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WEAKENED GALLAI-RAMSEY NUMBERS

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Abstract. In the Ramsey theory of graphs, one seeks to determine the value of the Ramsey number $r^t(n)$, defined to be the least natural number p such that every coloring of the edges of K_p using t colors results in a monochromatic copy of K_n in some color. In this paper, we demonstrate the standard techniques used for finding bounds for Ramsey numbers by combining two standard generalizations of $r^t(n)$. First, we restrict our attention to Gallai colorings: those that avoid rainbow triangles. Within this setting, we then focus on finding subgraphs isomorphic to K_n that are spanned by edges using at most $s \leq t-1$ colors. The resulting generalization of $r^t(n)$ is called a weakened Gallai-Ramsey number, denoted $gr^t_s(n)$. As such, we determine several explicit small values and prove a few general properties of such numbers.

1 Introduction

The focus of Ramsey theory is to demonstrate in a precise manner how quantity produces structure. In its application to the theory of graphs, Ramsey numbers are the primary objects of interest, offering necessary and sufficient conditions under which monochromatic subgraphs exist. In this exposition, we seek to introduce the reader to the standard techniques used in this area by studying the intersection of two distinct generalizations of Ramsey numbers. First, we must establish the notations and definitions needed for our excursion.

For $t \geq 2$ and $n \geq 1$, we define the t-colored Ramsey number $r^t(n)$ to be the least natural number p such that every coloring of the edges of K_p (a complete graph on p vertices) using at most t distinct colors results in a monochromatic subgraph isomorphic to K_n in some color. Such a coloring of the edges in K_n is called a t-coloring and is not assumed to be proper; that is, we allow adjacent edges to have the same color. The fact that the numbers $r^t(n)$ exist is implied by Frank Ramsey's foundational work [16] on the subject.

When $n = 1$, a K_1 consists of a single vertex, so it is trivially contained in any t-coloring of K_1. Of course, at least one vertex is required to have a K_1, implying that $r^t(1) = 1$ for all t. Similarly, at least two vertices are needed in order to have

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a K_2. When considering a t-coloring of K_2, the one edge must receive some color, producing a monochromatic K_2. Hence, $r^t(2) = 2$ for all t. The Ramsey numbers $r^t(1)$ and $r^t(2)$ are often referred to as trivial Ramsey numbers since we are able to determine their values using such simple arguments.

The only other values of t and n for which we currently know the exact values of $r^t(n)$ are the following: $r^2(3) = 6$, $r^2(4) = 18$, and $r^3(3) = 17$. These three evaluations were all originally proven by Greenwood and Gleason [11] in 1955, and their paper was among the first to present Ramsey's ideas in the framework of edge colorings of graphs. It should be noted, however, that Greenwood and Gleason were not intentionally developing Ramsey's results, as they did not cite his work or rely on it anywhere in their paper. The evaluations of these three Ramsey numbers established the usual methodology used in the subject in which an optimal t-coloring of a complete graph that lacks a monochromatic K_n provides a lower bound for $r^t(n)$. For the upper bound, a theoretical argument is used. While the upper and lower bounds agree in these three cases, in general, such methods only provide an interval of possible values for a given Ramsey number.

Over the past nine decades, many generalizations of the classical Ramsey number $r^t(n)$ have been considered. While a variety of techniques are employed in different settings, the approach of identifying lower bounds by constructing optimal examples and using theoretical arguments for the upper bounds is standard. For this reason, the subject has appealed to both mathematicians and computer scientists, given the computational and theoretical natures of these techniques. We refer the reader to Radziszowski's dynamic survey [15] for the current state of knowledge on $r^t(n)$ and many of its generalizations. In this paper, we demonstrate these standard approaches to the reader by combining two different generalizations of $r^t(n)$.

In Section 2 of this paper, we provide some background on two distinct generalizations of $r^t(n)$: weakened Ramsey numbers and Gallai-Ramsey numbers. In the former case, we study the existence of subgraphs spanned by a limited number of colors, while in the latter case, we restrict ourselves to t-colorings that lack rainbow triangles. Both of these generalizations have been extensively studied, but at this time, few results exist involving their conglomeration (e.g., see [7]). As such, in Section 3, we define the concept of weakened Gallai-Ramsey numbers and prove some of their initial bounds. Section 4 offers two general constructions using the lexicographic product of graphs and finally, Section 5 concludes by describing some open problems for the motivated reader.

2 Two Generalizations of $r^t(n)$

The first generalization we consider comes from a restriction on the t-colorings when $t \geq 3$. Specifically, we consider only t-colorings that lack rainbow triangles. A rainbow triangle consists of three vertices, x, y, and z such that the edges xy, yz, and xz receive distinct colors. These restricted colorings are named for Tibor Gallai, whose 1967 paper [10] provided a result equivalent to the following classification of such colorings (see Theorem A in [12]).

Theorem 1. *For $t \geq 3$, every Gallai t-coloring of K_p can be obtained by substituting complete graphs with Gallai colorings into the vertices of 2-colored complete graphs.*

For $t \geq 2$, the Gallai-Ramsey number $gr^t(n)$ is defined to be the least natural number p such that every Gallai t-coloring (using at most t colors and lacking rainbow triangles) of K_p results in a monochromatic subgraph isomorphic to K_n in one of the colors. When $t = 2$, it is not possible for any t-coloring to contain a rainbow triangle. So, we define $gr^2(n) = r^2(n)$ in this case.

To demonstrate how one can obtain a Gallai 3-coloring from known bounds on $r^t(n)$, consider the Ramsey number $r^2(3) = 6$. From this number, it follows that there exists a 2-coloring of K_5 that lacks monochromatic triangles. In Figure 1, we have constructed two disjoint copies of a 2-colored K_5. We then colored all interconnecting edges using a third color. Within the context of Theorem 1, we have substituted (Gallai) 2-colorings of K_5 into the vertices of K_2, and one can verify that there are no rainbow triangles. From this construction, it is clear that

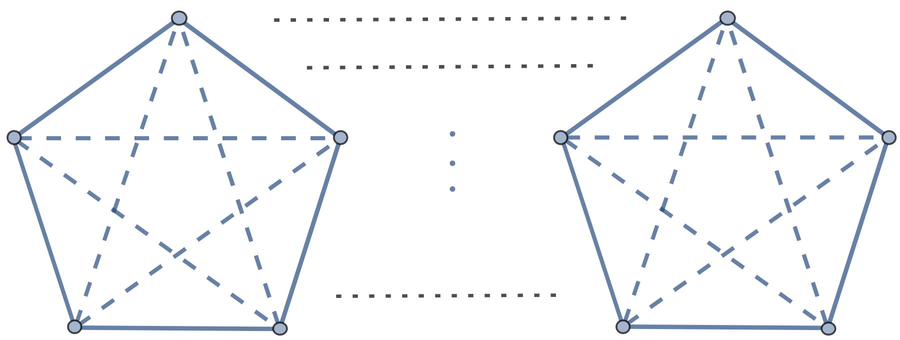

Figure 1: A Gallai 3-coloring of the edges of K_{10} that lacks monochromatic triangles.

no monochromatic triangle exists entirely within either copy of K_5. Every triangle formed using vertices from both copies of K_5 must have two edges in color 3 and one edge in either color 1 or 2. Hence, no monochromatic triangle exists in this 3-coloring of K_{10}. Thus, we find that $gr^3(3) \geq 11$.

In fact, it is known that $gr^3(3) = 11$. This follows from the work of Chung and Graham [2], in which they proved a result equivalent to the following theorem.

Theorem 2 (Chung and Graham)**.** *For $t \geq 2$,*

$$gr^t(3) = \begin{cases} 5^{t/2} + 1 & \text{if } t \text{ is even} \\ 2 \cdot 5^{(t-1)/2} + 1 & \text{if } t \text{ is odd.} \end{cases}$$

Since Gallai t-colorings form a proper subset of the set of all t-colorings of a complete graph, it follows that

$$gr^t(n) \leq r^t(n) \quad \text{for all } n, t \geq 3.$$

Of course, the construction presented above in Figure 1 can be generalized. For $p = gr^{t-1}(n)$, consider $n-1$ copies of a Gallai $(t-1)$-coloring of K_{p-1} that lacks a monochromatic K_n. Interconnect the copies of K_{p-1} with edges in color t. As with our previous argument, this construction produces a Gallai t-coloring of $K_{(p-1)(n-1)}$ that lacks a monochromatic K_n. Hence,

$$(gr^{t-1}(n) - 1)(n-1) \leq gr^t(n).$$

For more specifics on known results concerning Gallai-Ramsey numbers, the reader should consult [2], [5], [6], [7], [8], [9], and [12].

Now we shift our attention to the second generalization of $r^t(n)$. Given a t-coloring of the edges of a complete graph using $t \geq 3$ colors, a different (weakened) generalization of $r^t(n)$ can be obtained by asking how many vertices are necessary to guarantee that there exists a K_n spanned by edges using at most s of the colors, where $1 \leq s \leq t-1$. We denote this minimum number of vertices by $r_s^t(n)$, and call it a weakened Ramsey number. Such numbers were first considered by Chung, Chung, and Liu [3] and Chung and Liu [4] in a slightly more general setting than that which is presented here.

As an example, consider the weakened Ramsey number $r_2^3(3)$. Figure 2 provides a 3-coloring of K_4 that lacks a triangle spanned by edges using at most two colors. In particular, every triangle is a rainbow triangle. Hence, $r_2^3(3) \geq 5$. On the other hand, consider an arbitrary 3-coloring of K_5. Any fixed vertex x is incident with four edges. By the pigeonhole principle, at least two such edges are the same color. If xa and xb are the same color, then the subgraph induced by $\{x, a, b\}$ is a triangle spanned by edges using at most two colors. Hence, $r_2^3(3) \leq 5$, and from these two bounds, it follows that $r_2^3(3) = 5$.

Since every t-coloring of K_p that contains a K_n spanned by edges using at most s colors necessarily contains a K_n spanned by edges using at most s' colors, when $1 \leq s \leq s' \leq r-1$ we find that,

$$r_{s'}^t(n) \leq r_s^t(n) \leq r^t(n).$$

It is also worth noting that increasing the number of colors decreases the likelihood of having a K_n spanned by edges using at most s colors. Hence,

$$r_s^t \leq r_s^{t'}(n) \quad \text{for all } t \leq t'.$$

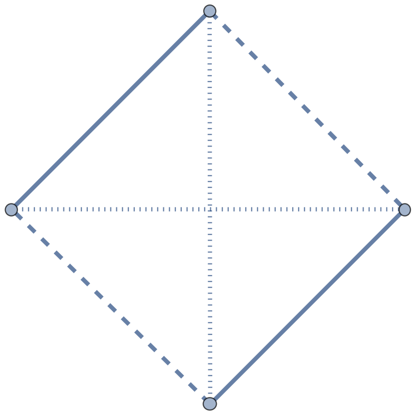

Figure 2: A 3-coloring of the edges of K_4 that lacks triangles spanned by edges using at most two colors.

For more background on the evaluation of weakened Ramsey numbers, the reader is referred to [1], [13], and [14].

At this point it is worth observing that t-colorings that avoid rainbow triangles are more likely to have complete subgraphs spanned by s colors, while t-colorings that avoid complete subgraphs spanned by a limited number of colors are more likely to contain rainbow triangles. This leads us to consider the more restrictive generalization of Ramsey numbers introduced in the next section.

3 Combining Generalizations

For $1 \leq s \leq t-1$, define the weakened Gallai-Ramsey number $gr_s^t(n)$ to be the least natural number p such that every Gallai t-coloring of K_p contains a subgraph isomorphic to K_n spanned by edges using at most s colors. This number can be viewed as a generalization of a Gallai-Ramsey number with $gr_1^t(n) = gr^t(n)$. As with weakened Ramsey numbers, for all $1 \leq s \leq s' \leq t-1$, it follows that

$$gr_{s'}^t(n) \leq gr_s^t(n) \leq gr_1^t(n) = gr^t(n).$$

Since every Gallai t-coloring of K_p is a t-coloring, it also follows that

$$gr_s^t(n) \leq r_s^t(n).$$

The following basic property of Gallai colorings was proved by Erdős, Simonovits, and Sós [5] in 1973, and it will serve us in evaluating $gr_s^t(n)$. For the sake of completeness, we offer a straightforward inductive proof of their result.

Theorem 3 (Erdős, Simonovits, and Sós)**.** *Every Gallai coloring of K_t $(t \geq 2)$ contains at most $t-1$ colors.*

Proof. We proceed by induction on t. As an initial case, note that every coloring of K_2 contains at most one color. Now, assume that every Gallai coloring of K_k contains at most $k-1$ colors and consider a Gallai coloring of K_{k+1}. Suppose x is some vertex in the K_{k+1} and consider the subgraph induced by the vertex set with x removed. By the inductive hypothesis, the resulting Gallai coloring of K_k contains at most $k-1$ colors. When reintroducing x, we find that at most one new color can be included in the edges incident with x. Otherwise, if xy and xz are edges given colors k and $k+1$, then the triangle xyz will be rainbow colored. Hence, the original Gallai coloring of K_{k+1} contains at most k colors. □

Corollary 4. *For all $t \geq 2$, $gr^t_{t-1}(t) = t$.*

Proof. In order to have a subgraph isomorphic to K_t, at least t vertices are necessary. Thus, $gr^t_{t-1}(t) \geq t$. Theorem 3 implies that every Gallai t-coloring of K_t contains at most $t-1$ colors. It follows that $gr^t_{t-1}(t) \leq t$. □

It is also worth noting that for every $t \geq 2$, there exists a Gallai $(t-1)$-coloring of K_t. Consider first a K_2, which can be trivially 1-colored. Denote the vertices of this K_2 by x_1 and x_2. When adding in a third vertex, x_3, to form a triangle, give both new edges color 2, preventing the construction of a rainbow triangle. Then add in vertex x_4, again giving all new edges color 3. The previous K_3 lacked rainbow triangles. Additionally, no triangle that includes vertex x_4 is rainbow since all edges incident with x_4 have the same color. In general, given a Gallai $(i-1)$-coloring of K_i, where $2 \leq i \leq t-1$, we construct a Gallai i-coloring of K_{i+1} by adding in vertex x_{i+1} and giving all edges incident with this vertex color i. This process produces a Gallai $(t-1)$-colored K_t. Figure 3 shows a Gallai 3-colored K_4 and a Gallai 4-colored K_5 constructed in this way. This general construction provides us with the lower bounds

$$gr^t_{t-2}(t) \geq t+2, \qquad \text{for all } t \geq 3. \tag{3.1}$$

General lower bounds for $gr^t_s(n)$ appear in [7].

Next, we turn our attention to the evaluation of $gr^t_{t-1}(t+1)$. In the case where $t=3$, the first graph in Figure 3 shows that $gr^3_2(4) \geq 5$. The following lemma will serve as a key element in proving that $gr^3_2(4) = 5$, as well as the more general result $gr^t_{t-1}(t+1) = t+2$, when $t \geq 3$. It provides a "step size" for how $gr^t_s(n)$ grows when n and s increase according to a fixed relationship.

Lemma 5. *Let $t \geq 3$, $1 \leq s \leq n$, $n' \leq n$, and $n - s = n' - s'$. Then*

$$gr^t_s(n) \leq gr^t_{s'}(n') + n - n'.$$

Proof. Let $m' = gr^t_{s'}(n')$ and consider a Gallai t-coloring of K_m, where $m = m' + n - n'$. Since $m \geq m'$, there exists an s'-colored $K_{n'}$ and we denote its vertex set

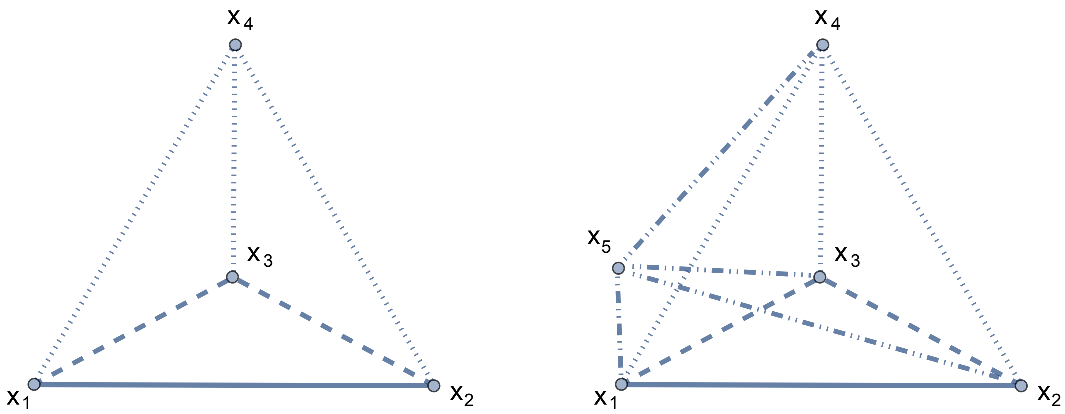

Figure 3: A Gallai 3-coloring of K_4 that lacks monochromatic triangles and a Gallai 4-coloring of K_5 that lacks a K_4 spanned by edges using at most 2 colors.

by $X = \{x_1, x_2, \dots, x_{n'}\}$. Let y_1 be some other vertex from the K_m that is not in X. As observed in the proof of Theorem 3, the edges joining y_1 to the vertices in X use at most one additional color beyond those used in the subgraph induced by X. This is due to the fact that if y_1x_i and y_1x_j $(i \neq j)$ are distinct colors different from those in X, then the subgraph induced by $\{y_1, x_i, x_j\}$ forms a rainbow triangle. Hence, the subgraph induced by $\{y_1\} \cup X$ is an $(s'+1)$-colored $K_{n'+1}$. In general, we add in vertex y_i, for $2 \leq i \leq n-n'$, with at most one new color added for each i. The resulting subgraph induced by $\{y_1, y_2, \dots, y_{n-n'}\} \cup X$ is a rainbow triangle-free complete graph of order m, spanned by edges using at most $s' + n - n' = s$ colors. Thus, $gr_s^t(n) \leq m$, resulting in the statement given in the lemma. □

The strength of this lemma becomes apparent in the proof of the following theorem. We will show that every Gallai t-coloring of K_{t+2} contains a $(t-1)$-colored K_{t+1}.

Theorem 6. *For all $t \geq 3$, $gr_{t-1}^t(t+1) = t+2$.*

Proof. From the construction derived from the examples in Figure 3, there exists a Gallai t-colored K_{t+1}. Thus,

$$gr_{t-1}^t(t+1) \geq t+2.$$

To obtain the reverse inequality, consider a Gallai t-coloring of K_{t+2}. Let x be some vertex in this K_{t+2}. If removing x produces a $(t-1)$-colored K_{t+1}, then we are done. Otherwise, we have a t-colored K_{t+1}. Since a Gallai colored K_t has at most $t-1$ colors, the removal of any vertex in this K_{t+1} must also remove some color. There exists $t+1$ vertices and only t colors. By the pigeonhole principle, there exists two distinct vertices (say y and z) whose removal deletes the same color (say red). This can only occur if all red edges are incident with both y and z (ie., yz is the only red edge in the K_{t+1}). Now returning to the original K_{t+2}, the vertex x may have also been incident with red edges. We must consider several cases.

<u>Case 1</u> Suppose that x is not incident with any red edges. Then removing y results in a $(t-1)$-colored K_{t+1}.

<u>Case 2</u> Suppose that x is adjacent to some other vertex w (distinct from y and z) via a red edge, but is not adjacent to y or z. Then the subgraph induced by $\{w, x, y, z\}$ is a 2-colored K_4. By Lemma 5, there exists a $(t-1)$-colored K_{t+1}.

<u>Case 3</u> Suppose that x is adjacent to some other vertex w and exactly one of y and z via red edges. Then the subgraph induced by $\{w, x, y, z\}$ is a 2-colored K_4, which by Lemma 5, results in a $(t-1)$-colored K_{t+1}.

<u>Case 4</u> Suppose that x is adjacent to exactly one of y and z via a red edge (assume y), and is not incident with any other red edges. Then the removal of y removes all red edges and we have a $(t-1)$-colored K_{t+1}.

<u>Case 5</u> Suppose that x is adjacent to both y and z via red edges. Then the subgraph induced by $\{x, y, z\}$ is a monochromatic triangle. By Lemma 5, there exists a $(t-1)$-colored K_{t+1}.

Thus, we have shown that every Gallai t-coloring of K_{t+2} contains a $(t-1)$-colored K_{t+1}. So,

$$gr^t_{t-1}(t+1) \leq t+2,$$

completing the proof of the theorem. □

Plugging in $t=4$ into inequality (3.1) gives $gr^4_2(4) \geq 6$. This observation leads us to the following theorem, giving a lower bound for $gr^t_2(4)$.

Theorem 7. *For all $t \geq 3$,*

$$gr^t_2(4) \geq t+2.$$

Proof. We proceed by induction on $t \geq 3$. The first two cases being

$$gr^3_2(4) \geq 5 \qquad \text{and} \qquad gr^4_2(4) \geq 6$$

follow from Theorem 6 and Inequality (3.1), respectively. Now suppose that

$$gr^{t-1}_2(4) \geq t+1$$

and consider a rainbow triangle-free $(t-1)$-coloring of K_t that lacks a K_4 spanned by edges using at most two colors. Add in a vertex x and color all edges joining x with the K_t using color t. No new triangles formed that include x can be rainbow since all edges incident with x use color t. Any new K_4 must include vertex x

and three vertices, say a, b, and c, from the K_t. By the inductive hypothesis, abc is not a rainbow triangle, and by Lemma 5, abc is not monochromatic. Hence, the K_4 induced by $\{x, a, b, c\}$ is spanned by edges using exactly three colors. This construction results a rainbow triangle-free t-coloring of K_{t+1} that lacks a K_4 spanned by edges using at most two colors. Hence, the statement of the theorem follows. $\square$

Theorem 8. *If $t \geq 4$ and $gr_2^{t-1}(4) \leq p$, then $gr_2^t(4) \leq t(p-1)+2$.*

Proof. Assume that $gr_2^{t-1}(4) \leq p$ and consider a Gallai t-coloring of $K_{t(p-1)+2}$. We will prove that such a coloring contains a K_4-subgraph spanned by edges using at most 2 colors. We assume this to be false. Then by Lemma 5, this coloring lacks monochromatic triangles. A fixed vertex x is incident with $t(p-1)+1$ edges, from which it follows that at least p edges have the same color. Without loss of generality, assume that $xx_1, xx_2, \ldots, xx_p$ are all red. Since the coloring lacks monochromatic triangles, the subgraph induced by $\{x_1, x_2, \ldots, x_p\}$ does not contain any red edges, and as such, is a Gallai $(t-1)$-coloring of K_p. Our assumption $gr_2^{t-1}(4) \leq p$ implies that this coloring contains a K_4 spanned by edges using at most 2 colors, contradicting the assumption that no such subgraph exists. Thus, the original Gallai t-coloring of $K_{t(p-1)+2}$ must have a K_4-subgraph spanned by edges using at most 2 colors. $\square$

Combining Theorems 7 and 8 gives the following range for $gr_2^t(4)$:

$$t+2 \leq gr_2^t(4) \leq t(gr_2^{t-1}(4)-1)+2 \quad \text{for all } t \geq 4.$$

It follows that

$$6 \leq g_2^4(4) \leq 18,$$

but these bounds prove to be somewhat weak for larger values of t. In the next section, we turn to general constructions of lower bounds.

4 Some General Constructions for $gr_s^t(n)$

In this section, we demonstrate some general constructions that imply lower bound for weakened Gallai-Ramsey numbers. As an initial example, consider the number $gr_2^3(5)$. Figure 4 shows a Gallai 3-colored K_8 that lacks a K_5 spanned by edges using at most two colors. This K_8 was formed by using two copies of a 2-colored K_4 that lacks monochromatic triangles (which exists since $gr_1^2(3) = r^2(3) = 6 > 4$), interconnected with edges in a third color. Note that this construction does not produce any rainbow triangles since every triangle formed using three vertices from the same copy of K_4 has at most two colors and every triangle that spans vertices from both copies of K_4 necessarily contains two edges in the third color.

In Figure 4, observe that every subgraph induced by a set of five distinct vertices necessarily includes vertices from both K_4-subgraphs and at least three vertices from

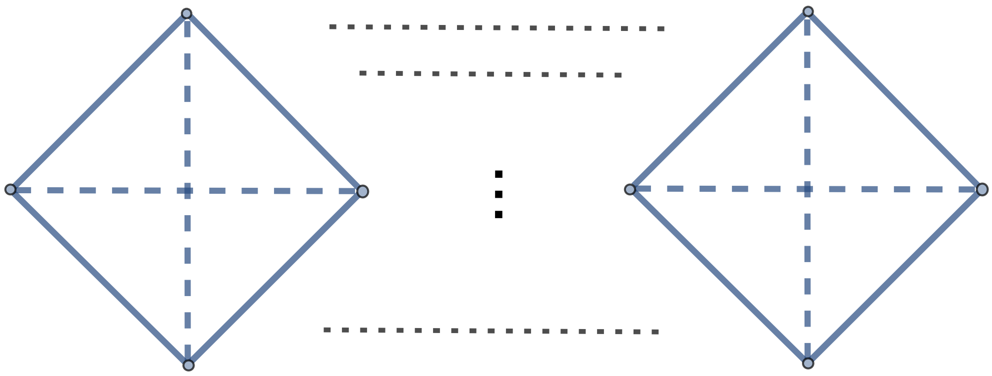

Figure 4: A Gallai 3-coloring of K_8 that lacks a K_5-subgraph spanned by edges using at most 2 colors.

one of the two copies of K_4. Thus, the three vertices includes in only one K_4 are two colored and the third color gets included with the interconnecting edges. Hence, every K_5-subgraph is 3-colored:

$$gr_2^3(5) \geq 9.$$

The following theorem generalizes this construction.

Theorem 9. *For all $t \geq 3$ and $1 \leq s \leq t-1$,*

$$gr_s^t(n) \geq p \quad \Longrightarrow \quad gr_{s+1}^{t+1}(k(n-1)+1) \geq k(p-1)+1,$$

for all $k \geq 2$ that satisfies $k(n-1)+1 > p-1$.

Proof. Assuming $gr_s^t(n) = p$, there exists a Gallai t-coloring of K_{p-1} that lacks a K_n-subgraph spanned by edges using at most s colors. Consider k copies of this coloring of K_{p-1} interconnected with edges using a $(t+1)^{st}$ color. No rainbow triangles exist in this coloring since every subgraph induced by three vertices coming from the same K_{p-1} are not rainbow and any triangle that includes vertices from at least two distinct copies of K_{p-1} contains at least two edges in the $(t+1)^{st}$ color. When considering a subgraph induced by any collection of $k(n-1)+1$ vertices, by the pigeonhole principle, there exists some copy of K_{p-1} that contains at least n of the vertices. Since $gr_s^t(n) = p$, the subgraph spanned by these n vertices is spanned by edges using at least $s+1$ colors. From the assumption that $k(n-1)+1 > p-1$, it follows that the subgraph induced by any collection of $k(n-1)+1$ vertices includes vertices from at least two distinct copies of K_{p-1}, and hence, is spanned by edges using at least $s+1$ colors. Thus, we have constructed a Gallai $(t+1)$-coloring of $K_{k(p-1)}$ that lacks a $K_{k(n-1)+1}$ spanned by edges using at most $s+1$ colors, from which the theorem follows. □

Now, we consider how a certain graph product can be used to obtain lower bounds for weakened Gallai-Ramsey numbers, applying the ideas introduced in Section 2 of [7]. Recall that the lexicographic product of graphs G_1 and G_2, denoted $G_1[G_2]$, is defined to have vertex set $V(G_1[G_2]) = V(G_1) \times V(G_2)$ and edge set

$$E(G_1[G_2]) = \{(a_1, b_1)(a_2, b_2) \mid a_1a_2 \in E(G_1) \text{ or } (a_1 = a_2 \text{ and } b_1b_2 \in E(G_2))\}.$$

Notice that the lexicographic product is not commutative.

Given a graph G, an independent set $I \subseteq V(G)$ is a subset of vertices in which no two elements in I are adjacent. The cardinality of a maximal independent set in G is called the independence number, and is denoted by $\beta(G)$. In 1975, Geller and Stahl [12] proved that

$$\beta(G_1[G_2]) = \beta(G_1)\beta(G_2).$$

The order of a maximal complete subgraph of a graph G is called the clique number of G and is denoted by $\omega(G)$. It is a simple exercise to check that

$$\overline{G_1[G_2]} = \overline{G_1}[\overline{G_2}], \tag{4.1}$$

from which it follows that

$$\omega(G_1[G_2]) = \omega(G_1)\omega(G_2). \tag{4.2}$$

This property of lexicographic products makes them useful in the construction of lower bounds in Ramsey theory. We now show how this can be applied to $gr_s^t(n)$.

Theorem 10. *If $gr_s^t(m) \geq n$, then*

$$gr_s^t((m-1)^2 + 1) \geq (n-1)^2 + 1.$$

Proof. Assuming that $gr_s^t(m) \geq n$, there exists an optimal Gallai t-coloring of K_{n-1} that lacks a K_m spanned by edges using at most s colors. So, the largest order of an s-colored complete subgraph is $m-1$. Consider the t-coloring of $K_{(n-1)^2}$ formed by taking an optimal coloring of K_{n-1} and replacing each of the vertices with an optimal coloring of K_{n-1}. More precisely, if we label the vertices in an optimal coloring of K_{n-1} by $x_1, x_2, \ldots, x_{n-1}$, then the vertices in the $K_{(n-1)^2}$ are identified with the set

$$\{(a_i, b_j) \mid i, j \in \{1, 2, \ldots, n-1\}\}.$$

The edge $(a_i, b_j)(a_k, b_\ell)$ is given the same color as $x_i x_k$ when $i \neq k$, and the same color as $x_j x_\ell$ when $i = k$. For example, see Figure 5. By (4.1), it follows that the largest complete subgraph spanned by edges using at most s colors is isomorphic to the lexicographic product of the corresponding complete subgraphs of those given colors in the original optimal coloring of K_{n-1}. By (4.2), the largest complete subgraph of our Gallai t-colored $K_{(n-1)^2}$ spanned by edges using at most s colors has order $(m-1)^2$. Thus, we have constructed a Gallai coloring of $K_{(n-1)^2}$ that lacks a $K_{(m-1)^2+1}$ spanned by edges using at most s colors. □

For example, Figure 5 shows the construction in Theorem 10 applied to the first graph from Figure 3. Specifically, we use a Gallai 3-colored K_4 that is not 2-colored ($gr_2^3(4) \geq 5$) to produce a 3-colored K_{16} that lacks a K_{10} spanned by edges using at most 2 colors. It follows that $gr_2^3(10) \geq 17$. The next theorem considers a similar construction to that of Theorem 10, while allowing for the addition of more colors.

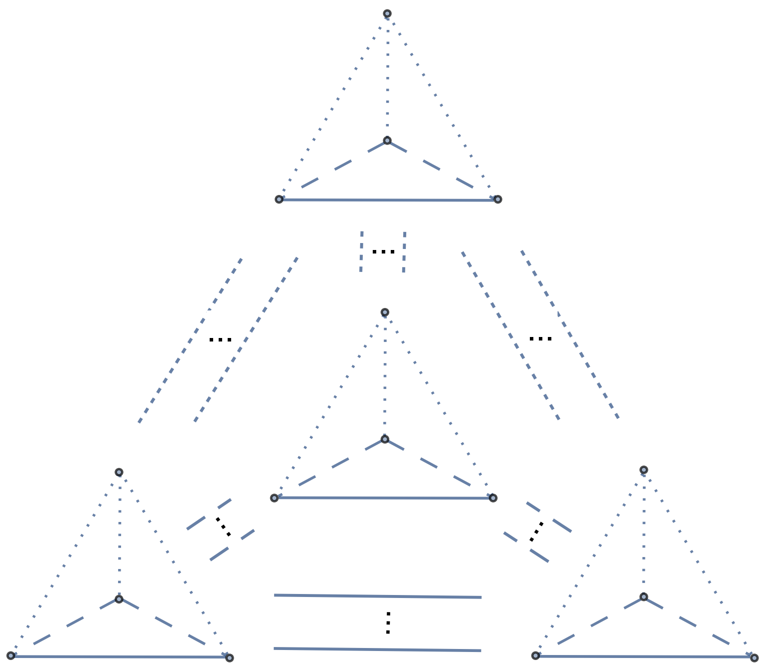

Figure 5: Applying Theorem 10 to a Gallai 3-colored K_4 that is not 2-colored.

Theorem 11. *If $gr_{t-1}^t(m) \geq n$, then*

$$gr_{2t-1}^{2t}((m-1)(n-1)+1) \geq (n-1)^2 + 1.$$

Proof. The construction here is similar to that of Theorem 10, except that a new collection of colors is utilized for the edges interconnecting the different copies of of the optimal Gallai t-colored K_{n-1} subgraphs. We leave the details of the proof to the reader to fill in since the proof mirrors that of Theorem 10. □

Returning to Figure 5, if the edges interconnecting the K_4 subgraphs use 3 new colors, then we obtain a Gallai 6-colored K_{16} that lacks a K_{13} spanned by edges using at most 5 colors. Hence, $gr_5^6(13) \geq 17$. In fact, if we apply Theorems 10 and 11 to the weakened Gallai-Ramsey number in Theorem 6, we obtain the following general inequalities:

$$gr_{t-1}^{t}(t^2+1) \geq (t+1)^2+1 \quad \text{and} \quad gr_{2t-1}^{2t}(t(t+1)+1) \geq (t+1)^2+1.$$

5 Some Open Problems

Finally, we conclude by describing a few directions for future study. We have laid down the basics for studying $gr_s^t(n)$, but surely our efforts can be further advanced by others. Besides the Ramsey numbers we have extended in this paper, we offer three other generalizations that warrant inquiry.

1. Consider subgraphs other than complete subgraphs. If G is any graph, then one can define $gr_s^t(G)$ to be the least natural number p such that every Gallai t-coloring of K_p results in a subgraph isomorphic to G that is spanned by edges using at most s colors. More generally, one can consider weakened Gallai-Ramsey numbers that are not diagonal (similar to [3] and [4]). For example, define $gr_2^3(G_1, G_2, G_3)$ to be the least natural number p such that every Gallai 3-coloring of K_p contains a copy of G_1 spanned by edges using only colors 1 and 2, a copy G_2 spanned by edges using only colors 2 and 3, or a copy of G_3 using only colors 1 and 3. This concept can be generalized to any t and s using $\binom{t}{s}$ subgraphs.

2. Instead of just avoiding rainbow triangles, consider only t-colorings that avoid rainbow K_r-subgraphs. Denote by $gr_s^t(K_r : K_n)$ the least natural number p such that every t-coloring of K_p that lacks rainbow K_r-subgraphs contains a K_n-subgraph spanned by edges using at most s colors.

3. Consider weakened Gallai-Ramsey numbers in the setting of r-uniform hypergraphs. An r-uniform hypergraph H consists of a set $V(H)$ of vertices and a set of hyperedges $E(H)$ that consists of different unordered r-tuples of distinct vertices. We denote the complete r-uniform hypergraph on n vertices by $K_n^{(r)}$.

> Define $gr_s^t(K_n^{(r)})$ to be the least natural number p such that every t-coloring of the hyperedges of $K_p^{(r)}$ that lacks rainbow $K_{r+1}^{(r)}$-subhypergraphs results in a $K_n^{(r)}$-subhypergraph spanned by hyperedges using at most s colors.

Of course, beyond studying the bounds of any of the above generalizations of $gr_s^t(n)$, one can also combine these ideas in in a similar fashion to the primary concepts introduced in this paper. It is our hope that we have provided enough encouragement and background to entice the reader to further advance our understanding of the many generalizations of Ramsey numbers.

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