Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 14 (2019), 341–354

VARIATIONS ON THE THEME EULER ANGLES

Clementina D. Mladenova and Ivaïlo M. Mladenov

Abstract. We discuss different parameterizations of the *Lie* group SO(3). The well-known *Rodrigues* formula describes the three dimensional orthogonal matrices in terms of their axes and angles of rotation. In particular, an arbitrary SO(3) element can be described by two real parameters and one angle. In [5] an alternative to *Rodrigues* representation in which an arbitrary rotation is expressed in terms of two angles and one real parameter is derived. This is done via the *Cayley* map applied to the canonical form of the $\mathfrak{so}(3)$ matrices. The relationships between the novel parameterization, the classical *Rodrigues* representation and the extended SO(3) vector parameterization are established. The composition law in the new coordinates is derived for the composition of two regular rotations. In this paper we cover all possible scenarios for the composition law, including the cases when at least one of the composed matrices is a half-turn. To do this the extended vector-parameter composition law in SO(3) is used.

1 Introduction

Lie groups and *Lie* algebras are fundamental fields of the contemporary mathematics. They appear as basic notion in mathematics, mechanics, physics and other theoretical sciences [7, 14]. Their development influence over the corresponding application areas. The *Lie* groups parameterizations are used in their study and application.

The group SO(3) is invariably connected with classical mechanics [12]. In the kinematics of the three-dimensional space $\mathbb{R}^3$, the well known theorem of *Chasle* that states that the rigid body motion may be described though a translation along a line and followed by a rotation around it, plays a basic role. From the both motions translation and rotation, the three-dimensional rotation is the nontrivial one. The group SO(3,$\mathbb{R}$) is represented as in the theoretical mechanics, so in its applications. In this aspect, one improvement or a new look at the group leads to

2010 Mathematics Subject Classification: 70B10; 70B15; 22E60; 22E70
Keywords:Lie algebras; Lie groups; rigid body kinematics; parameterizations; rotations; vector-parameter.

many improvements in different directions. Important problems are: the problem of the effective recovery and interpolation of a rotation matrix in different mechanical treatments, the problem of compositions of many rotations, and also the problem of decomposition around two or three pre-set axes.

A significant moment in SO(3) study is the vector-parameterization [6, 10], reminded in the text below. Instead of the exponential map over the *Lie* algebra $\mathfrak{so}(3)$ of the antisymmetric 3×3 matrices, it is used the *Cayley* map. In this way a rotation matrix is associated with a three dimensional vector of rotation along the rotation axis and its module is equal to the tangent of the half of the rotation angle called in our considerations "vector-parameter". These vectors make a group with a simple and a nice composition law. Except the fact that the formula of the composition law is very elegant, it is quite effective in computational aspect and it is the reason the elementary calculations to be reduced with more than 50%. Only 12 multiplications are necessary for a composition of two vectors, while the multiplications of two rotation matrices requires 27. As you will see later in the exposition, a disadvantage of this realization is that the half-turns can not be presented through regular vector-parameters, and the composition law is not well defined when its denominator is equal to zero.

The way of describing the motions of rigid bodies in the inertial space using a rotating and translating of a non-inertial frame of reference is important for different many problems appearing in many areas like vehicle and spacecraft dynamics, mechanisms, robotics and biomechanics. Recently, a lot of efforts have been made in order to include also the flexibility of joints and bodies in parallel with that one established within rigid body dynamics. In order that the transformation SO(3) matrix between the above frames be found, a number of different sets of parameters can be used. These sets of parameters are quite different according to the physical interpretation, the presence of singularities, the use of trigonometric or purely algebraic functions, the number of accompanying constraint equations, etc. The vector-parameter apparatus of *Lie* groups of small dimension and its important applications in the field of Analytical Mechanics can be found in Mladenova[10, 11, 12] and Mladenov[9], as well in [13] and etc. One of the applications is the achievement of general refined theorem about generalized Euler decomposition of a rotation matrix in [1]. The well known *Rodrigues* representation [2] and the corresponding composition of two finite rotations were introduced in 1840. In [5] an alternative to *Rodrigues* representation in which an arbitrary rotation is expressed in terms of two angles and one real parameter is derived. This is done via the *Cayley* map applied to the canonical form of the $\mathfrak{so}(3)$ matrices. The relationships between the novel parameterization, the classical *Rodrigues* representation and the extended SO(3) vector-parameterization are established. The composition law in the new coordinates is derived for the composition of two regular rotations.

Let $\mathcal{R} = \mathcal{R}(\mathbf{n}, \theta)$ be the matrix of a proper (i.e., not a half-turn) three-dimensional rotation in the axis-angle formalism. A convenient representation of $\mathcal{R}$ can be realized by the vector-parameter $\boldsymbol{c} = \tan\dfrac{\theta}{2}\mathbf{n}, \theta \neq \dfrac{\pi}{2}$, i.e.,

$$\mathcal{R}(\boldsymbol{c}) = \frac{2}{1+c^2}\begin{pmatrix} 1+c_1^2-c_2^2-c_3^2 & c_1c_2-c_3 & c_1c_3+c_2 \\ c_1c_2+c_3 & 1-c_1^2+c_2^2-c_3^2 & c_2c_3-c_1 \\ c_1c_3-c_2 & c_2c_3+c_1 & 1-c_1^2-c_2^2+c_3^2 \end{pmatrix}. \tag{1.1}$$

However, one has to be careful when half-turns occur because they can not be represented by regular *Gibbs* vectors. If $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ represent the proper rotations $\mathcal{R}(\boldsymbol{c}_1), \mathcal{R}(\boldsymbol{c}_2)$, the composition law in the vector-parameter form is given by the formulas

$$\mathcal{R}(\boldsymbol{c}_3) = \mathcal{R}(\boldsymbol{c}_2)\mathcal{R}(\boldsymbol{c}_1), \qquad \boldsymbol{c}_3 = \boldsymbol{c}_3(\boldsymbol{c}_2, \boldsymbol{c}_1) = \frac{\boldsymbol{c}_2 + \boldsymbol{c}_1 + \boldsymbol{c}_2 \times \boldsymbol{c}_1}{1 - \boldsymbol{c}_2.\boldsymbol{c}_1}. \tag{1.2}$$

This composition law is not well-defined when either one of the rotations is a half-turn or when $\boldsymbol{c}_2.\boldsymbol{c}_1 = 1$. The vector-parameterization of the covering group SU(2) and the corresponding composition law is presented in [3]. Also, half-turns are well defined as the composition law there has no singularities. Within this formalism, the natural covering map SU(2) $\to$ SO(3,$\mathbb{R}$) and its sections are written and studied. The technique developed in [3] is used in [4] to *extend* the composition law (1.2) in the cases when half-turns are involved or the result of the composition is a half-turn, i.e., when $\boldsymbol{c}_2.\boldsymbol{c}_1 = 1$. To do that, a half-turn $\mathcal{R}(\mathbf{n}, \pi) = \mathcal{O}(\mathbf{n})$ is represented as a *ray*, i.e., by the set of all three dimensional non-zero vectors, co-linear with the axis of rotation $\mathbf{n}$. This is an alternative description of SO(3,$\mathbb{R}$) and the corresponding composition laws are more intuitive and are computationally cheaper even than the quaternionic formalism [8] when it comes to the composition of rotations [12]. We will use the extended composition law to derive all the other cases for the composition law in the new representation.

2 Preliminaries

Substituting $\theta\mathbf{n}^\times$ where $\theta \in \mathbb{R}^* = \mathbb{R}\backslash\{\mathbf{0}\}$ and $\mathbf{n} = (u, v, w) \in \mathbb{R}^3, \mathbf{n}^2 = u^2+v^2+w^2 = 1$ for $\mathcal{C}$ in the exponential map $\exp : \mathfrak{so}(3) \to$ SO(3,$\mathbb{R}$) produces

$$\begin{aligned} \exp(\theta\mathbf{n}^\times) &= \mathcal{I} + \frac{\sin\theta}{\theta}\theta\mathbf{n}^\times + \frac{1-\cos\theta}{\theta^2}\theta^2(\mathbf{n}^\times)^2 \\ &= \mathcal{I} + \sin\theta\mathbf{n}^\times + (1-\cos\theta)(\mathbf{n}\otimes\mathbf{n}^t - \mathcal{I}) \\ &= \cos\theta\mathcal{I} + \sin\theta\mathbf{n}^\times + (1-\cos\theta)(\mathbf{n}\otimes\mathbf{n}^t) = \mathcal{R}(\mathbf{n}, \theta) \end{aligned} \tag{2.1}$$

where $\mathbf{n}^\times = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$, $\mathbf{n}\otimes\mathbf{n}^t$ denotes the dyadic product of $\mathbf{n}$ and $\mathbf{n}^t$. Of course, for all $\mathbf{n} \in \mathbb{R}^3$, $\exp(0\mathbf{n}^\times) = \exp(\mathbf{0}) = \mathcal{I}$. Formula (2.1) is exactly the *Rodrigues* formula [2] in which θ is the angle and $\mathbf{n}$ is the the axis of rotation. Where appropriate for typographical reasons we will denote the $\cos(\cdot)$ and $\sin(\cdot)$ functions as $\mathrm{c}(\cdot)$ and $\mathrm{s}(\cdot)$. For the matrix form of the classical *Rodrigues* representation we have

$$\mathcal{R}(\mathbf{n},\theta)=\begin{pmatrix} u^2(1-\mathrm{c}\theta)+\mathrm{c}\theta & uv(1-\mathrm{c}\theta)-w\mathrm{s}\theta & uw(1-\mathrm{c}\theta)+v\mathrm{s}\theta \\ uv(1-\mathrm{c}\theta)+w\mathrm{s}\theta & v^2(1-\mathrm{c}\theta)+\mathrm{c}\theta & vw(1-\mathrm{c}\theta)-u\mathrm{s}\theta \\ uw(1-\mathrm{c}\theta)-v\mathrm{s}\theta & vw(1-\mathrm{c}\theta)+u\mathrm{s}\theta & w^2(1-\mathrm{c}\theta)+\mathrm{c}\theta \end{pmatrix}. \tag{2.2}$$

Recall [6] that the *Cayley* map for $\mathfrak{so}(3)$ associates with $\boldsymbol{c}\cdot\mathbf{J}\in\mathfrak{so}(3)$ matrix

$$\mathcal{R}(\boldsymbol{c})=\mathrm{Cay}_{\mathfrak{so}(3)}(\mathcal{C})=(\mathcal{I}+\mathcal{C})(\mathcal{I}-\mathcal{C})^{-1}=(\mathcal{I}-\mathcal{C})^{-1}(\mathcal{I}+\mathcal{C}). \tag{2.3}$$

One checks immediately that $(\mathcal{I}-\mathcal{C})^{-1}=\mathcal{I}+\dfrac{1}{1+c^2}\mathcal{C}+\dfrac{1}{1+c^2}\mathcal{C}^2$ and that (2.4) can be expressed in the form

$$\mathrm{Cay}_{\mathfrak{so}(3)}(\mathcal{C})=\mathcal{I}+\frac{2}{1+c^2}\mathcal{C}+\frac{2}{1+c^2}\mathcal{C}^2 \tag{2.4}$$

for all $\mathcal{C}(\boldsymbol{c})\in\mathfrak{so}(3)$. Substitution of $\varphi\mathbf{n}^\times$ for $\mathcal{C}$ in (2.4) gives

$$\mathrm{Cay}(\varphi\mathbf{n}^\times)=\frac{1-\varphi^2}{1+\varphi^2}\mathcal{I}+\frac{2\varphi}{1+\varphi^2}\mathbf{n}^\times+\frac{2\varphi^2}{1+\varphi^2}(\mathbf{n}\otimes\mathbf{n}^t). \tag{2.5}$$

Comparing (2.1) with (2.5), one can obtain immediately that for $\theta\neq(2k+1)\pi$, $k\in\mathbb{Z}$

$$\exp\theta\mathbf{n}^\times=\mathrm{Cay}_{\mathfrak{so}(3)}(\varphi\mathbf{n}^\times),\qquad \varphi=\tan\frac{\theta}{2}\cdot \tag{2.6}$$

Recall that the composition law [3, 6] of SO(3,$\mathbb{R}$) expressed in terms of regular vector-parameters is

$$\mathcal{R}(\boldsymbol{c}_3)=\mathcal{R}(\boldsymbol{c}_2)\mathcal{R}(\boldsymbol{c}_1),\qquad \boldsymbol{c}_3=\langle\boldsymbol{c}_2,\boldsymbol{c}_1\rangle_{\mathrm{SO}(3,\mathbb{R})}=\frac{\boldsymbol{c}_2+\boldsymbol{c}_1+\boldsymbol{c}_2\times\boldsymbol{c}_1}{1-\boldsymbol{c}_2.\boldsymbol{c}_1} \tag{2.7}$$

provided that half-turns do not appear in the composition and the result itself is not a half-turn (the latter is equivalent to $\boldsymbol{c}_2.\boldsymbol{c}_1=1$). In [4] the composition law (2.7) is extended to cover the exhaustive list of all possible scenarios. To do this the *Cayley* map $\mathrm{Cay}_{\mathfrak{su}(2)}:\mathfrak{su}(2)\to\mathrm{SU}(2)$ is used to parameterize the covering group [3]. In this context, a regular rotation is denoted by $\mathcal{R}(\boldsymbol{c})$ whereas a half-turn about an axis $\mathbf{n}$ is denoted as $\mathcal{O}(\mathbf{n})$ and represented by SU(2) vector parameter $2\mathbf{n}$ of length 2. In the extended vector-parameterization of SO(3,$\mathbb{R}$) any half-turn is represented by the ray $[\mathbf{n}]$ consisting of all nonzero vectors proportional to $\mathbf{n}$. With any rotation $\mathcal{R}$ we have associated [4] the pair $\zeta=(\boldsymbol{c},\delta)\in\mathbb{R}^3\times\{0,1\}$ such that

$$\zeta=\zeta(\mathcal{R})=\begin{cases}(\boldsymbol{c},1), & \mathcal{R}\text{ is proper rotation}\\ (\lambda\mathbf{n},0), & \mathcal{R}\text{ is a half-turn, }\lambda\in\mathbb{R}^*\end{cases} \tag{2.8}$$

where $\boldsymbol{c}$ is the *Gibbs* vector-parameter if $\mathcal{R}$ is proper [6] while in the case when $\mathcal{R}$ is a half-turn the axis $\mathbf{n}$ is obtained by the columns of $\mathcal{R}+\mathcal{I}$, see [4, equation (6)]. In the latter case $\lambda\mathbf{n}$ is an element of the ray $[\mathbf{n}]$. Table 1 systematizes the results from the extended composition law in SO(3,$\mathbb{R}$) in vector-parameter form.

Table 1: All scenarios for the extended composition law in SO(3,$\mathbb{R}$).

Product of rotations	Result	Condition	Compound rotation
$\mathcal{R}(\boldsymbol{c}_2)\mathcal{R}(\boldsymbol{c}_1)$	$\boldsymbol{c}_3 = \dfrac{\boldsymbol{c}_2 + \boldsymbol{c}_1 + \boldsymbol{c}_2 \times \boldsymbol{c}_1}{1 - \boldsymbol{c}_2.\boldsymbol{c}_1}$,	$\boldsymbol{c}_2.\boldsymbol{c}_1 \neq 1$	$\mathcal{R}(\boldsymbol{c}_3)$
	$[\mathbf{n}_3] = [\boldsymbol{c}_2 + \boldsymbol{c}_1 + \boldsymbol{c}_2 \times \boldsymbol{c}_1]$,	$\boldsymbol{c}_2.\boldsymbol{c}_1 = 1$	$\mathcal{O}(\mathbf{n}_3)$
$\mathcal{R}(\boldsymbol{c}_2)\mathcal{O}(\mathbf{n}_1)$	$\boldsymbol{c}_3 = -\dfrac{\mathbf{n}_1 + \boldsymbol{c}_2 \times \mathbf{n}_1}{\boldsymbol{c}_2.\mathbf{n}_1}$,	$\boldsymbol{c}_2.\mathbf{n}_1 \neq 0$	$\mathcal{R}(\boldsymbol{c}_3)$
	$[\mathbf{n}_3] = [\mathbf{n}_1 + \boldsymbol{c}_2 \times \mathbf{n}_1]$,	$\boldsymbol{c}_2.\mathbf{n}_1 = 0$	$\mathcal{O}(\mathbf{n}_3)$
$\mathcal{O}(\mathbf{n}_2)\mathcal{R}(\boldsymbol{c}_1)$	$\boldsymbol{c}_3 = -\dfrac{\mathbf{n}_2 + \mathbf{n}_2 \times \boldsymbol{c}_1}{\mathbf{n}_2.\boldsymbol{c}_1}$,	$\mathbf{n}_2.\boldsymbol{c}_1 \neq 0$	$\mathcal{R}(\boldsymbol{c}_3)$
	$[\mathbf{n}_3] = [\mathbf{n}_2 + \mathbf{n}_2 \times \boldsymbol{c}_1]$,	$\mathbf{n}_2.\boldsymbol{c}_1 = 0$	$\mathcal{O}(\mathbf{n}_3)$
$\mathcal{O}(\mathbf{n}_2)\mathcal{O}(\mathbf{n}_1)$	$\boldsymbol{c}_3 = -\dfrac{\mathbf{n}_2 \times \mathbf{n}_1}{\mathbf{n}_2.\mathbf{n}_1}$,	$\mathbf{n}_2.\mathbf{n}_1 \neq 0$	$\mathcal{R}(\boldsymbol{c}_3)$
	$[\mathbf{n}_3] = [\mathbf{n}_2 \times \mathbf{n}_1]$,	$\mathbf{n}_2.\mathbf{n}_1 = 0$	$\mathcal{O}(\mathbf{n}_3)$

2.1 An alternative of the Rodrigues parameterization

Proposition 1. ([5]) *Let* $\varphi \in \mathbb{R}$ *and* $\mathbf{n} = (u, v, w) \in \mathbb{R}^3$ *be such that* $\mathbf{n}^2 = 1$ *and*

$$\mathcal{A} = \mathcal{A}(\mathbf{n}, \varphi) = (\varphi \mathbf{n})^{\times} = \varphi \mathbf{n}^{\times} = \varphi \begin{pmatrix} 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \end{pmatrix}. \tag{2.9}$$

Then the canonical form of $\mathcal{A}$ *is* $\Omega = \Omega(\varphi)$

$$\Omega = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\varphi \\ 0 & \varphi & 0 \end{pmatrix} = T_{\alpha,\beta} \mathcal{A} T_{\alpha,\beta}^{-1} \tag{2.10}$$

where $T_{\alpha,\beta} = \mathcal{R}_Y(\beta)\mathcal{R}_Z(\alpha)$ *for*

$$\begin{aligned} \alpha: \quad \cos\alpha &= \frac{u}{\sqrt{u^2+v^2}}, \qquad \sin\alpha = \frac{-v}{\sqrt{u^2+v^2}} \\ \beta: \quad \cos\beta &= \sqrt{u^2+v^2}, \qquad \sin\beta = -w \end{aligned} \tag{2.11}$$

provided that $\mathbf{n} \neq (0,0,1)$ *and* $T_{0,-\pi/2} = \mathcal{R}_Y(-\dfrac{\pi}{2})\mathcal{R}_Z(0)$ *otherwise.*

Let $\mathcal{R} = \mathcal{R}(\mathbf{n}, \theta) = \exp(\theta \mathbf{n})$ is the axis-angle representation of the regular rotation $\mathcal{R}$, i.e., which is not a half-turn. According to equation (2.6) we have $\varphi = \tan\dfrac{\theta}{2}$ and using the *Cayley* map on the canonical form of the basis element of $\mathfrak{so}(3)$ from Proposition 1, in [5] we derived the following representation

$$\mathcal{R} = \mathcal{R}(\mathbf{n}, \varphi) = T_{\alpha,\beta}^{-1} \mathcal{R}_{\Omega}(\varphi) T_{\alpha,\beta}. \tag{2.12}$$

In this way we have parameterized the set of regular rotations by the following three parameters: two angles α and β and one real number φ. For comparison, *Rodrigues* representation (2.1) is based on two real numbers (which define the axis of rotation $\mathbf{n}$) and one angle θ.

What happens when $\mathcal{R}$ is a half-turn, i.e., $\mathcal{R} = (\mathbf{n}, \pm\pi)$? We must take the limit $\theta \to \pm\pi$, i.e., $\varphi \to \pm\infty$ (for more discussion about the parameterization of half-turns see [3, 5]). It is immediate that $\lim_{\varphi\to\pm\infty} \mathcal{R}_\Omega = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)$. Thus, if we want to parameterize the whole group SO(3,$\mathbb{R}$) we need to add the point at infinity to the real parameter τ, i.e., $\tau \in S^1 = \mathbb{R} \cup \{\infty\}$. Actually, if we take $\theta \in [0, \pi]$ we can consider τ to be a non-negative real number, $\tau \in \mathbb{R}^+ \cup \{0\} \cup \{\infty\}$.

To obtain the analogue of *Rodrigues* formula we need to invert formulas (2.12). We have

$$\mathbf{n} = \mathbf{n}(\alpha, \beta) = \mathbf{n}_{\alpha,\beta} = (u, v, w) = (\cos\alpha\cos\beta, -\sin\alpha\cos\beta, -\sin\beta). \tag{2.13}$$

Following (2.12)[5] we are led to the final formula

$$\mathcal{R}(\tau, \alpha, \beta) = \mathcal{I} + \frac{2\tau}{1+\tau^2}\mathbf{n}^\times_{\alpha,\beta} + \frac{2\tau^2}{1+\tau^2}(\mathbf{n}_{\alpha,\beta} \otimes \mathbf{n}^t_{\alpha,\beta} - \mathcal{I}) \tag{2.14}$$

where $\mathbf{n}^\times_{\alpha,\beta} = \begin{pmatrix} 0 & \sin\beta & -\sin\alpha\cos\beta \\ -\sin\beta & 0 & -\cos\alpha\cos\beta \\ \sin\alpha\cos\beta & \cos\alpha\cos\beta & 0 \end{pmatrix}$. Some simple algebraic manipulations produce the matrix $\mathcal{R} = \mathcal{R}(\alpha, \beta, \tau)$, namely

$$\mathcal{R} = \frac{1}{1+\tau^2}\begin{pmatrix} 1-\tau^2+2\tau^2\mathrm{c}^2\alpha\mathrm{c}^2\beta & 2\tau\mathrm{s}\beta-\tau^2\mathrm{s}2\alpha\mathrm{c}^2\beta & -2\tau\mathrm{s}\alpha\mathrm{c}\beta-\tau^2\mathrm{c}\alpha\mathrm{s}2\beta \\ -2\tau\mathrm{s}\beta-\tau^2\mathrm{s}2\alpha\mathrm{c}^2\beta & 1-\tau^2+2\tau^2\mathrm{c}^2\beta\mathrm{s}^2\alpha & -2\tau\mathrm{c}\alpha\mathrm{c}\beta+\tau^2\mathrm{s}\alpha\mathrm{s}2\beta \\ 2\tau\mathrm{s}\alpha\mathrm{c}\beta-\tau^2\mathrm{c}\alpha\mathrm{s}2\beta & 2\tau\mathrm{c}\alpha\mathrm{c}\beta+\tau^2\mathrm{s}\alpha\mathrm{s}2\beta & 1-\tau^2+2\tau^2\mathrm{s}^2\beta \end{pmatrix} \tag{2.15}$$

In the case when the rotation is a half-turn, the matrix is obtained by taking the limit $\tau \to \infty$ in the above formula (2.15) and this gives

$$\mathcal{R}(\alpha, \beta, \infty) = \begin{pmatrix} 2\cos^2\alpha\cos^2\beta - 1 & -\sin 2\alpha\cos^2\beta & -\cos\alpha\sin 2\beta \\ -\sin 2\alpha\cos^2\beta & 2\sin^2\alpha\cos^2\beta - 1 & \sin\alpha\sin 2\beta \\ -\cos\alpha\sin 2\beta & \sin\alpha\sin 2\beta & 2\sin^2\beta - 1 \end{pmatrix}. \tag{2.16}$$

Let $\mathcal{R}(\mathbf{n}, \theta)$ be a rotation matrix which is not a half-turn, i.e., $\theta \neq \pi$. Then using equation (2.13) and the relationship between the axis-angle form of a rotation matrix and the vector-parameter form, the latter can be expressed in terms of α, β, τ

$$\boldsymbol{c} = \tan\frac{\theta}{2}\mathbf{n} = \tau(\cos\alpha\cos\beta, -\sin\alpha\cos\beta, -\sin\beta). \tag{2.17}$$

If $\tau = \infty$ then the ray $[\mathbf{n}]$ which corresponds [4] to $\mathcal{R}(\mathbf{n}, \pi)$ is

$$[\mathbf{n}] = [(\cos\alpha\cos\beta, -\sin\alpha\cos\beta, -\sin\beta)]. \tag{2.18}$$

Figure 1 shows the relations between the discussed parameterizations.

$$\begin{tikzcd}
\boldsymbol{c} \text{ or } [\mathbf{n}] \arrow[r, bend left, "{[4]}"] & \mathbf{n}, \theta \arrow[l, bend left, "\boldsymbol{c} = \tan\frac{\theta}{2}\mathbf{n}"] \arrow[r, bend left, "{\text{eq. (2.12)}, \ \tau = \tan\frac{\theta}{2}}"] & \alpha, \beta, \tau \arrow[l, bend left, "{\text{eq. (2.13)}, \ \theta = 2\arctan\tau}"]
\end{tikzcd}$$

Figure 1: Relations between the extended vector-parameterization of SO(3,$\mathbb{R}$), the classical axis-angle and the derived alternative representation.

3 The composition law in the (α, β, τ) representation

To obtain the (α, β, τ) parameters directly from a given rotational matrix $\mathcal{R}$ in any representation we make use of equation (2.15) and the fact that a rotational matrix is a half-turn if and only if it is symmetric. Calculation shows that if $\mathcal{R} \neq \mathcal{R}^t$ then $\operatorname{tr}\mathcal{R} = \dfrac{3-\tau^2}{1+\tau^2}$ and therefore we have also $\tau^2 = \begin{cases} \infty & \text{if } \ \mathcal{R} = \mathcal{R}^t \\ \dfrac{3 - \operatorname{tr}\mathcal{R}}{1+\operatorname{tr}\mathcal{R}} & \text{if } \ \mathcal{R} \neq \mathcal{R}^t. \end{cases}$

Again from (2.15) we obtain $\mathcal{R} - \mathcal{R}^t = \dfrac{4\tau}{1+\tau^2}\mathbf{n}^{\times}_{\alpha,\beta}$ and thus $\mathbf{n}^{\times} = \mathbf{n}^{\times}_{\alpha,\beta} = \dfrac{\mathcal{R}-\mathcal{R}^t}{1+\operatorname{tr}\mathcal{R}}$. Now α and β are determined by formulas (2.12). If however $\mathcal{R} = \mathcal{R}^t$ then $\mathcal{R}$ is a half-turn and the axis $\mathbf{n}$ can be obtained from the columns of $\mathcal{R} + \mathcal{I}$ (see [4]), from where using (2.12) we obtain α and β.

Let us define the scalar function

$$\nu(\alpha_1, \alpha_2, \beta_1, \beta_2, \tau_1, \tau_2) = 1 - \tau_1\tau_2(\mathrm{c}\beta_1\mathrm{c}\beta_2\mathrm{c}(\alpha_1 - \alpha_2) + \mathrm{s}\beta_1\mathrm{s}\beta_2). \tag{3.1}$$

In [5] the case when both the composition rotations are proper rotation is given by

Proposition 2. ([5]) *Let $\mathcal{R}_i = \mathcal{R}_i(\alpha_i, \beta_i, \tau_i), i = 1, 2$ be two regular rotations given in the (α, β, τ) convention. Let $\mathcal{R} = \mathcal{R}_2\mathcal{R}_1$ be their composition. Then $\mathcal{R} = \mathcal{R}(\alpha, \beta, \tau)$ in which*

$$\begin{aligned}
\tau &= \begin{cases} \dfrac{\sqrt{(1+\tau_1^2)(1+\tau_2^2) - \nu^2}}{|\nu|}, & \text{if } \nu \neq 0 \\ \infty, & \text{if } \nu = 0 \end{cases} \\
\alpha &= \arctan\frac{\tau_1\mathrm{s}\alpha_1\mathrm{c}\beta_1 + \tau_2\mathrm{s}\alpha_2\mathrm{c}\beta_2 + \tau_1\tau_2(\mathrm{c}\alpha_1\mathrm{c}\beta_1\mathrm{s}\beta_2 - \mathrm{c}\alpha_2\mathrm{c}\beta_2\mathrm{s}\beta_1)}{\tau_1\mathrm{c}\alpha_1\mathrm{c}\beta_1 + \tau_2\mathrm{c}\alpha_2\mathrm{c}\beta_2 + \tau_1\tau_2(\mathrm{s}\alpha_2\mathrm{s}\beta_1\mathrm{c}\beta_2 - \mathrm{s}\alpha_1\mathrm{c}\beta_1\mathrm{s}\beta_2)} \\
\beta &= \varepsilon_\mu \arcsin\frac{\tau_1\mathrm{s}\beta_1 + \tau_2\mathrm{s}\beta_2 + \tau_1\tau_2\mathrm{c}\beta_1\mathrm{c}\beta_2\mathrm{s}(\alpha_1 - \alpha_2)}{\sqrt{(1+\tau_1^2)(1+\tau_2^2) - \nu^2}}
\end{aligned} \tag{3.2}$$

where $\nu = \nu(\alpha_1, \alpha_2, \beta_1, \beta_2, \tau_1, \tau_2)$ from equation (3.1) and $\varepsilon_\nu = \operatorname{sgn}\nu$ provided that $\nu \neq 0$ and $\varepsilon_\nu = -1$ otherwise.

Now we are going to exhaust all the possible scenarios for composition of two SO(3,$\mathbb{R}$) rotations in the (α,β,τ) formalism using extensively the systematized results from Table 1.

Proposition 3. *Let $\mathcal{R}_1 = \mathcal{R}_1(\alpha_1,\beta_1,\infty)$ is a half-turn and $\mathcal{R}_2 = \mathcal{R}_2(\alpha_2,\beta_2,\tau_2)$ is a regular rotations. Let $\mathcal{R} = \mathcal{R}_2\mathcal{R}_1$ be their composition. Then $\mathcal{R} = \mathcal{R}(\alpha,\beta,\tau)$ in which*

$$\begin{aligned}
\tau &= \begin{cases} \dfrac{\sqrt{(1+\tau_2^2)-\mu^2}}{|\mu|}, & \text{if } \mu \neq 0 \\ \infty, & \text{if } \mu = 0 \end{cases} \\
\alpha &= \arctan\frac{\mathrm{s}\alpha_1\mathrm{c}\beta_1 + \tau_2(\mathrm{c}\alpha_1\mathrm{c}\beta_1\mathrm{s}\beta_2 - \mathrm{c}\alpha_2\mathrm{c}\beta_2\mathrm{s}\beta_1)}{\mathrm{c}\alpha_1\mathrm{c}\beta_1 + \tau_2(\mathrm{s}\alpha_2\mathrm{s}\beta_1\mathrm{c}\beta_2 - \mathrm{s}\alpha_1\mathrm{c}\beta_1\mathrm{s}\beta_2)} \\
\beta &= \varepsilon_\mu \arcsin\frac{\mathrm{s}\beta_1 + \tau_2\mathrm{c}\beta_2\mathrm{c}\beta_1\mathrm{s}(\alpha_1-\alpha_2)}{\sqrt{1+\tau_2^2-\mu^2}}
\end{aligned} \tag{3.3}$$

and where

$$\mu = \mu(\alpha_1,\alpha_2,\beta_1,\beta_2,\tau_2) := \tau_2(\cos\beta_2\cos\beta_1\cos(\alpha_2-\alpha_1) + \sin\beta_2\sin\beta_1) \tag{3.4}$$

and $\varepsilon_\mu = \operatorname{sgn}\mu$ provided that $\mu \neq 0$ and $\varepsilon_\mu = -1$ otherwise.

Proof. We associate the ray $[\mathbf{n}_1] = [(\cos\alpha_1\cos\beta_1, \sin\alpha_1\cos\beta_1, -\sin\beta_1)]$ and the vector-parameter $\boldsymbol{c}_2 = \tau_2(\cos\alpha_2\cos\beta_2, \sin\alpha_2\cos\beta_2, -\sin\beta_2)$. Directly from the second row in Table 1 we have that the condition for the resulting rotation to be a half turn is $\boldsymbol{c}_2.\mathbf{n}_1 = 0$ which is equivalent to $\tau_2(\cos\beta_2\cos\beta_1\cos(\alpha_2-\alpha_1)+\sin\beta_2\sin\beta_1) = \mu(\tau_2,\alpha_1,\alpha_2,\beta_1,\beta_2) = 0$. Because $\tau_2 \neq 0$ this reduces to $\cos\beta_2\cos\beta_1\cos(\alpha_2-\alpha_1) + \sin\beta_2\sin\beta_1 = 0$.

Let $\mu \neq 0$, i.e., the resulting rotation is proper and is described by the vector-parameter $\boldsymbol{c} = -\dfrac{\mathbf{n}_1 + \boldsymbol{c}_2 \times \mathbf{n}_1}{\boldsymbol{c}_2.\mathbf{n}_1}$ (see Table 1). It is direct to calculate

$$\tau^2 = (\boldsymbol{c},\boldsymbol{c}) = \frac{1+c^2-(\boldsymbol{c}_2.\mathbf{n}_1)^2}{(\boldsymbol{c}_2.\mathbf{n}_1)^2} = \frac{1+\tau_2^2-\mu^2}{\mu^2}, \qquad \tau = \frac{\sqrt{1+\tau_2^2-\mu^2}}{|\mu|}\cdot \tag{3.5}$$

Now from (2.17) we get $-\sin\beta = c_3$, i.e.,

$$-\frac{\sqrt{1+\tau_2^2-\mu^2}}{|\mu|}\sin\beta = -\frac{\sin\beta_1 + \tau_2\cos\beta_2\cos\beta_1\sin(\alpha_1-\alpha_2)}{\mu} \tag{3.6}$$

which immediately leads to (3.4). Following (2.17) we have again $-\tan\alpha = -\dfrac{c_2}{c_1}$ provided that $c_1 \neq 0$. Simple substitutions lead to (3.3).

Let $\mu = 0$, i.e., the resulting rotation is a half-turn, i.e. $\tau = \infty$. From Table 1 we have that $\mathcal{R}$ is represented by the ray $[\mathbf{n}] = [\mathbf{n}_1 + \boldsymbol{c}_2 \times \mathbf{n}_1]$. To find α and β we need to find the unit vector with the direction of the ray. We have $(|\mathbf{n}_1 + \boldsymbol{c}_2 \times \mathbf{n}_1|)^2 = 1 + \tau^2$ and thus from (2.18)

$$(\cos\alpha\cos\beta, -\sin\alpha\cos\beta, -\sin\beta) = \frac{\mathbf{n}_1 + \boldsymbol{c}_2 \times \mathbf{n}_1}{\sqrt{1+\tau_2^2}}. \tag{3.7}$$

After expressing the second fraction in terms of $\tau_2, \alpha_1, \alpha_2, \beta_1$ and β_2 and following similar logic as the first part of the proof we again get formulas (3.3) and (3.4).

Remark 4. *Note that in the proof the case $c_1 = 0$ should be treated separately but we omit the details here. In this case $\alpha = \pm\frac{\pi}{2}$ depending on the sign of the nominator in (3.3). The same applies for the next two propositions as well.*

Proposition 5. *Let $\mathcal{R}_1 = \mathcal{R}_1(\alpha_1, \beta_1, \tau_1)$ is a regular proper rotation and $\mathcal{R}_2 = \mathcal{R}_2(\alpha_2, \beta_2, \infty)$ is a half-turn. Let $\mathcal{R} = \mathcal{R}_2\mathcal{R}_1$ be their composition. Then $\mathcal{R} = \mathcal{R}(\alpha, \beta, \tau)$ is generated by*

$$\begin{aligned}
\tau &= \begin{cases} \dfrac{\sqrt{(1+\tau_1^2) - \xi^2}}{|\xi|}, & \text{if } \xi \neq 0 \\ \infty, & \text{if } \xi = 0 \end{cases} \\
\alpha &= \arctan \frac{\mathrm{s}\alpha_2\mathrm{c}\beta_2 + \tau_1(\mathrm{c}\alpha_2\mathrm{c}\beta_2\mathrm{s}\beta_1 - \mathrm{c}\alpha_1\mathrm{c}\beta_1\mathrm{s}\beta_2)}{\mathrm{c}\alpha_2\mathrm{c}\beta_2 + \tau_1(\mathrm{s}\alpha_2\mathrm{s}\beta_1\mathrm{c}\beta_2 - \mathrm{s}\alpha_1\mathrm{c}\beta_1\mathrm{s}\beta_2)} \\
\beta &= \varepsilon_\xi \arcsin \frac{\mathrm{s}\beta_2 + \tau_2\mathrm{c}\beta_2\mathrm{c}\beta_1\mathrm{s}(\alpha_1 - \alpha_2)}{\sqrt{1+\tau_1^2 - \xi^2}}
\end{aligned} \tag{3.8}$$

in which

$$\xi = \xi(\alpha_1, \alpha_2, \beta_1, \beta_2, \tau_1) := \tau_1(\cos\beta_2\cos\beta_1\cos(\alpha_2 - \alpha_1) + \sin\beta_2\sin\beta_1) \tag{3.9}$$

and where $\varepsilon_\xi = \operatorname{sgn}\xi$ provided that $\xi \neq 0$ and $\varepsilon_\xi = -1$ otherwise.

Proof. Similar to the one of Proposition 3.

Proposition 6. *Let $\mathcal{R}_1 = \mathcal{R}_1(\alpha_i, \beta_i, \infty)$, $i = 1, 2$ are half-turns. Let $\mathcal{R} = \mathcal{R}_2\mathcal{R}_1$ be their composition. Then $\mathcal{R} = \mathcal{R}(\alpha, \beta, \tau)$ is generated by*

$$\begin{aligned}
\tau &= \begin{cases} \dfrac{\sqrt{1 - \eta^2}}{|\eta|}, & \text{if } \eta \neq 0 \\ \infty, & \text{if } \eta = 0 \end{cases} \\
\alpha &= \arctan \frac{\mathrm{c}\alpha_2\mathrm{c}\beta_2\mathrm{s}\beta_1 - \mathrm{c}\alpha_1\mathrm{c}\beta_1\mathrm{s}\beta_2}{\mathrm{s}\alpha_2\mathrm{s}\beta_1\mathrm{c}\beta_2 - \mathrm{s}\alpha_1\mathrm{c}\beta_1\mathrm{s}\beta_2} \\
\beta &= \varepsilon_\eta \arcsin \frac{\mathrm{c}\beta_2\mathrm{c}\beta_1\mathrm{s}(\alpha_1 - \alpha_2)}{\sqrt{1 - \eta^2}}
\end{aligned} \tag{3.10}$$

where

$$\eta = \eta(\alpha_1, \alpha_2, \beta_1, \beta_2) := \cos\beta_2 \cos\beta_1 \cos(\alpha_2 - \alpha_1) + \sin\beta_2 \sin\beta_1 \tag{3.11}$$

and $\varepsilon_\eta = \operatorname{sgn}\eta$ *provided that* $\eta \neq 0$ *and* $\varepsilon_\eta = -1$ *otherwise.*

Proof. From the results in Table 1 we have that the condition for the resulting rotation to be proper is $\mathbf{n}_2.\mathbf{n}_1 = 0$ where $\mathbf{n}_i = (\cos\alpha_i \cos\beta_i, -\sin\alpha_i \cos\beta_i, -\sin\beta_i)$, $i = 1, 2$ are the associated directions of the rays that represent $\mathcal{R}_1$ and $\mathcal{R}_2$. It is clear that $\mathbf{n}_2.\mathbf{n}_1 = \eta = \eta(\alpha_1, \alpha_2, \beta_1, \beta_2)$.

Let $\eta \neq 0$, i.e., the resulting rotation is a half-turn. Then the resulting rotation is represented by the vector-parameter $\boldsymbol{c} = -\dfrac{\mathbf{n}_2 \times \mathbf{n}_1}{\mathbf{n}_2.\mathbf{n}_1}$ expressed in $\alpha_i, \beta_i, i = 1, 2$ terms in the following way

$$\boldsymbol{c} = \left(\frac{c\beta_1 s\alpha_1 s\beta_2 - c\beta_2 s\alpha_2 s\beta_1, c\alpha_1 c\beta_1 s\beta_2 - c\alpha_2 c\beta_2 s\beta_1, c\beta_1 c\beta_2 s(\alpha_1 - \alpha_2)}{c\beta_2 c\beta_1 c(\alpha_2 - \alpha_1) + s\beta_2 s\beta_1}\right) \tag{3.12}$$

Simple calculation shows that $c^2 = \dfrac{\mathbf{n}_1^2 \mathbf{n}_2^2 (1 - (\mathbf{n}_2.\mathbf{n}_1)^2)}{(\mathbf{n}_2.\mathbf{n}_1)^2}$ and thus $\tau = \dfrac{\sqrt{1-\eta^2}}{|\eta|}$. Now from (2.17) and (3.12) we get formulas (3.10) and (3.11).

In the second case, i.e., $\eta = 0$ the composite rotation $\mathcal{R}$ is a half-turn represented by the ray $[\mathbf{n}_2 \times \mathbf{n}_1]$. Thus

$$-\frac{\mathbf{n}_2 \times \mathbf{n}_1}{\sqrt{1-\eta^2}} = (\cos\alpha\cos\beta, -\sin\alpha\cos\beta, -\sin\beta) \tag{3.13}$$

and using $\alpha_i, \beta_i, i = 1, 2$ expressions for the numerator of $\mathbf{n}_2 \times \mathbf{n}_1$ from (3.12), we obtain the rest of the cases in the Proposition.

4 Systematized results

The results formulated in Proposition 2 and Proposition 6 can be systematized eventually as follows. In each of these cases the composition law in the (α, β, τ) formalism is presented explicitly and additionally one has: $\epsilon_X = \operatorname{sgn}X$ if $X \neq 0$ and -1 otherwise, where $X = \nu, \mu, \xi, \eta$ and the last numbers are defined correspondingly in (3.1), (3.4), (3.9) and (3.11).

CASE - 1 : Product of rotations:

$$\mathcal{R}(\alpha_2, \beta_2, \tau_2)\mathcal{R}(\alpha_1, \beta_1, \tau_1)$$

Result and condition:

$$\alpha = \arctan \frac{\tau_1 \mathrm{s}\alpha_1 \mathrm{c}\beta_1 + \tau_2 \mathrm{s}\alpha_2 \mathrm{c}\beta_2 + \tau_1\tau_2(\mathrm{c}\alpha_1\mathrm{c}\beta_1\mathrm{s}\beta_2 - \mathrm{c}\alpha_2\mathrm{c}\beta_2\mathrm{s}\beta_1)}{\tau_1 \mathrm{c}\alpha_1 \mathrm{c}\beta_1 + \tau_2 \mathrm{c}\alpha_2 \mathrm{c}\beta_2 + \tau_1\tau_2(\mathrm{s}\alpha_2\mathrm{s}\beta_1\mathrm{c}\beta_2 - \mathrm{s}\alpha_1\mathrm{c}\beta_1\mathrm{s}\beta_2)}$$

$$\beta = \varepsilon_\mu \arcsin \frac{\tau_1 \mathrm{s}\beta_1 + \tau_2 \mathrm{s}\beta_2 + \tau_1\tau_2 \mathrm{c}\beta_1\mathrm{c}\beta_2\mathrm{s}(\alpha_1-\alpha_2)}{\sqrt{(1+\tau_1^2)(1+\tau_2^2)-\nu^2}}$$

$$\tau = \begin{cases} \dfrac{\sqrt{(1+\tau_1^2)(1+\tau_2^2)-\nu^2}}{|\nu|}, & \text{if } \nu \neq 0 \\ \infty, & \text{if } \nu = 0. \end{cases}$$

CASE - 2 : Product of rotations:

$$\mathcal{R}(\alpha_2, \beta_2, \tau_1)\mathcal{R}(\alpha_1, \beta_1, \infty)$$

Result and condition:

$$\alpha = \arctan \frac{\mathrm{s}\alpha_1\mathrm{c}\beta_1 + \tau_2(\mathrm{c}\alpha_1\mathrm{c}\beta_1\mathrm{s}\beta_2 - \mathrm{c}\alpha_2\mathrm{c}\beta_2\mathrm{s}\beta_1)}{\mathrm{c}\alpha_1\mathrm{c}\beta_1 + \tau_2(\mathrm{s}\alpha_2\mathrm{s}\beta_1\mathrm{c}\beta_2 - \mathrm{s}\alpha_1\mathrm{c}\beta_1\mathrm{s}\beta_2)}$$

$$\beta = \varepsilon_\mu \arcsin \frac{\mathrm{s}\beta_1 + \tau_2\mathrm{c}\beta_2\mathrm{c}\beta_1\mathrm{s}(\alpha_1-\alpha_2)}{\sqrt{1+\tau_2^2-\mu^2}}$$

$$\tau = \begin{cases} \dfrac{\sqrt{(1+\tau_2^2)-\mu^2}}{|\mu|}, & \text{if } \mu \neq 0 \\ \infty, & \text{if } \mu = 0. \end{cases}$$

CASE - 3 : Product of rotations:

$$\mathcal{R}(\alpha_2, \beta_2, \infty)\mathcal{R}(\alpha_1, \beta_1, \tau_1)$$

Result and condition:

$$\alpha = \arctan \frac{\mathrm{s}\alpha_2\mathrm{c}\beta_2 + \tau_1(\mathrm{c}\alpha_2\mathrm{c}\beta_2\mathrm{s}\beta_1 - \mathrm{c}\alpha_1\mathrm{c}\beta_1\mathrm{s}\beta_2)}{\mathrm{c}\alpha_2\mathrm{c}\beta_2 + \tau_1(\mathrm{s}\alpha_2\mathrm{s}\beta_1\mathrm{c}\beta_2 - \mathrm{s}\alpha_1\mathrm{c}\beta_1\mathrm{s}\beta_2)}$$

$$\beta = \varepsilon_\xi \arcsin \frac{\mathrm{s}\beta_2 + \tau_2\mathrm{c}\beta_2\mathrm{c}\beta_1\mathrm{s}(\alpha_1-\alpha_2)}{\sqrt{1+\tau_1^2-\xi^2}}$$

$$\tau = \begin{cases} \dfrac{\sqrt{(1+\tau_1^2)-\xi^2}}{|\xi|}, & \text{if } \xi \neq 0 \\ \infty, & \text{if } \xi = 0. \end{cases}$$

CASE - 4 : Product of rotations:

$$\mathcal{R}(\alpha_2, \beta_2, \infty)\mathcal{R}(\alpha_1, \beta_1, \infty)$$

Result and condition:

$$\alpha = \arctan \frac{\mathrm{c}\alpha_2 \mathrm{c}\beta_2 \mathrm{s}\beta_1 - \mathrm{c}\alpha_1 \mathrm{c}\beta_1 \mathrm{s}\beta_2}{\mathrm{s}\alpha_2 \mathrm{s}\beta_1 \mathrm{c}\beta_2 - \mathrm{s}\alpha_1 \mathrm{c}\beta_1 \mathrm{s}\beta_2}$$

$$\beta = \varepsilon_\eta \arcsin \frac{\mathrm{c}\beta_2 \mathrm{c}\beta_1 \mathrm{s}(\alpha_1 - \alpha_2)}{\sqrt{1-\eta^2}}$$

$$\tau = \begin{cases} \dfrac{\sqrt{1-\eta^2}}{|\eta|}, & \text{if } \eta \neq 0 \\ \infty, & \text{if } \eta = 0. \end{cases}$$

5 Concluding remarks

The alternative forms of the *Rodrigues* representation [5] developed here along the full spectrum of cases for the composition law are not so computationally cheap and simple as the vector-parameter formalism. However, we find that this is important as the (α, β, τ) formalism is naturally connected to the spherical coordinates. As such, we believe, that there are mechanical or other physical problems in which the setting is such that the use of the alternative *Rodrigues* representation would be natural and more effective. Last but not least, it is curious if there are rotational representations related to other surfaces in classical differential geometry and in which problems they arise.

References

[1] D. Brezov, C. Mladenova and I. Mladenov, *Vector decompositions of rotations*, J. Geom. Symmetry Phys. **28** (2012), 59-95. MR3114816. Zbl 1291.15070.

[2] J. Dai, *Euler - Rodrigues formula variations, quaternion conjugation and intrinsicconnections*, Mech. and Machine Theory **92** (2015), 144-152. doi: 10.1016/j.mechmachtheory.2015.03.004.

[3] V. Donchev, C. Mladenova and I. Mladenov, *On vector parameter forms of* $\mathrm{SU}(1,1), \mathrm{SL}(2,\mathbb{R})$ *and their connection to* $\mathrm{SO}(2,1)$, Geom. Integrability & Quantization **17** (2016), 196-230. Zbl 1402.22023.

[4] V. Donchev, C. Mladenova, I. Mladenov, *On the compositions of rotations*, AIP Conf. Proc. **1684** (2015), 1-11. doi: 10.1063/1.4934315.

[5] V. Donchev, C. Mladenova and I. Mladenov, *Some alternatives of the Rodrigues axis-angle formula*, C. R. Acad. Bulg. Sci. **69** (2016), 697-706. MR3559577. Zbl 1402.22023.

[6] F. Fedorov, *The Lorentz Group* (in Russian), Nauka, Moscow 1979. MR575165. Zbl 0509.22012.

[7] R. Gilmore, *Relations among low-dimensional simple Lie groups*, J. Geom. Symmetry Phys. 28 (2012), 1-45. MR3114813. Zbl 1317.22005.

[8] J. Kuipers, *Quaternions and rotation sequences*, Geom. Integrability & Quantization **1** (2000), 127–143. MR1758157. Zbl 0993.53005.

[9] I. Mladenov, *Saxon-Hutner theorem for periodic multilayers*, C. R. Acad. Bulg. Sci. **40** (1987), 35-38. MR0940050. Zbl 0646.34053.

[10] C. Mladenova, *An approach to description of a rigid body motion*, C. R. Acad. Sci. Bulg. **38** (1985), 1657-1660. Zbl 0588.70006.

[11] C. Mladenova, *A contribution to the modeling and control of manipulators*, J. Intell. Rob. Syst. **3** (1990), 349-363. doi: 10.1007/BF00439423.

[12] C. Mladenova, *Group theory in the problems of modeling and control of multi-body systems*, J. Geom. Symmetry Phys. **8** (2006), 17-121. MR2336855. Zbl 1121.70007.

[13] C. Mladenova and I. Mladenov, *Vector decompositions of finite rotations*, Rep. Math. Phys. **68** (2011), 91-107. MR2846214. doi: 10.1016/S0034-4877(11)60030-X.

[14] A. Müller, *Group theoretical approaches to vector parameterization of rotations*, J. Geom. Symmetry Phys. **19** (2010), 43-72. MR2674961. Zbl 1376.70007.

Clementina D. Mladenova
Institute of Mechanics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Block 4
Sofia 1113, Bulgaria
e-mail: clem@imbm.bas.bg

Ivaïlo M. Mladenov
Inst. Biophys. & Biomed. Eng.
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Block 21
Sofia 1113, Bulgaria
e-mail: mladenov@bio21.bas.bg

License

This work is licensed under a Creative Commons Attribution 4.0 International License.