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EXISTENCE AND ATTRACTIVITY OF SOLUTIONS OF SEMILINEAR VOLTERRA TYPE INTEGRO-DIFFERENTIAL EVOLUTION EQUATIONS

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Abstract. In this paper, we prove a result on the existence and local attractivity of solutions of second order semilinear evolution equation. Our investigations will be situated on the Banach space of functions which are defined, continuous and bounded on the nonnegative real axis. The results are obtained by using the Mönch fixed point and the Kuratowski measure of noncompactness. An example is provided to illustrate the main result.

1 Introduction

In this paper, we investigate the existence and local attractivity of the mild solution, defined on a semi-infinite positive real interval $J = [0, \infty)$, for non-autonomous semilinear second order evolution equation of mixed type in a real Banach space. More precisely, we will consider the following problem

$$y''(t) - A(t)y(t) = f\left(t, y(t), \int_0^t K(t,s,y(s))ds\right), \ t \in J, \tag{1.1}$$

$$y(0) = y_0, \ y'(0) = y_1, \tag{1.2}$$

where $\{A(t)\}_{0\leq t<+\infty}$ is a family of linear closed operators from E into E, $f : J \times E \times E \to E$ is a Carathéodory function, $K : \Delta \times E \to E$ is a continuous function, $\Delta := \{(t,s) \in J \times J : s \leq t\}$, $y_0, y_1 \in E$ and $(E, |\cdot|)$ is a real Banach space.

Evolution equations arise in many areas of applied mathematics [2, 40]. This type of equations have received much attention in recent years [1]. Integro-differential equations on infinite intervals have attracted great interest due to their applications in characterizing many problems in physics, fluid dynamics, biological models and

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chemical kinetics see [5, 6, 15, 16, 36]. Qualitative properties such as the existence, uniqueness and stability for various functional differential and integro-differential equations have been extensively studied by many researchers (see, for instance, [7, 9, 11, 19, 25, 30]).

There are many results concerning the second-order differential equations, see for example [10, 18, 23, 26, 37, 38]. For the study of abstract second order equations, the existence of an evolution system $U(t,s)$ for the homogenous equation

$$y''(t) = A(t)y(t), \quad \text{for} \quad t \geq 0,$$

is useful. For this purpose there are many techniques to show the existence of $U(t,s)$ which has been developed by Kozak [29].

On the other hand, recently there has been an increasing interest in studying the abstract non-autonomous second order initial value problem

$$y''(t) - A(t)y(t) = f(t, y(t)),\ t \in [0,T] \text{ or } t \in [0,\infty) \tag{1.3}$$

$$y(0) = y_0,\ y'(0) = y_1. \tag{1.4}$$

The reader is referred to [14, 17, 22, 27] and the references therein.

In this paper we use the technique of measures of noncompactness. It is well known that this method provides an excellent tool for obtaining existence of solutions of nonlinear differential equation. This technique works fruitfully for both integral and differential equations. More details are found in Akhmerov *et al.* [3], Alv́ares [4], Aissani and Benchohra [8], Banaś and Goebel [12], Guo *et al.* [28], Olszowy and Wędrychowicz [32, 33], Zhang and Chen [41] and the references therein.

Motivated by the above-mentioned works, we derive some sufficient conditions for the existence of solutions of the system (1.1)-(1.2) by means of the Kuratowski measure of noncompactness and the fixed point theory.

This work is organized of as follows. In Section 2, we recall some definitions and facts about evolution systems. In Section 3, we give the existence of mild solutions to the problem (1.1)-(1.2). Section 4 is devoted to the attractivity of the solution of problem (1.1)-(1.2). An example is presented in Section 5 to illustrate the application of our results.

2 Preliminaries

In this section, we mention notations, definitions, lemmas and preliminary facts needed to establish our main results. Throughout this paper, we denote by E a Banach space with the norm $|\cdot|$. Let $BC(J,E)$ be the Banach space of all bounded and continuous functions y mapping J into E with the usual supremum norm

$$\|y\| = \sup_{t \in J} |y(t)|.$$

We set

$$B_R = \{y \in C(J,E) : \|y\| < R\}, \bar{B}_R = \{y \in C(J,E) : \|y\| \leq R\}$$

($R > 0$ is a constant).

In what follows, let $\{A(t),\ t \geq 0\}$ be a family of closed linear operators on the Banach space E with domain $D(A(t))$ which is dense in E and independent of t.

In this work the existence of solutions the problem (1.1)-(1.2) is related to the existence of an evolution operator $U(t,s)$ for the following homogeneous problem,

$$y''(t) = A(t)y(t), \qquad t \in J. \tag{2.1}$$

This concept of evolution operator has been developed by Kozak [29].

Definition 1. *A family U of bounded operators $U(t,s) : E \to E$, $(t,s) \in \Delta := \{(t,s) \in J \times J : s \leq t\}$, is called an evolution operator of the equation (2.1) if the following conditions hold:*

(e_1) *For any $x \in E$ the map $(t,s) \longmapsto U(t,s)x$ is continuously differentiable and*

- **(a)** *for any $t \in J$, $U(t,t) = 0$.*
- **(b)** *for all $(t,s) \in \Delta$ and for any $x \in E$, $\frac{\partial}{\partial t}U(t,s)x\big|_{t=s} = x$ and $\frac{\partial}{\partial s}U(t,s)x\big|_{t=s} = -x$.*

(e_2) *For all $(t,s) \in \Delta$, if $x \in D(A(t))$, then $\frac{\partial}{\partial s}U(t,s)x \in D(A(t))$, the map $(t,s) \longmapsto U(t,s)x$ is of class C^2 and*

- **(a)** $\frac{\partial^2}{\partial t^2}U(t,s)x = A(t)U(t,s)x,$
- **(b)** $\frac{\partial^2}{\partial s^2}U(t,s)x = U(t,s)A(s)x,$
- **(c)** $\frac{\partial^2}{\partial s \partial t}U(t,s)x\big|_{t=s} = 0.$

(e_3) *For all $(t,s) \in \Delta$, then $\frac{\partial}{\partial s}U(t,s)x \in D(A(t))$, there exist $\frac{\partial^3}{\partial t^2 \partial s}U(t,s)x$, $\frac{\partial^3}{\partial s^2 \partial t}U(t,s)x$ and*

- **(a)** $\frac{\partial^3}{\partial t^2 \partial s}U(t,s)x = A(t)\frac{\partial}{\partial s}(t)U(t,s)x.$

  Moreover, the map $(t,s) \longmapsto A(t)\frac{\partial}{\partial s}(t)U(t,s)x$ is continuous,

(b) $\dfrac{\partial^3}{\partial s^2 \partial t} U(t,s)x = \dfrac{\partial}{\partial t} U(t,s)A(s)x.$

Throughout this paper, we will use the following definition of the concept of Kuratowski measure of noncompactness [12].

Definition 2. *The Kuratowski measure of noncompactness α is defined by*

$$\alpha(D) = \inf\{r > 0 : D \text{ has a finite cover by sets of diameter} \leq r\},$$

for a bounded set D in any Banach space X.

Let us recall the basic properties of Kuratowski measure of noncompactness.

Lemma 3. *[12] Let X be a Banach space and $C, D \subset X$ be bounded, then the following properties hold:*

(i_1) $\alpha(D) = 0$ *if only if D is relatively compact,*

(i_2) $\alpha(\overline{D}) = \alpha(D)$ *; $\overline{D}$ denotes the closure of D,*

(i_4) $\alpha(C) \leq \alpha(D)$ *when $C \subset D$,*

(i_4) $\alpha(C + D) \leq \alpha(C) + \alpha(D)$ *where $C + D = \{x \mid x = y + z; y \in C; z \in D\}$,*

(i_5) $\alpha(aD) = |a|\alpha(D)$ *for any $a \in \mathbb{R}$,*

(i_6) $\alpha(ConvD) = \alpha(D)$, *where $ConvD$ is the convex hull of D,*

(i_7) $\alpha(C \cup D) = \max(\alpha(C), \alpha(D))$,

(i_8) $\alpha(C \cup \{x\}) = \alpha(C)$ *for any $x \in E$.*

Denote by $\omega^T(y, \varepsilon)$ the modulus of continuity of y on the interval $[0, T]$ i.e.

$$\omega^T(y, \varepsilon) = \sup\{|y(t) - y(s)|; t, s \in [0, T], |t - s| \leq \varepsilon\}.$$

Moreover, let us put

$$\omega^T(D, \varepsilon) = \sup\left\{\omega^T(y, \varepsilon); y \in D\right\},$$

$$\omega_0^T(D) = \lim_{\varepsilon \to 0} \omega^T(D, \varepsilon).$$

Lemma 4. *[26] If $H = \{u_n\}_{n=0}^{\infty} \subset L^1([0;T], E)$ is uniformly integrable, then the function $s \to \alpha(H(s))$ is measurable and*

$$\alpha\left\{\int_0^t u_n(s))ds\right\}_{n=0}^{\infty} \leq 2\int_0^t \alpha(H(s))ds, \qquad t \in [0;T].$$

We recall that a subset $B \subset L^1([0;T];E)$ *is uniformly integrable if there exists* $\xi \in L^1([0;T];\mathbb{R}^+)$ *such that*

$$\|x(s)\| \le \xi(s) \text{ for } x \in B \text{ and a.e. } s \in [0;T].$$

Lemma 5. *[34],([35], p. 35). Let* $u(t)$, $h(t)$, $p(t)$ *and* $q(t)$ *be real valued nonnegative integrable functions defined on* $\mathbb{R}^+$, *for which the inequality*

$$u(t) \le h(t) + \int_0^t p(s)\left[u(s) + \int_0^s q(\tau)u(\tau)d\tau\right] ds,$$

holds for all $t \in \mathbb{R}^+$, *then*

$$u(t) \le h(t) + \int_0^t p(s)\left[h(s) + \int_0^s h(\tau)(p(\tau)+q(\tau))\exp\left(\int_\tau^s (p(\delta)+q(\delta)d\delta)\right)d\tau\right] ds,$$

for all $t \in \mathbb{R}^+$.

We introduce now the concept of attractivity (stability) of solutions of operator equations in the space $BC(J,E)$. To this end, assume that $\mathcal{E}$ is a nonempty subset of the space $BC(J,E)$. Moreover, let Q be an operator defined on $\mathcal{E}$ with values in $BC(J,E)$. Let us consider the operator equation of the form

$$y(t) = (Qy)(t) \tag{2.2}$$

Definition 6. *[20] We say that solutions of (2.2) are locally attractive if there exists a ball* $B(y^*,r)$ *in the space* $BC(J,E)$ *such that* $\bar{B}(y^*,r) \cap \mathcal{E} \ne 0$, *and for arbitrary solutions* y_1 *and* y_2 *of (2.2) belonging to* $\bar{B}(y^*,r)\cap\mathcal{E}$ *we have*

$$\lim_{t\to+\infty}(y_2(t) - y_1(t)) = 0.$$

In the case when this limit (2.2) is uniform with respect to the set $\bar{B}(y^*,r)\cap\mathcal{E}$ *i.e. when for each* $\varepsilon > 0$ *there exists a* $T > 0$ *such that*

$$|y_2(t) - y_1(t)| \le \varepsilon$$

for all $y_2, y_1 \in \bar{B}(y^*,r)\cap\mathcal{E}$ *being solutions of equation (2.2)and for* $t \ge T$, *we will say that solutions of equation (2.2) are uniformly locally attractive.*

The concept of uniform local attractivity of solutions is equivalent to the concept of asymptotic stability of solutions (introduced in the paper [13]).

Theorem 7 (Mönch fixed point theorem)**.** *[21] Let* X *be a Banach space,* Ω *is bounded open subset of* X *with* $0 \in \Omega$. *Let* $F : \bar{\Omega} \to X$ *be a continuous operator satisfying*

($\imath$) *If* $H \subset \bar{\Omega}$ *is countable and* $H \subset \overline{Conv}(\{0\} \cup F(H))$; *then* H *is relatively compact.*

($\imath\imath$) $y \ne \lambda F y; \forall\lambda \in [0;1]; y \in \partial\Omega$,

Then F *has a fixed point in* $\bar{\Omega}$.

3 Existence of solutions

Definition 8. *A function $y \in BC(J, E)$ is called a mild solution to the problem (1.1)-(1.2) if y satisfies the integral equation*

$$y(t) = -\frac{\partial}{\partial s}U(t,0)y_0 + U(t,0)y_1 + \int_0^t U(t,s)f\left(s, y(s), \int_0^s K(s,\tau,y(\tau))d\tau\right) ds. \tag{3.1}$$

For the forthcoming analysis, we need the following assumptions:

(H_1) There exist constants $M \geq 1$ and $\omega > 0$, such that

$$\|U\|_{B(E)} \leq Me^{-\omega(t-s)} \quad \text{for any } (t,s) \in \Delta.$$

(H_2) There exist constants $\tilde{M} \geq 1$ and $\varpi > 0$, such that:

$$\|\frac{\partial}{\partial s}U(t,s)\|_{B(E)} \leq \tilde{M}e^{-\varpi(t-s)} \quad \text{for any } (t,s) \in \Delta.$$

(H_3) The function $f : J \times E \times E \to E$ is Carathéodory and satisfies the following:

(a)
$$\lim_{t\to+\infty} \int_0^t e^{-\omega(t-s)}|f(s,0,0)|ds = 0,$$
.

(b) There exists an integrable function $p : J \to \mathbb{R}^+$, such that:

$$|f(t,u_2,v_2) - f(t,u_1,v_1)| \leq p(t)(1 + |u_2 - u_1| + |v_2 - v_1|)$$

$$\text{for a.e } t \in J \text{ and each } u_i, v_i \in E, (i = 1,2),$$

and
$$\lim_{t\to+\infty} \int_0^t e^{-\omega(t-s)}p(s)ds = 0.$$

(c) There exist locally integrable functions $\sigma_i : J \to \mathbb{R}^+, (i = 1,2)$ such that:

$$\alpha(f(t,D_1,D_2)) \leq \sigma_1(t)\alpha(D_1)+\sigma_2(t)\alpha(D_2) \text{ for a.e } t \in J \text{ and } D_1, D_2 \subset E.$$

(H_4) The function $K : \Delta \times E \to E$ satisfies the following:

(a) There exists an integrable function $q : J \to \mathbb{R}_+$, such that:

$$|K(t,s,u) - K(t,s,v)| \leq q(t)\,|u - v| \text{ for a.e } (t,s) \in \Delta \text{ and each } u, v \in E.$$

(b) There exist constants $K \geq 0$ and $\gamma > 0$, such that:

$$|K(t,s,0)| < Ke^{-\gamma(t-s)} \quad \text{for any } (t,s) \in \Delta.$$

(c) There exists a constant $K^* > 0$, such that

$$\alpha(K(t,s,D)) \leq K^* \alpha(D) \text{ for a.e } (t,s) \in \Delta \text{ and } D \subset E.$$

Remark 9. *Notice that if the hypothesis* (H_3) *holds, then there exist constants* $f^*, p^* > 0$ *such that:*

$$f^* = \sup_{t \in J} \int_0^t e^{-\omega(t-s)} |f(s,0,0)| ds, \qquad p^* = \sup_{t \in J} \int_0^t e^{-\omega(t-s)} p(s) ds.$$

Theorem 10. *Assume that the hypotheses* $(H_1) - (H_4)$ *are satisfied. Then the problem (1.1)-(1.2) admits at least one mild solution, which is uniformly locally asymptotically attractive.*

Proof. Consider the operator $N : BC(J,E) \to BC(J,E)$ defined by

$$(Ny)(t) = -\frac{\partial}{\partial s} U(t,0) y_0 + U(t,0) y_1 + \int_0^t U(t,s) f\left(s, y(s), \int_0^s K(s,\tau,y(\tau)) d\tau\right) ds.$$

We notice that the fixed points of the operator N are mild solutions of the problem (1.1)-(1.2).

Step 1. $N(y) \in BC(J,E)$ for any $y \in BC(J,E)$.

Let $y \in BC(J,E)$, then for $t \in J$, we have

$$\begin{aligned}
&|Ny(t)| \\
\leq\ & \left\|\frac{\partial}{\partial s}U(t,0)\right\|_{B(E)} |y_0| + \|U(t,s)\|_{B(E)}\,|y_1| \\
+\ & \|U(t,s)\|_{B(E)} \int_0^t \left| f\left(s, y(s), \int_0^s K(s,\tau,y(\tau))d\tau\right)\right| ds \\
\leq\ & \tilde{M}\,|y_0| + M\,|y_1| \\
+\ & M\int_0^t e^{-\omega(t-s)} \left| f\left(s, y(s), \int_0^s K(s,\tau,y(\tau))d\tau\right) - f(s,0,0) + f(s,0,0)\right| ds \\
\leq\ & \tilde{M}\,|y_0| + M\,|y_1| \\
+\ & M\int_0^t e^{-\omega(t-s)} p(s)\left(1 + |y(s)| + \int_0^s |K(s,\tau,y(\tau))| d\tau\right) ds \\
+\ & M\int_0^t e^{-\omega(t-s)} |f(s,0,0)| ds \\
\leq\ & \tilde{M}\,|y_0| + M\,|y_1| \\
+\ & M\int_0^t e^{-\omega(t-s)} p(s)\left(1 + |y(s)| + \int_0^s |K(s,\tau,y(\tau)) - K(s,\tau,0) + K(s,\tau,0)| d\tau\right) ds \\
\leq\ & \tilde{M}\,|y_0| + M\,|y_1| \\
+\ & M\int_0^t e^{-\omega(t-s)} p(s)\left(1 + |y(s)| + \int_0^s q(\tau)|y(\tau)|\right) ds \\
+\ & M\int_0^t e^{-\omega(t-s)} |f(s,0,0)| ds \\
+\ & M\int_0^t\int_0^s e^{-\omega(t-s)} p(s)|k(s,\tau,0)| d\tau ds \\
\leq\ & \tilde{M}\,|y_0| + M\,|y_1| + Mp^*(1+\|q\|_{L^1})\sup_{t\in J} y(t) + Mf^* + M\left(1+\frac{K}{\gamma}\right)p^* \\
<\ & +\infty.
\end{aligned}$$

Consequently, $N(y) \in BC(J,E)$.

Step 2. N is continuous.

Let $(y_n)_{n\in N}$ be a sequence in $BC(J,E)$ such that $y_n \to y$ in $BC(J,E)$.

Case 1. If $t \in [0,T]$; $T > 0$, then, we have

$$\begin{aligned}
&|(Ny_n)(t) - (Ny)(t)| \\
&\leq M\int_0^t \left| f(s, y_n(s), \int_0^s K(s,\tau,y_n(\tau))d\tau) - f(s,y(s),\int_0^s K(s,\tau,y(\tau))d\tau)\right| ds.
\end{aligned} \tag{3.2}$$

Hence, since the functions f is Carathéodory and K is continuous function, the Lebesgue dominated convergence theorem implies that

$$\|Ny_n - Ny\| \to 0 \qquad \text{as} \qquad n \to +\infty.$$

Case 2. If $t \in (T, \infty)$, $T > 0$.
Since $y_n \to y$ as $n \to \infty$, we conclude that for $\varepsilon \geq 0$, there is a real number $\mathbb{T} \geq 0$ such that

$$\|y_n(t) - y(t)\| \leq \varepsilon, \text{ for any } t \geq \mathbb{T}.$$

We choose $T \geq \mathbb{T}$, then (3.2) and the hypotheses imply that

$$\begin{aligned} &|Ny_n(t) - Ny(t)| \\ &\leq M \int_0^t e^{-\omega(t-s)} p(s) \left(1 + |y_n(s) - y(s)| + \int_0^s q(\tau)(|y_n(\tau) - y(\tau)|) d\tau\right) ds \\ &\leq M(1 + \varepsilon(1 + \|q\|_{L^1})) \int_0^t e^{-\omega(t-s)} p(s) ds. \end{aligned} \tag{3.3}$$

Since (H_3), then the inequality (3.3) reduces to

$$\|N(y_n) - N(y)\| \to 0 \quad \text{as } n \to \infty.$$

So N is continuous.
Step 3: $N(\bar{B}_R)$ is equicontinuous.
Let $t_1, t_2 \in [0, T]$ with $t_2 > t_1$ and $y \in B_R$. Then, we have

$$\begin{aligned} &|(Ny)(t_2) - (Ny)(t_1)| \\ &= \left|\int_0^{t_1} (U(t_2, s) - U(t_1, s)) f\left(s, y(s), \int_0^s K(s, \tau, y(\tau)) d\tau\right) ds\right| \\ &+ \left|\int_{t_1}^{t_2} U(t_2, \tau) f\left(s, y(s), \int_0^s K(s, \tau, y(\tau)) d\tau\right) ds\right| \\ &\leq \int_0^{t_1} \|U(t_2, \tau) - U(t_1, \tau)\|_{B(E)}\; p(\tau) \left(1 + |y(s)| + \int_0^s q(\tau)|y(\tau)| d\tau\right) ds \\ &+ \int_0^{t_1} \|U(t_2, \tau) - U(t_1, \tau)\|_{B(E)} |f(s, 0, 0)| ds \\ &+ \int_0^{t_1} \int_0^s \|U(t_2, \tau) - U(t_1, \tau)\|_{B(E)} p(s) |K(s, \tau, 0)| d\tau ds \\ &+ M \int_{t_1}^{t_2} p(s) e^{-\omega(t-s)} \left(1 + |y(s)| + \int_0^s q(\tau)|y(\tau)| d\tau\right) ds \\ &+ M \int_{t_1}^{t_2} e^{-\omega(t-s)} |f(s, 0, 0)| ds \\ &+ M \int_{t_1}^{t_2} \int_0^s e^{-\omega(t-s)} p(s) |K(s, \tau, 0)| d\tau ds. \end{aligned} \tag{3.4}$$

We get

$$
\begin{aligned}
&|(Ny)(t_2)-(Ny)(t_1)|\\
&\le \int_0^{t_1}\|U(t_2,\tau)-U(t_1,\tau)\|_{B(E)}p(\tau)\left(1+R+R\int_0^s q(\tau)d\tau\right)ds\\
&+\int_0^{t_1}\|U(t_2,\tau)-U(t_1,\tau)\|_{B(E)}\|f(s,0,0)|ds\\
&+\int_0^{t_1}\int_0^s\|U(t_2,\tau)-U(t_1,\tau)\|_{B(E)}p(s)|K(s,\tau,0)|ds\\
&+M\int_{t_1}^{t_2}e^{-\omega(t-s)}\ p(s)ds.\\
&+MR\int_{t_1}^{t_2}\ p(s)e^{-\omega(t-s)}\left(1+\int_0^s q(\tau)d\tau\right)ds\\
&+M\int_{t_1}^{t_2}e^{-\omega(t-s)}|f(s,0,0)|ds\\
&+M\int_{t_1}^{t_2}\int_0^s e^{-\omega(t-s)}p(s)|K(s,\tau,0)|d\tau ds.
\end{aligned}
$$

The right-hand side of the above inequality tends to zero as $t_2-t_1\to 0$, which implies that $N(\bar{B}_R)$ is equicontinuous.

Consider the measure of noncompacteness $\mu(B)$ defined on the family of bounded subsets of the space $BC(J,E)$ by

$$\mu(B)=\omega_0^T(B)+\sup_{t\in J}e^{-\tau\sigma(t)}\overline{\alpha}(B(t))+\lim_{t\to+\infty}\sup_{t\in J}|y(t)|,$$

where

$$\sigma(t)=4M\int_0^t(\sigma_1(s)+2K^*s\sigma_2(s))ds,\tau\ge 1,\qquad \overline{\alpha}(B(t))=\sup_{s\in[0,t]}\alpha(B(s)).$$

Now, we will show that the operator N satisfies the conditions $(\imath)$ and $(\imath\imath)$ of Mönch's fixed point theorem. Suppose $B\subset BC(J,E)$ is countable and $B\subset\overline{Conv}(\{0\}\cup N(B))$.

Step 4. B is relatively compact.

Claim 1. $\omega_0^T(B)=0$

Using the properties of $\omega_0^T(\cdot)$(see [31]), and $N(\bar{B}_R)$ is equicontinuous, we get

$$\omega_0^T(B)\le\omega_0^T(\overline{Conv}(\{0\}\cup N(B)))=\omega_0^T(N(B))=0.$$

So we deduce $\omega_0^T(B)=0$.

Claim 2. $\sup_{t\in J}e^{-\tau\sigma(t)}\overline{\alpha}(B(t))=0$.

Using the properties of α, Lemma 4 and assumptions (H_1), (H_3) and (H_4), we get

$$
\begin{aligned}
&\alpha(B(t))\\
&\leq \alpha(\overline{Conv}(\{0\}\cup N(B)(t))) = \alpha(NB(t))\\
&\leq \alpha\left(\int_0^t U(t,s)f\left(s, B(s), \int_0^s K(s,\tau,B(s))d\tau\right)ds\right)\\
&\leq 2M\int_0^t \alpha\left(f\left(s, B(s), \int_0^s K(s,\tau,B(\tau))d\tau\right)ds\right)ds\\
&\leq 2M\int_0^t \left(\sigma_1(s)\alpha(B(s)) + \sigma_2(s)\alpha\left(\int_0^s K(s,\tau,B(\tau))d\tau\right)\ \right)ds\\
&\leq 2M\int_0^t \left(\sigma_1(s)\alpha(B(s)) + 2K^*\sigma_2(s)\int_0^s \alpha(B(\tau)d\tau\right)ds.\\
&\leq 2M\int_0^t \left(\sigma_1(s)\alpha(B(s)) + 2K^*\sigma_2(s)\int_0^s \alpha(B(\tau)d\tau\right)ds.\\
&\leq 2M\int_0^t \left(\sigma_1(s)\sup_{s\in[0,t]}\alpha(B(s)) + 2K^*\sigma_2(s)s\sup_{\tau\in[0,s]}\alpha(B(\tau))\right)ds.\\
&\leq 2M\int_0^t \left(\sigma_1(s)\sup_{s\in[0,t]}\alpha(B(s)) + 2K^*\sigma_2(s)s\sup_{s\in[0,t]}\alpha(B(s))\right)ds.\\
&\leq 2M\int_0^t (\sigma_1(s) + 2K^*\sigma_2(s)s)\sup_{s\in[0,t]}\alpha(B(s))ds.
\end{aligned}
$$

Therefore, we have

$$\alpha(B(t)) \quad\leq\quad 2M\int_0^t (\sigma_1(s) + 2K^* s\sigma_2(s))e^{\tau\sigma(s)}e^{-\tau\sigma(s)}\overline{\alpha}(B(s))ds,$$

then

$$e^{-\tau\sigma(t)}\alpha(B(t)) \leq \frac{1}{\tau}\sup_{t\in J} e^{-\tau\sigma(t)}\overline{\alpha}(B(t)).$$

hence

$$e^{-\tau\sigma(t)}\sup_{t\in J}\alpha(B(t)) \leq \frac{1}{\tau}\sup_{t\in J} e^{-\tau\sigma(t)}\overline{\alpha}(B(t)).$$

Since

$$e^{-\tau\sigma(t)}\sup_{s\in[0,t]}\alpha(B(s)) \leq e^{-\tau\sigma(t)}\sup_{t\in J}\alpha(B(t)),$$

we get

$$e^{-\tau\sigma(t)}\sup_{s\in[0,t]}\alpha(B(s)) \leq \frac{1}{\tau}\sup_{t\in J} e^{-\tau\sigma(t)}\overline{\alpha}(B(t)).$$

Then

$$\sup_{t\in J} e^{-\tau\sigma(t)}\overline{\alpha}(B(t)) \leq \frac{1}{\tau}\sup_{t\in J} e^{-\tau\sigma(t)}\overline{\alpha}(B(t)). \tag{3.5}$$

Since $\tau > 1$ and inequality (3.5), we obtain

$$\sup_{t\in J} e^{-\tau\sigma(t)}\overline{\alpha}(B(t)) = 0.$$

Claim 3. $\lim_{t\to+\infty}\sup_{t\in J}|y(t)| = 0$.

We have

$$\begin{aligned}
|y(t)| &\leq \tilde{M}|y_0|e^{-\varpi t} + M|y_1|e^{-\omega t}\\
&+ M\int_0^t e^{-\omega(t-s)}p(s)\left[1+|y(t)|+\int_0^s q(\tau)|y(t)|d\tau\right]ds\\
&+ M\int_0^t e^{-\omega(t-s)}|f(s,0,0)|ds\\
&+ M\int_0^t\int_0^s e^{-\omega(t-s)}p(s)|K(s,\tau,0)|d\tau ds\\
&\leq \tilde{M}|y_0|e^{-\varpi t} + +M|y_1|e^{-\omega t}\\
&+ M\int_0^t e^{-\omega(t-s)}|f(s,0,0)|ds\\
&+ M\left(1+\frac{K}{\gamma}\right)\int_0^t e^{-\omega(t-s)}p(s)ds\\
&+ M\int_0^t e^{-\omega(t-s)}p(s)\left[|y(t)|+\int_0^s q(\tau)|y(t)|d\tau\right]ds.
\end{aligned}$$

By Lemma 5, we have

$$\begin{aligned}
&|y(t)|\\
&\leq h(t) + \int_0^t Me^{-\omega(t-s)}p(s)\\
&\times\left[h(s)+\int_0^s h(\tau)(Me^{-\omega(t-s)}p(\tau)+q(\tau))\exp\left(\int_\tau^s (Me^{-\omega(t-s)}p(\delta)+q(\delta)d\delta)\right)d\tau\right]ds,
\end{aligned}$$

where

$$\begin{aligned}
h(t) &= \tilde{M}|y_0|e^{-\varpi t} + M|y_1|e^{-\omega t}\\
&+ M\int_0^t e^{-\omega(t-s)}|f(s,0,0)|ds\\
&+ M\left(1+\frac{K}{\gamma}\right)\int_0^t e^{-\omega(t-s)}p(s)ds.
\end{aligned}$$

Then

$$|y(t)| \quad \leq h(t) + \xi \int_0^t e^{-\omega(t-s)} p(s)ds,$$

where

$$\xi = \left[\tilde{M}\,|y_0| + M\,|y_1| + Mf^* + M\left(1+\frac{K}{\gamma}\right)p^*\right][1+p^*(Mp^* + \|q\|_{L^1})]\exp\left(Mp^* + \|q\|_{L^1}\right).$$

It follows immediately by assumptions $(H1)-(H4)$ that

$$\lim_{t\to+\infty}\sup_{t\in J}|y(t)| = 0.$$

From **Claims 1, 2, 3,** we obtain

$$\mu(B) = 0.$$

Thus, we find that B is relatively compact.
Step 5. A priori bounds.
We now show there exists an open set $Y \subseteq B$ with $y \neq \lambda N(y)$, for $\lambda \in (0,1)$ and $y \in \partial Y$. Let $y \in B$ and $y = \lambda N(y)$ for some $0 < \lambda < 1$. Then

$$y(t) = -\lambda\frac{\partial}{\partial s}U(t,0)y_0 + \lambda U(t,0)y_1 + \lambda\int_0^t U(t,s)f\left(s, y(s), \int_0^s K(s,\tau,y(\tau))d\tau\right)ds.$$

This implies by $(H1)-(H4)$ that, for each $t \in J$, we have

$$\begin{aligned}
|y(t)| \quad &\leq \quad \left\|\frac{\partial}{\partial s}U(t,0)\right\|_{B(E)}|y_0| + \|U(t,s)\|_{B(E)}\,|y_1| \\
&+ \quad \|U(t,s)\|_{B(E)}\int_0^t p(s)e^{-\omega(t-s)}\left(1+|y(s)|+\int_0^s q(\tau)|y(\tau)|)d\tau\right)ds \\
&+ \quad M\int_0^t e^{-\omega(t-s)}|f(s,0,0)|ds \\
&+ \quad M\int_0^t\int_0^s e^{-\omega(t-s)}p(s)|k(s,\tau,0)|dsdt \\
&\leq \quad \tilde{M}\,|y_0| + M\,|y_1| + Mf^* + M\left(1+\frac{K}{\gamma}\right)p^* \\
&+ \quad M\int_0^t p(s)e^{-\omega(t-s)}\left(|y(s)|+\int_0^s q(\tau)|y(\tau)|d\tau\right)ds.
\end{aligned}$$

By Lemma 5, we have

$$\begin{aligned} |y(t)| &\leq h(t) + \xi \int_0^t e^{-\omega(t-s)} p(s) ds. \\ &\leq \tilde{M}\,|y_0| + M\,|y_1| + Mf^* + M\left(1 + \frac{K}{\gamma} + \xi\right) p^* = \Lambda. \end{aligned}$$

Set

$$Y = \{y \in BC(J, E) : \|y\| < \Lambda + 1\}.$$

By the choice of Y, there is no $y \in \partial Y$ such that $y = \lambda N(y)$, for $\lambda \in (0, 1)$. Thus by Mönch fixed point theorem, the operator $N : \bar{Y} \to BC(J, E)$ has at least one fixed point which is a mild solution of problem (1.1)-(1.2).

4 Attractivity of solutions

Now we investigate the uniform local attractivity for solutions of problem (1.1)-(1.2). Let y^* be a solution to problem (1.1)-(1.2) and $\bar{B}(y^*, r_0)$ with $r_0 \geq \dfrac{Mp^*}{1 - Mp^*(1 + \|q\|_{L^1})}$ the closed ball in $BC(J, E)$. Then, for $y \in \bar{B}(y^*, r_0)$ by (H_1)-(H_4) , we have

$$\begin{aligned} &|Ny(t) - y^*(t)| = |Ny(t) - Ny^*(t)| \\ &\leq \int_0^t \|U(t,s)\|_{B(E)} \left| f\left(s, y(s), \int_0^s K(s,\tau,y(\tau))d\tau\right) - f\left(s, y^*(s), \int_0^s K(s,\tau,y^*(\tau))d\tau\right) \right| ds \\ &\leq M \int_0^t e^{-\omega(t-s)} p(s) \left(1 + |y_2(s) - y_1(s)| + \int_0^s |K(s,\tau,y_2(\tau)) - K(s,\tau,y_1(\tau))|)d\tau\right) ds \\ &\leq \|U(t,s)\|_{B(E)} \int_0^t e^{-\omega(t-s)} p(s) \left(1 + |y(s) - y^*(s)| + \int_0^s q(\tau)(|y(\tau) - y^*(\tau)|)d\tau\right) ds \\ &\leq M \int_0^t e^{-\omega(t-s)} p(s) ds \\ &+ Mr_0 \int_0^t e^{-\omega(t-s)} p(s) \left(1 + \int_0^s q(\tau)d\tau\right) ds \\ &\leq Mp^* + Mp^*(1 + \|q\|_{L^1})r_0 \\ &< r_0. \end{aligned}$$

Therefore, we get $N(\bar{B}(y^*, r_0)) \subset \bar{B}(y^*, r_0)$.
So, for any solution $y_1, y_2 \in \bar{B}(y^*, r_0)$ to problem (1.1)-(1.2) and $t \in J$, we have

$$
\begin{aligned}
&|Ny_2(t) - Ny_1(t)| \\
&\leq \int_0^t \|U(t,s)\|_{B(E)} \|f(s, y_2(s), \int_0^s K(s,\tau,y_2(\tau))d\tau) - f(s, y_1(s), \int_0^s K(s,\tau,y_1(\tau))d\tau)\Big| ds \\
&\leq M \int_0^t e^{-\omega(t-s)} p(s) \left(1 + |y_2(s) - y_1(s)| + \int_0^s |K(s,\tau,y_2(\tau)) - K(s,\tau,y_1(\tau))|)d\tau\right) ds \\
&\leq M \int_0^t e^{-\omega(t-s)} p(s) \left(1 + |y_2(s) - y_1(s)| + \int_0^s q(\tau)(|y_2(\tau) - y_1(\tau)|)d\tau\right) ds \\
&\leq M \int_0^t e^{-\omega(t-s)} p(s) ds \\
&+ M r_0 \int_0^t e^{-\omega(t-s)} p(s) \left[1 + \int_0^s q(\tau)d\tau\right] ds \\
&\leq (M + M(1 + \|q\|_{L^1}) r_0) \int_0^t e^{-\omega(t-s)} p(s) ds.
\end{aligned}
$$

Hence, from (H_3), we conclude that for $\varepsilon \geq 0$, there are real numbers $T \geq 0$ such that

$$
\int_0^t e^{-\omega(t-s)} p(s) ds \leq \frac{\varepsilon}{M + M(1 + \|q\|_{L^1}) r_0}, \text{ for all } t \geq T,
$$

Then from the above inequality it follows that

$$
|y_2(t) - y_1(t)| \leq \varepsilon \text{ for all } t \geq T.
$$

Consequently, the solutions of problem (1.1)-(1.2) are uniformly locally attractive.

5 Example

Let us consider the following class of partial differential equations;

$$
\begin{cases}
\dfrac{\partial^2}{\partial t^2} z(t,\tau) &= \dfrac{\partial^2}{\partial \tau^2} z(t,\tau) + a(t) \dfrac{\partial}{\partial t} z(t,\tau) \\
& + \dfrac{\sin(t) e^{-|z(t,\tau)| - \nu t}}{t^2 + 1} \\
& + \dfrac{\ln(1 + 2e^{-\nu t}) z(t,\tau)}{(t^2+1)(1 + |z(t,\tau)|)} \\
& + \dfrac{\sin(e^{\nu t})}{(t^2+1)^2} \displaystyle\int_0^t \dfrac{\ln(e^{-s} + 2t) \cos(z(s,\tau)) e^{-\nu(t-s)}}{(2 + 2t^2 + s^2)^3} ds, \quad t \in J, \ \tau \in [0,\pi], \\
z(t,0) = & z(t,\pi) = 0 \quad t \in J, \\
\dfrac{\partial}{\partial t} z(0,\tau) &= \psi(\tau) \quad \tau \in [0,\pi],
\end{cases}
\tag{5.1}
$$

where $a : J \to \mathbb{R}$ is a Hölder continuous function and ν is a positive constant such that $\nu > 1$.
Let $E = L^2([0,\pi],\mathbb{R})$ be the space of 2-integrable functions from $[0,\pi]$ into $\mathbb{R}$, and let $H^2([0,\pi],\mathbb{R})$ be the Sobolev space of functions $x : [0,\pi] \to \mathbb{R}$, such that $x'' \in L^2([0,\pi],\mathbb{R})$. We consider the operator $A_1 y(\tau) = y''(\tau)$ with domain $D(A_1) = H^2(\mathbb{R},\mathbb{C})$, infinitesimal generator of strongly continuous cosine function $C(t)$ on E. Moreover, we take $A_2(t)y(s) = a(t)y'(s)$, defined on $H^1([0,\pi],\mathbb{R})$, and consider the closed linear operator $A(t) = A_1 + A_2(t)$ which, generates an evolution operator U, defined by

$$U(t,s) = \sum_{n\in\mathbb{Z}} z_n(t,s)\langle x, w_n\rangle w_n,$$

where z_n is a solution to the following scalar initial value problem,

$$\begin{cases} z''(t) = -n^2 z(t) + ina(t)z(t) \\ z(0) = 0, \qquad z'(0) = 1. \end{cases}$$

It follows from this representation that

$$\|U(t,s)\|_{B(E)} \le e^{-(t-s)}, \text{ for every } (t,s) \in \Delta.$$

Set

$$z(t)(\tau) = w(t)(\tau),\ t \ge 0,\ \tau \in [0,\pi],$$

$$f(t,u,v)(\tau) = \frac{\sin(t)e^{-|u(t,\tau)|-\nu t}}{t^2+1} + \frac{\ln(1+2e^{-\nu t}))u(t,\tau)}{(t^2+1)(1+|u(t,\tau)|)} + \frac{\sin(e^{-\nu t})}{(t^2+1)^2}v(t,\tau),$$

$$k(t,s,u)(\tau) = \frac{\ln(2t+e^{-s})\cos(u(t,s))e^{-\nu(t-s)}}{(1+t^2+s^2)^3},$$

and

$$\frac{\partial}{\partial t}z(0)(\tau) = \frac{d}{dt}w(0)(\tau),\ \tau \in [0,\pi].$$

Moreover, applying the inequalities

$$\ln(1+x) \le x, \qquad \sin x \le x,$$

We have

$$\begin{aligned} &|f(t,u_2,v_2)(\tau) - f(t,u_1,v_1)(\tau)| \\ &\le \frac{e^{-\nu t}}{t^2+1} + \frac{\ln(1+e^{-\nu t})}{(t^2+1)}|u_2(t,\tau) - u_1(t,\tau)| \\ &+ \frac{e^{-\nu t}}{(t^2+1)^2}|v_2(t,\tau) - v_1(t,\tau)| \\ &\le \frac{e^{-\nu t}}{t^2+1}(1 + |u_2(t,\tau) - u_1(t,\tau)| + |v_2(t,\tau) - v_1(t,\tau)|), \end{aligned} \tag{5.2}$$

and

$$|K(t,s,u)(\tau)-K(t,s,v)(\tau)|\leq\frac{\ln(1+2t)}{(t^2+1)^3}|u(t,\tau)-v(t,\tau)|. \tag{5.3}$$

Hence conditions $(H3)(a)$ and $(H4)(a)$ are satisfied with

$$p(t)=\frac{e^{-\nu t}}{t^2+1},\quad q(t)=\frac{\ln(1+2t)}{(t^2+1)^3}.$$

Also, we have

$$\begin{aligned}\int_0^t e^{-(t-s)}|f(s,0,0)|ds &= \int_0^t e^{-(t-s)}\frac{e^{-\nu s}\sin(s)}{s^2+1}ds\\ &= e^{-t}\int_0^t\frac{1}{s^2+1}ds\\ &\leq e^{-t}\arctan(t)\ \longrightarrow 0\ as\ t\to\infty,\end{aligned}$$

,

$$\begin{aligned}\int_0^t e^{-(t-s)}p(s)ds &= \int_0^t\frac{e^{-(t-s)}e^{-\nu s}}{s^2+1}ds\\ &\leq e^{-t}\arctan(t)\ \longrightarrow 0\ as\ t\to\infty,\end{aligned}$$

and

$$\begin{aligned}|k(t,s,0)| &\leq \frac{\ln(2t+e^{-s})e^{-\nu(t-s)}}{(1+t^2+s^2)^3}\\ &\leq \frac{\ln(1+2t)e^{-\nu(t-s)}}{(t^2+1)^3}\\ &\leq \frac{2te^{-\nu(t-s)}}{(t^2+1)^3}\\ &\leq \frac{25\sqrt{5}}{108}e^{-\nu(t-s)}.\end{aligned}$$

By (5.2), for any bounded sets $D_1,D_2\subset E$, we get

$$\alpha(f(t,D_1,D_2))\leq\frac{\ln(1+e^{-\nu t})}{t^2+1}\alpha(D_1)+\frac{\sin(e^{-\nu t})}{(t^2+1)^2}\alpha(D_2)\text{ for a.e } t\in J.$$

By (5.3), for any bounded sets $D\subset E$, we get

$$\alpha(K(t,s,D))\leq\frac{25\sqrt{5}}{108}\alpha(D)\text{ for a.e } t\in J.$$

Hence $(H3)(c)$ and $(H4)(c)$ are satisfied.

Consequently, (5.1) can be written in the abstract form (1.1)-(1.2). The existence of a mild solutions can be deduced from an application of Theorem 10. Moreover, these solutions are uniformly locally attractive.

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