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# AN APPLICATION OF THE DISTRIBUTION SERIES FOR CERTAIN ANALYTIC FUNCTION CLASSES

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**Abstract**. For the generalized distribution with the Pascal model defined by

$$P(\mathcal{X}=j)=\binom{j+\mathfrak{t}-1}{\mathfrak{t}-1}p^{j}(1-p)^{\mathfrak{t}} \quad j\in\{0,1,2,3,...\},$$

let $\mathcal{U}P(\lambda,\alpha,\mu)$ and $\mathcal{H}P(\lambda,\alpha)$ represent the analytic function classes in the open unit disk $\mathcal{D}=\{z:z\in\mathbb{C} \text{ and } |z|<1\}$. The main aim of this paper is to derive the sufficient conditions for functions in these classes.

## 1 Introduction and motivation

The elementary distributions such as the Poisson, the Binomial, the Pascal, the Logarithmic arise the most well-utilized discrete distributions in various fields of science. For example, a firm footing of the usage of the elementary distributions was first used in Geometric Function Theory by Porwal [8]. Since then the distributions have received attention in the Geometric Function Theory (see [1, 2, 5, 6]).

Let us consider a non-negative discrete random variable $\mathcal{X}$ with a Pascal probability generating function

$$P(\mathcal{X}=j)=\binom{j+\mathfrak{t}-1}{\mathfrak{t}-1}p^{j}(1-p)^{\mathfrak{t}}, \quad j\in\{0,1,2,3,...\},$$

where $p$, $\mathfrak{t}$ are called the parameters.

Let $\mathcal{A}$ represent the class of functions $f$ of the form

$$f(z)=z+\sum_{j=2}^{\infty}a_jz^j, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{D}$ and let also $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions which are univalent in $\mathcal{D}$.

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Now, we recall a very concise overview of well-known definitions.

**Definition 1.** *(See [3]) A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{U}\mathrm{P}(\lambda, \alpha, \mu)$ if it satisfies the following condition*

$$\Re\left(\frac{z\psi'(z)}{\psi(z)}\right) > \mu \quad (0 \le \alpha \le \lambda \le 1,\ 0 \le \mu < 1,\ z \in \mathcal{D}),$$

*and*

$$\psi(z) = \lambda\alpha z^2 f''(z) + (\lambda - \alpha) z f'(z) + (1 - \lambda + \alpha) f(z).$$

The function class $\mathcal{U}P(\lambda, \alpha, \mu)$ is of notable interest and it comprises many common classes of univalent functions (see [9]). Further we get

$$U(0, 0, \mu) = T^*(\mu) \quad \text{and} \quad U(1, 0, \mu) = C(\mu).$$

**Definition 2.** *(See [7]) A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{H}\mathrm{P}(\lambda, \alpha)$ if it satisfies the following condition*

$$\Re\left\{\frac{zf'(z)}{f(z)} + \frac{\lambda z^2 f''(z)}{f(z)}\right\} > \alpha \quad (0 \le \alpha < 1,\ z \in \mathcal{D})$$

*for some $\lambda\,(\lambda \ge 0)$ and $\frac{f(z)}{z} \ne 0$.*

The aim of this paper is to investigate the Pascal distribution for the analytic function classes $\mathcal{U}P(\lambda, \alpha, \mu)$ and $\mathcal{H}P(\lambda, \alpha)$.

## 2 Method of estimation

We recall here the following proved lemmas.

**Lemma 3.** *(See [3]) A function $f \in \mathcal{A}$ given by (1.1) is in the class $\mathcal{U}\mathrm{P}(\lambda, \alpha, \mu)$ if*

$$\sum_{j=2}^{\infty} (j - \mu)\left[(j - 1)(j\lambda\alpha + \lambda - \alpha) + 1\right]|a_j| \le 1 - \mu. \tag{2.1}$$

**Lemma 4.** *(See [4]) Let $f \in \mathcal{A}$ be of the form (1.1), then $f \in \mathcal{H}\mathrm{P}(\lambda, \alpha)$, if*

$$\sum_{j=2}^{\infty} \left[(j - 1)(1 + j\lambda) + (1 - \alpha)\right]|a_j| \le 1 - \alpha. \tag{2.2}$$

Now, based upon the Pascal distribution, consider the following power series:

$$\mathcal{K}(\mathfrak{t}, p, z) = z + \sum_{j=2}^{\infty} \binom{j + \mathfrak{t} - 2}{\mathfrak{t} - 1} p^{j-1}(1 - p)^{\mathfrak{t}} z^j \quad (\mathfrak{t} \ge 1, 0 \le p < 1, z \in \mathcal{D}). \tag{2.3}$$

By ratio test we conclude that the radius of convergence of the above power series is infinity.

By considering above definitions and lemmas, we have the following sufficient conditions for the function $\mathcal{K}$.

**Theorem 5.** *A sufficient condition for the function $\mathcal{K}$ given by (2.3) to be in the class $\mathcal{U}\mathrm{P}(\lambda,\alpha,\mu)$ is*

$$\begin{aligned} &\frac{\lambda\alpha\mathfrak{t}\,(\mathfrak{t}+1)\,(\mathfrak{t}+2)\,p^3}{(1-p)^3}+\frac{(\lambda-\alpha+5\lambda\alpha-\mu\lambda\alpha)\,\mathfrak{t}\,(\mathfrak{t}+1)\,p^2}{(1-p)^2} \\ &+\frac{(4\lambda\alpha+2\lambda-2\mu\lambda\alpha-\mu\lambda+\mu\alpha-2\alpha+1)\,\mathfrak{t}p}{1-p} \\ &+(\mu\alpha-\mu\lambda+1)-(\mu\alpha-\mu\lambda+1)\,(1-p)^{\mathfrak{t}} \\ \leq\;& 1-\alpha, \quad (0\leq p<1). \end{aligned} \tag{2.4}$$

**Theorem 6.** *A sufficient condition for the function $\mathcal{K}$ given by (2.3) to be in the class $\mathcal{H}\mathrm{P}(\lambda,\alpha)$ is*

$$\frac{\lambda\mathfrak{t}\,(\mathfrak{t}+1)\,p^2}{(1-p)^2}+\frac{(1+2\lambda)\,\mathfrak{t}p}{1-p}+(1-\alpha)-(1-\alpha)\,(1-p)^{\mathfrak{t}}\leq 1-\alpha, \quad (0\leq p<1). \tag{2.5}$$

## 3 Proof of theorems

*Proof of Theorem 5.* According to Lemma 3, we must show that

$$\sum_{j=2}^{\infty}\binom{j+\mathfrak{t}-2}{\mathfrak{t}-1}(j-\mu)\left[(j-1)\,(j\lambda\alpha+\lambda-\alpha)+1\right]p^{j-1}\,(1-p)^{\mathfrak{t}}\leq 1-\mu.$$

Therefore, by combining the relation (2.3) and the implication (2.4), we have the equality

$$\sum_{j=2}^{\infty}\binom{j+\mathfrak{t}-2}{\mathfrak{t}-1}(j-\mu)\left[(j-1)(j\lambda\alpha+\lambda-\alpha)+1\right]p^{j-1}(1-p)^{\mathfrak{t}}$$

$$=\sum_{j=2}^{\infty}\binom{j+\mathfrak{t}-2}{\mathfrak{t}-1}\left[j^{3}\lambda\alpha+j^{2}(\lambda-\alpha-\lambda\alpha-\mu\lambda\alpha)\right.$$

$$\left.+j(\mu\lambda\alpha-\mu\lambda+\mu\alpha-\lambda+\alpha+1)+\mu(\lambda-\alpha-1)\right]p^{j-1}(1-p)^{\mathfrak{t}}$$

$$=\lambda\alpha\sum_{j=2}^{\infty}\binom{j+\mathfrak{t}-2}{\mathfrak{t}-1}\left[(j-1)(j-2)(j-3)+6(j-1)(j-2)+7(j-1)+1\right]p^{j-1}(1-p)^{\mathfrak{t}}$$

$$+(\lambda-\alpha-\lambda\alpha-\mu\lambda\alpha)\sum_{j=2}^{\infty}\binom{j+\mathfrak{t}-2}{\mathfrak{t}-1}\left[(j-1)(j-2)+3(j-1)+1\right]p^{j-1}(1-p)^{\mathfrak{t}}$$

$$+(\mu\lambda\alpha-\mu\lambda+\mu\alpha-\lambda+\alpha+1)\sum_{j=2}^{\infty}\binom{j+\mathfrak{t}-2}{\mathfrak{t}-1}\left[(j-1)+1\right]p^{j-1}(1-p)^{\mathfrak{t}}$$

$$+\mu(\lambda-\alpha-1)\sum_{j=2}^{\infty}\binom{j+\mathfrak{t}-2}{\mathfrak{t}-1}p^{j-1}(1-p)^{\mathfrak{t}}$$

$$=\lambda\alpha(1-p)^{\mathfrak{t}}\sum_{j=4}^{\infty}\binom{j+\mathfrak{t}-2}{\mathfrak{t}+2}\mathfrak{t}(\mathfrak{t}+1)(\mathfrak{t}+2)p^{j-4}p^{3}$$

$$+6\lambda\alpha(1-p)^{\mathfrak{t}}\sum_{j=3}^{\infty}\binom{j+\mathfrak{t}-2}{\mathfrak{t}+1}\mathfrak{t}(\mathfrak{t}+1)p^{j-3}p^{2}$$

$$+7\lambda\alpha(1-p)^{\mathfrak{t}}\sum_{j=2}^{\infty}\binom{j+\mathfrak{t}-2}{\mathfrak{t}}\mathfrak{t}p^{j-2}p+\lambda\alpha(1-p)^{\mathfrak{t}}\sum_{j=2}^{\infty}\binom{j+\mathfrak{t}-2}{\mathfrak{t}-1}p^{j-1}$$

$$+(\lambda-\alpha-\lambda\alpha-\mu\lambda\alpha)(1-p)^{\mathfrak{t}}\sum_{j=3}^{\infty}\binom{j+\mathfrak{t}-2}{\mathfrak{t}+1}\mathfrak{t}(\mathfrak{t}+1)p^{j-3}p^{2}$$

$$+3(\lambda-\alpha-\lambda\alpha-\mu\lambda\alpha)(1-p)^{\mathfrak{t}}\sum_{j=2}^{\infty}\binom{j+\mathfrak{t}-2}{\mathfrak{t}}\mathfrak{t}p^{j-2}p$$

$$+(\lambda-\alpha-\lambda\alpha-\mu\lambda\alpha)(1-p)^{\mathfrak{t}}\sum_{j=2}^{\infty}\tbinom{j+\mathfrak{t}-2}{\mathfrak{t}-1}p^{j-1}$$

$$+(\mu\lambda\alpha-\mu\lambda+\mu\alpha-\lambda+\alpha+1)(1-p)^{\mathfrak{t}}\sum_{j=2}^{\infty}\tbinom{j+\mathfrak{t}-2}{\mathfrak{t}}\mathfrak{t}p^{j-2}p$$

$$+(\mu\lambda\alpha-\mu\lambda+\mu\alpha-\lambda+\alpha+1)(1-p)^{\mathfrak{t}}\sum_{j=2}^{\infty}\tbinom{j+\mathfrak{t}-2}{\mathfrak{t}-1}p^{j-1}$$

$$+\mu(\lambda-\alpha-1)\sum_{j=2}^{\infty}\tbinom{j+\mathfrak{t}-2}{\mathfrak{t}-1}p^{j-1}(1-p)^{\mathfrak{t}}$$

$$= \lambda\alpha\mathfrak{t}(\mathfrak{t}+1)(\mathfrak{t}+2)p^3(1-p)^{\mathfrak{t}}\sum_{j=0}^{\infty}\binom{j+\mathfrak{t}+2}{\mathfrak{t}+2}p^j$$

$$+(\lambda-\alpha+5\lambda\alpha-\mu\lambda\alpha)\mathfrak{t}(\mathfrak{t}+1)p^2(1-p)^{\mathfrak{t}}\sum_{j=0}^{\infty}\binom{j+\mathfrak{t}+1}{\mathfrak{t}+1}p^j$$

$$+(4\lambda\alpha+2\lambda-2\mu\lambda\alpha-\mu\lambda+\mu\alpha-2\alpha+1)\mathfrak{t}p(1-p)^{\mathfrak{t}}\sum_{j=0}^{\infty}\binom{j+\mathfrak{t}}{\mathfrak{t}}p^j$$

$$+(1+\mu\alpha-\mu\lambda)(1-p)^{\mathfrak{t}}\sum_{j=0}^{\infty}\binom{j+\mathfrak{t}-1}{\mathfrak{t}-1}p^j-(1+\mu\alpha-\mu\lambda)(1-p)^{\mathfrak{t}}$$

$$= \frac{\lambda\alpha\mathfrak{t}(\mathfrak{t}+1)(\mathfrak{t}+2)p^3}{(1-p)^3}+\frac{(\lambda-\alpha+5\lambda\alpha-\mu\lambda\alpha)\mathfrak{t}(\mathfrak{t}+1)p^2}{(1-p)^2}$$

$$+\frac{(4\lambda\alpha+2\lambda-2\mu\lambda\alpha-\mu\lambda+\mu\alpha-2\alpha+1)\mathfrak{t}p}{1-p}$$

$$+(\mu\alpha-\mu\lambda+1)-(\mu\alpha-\mu\lambda+1)(1-p)^{\mathfrak{t}}$$

$$\leq 1-\alpha.$$

Thus the proof of Theorem 5 is now completed. □

*Proof of Theorem 6.* To prove that $\mathcal{K}\in\mathcal{H}P(\lambda,\alpha)$, we must show that

$$\frac{\lambda\mathfrak{t}(\mathfrak{t}+1)p^2}{(1-p)^2}+\frac{(1+2\lambda)\mathfrak{t}p}{1-p}+(1-\alpha)-(1-\alpha)(1-p)^{\mathfrak{t}}\leq 1-\alpha.$$

From the relation (2.3) and the implication (2.5), we get

$$\sum_{j=2}^{\infty}\binom{j+\mathfrak{t}-2}{\mathfrak{t}-1}\left[(j-1)(1+j\lambda)+(1-\alpha)\right]p^{j-1}(1-p)^{\mathfrak{t}}$$

$$=\sum_{j=2}^{\infty}\binom{j+\mathfrak{t}-2}{\mathfrak{t}-1}\left[(j-1)(j-2)\lambda+(j-1)(1+2\lambda)+(1-\alpha)\right]p^{j-1}(1-p)^{\mathfrak{t}}$$

$$=\lambda(1-p)^{\mathfrak{t}}\sum_{j=2}^{\infty}\binom{j+\mathfrak{t}-2}{\mathfrak{t}-1}(j-1)(j-2)p^{j-1}$$

$$+(1+2\lambda)(1-p)^{\mathfrak{t}}\sum_{j=2}^{\infty}\binom{j+\mathfrak{t}-2}{\mathfrak{t}-1}(j-1)p^{j-1}+(1-\alpha)(1-p)^{\mathfrak{t}}\sum_{j=2}^{\infty}\binom{j+\mathfrak{t}-2}{\mathfrak{t}-1}p^{j-1}$$

$$=\lambda(1-p)^{\mathfrak{t}}\sum_{j=3}^{\infty}\binom{j+\mathfrak{t}-2}{\mathfrak{t}+1}\mathfrak{t}(\mathfrak{t}+1)p^{j-3}p^{2}$$

$$+(1+2\lambda)(1-p)^{\mathfrak{t}}\sum_{j=2}^{\infty}\binom{j+\mathfrak{t}-2}{\mathfrak{t}}\mathfrak{t}p^{j-2}p+(1-\alpha)(1-p)^{\mathfrak{t}}\sum_{j=2}^{\infty}\binom{j+\mathfrak{t}-2}{\mathfrak{t}-1}p^{j-1}$$

$$=\lambda\mathfrak{t}(\mathfrak{t}+1)p^{2}(1-p)^{\mathfrak{t}}\sum_{j=0}^{\infty}\binom{j+\mathfrak{t}+1}{\mathfrak{t}+1}p^{j}+(1+2\lambda)\mathfrak{t}p(1-p)^{\mathfrak{t}}\sum_{j=0}^{\infty}\binom{j+\mathfrak{t}}{\mathfrak{t}}p^{j}$$

$$+(1-\alpha)(1-p)^{\mathfrak{t}}\sum_{j=0}^{\infty}\binom{j+\mathfrak{t}-1}{\mathfrak{t}-1}p^{j}-(1-\alpha)(1-p)^{\mathfrak{t}}$$

$$=\frac{\lambda\mathfrak{t}(\mathfrak{t}+1)p^{2}}{(1-p)^{2}}+\frac{(1+2\lambda)\mathfrak{t}p}{1-p}+(1-\alpha)-(1-\alpha)(1-p)^{\mathfrak{t}}\leq 1-\alpha.$$

Thus, according to Lemma 4, we conclude that $\mathcal{K}\in\mathcal{HP}(\lambda,\alpha)$. □

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