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SOME STABILITY RESULTS FOR COUPLED FIXED POINT ITERATIVE PROCESS IN A COMPLETE METRIC SPACE

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Abstract. In the paper [M. O. Olatinwo, Stability of coupled fixed point iterations and the continuous dependence of coupled fixed points, Communications on Applied Nonlinear Analysis 19 (2012), 71-83], the author has extended the notion of stability of fixed point iterative procedures contained in the paper [A. M. Harder and T. L. Hicks, Stability results for fixed point iteration procedures, Math. Japonica 33 (1988), 693-706], as well as the continuous dependence of fixed points to the coupled fixed point settings by employing the contractive conditions and the coupled fixed point iteration in the article [F. Sabetghadam, H. P. Masiha and A. H. Sanatpour, Some coupled fixed point theorems in cone metric spaces, Fixed Point Theory and Applications, Article ID 125426 (2009)]. In the present paper, we obtain some results on stability of coupled fixed point iterative procedures by using rational type contractive conditions.

1 Introduction

Let (X, d) be a complete metric space and $T\colon X \to X$. Ostrowski [20] gave a pioneering result on the stability of iterative procedure in metric space for Picard iteration.
Harder and Hicks [11] proved some stability theorems for the Picard, Mann and Kirk's iterative processes by employing some contractive-type conditions.
We now state the first formal definition of stability for general iterative scheme due to Harder and Hicks [11]:

Definition 1 (Harder and Hicks [11]). *Let (X, d) be a complete metric space and $T\colon X \to X$. Let $F(T) = \{p \in X \mid Tp = p\}$ denote the set of fixed points of T. Let $\{x_n\}_{n=0}^{\infty} \subset X$ be the sequence generated by an iterative procedure involving the*

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operator T, that is,

$$x_{n+1} = f(T, x_n),\ n = 0,\ 1,\ 2, \cdots, \tag{1.1}$$

where $x_0 \in X$ is the initial approximation and f is some function. Suppose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty} \subset X$ and set $\epsilon_n = d(y_{n+1}, f(T, y_n))$, $(n = 0,\ 1,\ 2, \cdots)$. Then, the iterative procedure (1.1) is said to be $T-$stable, or, stable with respect to T if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = p$.

The following contractive condition was employed by Harder and Hicks [11]: For $T\colon X \to X$, there exists $\alpha \in [0, 1)$ such that, $\forall\ x,\ y \in X$, we have

$$d(Tx, Ty) \leq \alpha d(x, y). \tag{1.2}$$

In addition, the following contractive definition was considered by Harder and Hicks [11]: For $T\colon X \to X$, there exist some real numbers $0 \leq \alpha < 1,\ 0 \leq \beta < \frac{1}{2},\ 0 \leq \gamma < \frac{1}{2}$, such that, $\forall\ x,\ y \in X$, then

$$\left.\begin{array}{l} d(Tx, Ty) \leq \alpha d(x, y) \\ d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)] \\ d(Tx, Ty) \leq \gamma[d(x, Ty) + d(y, Tx)]. \end{array}\right\} \tag{1.3}$$

The contractive conditions in (1.2) and (1.3) were both used by Harder and Hicks [11] to establish stability results for various iterative processes.

Rhoades [21] extended the results of Harder and Hicks [11] by employing the following contractive condition: For $T\colon X \to X$, there exists $c \in [0, 1)$ such that, $\forall\ x,\ y \in X$, we have

$$d(Tx, Ty) \leq c \max\{d(x, y), d(x, Ty), d(y, Tx)\}. \tag{1.4}$$

Also, Rhoades [22] obtained generalizations and extensions of the results of [21] by using the following contractive condition: For $T\colon X \to X$, there exists $c \in [0, 1)$ such that, $\forall\ x,\ y \in X$,

$$d(Tx, Ty) \leq c \max\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, d(x, Ty), d(y, Tx)\}. \tag{1.5}$$

Furthermore, Osilike [18] generalized and extended some of the results of Rhoades [21, 22] for a larger class of contractive-type operators. In [18], he employed the following contractive condition: For $T\colon X \to X$, there exist $\lambda \in [0, 1),\ L \geq 0$, such that,

$$d(Tx, Ty) \leq Ld(x, Tx) + \lambda d(x, y),\ \forall\ x,\ y \in X. \tag{1.6}$$

Harder and Hicks [11], Rhoades [21, 22] and Osilike [18] used the method of the summability theory of infinite matrices to prove various stability results for certain contractive definitions. However, Osilike and Udomene [19] introduced a shorter method to prove stability results for various iterative processes using the condition (1.6).

However, using the same method of proof as in [19] and the same contractive conditions as in Harder and Hicks [11], Berinde [4] also established some stability results for the same iterative processes for which the authors of [11] had proved their results. Imoru and Olatinwo [12] extended some of the results of Harder and Hicks [11], Rhoades [21, 22], Berinde [4], Osilike [18], Osilike and Udomene [19] and others to a much more larger class of operators than those satisfying the contractive condition (1.6). In [12], the following contractive condition was used: For $T\colon X \to X$, there exist $\lambda \in [0, 1)$ and a monotone increasing function $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$, such that

$$d(Tx, Ty) \leq \varphi(d(x, Tx)) + \lambda d(x, y),\ \forall\ x,\ y \in X. \tag{1.7}$$

We give the following definition which will be considered in the sequel.

Definition 2 (Berinde [5, 6]). *Consider a function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfying*

(i) *ψ is monotone increasing;*

(ii) *$\psi^n(t) \to\ 0,\ $ as $n \to \infty$;*

(iii) *$\sum_{n=0}^{\infty} \psi^n(t)$ converges for all $t > 0$.*

1. *A function ψ satisfying (i) and (ii) above is called a comparison function.*
2. *A function ψ satisfying (i) and (iii) above is called a (c)-comparison function.*

Remark 3. *In [5, 6], we have the following:*

(i) *Any (c)-comparison function is a comparison function.*

(ii) *Every comparison function satisfies $\psi(0) = 0$.*

2 Preliminaries

In this section, we shall consider some basic definitions and results on coupled fixed point theorems:

Definition 4. *[9, 10, 14, 23] Let (X, d) be a metric space. An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $T\colon X \times X \to X$ if $T(x, y) = x$ and $T(y, x) = y$.*

Interested readers can also see the articles of the author [15, 16, 17] on the concept of coupled fixed points.

Let (X, d) be a metric space and $T\colon X \times X \to X$ a mapping. For $(x_0, y_0) \in X \times X$, the sequence $\{(x_n, y_n)\}_{n=0}^{\infty} \subset X \times X$ defined iteratively by

$$x_{n+1} = T(x_n, y_n), \quad y_{n+1} = T(y_n, x_n), \; n = 0, 1, 2, \cdots, \tag{2.1}$$

is said to be a *coupled fixed point iterative procedure*, according to [17].

Furthermore, to the best of our knowledge, the pioneering and formal definition of stability of coupled fixed point iteration is the following due to Olatinwo [17]:

Definition 5. *[Olatinwo [17]] Let (X, d) be a complete metric space. Suppose that*

$$C_{fix}(T) = \{(x^*, y^*) \in X \times X \mid T(x^*, y^*) = x^*, \; T(y^*, x^*) = y^*\}$$

is the set of coupled fixed points of T. Let $\{(x_n, y_n)\}_{n=0}^{\infty} \subset X \times X$ be the sequence generated by an iterative procedure involving T defined by

$$x_{n+1} = f(T, (x_n, y_n)), \quad y_{n+1} = f(T, (y_n, x_n)), \; n = 0, 1, 2, \cdots, \tag{2.2}$$

where $(x_0, y_0) \in X \times X$ is the initial approximation and f is some function. Suppose $\{(x_n, y_n)\}_{n=0}^{\infty} \subset X \times X$ converges to a coupled fixed point (x^*, y^*) of T. Let $\{(u_n, v_n)\}_{n=0}^{\infty}$ be a sequence in $X \times X$ and set

$$\epsilon_n = d(u_{n+1}, f(T, (u_n, v_n)), \delta_n = d(v_{n+1}, f(T, (v_n, u_n)), \; (n = 0, 1, 2, \cdots).$$

Then, the coupled fixed point iterative procedure (M) is said to be $T-$stable, or, stable with respect to T if and only if $\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} \delta_n = 0$ implies $\lim_{n\to\infty} u_n = x^*$ and $\lim_{n\to\infty} v_n = y^*$.

Remark 6. *If in Eqn. (M), $f(T, (x_n, y_n)) = T(x_n, y_n)$ and $f(T, (y_n, x_n)) = T(y_n, x_n)$. then we obtain the coupled fixed point iterative procedure of [23].*

Bhaskar and Lakshmikantham [7] proved a coupled fixed point theorem in a metric space endowed with partial order by employing a weak contractive type condition. For excellent study on coupled fixed point theorems, we implore our interested readers to consult Abbas and Beg [1], Beg *et al.* [3], Chang and Ma [8], Ciric and Lakshmikantham [10], Lakshmikantham and Ciric [14] and Sabetghadam *et al.* [23], in addition to [7] earlier mentioned.

In Olatinwo [17], stability results have been proved for the following three contractive conditions for which the existence of a unique coupled fixed point has been established by Sabetghadam *et al.* [23]. Let (X, d) be a metric space. Then, we have the following:

(i) A mapping $T\colon X\times X\to X$ is said to be a $(k,\mu)-$contraction if and only if there exist two constants $k\geq 0,\ \mu\geq 0,\ k+\mu<1$, such that, $\forall\ x,\ y,\ u,\ v\in X$, we have

$$d(T(x,y),T(u,v))\leq kd(x,u)+\mu d(y,v). \tag{2.3}$$

(ii) For a mapping $T\colon X\times X\to X$, there exist constants $k\geq 0,\ \mu\in[0,\frac{1}{2}),\ k+\mu<1$, such that

$$d(T(x,y),T(u,v))\leq kd(T(x,y),x)+\mu d(T(u,v),u),\ \forall\ x,\ y,\ u,\ v\in X. \tag{2.4}$$

(iii) For a mapping $T\colon X\times X\to X$, there exist constants $k\geq 0,\ \mu\geq 0,\ k+\mu<1$, such that

$$d(T(x,y),T(u,v))\leq kd(T(x,y),u)+\mu d(T(u,v),x),\ \forall\ x,\ y,\ u,\ v\in X. \tag{2.5}$$

We present the following lemmas which will be used in the sequel.

Lemma 7 (Berinde [4, 5, 6]). *If γ is a real number such that $0\leq\gamma<1$, and $\{b_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim\limits_{n\to\infty}b_n=0$, then for any sequence of positive numbers $\{a_n\}_{n=0}^{\infty}$ satisfying*

$$a_{n+1}\leq\gamma a_n+b_n,(n=0,1,2,\cdots),$$

we have $\lim\limits_{n\to\infty}a_n=0$.

Lemma 8 (Imoru *et al.* [13]). *If $\psi\colon I\!\!R^+\to I\!\!R^+$ is a subadditive comparison function and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim\limits_{n\to\infty}\epsilon_n=0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying*

$$u_{n+1}\leq\sum_{k=0}^{m}\delta_k\psi^k(u_n)+\epsilon_n,\ n=0,1,2,\cdots,$$

where $\delta_k\in[0,1),\ k=0,1,\cdots,m,\ 0\leq\sum_{k=0}^{m}\delta_k\leq 1$, we have $\lim\limits_{n\to\infty}u_n=0$.

We now establish some stability results for certain contractive conditions.

3 Main Results

Theorem 9. *Let (X,d) be a complete metric space and $T\colon X\times X\to X$ a mapping satisfying the rational type contractive condition*

$$d(T(x,y),T(u,v))\leq\frac{\alpha d(x,T(x,y)).d(u,T(u,v))}{d(x,u)}+\beta d(x,u), \tag{3.1}$$

$\forall\ x,\ y,\ u,\ v,\ x \neq\ u,\ \alpha \geq 0,\ \beta \geq 0,\ \alpha+\beta < 1$. *Suppose T has a coupled fixed point (x^*, y^*). For $(x_0, y_0) \in X \times X$, let $\{(x_n, y_n)\}_{n=0}^{\infty} \subset X \times X$ be the coupled fixed point iterative procedure defined by $(S1)$. Then, the coupled fixed point iterative procedure is $T-$stable.*

Proof. Let $\{x_n\}_{n=0}^{\infty},\ \{y_n\}_{n=0}^{\infty} \subset X,\ \epsilon_n = d(u_{n+1}, T(u_n, v_n))$ and

$$\delta_n = d(v_{n+1}, T(v_n, u_n)).$$

Assume also that $\lim\limits_{n\to\infty} \epsilon_n = \lim\limits_{n\to\infty} \delta_n = \lim\limits_{n\to\infty} (\epsilon_n + \delta_n) = 0$.

Then, we shall establish that $\lim\limits_{n\to\infty} u_n = x^*$ and $\lim\limits_{n\to\infty} v_n = y^*$. Therefore, by using (3.1), we obtain

$$\begin{aligned} d(u_{n+1}, x^*) &\leq d(u_{n+1}, T(u_n, v_n)) + d(T(u_n, v_n), x^*), \\ &= d(T(u_n, v_n), T(x^*, y^*)) + \epsilon_n, \\ &\leq \tfrac{\alpha.d(u_n, T(u_n, v_n)).d(x^*, T(x^*, y^*))}{d(u_n, x^*)} + \beta d(u_n, x^*) + \epsilon_n \\ &= \tfrac{\alpha.d(u_n, T(u_n, v_n)).d(x^*, x^*)}{d(u_n, x^*)} + \beta d(u_n, x^*) + \epsilon_n \\ &= \beta d(u_n, x^*) + \epsilon_n. \end{aligned} \tag{3.2}$$

Similarly,

$$\begin{aligned} d(v_{n+1}, y^*) &\leq d(v_{n+1}, T(v_n, u_n)) + d(T(v_n, u_n), y^*), \\ &= d(T(v_n, u_n), T(y^*, x^*)) + \delta_n, \\ &\leq \tfrac{\alpha.d(v_n, T(v_n, u_n)).d(y^*, T(y^*, x^*))}{d(v_n, y^*)} + \beta d(v_n, y^*) + \delta_n \\ &= \tfrac{\alpha.d(v_n, T(v_n, u_n)).d(y^*, y^*)}{d(v_n, y^*)} + \beta d(v_n, y^*) + \delta_n \\ &= \beta d(v_n, y^*) + \delta_n. \end{aligned} \tag{3.3}$$

Adding (3.2) and (3.3) gives

$$d(u_{n+1}, x^*) + d(v_{n+1}, y^*) \leq \beta[d(u_n, x^*) + d(v_n, y^*)] + \epsilon_n + \delta_n. \tag{3.4}$$

In (3.4), letting $a_n = d(u_n, x^*) + d(v_n, y^*),\ b_n = \epsilon_n + \delta_n$, we have $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} (\epsilon_n + \delta_n) = 0,\ 0 \leq \gamma = \beta < 1$, then the conditions of Lemma 7 are satisfied. Therefore, using Lemma 7 in (3.4) yields $\lim\limits_{n\to\infty} [d(u_n, x^*) + d(v_n, y^*)] = 0$. That is, $\lim\limits_{n\to\infty} d(u_n, x^*) = 0$ and $\lim\limits_{n\to\infty} d(v_n, y^*) = 0$ (or, $\lim\limits_{n\to\infty} u_n = x^*$ and $\lim\limits_{n\to\infty} v_n = y^*$). Conversely, let $\lim\limits_{n\to\infty} d(u_n, x^*) = \lim\limits_{n\to\infty} d(v_n, y^*) = 0$ and $\lim\limits_{n\to\infty} (d(u_n, x^*) + d(v_n, y^*)) = 0$. Then, using (3.1) again, we have

$$\begin{aligned} \epsilon_n + \delta_n &= d(u_{n+1}, T(u_n, v_n)) + d(v_{n+1}, T(v_n, u_n)) \\ &\leq d(u_{n+1}, x^*) + d(x^*, T(u_n, v_n)) \\ &\quad + d(v_{n+1}, y^*) + d(y^*, T(v_n, u_n)) \\ &= d(u_{n+1}, x^*) + d(v_{n+1}, y^*) \\ &\quad + d(T(x^*, y^*), T(u_n, v_n)) + d(T(y^*, x^*), T(v_n, u_n)) \\ &\leq d(u_{n+1}, x^*) + d(v_{n+1}, y^*) + \tfrac{\alpha.d(x^*, T(x^*, y^*)).d(u_n, T(u_n, v_n))}{d(x^*, u_n)} \\ &\quad + \beta d(x^*, u_n) + \tfrac{\alpha.d(y^*, T(y^*, x^*)).d(v_n, T(v_n, u_n))}{d(y^*, v_n)} + \beta d(y^*, v_n) \end{aligned}$$

$$= d(u_{n+1},x^*) + d(v_{n+1},y^*) + \frac{\alpha.d(x^*,x^*).d(u_n,T(u_n,v_n))}{d(x^*,u_n)}$$
$$+\beta d(x^*,u_n) + \frac{\alpha.d(y^*,y^*).d(v_n,T(v_n,u_n))}{d(y^*,v_n)} + \beta d(y^*,v_n)$$
$$= d(u_{n+1},x^*) + d(v_{n+1},y^*) + \beta d(x^*,u_n) + \beta d(y^*,v_n) \to \ 0 \text{ as } n \to \infty,$$

from which it follows that $\lim_{n\to\infty}(\epsilon_n + \delta_n) = 0$, that is, $\lim_{n\to\infty}\epsilon_n = \lim_{n\to\infty}\delta_n = 0$. □

Theorem 10. *Let (X, d) be a complete metric space and $T\colon X \times X \to X$ a mapping satisfying the rational type contractive condition*

$$d(T(x,y),T(u,v)) \leq \alpha\frac{d(x,T(u,v).d(x,T(x,y)).d(u,T(u,v))}{d(x,u)+d(u,T(u,v))} + \beta d(x,u), \tag{3.5}$$

$\forall\ x,\ y,\ u,\ v,\ \alpha \geq 0,\ \beta \in [0,1)$ and $d(x,u) + d(u,T(u,v) > 0$. Suppose T has a coupled fixed point (x^*,y^*). For $(x_0,y_0) \in X \times X$, let $\{(x_n,y_n)\}_{n=0}^{\infty} \subset X \times X$ be the coupled fixed point iterative procedure defined by $(S1)$. Then, the coupled fixed point iterative procedure is $T-$stable.

Proof. Let $\{x_n\}_{n=0}^{\infty},\ \{y_n\}_{n=0}^{\infty} \subset X,\ \epsilon_n = d(u_{n+1},T(u_n,v_n))$ and

$$\delta_n = d(v_{n+1},T(v_n,u_n)).$$

Assume also that $\lim_{n\to\infty}\epsilon_n = \lim_{n\to\infty}\delta_n = \lim_{n\to\infty}(\epsilon_n+\delta_n) = 0$. Then, we shall establish that $\lim_{n\to\infty}u_n = x^*$ and $\lim_{n\to\infty}v_n = y^*$.

Therefore, by using (3.5), we obtain

$$\begin{aligned} d(u_{n+1},x^*) &\leq d(u_{n+1},T(u_n,v_n)) + d(T(u_n,v_n),x^*) \\ &= d(T(u_n,v_n),T(x^*,y^*)) + \epsilon_n, \\ &\leq \frac{\alpha.d(u_n,T(x^*,y^*)).d(u_n,T(u_n,v_n)).d(x^*,T(x^*,y^*))}{d(u_n,x^*)+d(x^*,T(x^*,y^*))} + \beta d(u_n,x^*) + \epsilon_n \\ &= \frac{\alpha.d(u_n,x^*).d(u_n,T(u_n,v_n)).d(x^*,x^*)}{d(u_n,x^*)+d(x^*,x^*)} + \beta d(u_n,x^*) + \epsilon_n \\ &= \beta d(u_n,x^*) + \epsilon_n \end{aligned} \tag{3.6}$$

Similarly,

$$\begin{aligned} d(v_{n+1},y^*) &\leq d(v_{n+1},T(v_n,u_n)) + d(T(v_n,u_n),y^*), \\ &= d(T(v_n,u_n),T(y^*,x^*)) + \delta_n, \\ &\leq \frac{\alpha.d(v_n,T(y^*,x^*)).d(v_n,T(v_n,u_n)).d(y^*,T(y^*,x^*)}{d(v_n,y^*)+d(y^*,T(y^*,x^*))} + \beta d(v_n,y^*) + \delta_n \\ &= \frac{\alpha.d(v_n,y^*)d(v_n,T(v_n,u_n)).d(y^*,y^*)}{d(v_n,y^*)+d(y^*,y^*)} + \beta d(v_n,y^*) + \delta_n \\ &= \beta d(v_n,y^*) + \delta_n \end{aligned} \tag{3.7}$$

Adding (3.6) and (3.7) gives

$$\begin{aligned} d(u_{n+1},x^*) + d(v_{n+1},y^*) &\leq \beta d(u_n,x^*) + \beta d(v_n,y^*) + \epsilon_n + \delta_n \\ &= \beta[d(u_n,x^*) + d(v_n,y^*)] + \epsilon_n + \delta_n. \end{aligned} \tag{3.8}$$

In (3.8), letting $a_n = d(u_n, x^*) + d(v_n, y^*)$, $b_n = \epsilon_n + \delta_n$, we have $\lim_{n\to\infty} b_n = \lim_{n\to\infty}(\epsilon_n + \delta_n) = 0$, $0 \leq \gamma = \beta < 1$, then the hypotheses of Lemma 7 are satisfied. Therefore, using Lemma 7 in (3.8) yields $\lim_{n\to\infty}[d(u_n, x^*) + d(v_n, y^*)] = 0$. That is, $\lim_{n\to\infty} d(u_n, x^*) = 0$ and $\lim_{n\to\infty} d(v_n, y^*) = 0$ (or, $\lim_{n\to\infty} u_n = x^*$ and $\lim_{n\to\infty} v_n = y^*$). Conversely, let $\lim_{n\to\infty} d(u_n, x^*) = \lim_{n\to\infty} d(v_n, y^*) = \lim_{n\to\infty}(d(u_n, x^*) + d(v_n, y^*)) = 0$. Then by using (3.5) again,

$$\begin{aligned}
\epsilon_n + \delta_n &= d(u_{n+1}, T(u_n, v_n)) + d(v_{n+1}, T(v_n, u_n)) \\
&\leq d(u_{n+1}, x^*) + d(x^*, T(u_n, v_n)) + d(v_{n+1}, y^*) + d(y^*, T(v_n, u_n)) \\
&= d(u_{n+1}, x^*) + d(v_{n+1}, y^*) + d(T(x^*, y^*), T(u_n, v_n)) + d(T(y^*, x^*), T(v_n, u_n)) \\
&\leq d(u_{n+1}, x^*) + d(v_{n+1}, y^*) + \tfrac{\alpha.d(x^*,T(u_n,v_n)).d(x^*,T(x^*,y^*)).d(u_n,T(u_n,v_n))}{d(x^*,u_n)+d(u_n,T(u_n,v_n)} \\
&\quad + \beta d(x^*, u_n) + \tfrac{\alpha.d(y^*,T(v_n,u_n).d(y^*,T(y^*,x^*)).d(v_n,T(v_n,u_n))}{d(y^*,v_n)+d(v_n,T(v_n,u_n)} + \beta d(y^*, v_n) \\
&= d(u_{n+1}, x^*) + d(v_{n+1}, y^*) + \tfrac{\alpha.d(x^*,T(u_n,v_n)d(x^*,x^*).d(u_n,T(u_n,v_n))}{d(x^*,u_n)+d(u_n,T(u_n,v_n)} \\
&\quad + \beta d(x^*, u_n) + \tfrac{\alpha.d(y^*,T(v_n,u_n)).d(y^*,y^*).d(v_n,T(v_n,u_n))}{d(y^*,v_n)} + \beta d(y^*, v_n) \\
&= d(u_{n+1}, x^*) + d(v_{n+1}, y^*) + \beta d(x^*, u_n) + \beta d(y^*, v_n) \to\ 0 \text{ as } n \to \infty,
\end{aligned}$$

from which we have that $\lim_{n\to\infty}(\epsilon_n + \delta_n) = 0$, that is, $\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} \delta_n = 0$. □

Theorem 11. *Let (X, d) be a complete metric space and $T : X \times X \to X$ a mapping satisfying the rational type contractive condition*

$$d(T(x, y), T(u, v)) \leq \frac{\alpha d(x, T(x, y)[d(x, T(u, v))]^q.d(u, T(u, v))}{\gamma d(u, T(u, v)) + d(x, u)} + \psi(d(x, u)), \quad (3.9)$$

where $\alpha \geq 0$, $\gamma \geq 0, q \geq 0$, $\gamma d(u, T(u, v)) + d(x, u) > 0\ \forall\ x,\ y,\ u,\ v \in X$. Let $\psi\colon I\!R^+ \to I\!R^+$ be a subadditive comparison function. Suppose T has a coupled fixed point (x^*, y^*). For $(x_0, y_0) \in X \times X$, let $\{(x_n, y_n)\}_{n=0}^{\infty} \subset X \times X$ be the coupled fixed point iterative procedure defined by $(S1)$. Then, the coupled fixed point iterative procedure is $T-$stable.

Proof. Let $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty} \subset X$, $\epsilon_n = d(u_{n+1}, T(u_n, v_n))$ and

$$\delta_n = d(v_{n+1}, T(v_n, u_n)).$$

Suppose that $\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty\infty} \delta_n = \lim_{n\to\infty}(\epsilon_n + \delta_n) = 0$. Then, we shall establish that $\lim_{n\to\infty} u_n = x^*$ and $\lim_{n\to\infty} v_n = y^*$. Therefore, by using (3.9), we obtain

$$\begin{aligned}
d(u_{n+1}, x^*) &\leq d(u_{n+1}, T(u_n, v_n)) + d(T(u_n, v_n), x^*) \\
&= d(T(u_n, v_n), T(x^*, y^*)) + \epsilon_n \\
&\leq \tfrac{\alpha d(u_n,T(u_n,v_n).[d(u_n,T(x^*,y^*))]^q.d(x^*,T(x^*,y^*))}{\gamma d(x^*,T(x^*,y^*))+d(u_n,x^*)} + \psi(d(u_n, x^*)) + \epsilon_n \\
&= \tfrac{\alpha d(u_n,T(u_n,v_n).[d(u_n,x^*)]^q.d(x^*,x^*)}{\gamma d(x^*,x^*)+d(u_n,x^*)} + \psi(d(u_n, x^*)) + \epsilon_n \\
&= \psi(d(u_n, x^*)) + \epsilon_n,
\end{aligned}$$

that is,

$$d(u_{n+1},x^*) \le \psi(d(u_n,x^*)) + \epsilon_n. \tag{3.10}$$

Using Lemma 8 in (3.10) gives $\lim_{n\to\infty} d(u_n,x^*) = 0$. That is, $\lim_{n\to\infty} u_n = x^*$.

In a similar manner, we have

$$\begin{aligned} d(v_{n+1},y^*) &\le d(v_{n+1},T(v_n,u_n)) + d(T(v_n,u_n),y^*), \\ &= d(T(v_n,u_n),T(y^*,x^*)) + \delta_n, \\ &\le \frac{\alpha.d(v_n,T(v_n,u_n).[d(v_n,T(y^*,x^*))]^q.d(y^*,T(y^*,x^*)}{\gamma d(y^*,T(y^*,x^*))+d(v_n,y^*)} + \psi(d(v_n,y^*)) + \delta_n \\ &= \frac{\alpha d(v_n,T(v_n,u_n).[d(v_n,y^*)]^q.d(y^*,y^*)}{\gamma d(y^*,y^*)+d(v_n,y^*)} + \psi(d(v_n,y^*)) + \delta_n \\ &= \psi(d(v_n,y^*)) + \delta_n, \end{aligned}$$

which yields

$$d(v_{n+1},x^*) \le \psi(d(v_n,x^*)) + \epsilon_n. \tag{3.11}$$

Again, using Lemma 8 in (3.11) gives $\lim_{n\to\infty} d(v_n,x^*) = 0$. That is, $\lim_{n\to\infty} v_n = x^*$.

Conversely, let $\lim_{n\to\infty} d(u_n,x^*) = \lim_{n\to\infty} d(v_n,y^*) = \lim_{n\to\infty} (d(u_n,x^*) + d(v_n,y^*)) = 0$. Then, by using (3.9) again, we obtain

$$\begin{aligned} \epsilon_n + \delta_n &= d(u_{n+1},T(u_n,v_n)) + d(v_{n+1},T(v_n,u_n)) \\ &\le d(u_{n+1},x^*) + d(x^*,T(u_n,v_n)) + d(v_{n+1},y^*) + d(y^*,T(v_n,u_n)) \\ &= d(u_{n+1},x^*) + d(v_{n+1},y^*) + d(T(x^*,y^*),T(u_n,v_n)) + d(T(y^*,x^*),T(v_n,u_n)) \\ &\le d(u_{n+1},x^*) + d(v_{n+1},y^*) \\ &\quad + \frac{\alpha d(x^*,T(x^*,y^*))[d(x^*,T(u_n,v_n))]^q.d(u_n,T(u_n,v_n))}{\gamma d(u_n,T(u_n,v_n))+d(x^*,u_n)} \\ &\quad + \psi(d(x^*,u_n)) + \frac{\alpha d(y^*,T(y^*,x^*)).[d(y^*,T(v_n,u_n))]^q.d(v_n,T(v_n,u_n))}{\gamma d(v_n,T(v_n,u_n))+d(y^*,v_n)} \\ &\quad + \psi(d(y^*,v_n)) \\ &= d(u_{n+1},x^*) + d(v_{n+1},y^*) + \psi(d(x^*,u_n)) + \psi(d(y^*,v_n)) \to 0 \text{ as } n\to\infty, \end{aligned}$$

from which we obtain $\lim_{n\to\infty}(\epsilon_n + \delta_n) = 0$, that is, $\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} \delta_n = 0$. □

Remark 12. *Theorem 9 - Theorem 11 are generalizations of Theorem 2.1 - Theorem 2.6 of Olatinwo [17]. Also, Theorem 9 - Theorem 11 are extensions of a multitude of stability results from fixed point consideration to the coupled fixed point setting.*

Remark 13. **(i)** *The contractive condition (3.9) reduces to that in (3.5) if $\gamma = q = 1$ and $\psi(t) = \beta t,\ t \in I\!R^+$.*

(ii) *The contractive condition (3.9) reduces to that in (3.1) if $\gamma = q = 0$ and $\psi(t) = \beta t,\ t > 0$.*

Example 14. The following example shows that $T\colon X \times X \to X$ satisfies both the contractive condition (3.5) of Theorem 10 and the contractive condition (3.9) of Theorem 11:

Let $X = [0,1] \subset \mathbb{R}$ and assume the usual metric (that is, $d(x,y) = |x-y|$, $x,\ y \in X$). Define $T\colon X \times X \to X$ by

$$T(x,y) = \left\{ \begin{array}{ll} \frac{1}{4}, & \text{if } x,\ y \in [0, \frac{1}{2}) \\ 1 - \frac{1}{2}x - \frac{1}{2}y, & \text{if } x,\ y \in [\frac{1}{2}, 1], \end{array} \right.$$

and let a comparison function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\psi(t) = \frac{3}{4}t,\ t \in \mathbb{R}^+$. Then, T satisfies the contractive condition (3.5) of Theorem 10 as well as the contractive condition (3.9) of Theorem 11.

Solution

Case 1: We now show that T satisfies the contractive condition (3.5) as follows:

Let $\alpha = 1,\ x = \frac{1}{16},\ y = \frac{1}{8},\ u = \frac{1}{2}$ and $v = \frac{3}{4}$. Then, we obtain
$T(x,y) = \frac{1}{4},\ d(x,u) = \frac{7}{16},\ d(x, T(x,y)) = \frac{3}{16}$,
$T(u,v) = 1 - \frac{1}{4} - \frac{3}{8} = \frac{3}{8},\ d(x, T(u,v)) = \frac{5}{16},\ d(u, T(u,v)) = \frac{1}{8}$, and
$d(T(x,y), T(u,v)) = \frac{1}{8}$.

But,

$$\begin{aligned} \tfrac{1}{8} = d(T(x,y), T(u,v)) &\leq \alpha \tfrac{d(x,T(u,v).d(x,T(x,y)).d(u,T(u,v))}{d(x,u)+d(u,T(u,v))} + \beta d(x,u)) \\ &= \tfrac{(\frac{5}{16}).(\frac{3}{16}).(\frac{1}{8})}{\frac{9}{16}} + \tfrac{7}{16}\beta \\ &= (\tfrac{5}{16}).(\tfrac{3}{16}).(\tfrac{1}{8}).(\tfrac{16}{9}) + \tfrac{7}{16}\beta, \end{aligned}$$

from which we have that $\beta \geq \frac{43}{168}$. That is, $\beta \in [0,1)$.

Thus, T satisfies the contractive condition (3.5) of Theorem 10.

Case 2: We now show that T satisfies the contractive condition (3.9) too as in the following: We assume that $\alpha = q = \gamma = 1,\ x = \frac{1}{16},\ y = \frac{1}{8},\ u = \frac{1}{2}$ and $v = \frac{3}{4}$. Then, we obtain $T(x,y) = \frac{1}{4},\ d(x,u) = \frac{7}{16},\ d(x,T(x,y)) = \frac{3}{16}$,
$T(u,v) = 1 - \frac{1}{4} - \frac{3}{8} = \frac{3}{8},\ d(x,T(u,v)) = \frac{5}{16},\ d(u,T(u,v)) = \frac{1}{8}$, and
$d(T(x,y),T(u,v)) = \frac{1}{8}$. Also, $\psi(d(x,u)) = \frac{21}{64}$. Now,

$$\begin{aligned} \alpha \tfrac{[d(x,T(u,v)]^q.d(x,T(x,y)).d(u,T(u,v))}{d(x,u)+\gamma d(u,T(u,v))} + \psi(d(x,u)) &= \tfrac{(\frac{5}{16}).(\frac{3}{16}).(\frac{1}{8})}{\frac{9}{16}} + \psi(d(x,u)) \\ &= (\tfrac{5}{16}).(\tfrac{1}{24}) + \tfrac{21}{64} \\ &= \tfrac{131}{384} > \tfrac{48}{384} = \tfrac{1}{8} = d(T(x,y),T(u,v)), \end{aligned}$$

from which it follows therefore, that T satisfies the contractive condition (3.9) of Theorem 11. Indeed, the coupled fixed point of T is $(\frac{1}{2}, \frac{1}{2})$. That is, $T(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$.

Alternatively, since ψ is a comparison function, we can prove that T satisfies the contractive condition (3.9) by showing that $0 \leq \psi(t) < 1,\ t \in \mathbb{R}^+$, as demonstrated below: We have $\psi(d(x,u)) = \frac{21}{64}$ and

$$\begin{aligned} \tfrac{1}{8} = d(T(x,y),T(u,v)) &\leq \alpha \tfrac{[d(x,T(u,v)]^q.d(x,T(x,y)).d(u,T(u,v))}{d(x,u)+\gamma d(u,T(u,v))} + \psi(d(x,u)) \\ &= \tfrac{(\frac{5}{16}).(\frac{3}{16}).(\frac{1}{8})}{\frac{9}{16}} + \psi(d(x,u)), \end{aligned}$$

from which we have

$$\frac{21}{64} = \frac{126}{384} = \psi(d(x,u)) \geq \frac{1}{8} - (\frac{5}{16}).(\frac{1}{24}) = \frac{43}{384},$$

that is, we obtain $\frac{43}{384} \leq \psi(d(x,u)) = \frac{21}{64} < 1$.

Conflict of Interest: On behalf of both authors, the corresponding author states that there is no conflict of interest.

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