x)) = 0. \end{aligned}$$

This completes the proof of the estimates for the kernel of the operator Q_tTP_t. As we said before, the estimates for the kernel of P_tTQ_t follow, since T and T^t have the same properties.

Let $A_t = Q_tTP_t$. To prove that (6.1) and (6.2) hold, we will estimate $\left\|A_t^t A_s\right\|_{L(L^2)}$ for $t \leq s$, since the other estimates are proved in a similar fashion. As always, the upper index t denotes the transpose of the operator. The condition $t \leq s$ refers to the subindexes.

Let $\mathcal{A}(x, y, t, s)$ be the kernel of the operator $A_t^t A_s$. If we obtain estimates for

$$\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |\mathcal{A}(x, y, t, s)| \, dy$$

and

$$\sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} |\mathcal{A}(x, y, t, s)| \, dx,$$

these estimates will also serve as bounds for the norm of $A_t^t A_s$ as a linear and continuous operator from L^∞ to itself and from L^1 to itself, respectively. Thus, by Marcinkiewicz interpolation, we will have an estimate for $\left\|A_t^t A_s\right\|_{L(L^2)}$.

Now,

$$\int_{\mathbb{R}^n} |\mathcal{A}(x, y, t, s)| \, dy = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \mathcal{K}^t(x, z, t) \, \mathcal{K}(z, y, s) \, dz \right| dy,$$

and, since $\int_{\mathbb{R}^n} \mathcal{K}^t(x, z, t) \, dz = 0$, we can write the integral above as

$$\begin{aligned} &\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \mathcal{K}^t(x, z, t) \left(\mathcal{K}(z, y, s) - \mathcal{K}(x, y, s)\right) dz \right| dy \\ &\leq \int_{\mathbb{R}^n} \int_{2|x-z| \leq s} \left|\mathcal{K}^t(x, z, t)\right| \left|\left(\mathcal{K}(z, y, s) - \mathcal{K}(x, y, s)\right)\right| dz dy \qquad (6.9) \\ &+ \int_{\mathbb{R}^n} \int_{2|x-z| \geq s} \left|\mathcal{K}^t(x, z, t)\right| \left(|\mathcal{K}(z, y, s)| + |\mathcal{K}(x, y, s)|\right) dz dy. \qquad (6.10) \end{aligned}$$

We estimate (6.9) and (6.10), separately, assuming $t \leq s$. For (6.9), we use i) in the first factor and ii) in the second, observing that $2|x - z| \leq s$ implies

$2|x-z| \le |y-z| + s$. Thus, (6.9) is bounded by

$$C \int_{\mathbb{R}^n} \int_{2|x-z|\le s} \frac{t^\delta}{(|x-z|+t)^{n+\delta}} \frac{|x-z|}{s} \frac{s^\delta}{(|y-z|+s)^{n+\delta}} dzdy$$
$$= C \int_{2|x-z|\le s} \frac{t^\delta}{(|x-z|+t)^{n+\delta}} \frac{|x-z|}{s} \left(\int_{\mathbb{R}^n} \frac{s^\delta}{(|y-z|+s)^{n+\delta}} dy \right) dz$$
$$= \frac{C}{s} \int_{2|x-z|\le s} \frac{t^\delta |x-z|}{(|x-z|+t)^{n+\delta}} dz = \frac{C}{s} \int_0^{\frac{s}{2}} \frac{t^\delta r^n}{(r+t)^{n+\delta}} dr$$
$$= C\frac{t}{s} \int_0^{\frac{s}{2t}} \frac{r^n}{(1+r)^{n+\delta}} dr \le C\frac{t}{s} \le C \left(\frac{t}{s}\right)^\delta,$$

where we recall that $0 < \delta < 1$.

Now, we use i) to estimate (6.10), obtaining the following bound:

$$C \int_{\mathbb{R}^n} \int_{2|x-z|\ge s} \frac{t^\delta}{(|x-z|+t)^{n+\delta}} \left(\frac{s^\delta}{(|y-z|+s)^{n+\delta}} + \frac{s^\delta}{(|x-y|+s)^{n+\delta}} \right) dzdy. \tag{6.11}$$

By a change of variables, we have

$$\int_{\mathbb{R}^n} \left(\frac{s^\delta}{(|y-z|+s)^{n+\delta}} + \frac{s^\delta}{(|x-y|+s)^{n+\delta}} \right) dy = C,$$

so, (6.11) reduces to

$$C \int_{2|x-z|\ge s} \frac{t^\delta}{(|x-z|+t)^{n+\delta}} dz \le C \int_{\frac{s}{2t}}^\infty \xi^{-1-\delta} d\xi$$
$$= C \left(\frac{t}{s}\right)^\delta.$$

Since the estimates in all the other cases will be the same, we finally obtain

$$h_1^2(t,s) = h_2^2(t,s) = \begin{cases} C\left(\frac{t}{s}\right)^\delta & \text{for } t \le s, \\ C\left(\frac{s}{t}\right)^\delta & \text{for } t \ge s, \end{cases} \tag{6.12}$$

which proves a) in the hypotheses of Lemma 38.

To show that b) holds, we need to consider

$$h(t,s) = \int_0^\infty h_1(t,r)\, h_2(r,s) \frac{dr}{r}.$$

We want to show that $h(t,s)$ defines, by integration, an operator B that is continuous from $L^2\left((0,\infty), \frac{dt}{t}\right)$ into itself. As before, it suffices to estimate

$$\sup_{t>0} \int_0^\infty |h(t,s)| \frac{ds}{s}$$

and

$$\sup_{s>0}\int_0^\infty |h(t,s)|\,\frac{dt}{t},$$

using interpolation afterwards.

Let us first estimate $h(t,s)$. According to (6.12), if $t \leq s$,

$$\begin{aligned} h(t,s) &= C\int_0^t \left(\frac{r}{t}\right)^{\frac{\delta}{2}}\left(\frac{r}{s}\right)^{\frac{\delta}{2}}\frac{dr}{r} \\ &+ C\int_t^s \left(\frac{t}{r}\right)^{\frac{\delta}{2}}\left(\frac{r}{s}\right)^{\frac{\delta}{2}}\frac{dr}{r} + C\int_s^\infty \left(\frac{t}{r}\right)^{\frac{\delta}{2}}\left(\frac{s}{r}\right)^{\frac{\delta}{2}}\frac{dr}{r} \\ &\leq C\left(\frac{t}{s}\right)^{\frac{\delta}{2}}\ln\left(\frac{s}{t}+1\right). \end{aligned}$$

On the other hand, if $t \geq s$,

$$\begin{aligned} h(t,s) &= C\int_0^s \left(\frac{r}{t}\right)^{\frac{\delta}{2}}\left(\frac{r}{s}\right)^{\frac{\delta}{2}}\frac{dr}{r} + C\int_s^t \left(\frac{r}{t}\right)^{\frac{\delta}{2}}\left(\frac{s}{r}\right)^{\frac{\delta}{2}}\frac{dr}{r} \\ &+ C\int_t^\infty \left(\frac{t}{r}\right)^{\frac{\delta}{2}}\left(\frac{s}{r}\right)^{\frac{\delta}{2}}\frac{dr}{r} \\ &\leq C\left(\frac{s}{t}\right)^{\frac{\delta}{2}}\ln\left(\frac{t}{s}+1\right). \end{aligned}$$

Thus,

$$\int_0^\infty |h(t,s)|\,\frac{dt}{t} \leq C\int_0^s \left(\frac{t}{s}\right)^{\frac{\delta}{2}}\ln\left(\frac{s}{t}+1\right)\frac{dt}{t} + C\int_s^\infty \left(\frac{s}{t}\right)^{\frac{\delta}{2}}\ln\left(\frac{t}{s}+1\right)\frac{dt}{t} \leq C.$$

and, likewise,

$$\int_0^\infty |h(t,s)|\,\frac{ds}{s} \leq C,$$

where the positive constant C only depends on specific parameters such as n, δ, etc. So, we have proved b).

Then, Lemma 38 tells us that, for each $m \geq 1$, the operator $T_m = \int_{\frac{1}{m}}^{m} Q_t T P_t \frac{dt}{t}$ is continuous on L^2 and $\|T_m\|_{L(L^2)} \leq \sqrt{C}$, independently of m.

Since we have proved already (see (6.6)) that the sequence $\{T_m\}_{m\geq 1}$ converges in the weak sense to T, we conclude that T is also continuous on L^2. That is to say, T is a CZO.

Thus, we have completed the proof of the T(1) Theorem. □

7 The applicability of the T(1) Theorem

For the examples discussed in Section 3, the proof of the L^2 continuity relies, for the most part, on *ad-hoc* techniques. Thus, one main interest of the T(1) Theorem lies on its role as a unifier of many particular cases. Let us look at some of them, in the context of this theorem.

We begin with the case of pseudo-differential operators in $L^0_{1,1}$. Taking $\delta = 1$, makes this, so called "wrong" class, quite special ([9], [53] and [54]).

Summing up what we have said in previous sections, an operator L in $L^0_{1,1}$ will be continuous on L^2 if and only if $L^t(1) \in BMO$ (see Remark 30). In particular, if we look once again at the symbol (4.5) of the operator due to Ching, we have $L(1) = p(x,0) = 0$. As for the condition $L^t(1) \in BMO$, we will first calculate $L^t(1)$ using Remark 26. According to (4.11), if we fix $g \in \mathcal{D}_0$,

$$\left(L^t(1), g\right) = \lim_{j\to\infty} (\theta_j, L(g)),$$

where we assume that $\theta \in \mathcal{D}$ is a radial function, $\theta(x) = 1$ for $|x| < 1$, $\theta(x) = 0$ for $|x| > 2$, and $\theta_j(x) = \theta\left(\frac{x}{j}\right)$. Observe that for some j_0, the function θ will be identically one on a neighborhood of $supp\,(g)$. In order to drop the factor e_1 from the exponent in (4.5), we will work in $\mathbb{R}$. For $j \geq j_0$ fixed, we write

$$(\theta_j, T(g)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i x\xi}\left[\sum_{k\geq 0} a_k e^{2\pi i 5^k 3x}\psi\left(\frac{\xi}{5^k}\right)\right]\widehat{g}(\xi)\,\theta\left(\frac{x}{j}\right)d\xi dx. \tag{7.1}$$

The series (4.5) converges, for instance, in the sense of $\mathcal{O}_M$, the space of functions slowly increasing at infinity ([6], p. 128). So, (7.1) is equal to

$$\sum_{k\geq 0} a_k \int_{-\infty}^{\infty} j\widehat{\theta}(j\eta)\,\psi\left(\frac{\eta}{5^k}+3\right)\widehat{g}\left(\eta+5^k 3\right)d\eta. \tag{7.2}$$

Since

$$\sum_{l\geq 0}\frac{1}{\left(1+|\eta-l|^2\right)^s} \leq C$$

for $\eta \in \mathbb{R}$ and $s > \frac{n}{2}$, we can write (7.2) as

$$\int_{-\infty}^{\infty} j\widehat{\theta}(j\eta)\left(\sum_{k\geq 0} a_k\psi\left(\frac{\eta}{5^k}+3\right)\widehat{g}\left(\eta+5^k 3\right)\right)d\eta,$$

where the series, uniformly convergent on compact subsets of $\mathbb{R}$, defines a continuous function of η. Thus, the integral above converges, as $j \to \infty$, to

$$\sum_{k\geq 0} a_k\widehat{g}\left(5^k 3\right) = \sum_{k\geq 0} a_k \int_{-\infty}^{\infty} e^{2\pi i x 5^k 3} g(x)\,dx.$$

So, at least in the sense of $\mathcal{D}'$,

$$L^t(1) = \sum_{k\geq 0} a_k e^{2\pi i x 5^k 3}, \tag{7.3}$$

where the series converges in the sense of $\mathcal{D}'$. It is a trigonometric lacunary series. That is to say, is a series of the form

$$\sum_{k\geq 0} a_k e^{2\pi i x m_k},$$

where $\frac{m_{k+1}}{m_k} \geq 1+\varepsilon$, for $k \geq k_0$ and some $\varepsilon > 0$. For a brief account on lacunary series, or functions, we refer to [91], p. 98, 5.501). Going back to (7.3), it defines a function in BMO if and only if the sequence $\{a_k\}_{k\geq 0}$ is square summable (see [77], p. 178, further results 6.3). As a consequence, if $\sum\limits_{k\geq 0} |a_k|^2 = \infty$, the operator L satisfies conditions 1) and 2) but not 3), while L^t satisfies conditions 1) and 3) but not 2) and, finally, $L + L^t$ neither satisfies 2) nor 3), in the statement of Theorem 29.

Many of the particular operators known in the late seventies to be continuous on L^2, are associated with antisymmetric standard kernels (see Section 3). At that time, Calderón asked under what conditions, an operator associated with an antisymmetric standard kernel, is continuous on L^2. We will look at this question from the point of view of the T(1) Theorem.

To begin, an antisymmetric standard kernel, extends to the distribution kernel of the operator, as a principal value integral. Indeed, if we fix $\varphi, \psi \in \mathcal{D}$, we can write, for $\varepsilon > 0$ fixed,

$$\begin{aligned} &\int_{|x-y|>\varepsilon} k(x,y)\,\varphi(y)\,\psi(x)\,dydx \\ &= \frac{1}{2}\int_{|x-y|>\varepsilon} k(x,y)\,(\varphi(y)\,\psi(x) - \varphi(x)\,\psi(y))\,dydx. \end{aligned} \tag{7.4}$$

We have the estimate

$$|k(x,y)\,(\varphi(y)\,\psi(x) - \varphi(x)\,\psi(y))| \leq C\frac{|x-y|}{|x-y|^n}\,\|\nabla\varphi\|_{L^\infty}\,\|\nabla\psi\|_{L^\infty}\,\chi(x,y),$$

where χ denotes the characteristic function of the support of $\varphi(y)\,\psi(x) - \varphi(x)\,\psi(y)$ in $\mathbb{R}^n \times \mathbb{R}^n$. This shows that (7.4) has limit as $\varepsilon \to 0$. In a similar manner, we could prove that the operator thus defined, has the WBP. Since T^t is essentially the same as T, the T(1) Theorem will then say that T is a CZO if and only if $T(1) \in BMO$.

Not for every operator T associated with an antisymmetric kernel is true that $T(1) \in BMO$. For example, we can consider an operator closely related to the operator L defined by (5.6) in Section 5. We take a equal, for instance, to the series in (7.3), which does not define a locally integrable function, so it cannot belong to BMO. For the details, we refer to ([25], p. 50).

Most of the examples considered in Section 3, fall into the category of operators associated with antisymmetric kernels. For instance, let us consider the m-th commutator (3.12)

$$\mathcal{C}_m(h)(x) = pv \int_{-\infty}^{\infty} \left(\frac{a(x) - a(y)}{x - y} \right)^m \frac{h(y)}{x - y} xdy,$$

where, to avoid confusion, we have ignored the factor C_m. The principal value is interpreted in the sense explained above. We will show by induction that $\mathcal{C}_m$ is a CZO, for every $m = 0, 1, 2, ...$

First of all, $\mathcal{C}_0$ differs in a multiplicative constant from the Hilbert transform H and $H(1) = 0$. In fact, we proceed as above.

$$\begin{aligned}(\theta_j, H(g)) &= i \int_{-\infty}^{\infty} \theta_j(x) \int_{-\infty}^{\infty} e^{-2\pi i x \xi} sgn(\xi) \widehat{g}(\xi) d\xi \\ &= i \int_{-\infty}^{\infty} j\widehat{\theta}(j\xi) sgn(\xi) \widehat{g}(\xi) d\xi,\end{aligned}$$

which converges, for $j \to \infty$, to $i\widehat{g}(0) = 0$.

Similarly, we could show that

$$\mathcal{C}_1(1) = \mathcal{C}_0(a').$$

Since $a' \in L^\infty(\mathbb{R})$ and $\mathcal{C}_0$ is CZO, Remark 31 shows that $\mathcal{C}_0(a')$ is well defined, in the appropriate sense, and it belongs to BMO. So, $\mathcal{C}_1$ is a CZO.

In general, if $m \geq 2$, it is true ([38], p. 381) that

$$\mathcal{C}_m(1) = \mathcal{C}_{m-1}(a').$$

So, assuming that $\mathcal{C}_{m-1}$ is a CZO, we conclude that $\mathcal{C}_m(1) \in BMO$, so $\mathcal{C}_m$ is a CZO as well. Furthermore ([25], p. 56, Theorem 7), there exists $C > 0$ so that

$$\|\mathcal{C}_m\|_{L(L^2(\mathbb{R}))} \leq C^m \|a'\|_{L^\infty(\mathbb{R})}^m. \tag{7.5}$$

We showed in Section 3 how several of the examples could be reduced to combinations of commutators. However, this approach does not always yield the best results on L^2 continuity. Let us consider, for instance, the Cauchy integral on a Lipschitz curve

$$\mathcal{C}(f)(x) = pv \int_{-\infty}^{\infty} \frac{f(y)}{y - x} \frac{dy}{1 + i\frac{a(y) - a(x)}{y - x}},$$

where, once again, the principal value is interpreted in the sense explained above.

Formally,

$$\mathcal{C}(f)(x) = -i^m \sum_{m\geq 0} \mathcal{C}_m(f)(x).$$

We just proved that each term is a CZO. However, (7.5) will not yield a convergent series, unless $\|a'\|_{L^\infty(\mathbb{R})}$ is sufficiently small. From there, a localization argument will give the general result [36].

David, Journé and Semmes [39] have proved a version of the T(1) Theorem, in which the operator and its transpose are tested on a para-accretive function b. This is the so called T(b) Theorem (see also [25], Chapter IV (5)). In the case of the Cauchy integral, the previously neglected factor $1 + ia'$ plays a central role as the para-accretive function. For more on the Cauchy integral, we refer to [68], [34] and [37].

This ends the section on the applicability of the T(1) Theorem. We dedicate a last section to ...

8 The conditions on the kernel

As we mentioned in the introduction, the conditions on the kernel k given in Definition 1 are one of the possible classical versions (see, for instance, [4], [72]). However, other formulations have been used in the literature (see, for instance, [38], [64], [65], [44]), not only to prove variations and extensions of the T(1) Theorem, but for other purposes as well. We believe that it would be of interest to make precise the relationship between a few of the most common formulations, so we dedicate this last section to that purpose. We begin with a definition that gathers three pointwise versions.

Definition 39. *1. A continuous function $k : \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \to \mathbb{C}$ satisfies condition (R) if ([72])*

$$|k(x,y) - k(x,z)| \leq C_1 \frac{|y-z|^\delta}{|x-z|^{n+\delta}},$$

for a fixed $0 < \delta \leq 1$ and for $|x-z| \geq c_1 |y-z|$, with $C_1 > 0, c_1 > 1$.

2. A continuous function $k : \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \to \mathbb{C}$ satisfies condition (D) if ([38])

$$\left|k(x,y) - k\left(x,y'\right)\right| \leq C_2 \frac{|y-y'|^\delta}{|x-y|^{n+\delta}},$$

for a fixed $0 < \delta \leq 1$ and for $|x-y| \geq c_2 |y-y'|$, with $C_2 > 0, c_2 > 1$.

3. A continuous function $k : \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \to \mathbb{C}$, satisfies condition (G) if ([44])

$$\left|k(x,y) - k\left(x,y'\right)\right| \leq C_3 \frac{|y-y'|^\delta}{(|x-y| + |x-y'|)^{n+\delta}},$$

for a fixed $0 < \delta \leq 1$ *and for* $\max(|x-y|, |x-y'|) \geq c_3 |y-y'|$*, with* $C_3 > 0, c_3 > 1$.

As indicated in Definition 39, the original proof of the T(1) Theorem uses condition (D) on k and k^t. The proposition that follows makes precise the relationship between these conditions. The proof involves quite a few elementary calculations, that we present in some detail.

Proposition 40. *Any of the conditions* (R)*,* (D) *or* (G) *implies any other of these conditions, for a fixed* $0 < \delta \leq 1$ *and appropriate constants* $C_i > 0, c_i > 1$*,* $i = 1, 2, 3$.

Proof. Let us assume that k satisfies condition (R) and, for a constant $c_2 > 1$ to be chosen, let us fix x, y, y' with $|x-y| \geq c_2 |y-y'|$. Then, if we pick $z = y'$,

$$\begin{aligned} |x-z| = \left|x-y'\right| &\geq |x-y| - \left|y-y'\right| \geq \left(1 - \frac{1}{c_2}\right)|x-y| \\ &\geq c_2\left(1-\frac{1}{c_2}\right)\left|y-y'\right| = (c_2-1)|y-z| \\ &\geq c_1 |y-z|, \end{aligned}$$

provided that $c_2 \geq c_1 + 1$. So, we can use condition (R) to write

$$\left|k(x,y) - k\left(x,y'\right)\right| \leq C_1 \frac{|y-y'|^{\delta}}{|x-y'|^{n+\delta}}.$$

Since we showed that $|x-y'| \geq \left(1-\frac{1}{c_2}\right)|x-y|$, we can write

$$\left|k(x,y) - k\left(x,y'\right)\right| \leq \frac{c_2 C_1}{c_2 - 1}\frac{|y-y'|^{\delta}}{|x-y|^{n+\delta}},$$

which is condition (D), with $C_2 \geq \frac{c_2 C_1}{c_2-1}$.

Next, we assume that k satisfies condition (D) and, for a constant $c_3 > 1$ to be chosen, we fix x, y, y' so $\max(|x-y|, |x-y'|) \geq c_3 |y-y'|$. If $|x-y| \geq |x-y'|$, then $|x-y| \geq c_3 |y-y'|$, so using condition (D),

$$\left|k(x,y) - k\left(x,y'\right)\right| \leq C_2 \frac{|y-y'|^{\delta}}{|x-y|^{n+\delta}}.$$

Now,

$$|x-y| \geq \frac{1}{2}\left(|x-y| + \left|x-y'\right|\right),$$

so,

$$\left|k(x,y) - k\left(x,y'\right)\right| \leq 2^{n+\delta} C_2 \frac{|y-y'|^{\delta}}{(|x-y| + |x-y'|)^{n+\delta}}.$$

If $|x-y'| \geq |x-y|$, then, $|x-y'| \geq c_3 |y-y'|$. Consequently,

$$|x-y| \geq |x-y'| - |y-y|' \geq (c_3-1)|y-y'|.$$

If $c_3 \geq c_2+1$, we can use condition (D),

$$|k(x,y)-k(x,y')| \leq C_2 \frac{|y-y'|^\delta}{|x-y|^{n+\delta}}.$$

But,

$$|x-y| \geq |x-y'| - |y-y'| \geq \left(1-\frac{1}{c_3}\right)|x-y'|,$$

so we can write,

$$\begin{aligned}|k(x,y)-k(x,y')| &\leq C_2 \frac{c_3}{c_3-1}\frac{|y-y'|^\delta}{|x-y'|^{n+\delta}} \\ &\leq 2^{n+\delta} C_2 \frac{c_3}{c_3-1}\frac{|y-y'|^\delta}{(|x-y|+|x-y'|)^{n+\delta}},\end{aligned}$$

So, condition (G) holds, with $C_3 \geq 2^{n+\delta} C_2 \frac{c_3}{c_3-1}$.

Finally, let us assume that condition (G) hold and, for a constant $c_1 > 1$ to be chosen, let us fix x, y, z with $|x-z| \geq c_1 |y-z|$. Then,

$$|x-y| \geq |x-z| - |y-z| \geq (c_1-1)|y-z|.$$

So, if we pick $y' = z$ and we assume that $|x-y| \geq |x-y'|$, we have $\max(|x-y|, |x-y'|) \geq c_3 |y-y'|$, provided that $c_1 \geq c_3+1$. So,

$$\begin{aligned}|k(x,y)-k(x,y')| &\leq C_3 \frac{|y-y'|^\delta}{(|x-y|+|x-y'|)^{n+\delta}} \\ &\leq C_3 \frac{|y-y'|^\delta}{|x-y'|^{n+\delta}}.\end{aligned}$$

If $|x-y'| \geq |x-y|$, then by hypothesis, $\max(|x-y|, |x-y'|) = |x-y'| \geq c_1 |y-y'| \geq c_3 |y-y'|$, under the previous assumption, $c_1 \geq c_3+1$. So, again,

$$\begin{aligned}|k(x,y)-k(x,y')| &\leq C_3 \frac{|y-y'|^\delta}{(|x-y|+|x-y'|)^{n+\delta}} \\ &\leq C_3 \frac{|y-y'|^\delta}{|x-y'|^{n+\delta}}.\end{aligned}$$

That is to say, condition (R) holds for any constant $C_1 \geq C_3$.

This completes the proof of Proposition 40 □

Remark 41. *If we consider* $k^t(x,y) = k(y,x)$*, then Proposition 40 applies to the function* $k^t(x,y)$*.*

As we mentioned earlier, there are other versions and extensions of the T(1) Theorem (see, for instance, [64], [65], [90]), that do not use pointwise conditions on the kernel, replacing them, instead, with one of several integral conditions. Some of these integral conditions resemble those formulated in [72], which we now define. For comparison purposes, we will still assume that the function $k : \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \to \mathbb{C}$ is continuous, although local integrability suffices for this definition.

In what follows, l^1 will denote the space of sequences $\{a_j\}_{j\geq 1}$ for which $\sum_{j\geq 1} |a_j| < \infty$.

Definition 42. *[72] Given* $1 \leq r \leq \infty$*, a continuous function* $k : \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \to \mathbb{C}$ *satisfies the condition* (D_r)*, in short it satisfies* (D_r)*, if there exists a sequence* $\{C_j\}_{j\geq 1}$ *in* l^1 *such that*

$$\left\{ \int_{x\in S_j(x,z)} |k(x,y) - k(x,z)|^r \, dx \right\}^{1/r} \leq C_j \left|S_j(y,z)\right|^{-1/r'},$$

for each $j \geq 1$ *and* $y, z \in \mathbb{R}^n$*, where*

$$S_j(y,z) = \left\{ x \in \mathbb{R}^n ; 2^j |y-z| \leq |x-z| \leq 2^{j+1} |y-z| \right\}$$

and $|S_j(y,z)|$ *indicates the Lebesgue measure of* $S_j(y,z)$*.*

The function k *satisfies condition* (D'_r)*, in short it satisfies* (D'_r)*, if* $k^t(x,y) = k(y,x)$ *satisfies* (D_r)*.*

Lemma 43. *Given* $1 \leq q \leq p \leq \infty$*, if the function* k *satisfies* (D_p)*, then it satisfies* (D_q)*.*

1. *If* k *satisfies condition* (R) *(see Definition 39), then* k *satisfies* (D_∞)*.*

Proof. The proof of 1) relies on a straightforward use of Hölder's inequality and it will be omitted. As for the proof of 2), let us observe that $x \in S_j(y,z)$ implies $|x-z| \geq 2^j |y-z| \geq 2|y-z|$ for $k \geq 1$. So, if the function k satisfies condition (R), we can write, for $x \in S_j(y,z)$,

$$|k(x,y) - k(x,z)| \leq C \frac{|y-z|^\delta}{|x-z|^{n+\delta}}.$$

Now, for $x \in S_j(y,z)$,

$$|x-z|^{-n-\delta} \leq 2^{-j(n+\delta)} |y-z|^{-n-\delta},$$

so,

$$C\frac{|y-z|^{\delta}}{|x-z|^{n+\delta}} \leq C2^{-j(n+\delta)}\,|y-z|^{\delta}\,|y-z|^{-n-\delta}$$
$$= C2^{-j(n+\delta)}\,|y-z|^{-n}\,.$$

Finally,

$$\begin{aligned}|S_j(y,z)| &= C_n\left[\left(2^{j+1}|y-z|\right)^n - \left(2^j|y-z|\right)^n\right]\\ &= C_n 2^{nj}\left(2^n-1\right)|y-z|^n\,,\end{aligned}$$

or

$$|y-z|^{-n} = C_n 2^{nj}\,|S_j(y,z)|^{-1}\,.$$

Thus,

$$C2^{-j(n+\delta)}\,|y-z|^{-n} = CC_n 2^{-j\delta}\,|S_j(y,z)|^{-1}\,.$$

That is to say, k satisfies (D_∞).

This completes the proof of the lemma. □

This lemma shows, in particular, that (D_1) is the weakest of the (D_r) conditions, while (D_∞) is the strongest.

Remark 44. *The converse of 2) in Lemma 43 holds as well, for some* $0<\delta\leq 1$ *such that* $\left\{2^{j\delta}C_j\right\}_{j\geq 1}\in l^1$. *The proof is similar to the proof of 2) in Lemma 43.*

The following result can be seen as a dual version of Lemma 43. The proof is straightforward and it will be omitted.

Lemma 45. *If k satisfies (D_1), meaning*

$$\int_{x\in S_j(x,z)} |k(x,y)-k(x,z)|\,dx \leq C_j$$

with $\{C_j\}_{j\geq 1}\in l^1$, *then k satisfies the so called Hörmander's condition,*

$$\int_{|x-z|\geq 2|y-z|} |k(x,y)-k(x,z)|\,dx \leq C, \tag{8.1}$$

for some $C>0$.

Remark 46. *The interest of the Hörmander's condition in our context rests on several observations. Firstly, while it is the weakest of all the conditions we have examined, $T^t(1)$ still makes sense as a distribution acting on $\mathcal{D}_0$. Indeed, as we mentioned already in the proof of Proposition 25, applied to T^t, to estimate (4.6) it* *suffices to use the Hörmander's condition (8.1) for the transpose kernel. The same is true in the proof of the continuity from* L^∞ *to* BMO*, as shown in Remark 31. Finally, it is not known whether operators that satisfy the WBP and are associated with kernels* k *for which* k *and* k^t *satisfy such condition, are continuous on* L^2 *provided that* $T(1)$ *and* $T^t(1)$ *belong to* BMO*. The Hörmander's condition is of interest, as well, to proving other classical continuity properties (see, for instance, [5]).*

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