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CATEGORICAL MODELS AND QUASIGROUP HOMOTOPIES

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ABSTRACT. As is pointed out in [Smith (1997)], in many applications of quasigroups isotopies and homotopies are more important than isomorphisms and homomorphisms. In this paper, the way homotopies may arise in the context of categorical quasigroup model theory is investigated. In this context, the algebraic structures are specified by diagram-based logics, such as sketches, and categories of models become functor categories. An idea, pioneered in [Gvaramiya & Plotkin (1992)], is used to give a construction of a model category naturally equivalent to the category of quasigroups with homotopies between them.

1. Introduction

Traditionally, when the categorical study of classes of algebras is undertaken the role of morphisms in the categories under investigation is played by the ordinary homomorphisms of universal algebra (see, e.g., [Borceux (1994)], Vol. II, Chapter 3). However, in the case of quasigroups, a different kind of morphism may be more important, depending on the application at hand, than ordinary quasigroup homomorphisms [Smith (1997)]. These are quasigroup homotopies. A *homotopy* from a quasigroup $\mathbf{Q} = \langle Q, \cdot, /, \backslash \rangle$ to a quasigroup $\mathbf{P} = \langle P, \cdot, /, \backslash \rangle$ is a triple (h_1, h_2, h_3) of set maps from Q to P, such that, for all $x, y \in Q$, $h_3(x \cdot y) = h_1(x) \cdot h_2(y)$ (see, e.g., [Smith & Romanowska (1999)], I.4).

An alternative to equational logic for specifying algebraic structures is provided by graph-based logics, such as sketches. [Barr & Wells (1990)] and [Coppey & Lair (1984), Coppey & Lair (1988)] provide very readable introductions to sketches and a good part of the first two sections of the present paper heavily draws on their treatment. In addition, the 6th Lesson of [Coppey & Lair (1988)] contains sketches specifying many of the best known algebraic and graph-based structures. Following [Barr & Wells (1990)] and [Coppey & Lair (1984), Coppey & Lair (1988)], a sketch for quasigroups is given in the following section. Model morphisms in model categories of sketches being natural transformations, they are really tailored to capture the ordinary homomorphisms of universal algebra. Thus, as it is shown in Section 3, the category of models of the sketch for quasigroups is the category of quasigroups with homomorphisms between them.

This introduction motivates the main question we are faced with: What is the role of homotopies in categorical quasigroup model theory, or, rephrasing, how may one capture homotopies of quasigroups in the context of categorical quasigroup model theory? One

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feature of the sketch for quasigroups that spoils the admissibility of homotopies is the use of direct squares and of the accompanying projections. Roughly speaking, these force a natural transformation η from a model M to a model N to obey commutativity of

$$\begin{tikzcd}
M(Q^2) \arrow[r, "\eta_{Q^2}"] \arrow[d, "M(p_1)"'] & N(Q^2) \arrow[d, "N(p_1)"] & M(Q^2) \arrow[r, "\eta_{Q^2}"] \arrow[d, "M(p_2)"'] & N(Q^2) \arrow[d, "N(p_2)"] \\
M(Q) \arrow[r, "\eta_Q"'] & N(Q) & M(Q) \arrow[r, "\eta_Q"'] & N(Q)
\end{tikzcd} \qquad (1)$$

whence $\eta_{Q^2} = \eta_Q \times \eta_Q$ and η_Q becomes necessarily a homomorphism. If this problem is to be overcome, one has to get rid of the direct squares which entails approaching the design of the sketch with a different philosophy in mind.

What comes to the rescue is an idea, first exploited by Gvaramiya and Plotkin in [Gvaramiya & Plotkin (1992)]. Its cornerstone is the introduction of different sorts for each of the arguments of the quasigroup multiplication. In the present context, this has the effect of transforming direct powers to direct products of different objects and, thus, dissolves the difficulty imposed by the previous requirement that (1) commute. Based on this idea, the notion of a $*$-automaton was defined in [Gvaramiya & Plotkin (1992)] and it was shown that every quasigroup gives rise to an invertible $*$-automaton and that every invertible $*$-automaton is isomorphic to one derived in this way by some quasigroup.

Adapting this idea to the present context, a modified sketch for quasigroups with homotopies is presented, such that its models in the category of sets are the invertible $*$-automata of [Gvaramiya & Plotkin (1992)] and the model morphisms between them, which are homomorphisms of the multi-sorted algebras, correspond to homotopies between the associated quasigroups. It is then shown in the last section that this model category and the category of quasigroups with homotopies between them are naturally isomorphic categories.

In [Smith (1997)], Smith showed, using a "semisymmetrization technique", that the category of quasigroups with homotopies is isomorphic to a category of homomorphisms between semisymmetric quasigroups, i.e., quasigroups satisfying the semisymmetric identity $(y{\cdot}x){\cdot}y \approx x$. The question remains open of whether Smith's result may be exploited, in the present context, so that a sketch be obtained having as its model category a category isomorphic to the category of quasigroups with homotopies.

2. Sketching Quasigroups

[Barr & Wells (1990)] and [Coppey & Lair (1984), Coppey & Lair (1988)] "sketch" some of the most commonly encountered algebraic structures. They are the source of the graph-theoretic and categorical definitions that are used in this and the next section in developing the standard sketch for quasigroups and showing that it corresponds to the category of quasigroups with homomorphisms between them. Definitions that pertain directly to quasigroups and their morphisms may be found in [Smith & Romanowska (1999)].

2.1. Definition. *A* **(directed) graph** $G = \langle V, E, s, t\rangle$ *consists of a set* V *of* **nodes** *or* **vertices***, a set* E *of* **edges,** *and two functions* $s, t : E \to V$*, associating with each edge* e *its* **source vertex** $s(e)$ *and its* **target vertex** $t(e)$*, respectively. One writes* $e : s(e) \to t(e)$ *in this case. Let* $G = \langle V, E, s, t\rangle$ *and* $G' = \langle V', E', s', t'\rangle$ *be graphs. A* **graph morphism** $h : G \to G'$ *is a pair* $h = \langle h_1, h_2\rangle$*, with* $h_1 : V \to V'$ *and* $h_2 : E \to E'$ *satisfying* $s'(h_2(e)) = h_1(s(e))$ *and* $t'(h_2(e)) = h_1(t(e))$*, for all* $e \in E$.

As an example and for future reference we introduce the graph G_q with $V_q = \{Q_1, Q_2\}$, $E_q = \{p_1, p_2, m, l, r, \langle p_1, m\rangle, \langle m, p_2\rangle, \langle p_1, l\rangle, \langle r, p_2\rangle\}$, where s_q and t_q are given diagrammatically as follows:

$$p_1, p_2, m, l, r : Q_2 \to Q_1,$$

$$\langle p_1, m\rangle, \langle m, p_2\rangle, \langle p_1, l\rangle, \langle r, p_2\rangle : Q_2 \to Q_2.$$

2.2. Definition. *Let* G *be a graph. A* **path** *in* G *is a sequence* $(e_1, \ldots, e_n)$ *of edges in* G*, such that, for all* $i = 1, \ldots, n-1$, $t(e_i) = s(e_{i+1})$.

Two paths $p = (e_1, \ldots, e_n)$ *and* $q = (f_1, \ldots, f_m)$ *in* G *are said to be* **parallel** *if* $s(e_1) = s(f_1)$ *and* $t(e_n) = t(f_m)$.

An **equation** *in* G *is a pair of parallel paths* p *and* q *as above, and is usually denoted by* $e_n e_{n-1} \ldots e_1 \approx f_m f_{m-1} \ldots f_1$.

The following are equations in the graph G_q defined previously.

$$\begin{array}{llll} p_1\langle p_1, m\rangle \approx p_1 & p_2\langle p_1, m\rangle \approx m & p_1\langle m, p_2\rangle \approx m & p_2\langle m, p_2\rangle \approx p_2 \\ p_1\langle p_1, l\rangle \approx p_1 & p_2\langle p_1, l\rangle \approx l & p_1\langle r, p_2\rangle \approx r & p_2\langle r, p_2\rangle \approx p_2 \end{array}$$

Also

$$\begin{array}{ll} l\langle p_1, m\rangle \approx p_2 & r\langle m, p_2\rangle \approx p_1 \\ m\langle p_1, l\rangle \approx p_2 & m\langle r, p_2\rangle \approx p_1 \end{array}$$

2.3. Definition. *Let* G *be a graph. A* **diagram** d *in* G *is a graph morphism* $d : D \to G$*, where* $D = \langle U, F, \sigma, \tau\rangle$ *is the* **shape graph** *of* d.

A **cone** $v \lhd d$ *in* G *with* **vertex** v *and* **base** d *consists of a diagram* d *in* G*, a vertex* $v \in V$ *and a collection of edges* $\{e_u : u \in U\}$*, called* **projections***, such that* $s(e_u) = v$ *and* $t(e_u) = d(u)$*, for all* $u \in U$*. The cone* $v \lhd d$ *is said to be* **discreet** *or a* **product cone** *if* $F = \emptyset$ *and a* **finite product cone** *if, in addition,* $|U| < \omega$.

Let, for instance, $D_q = \langle U_q, F_q, \sigma_q, \tau_q\rangle$ be the graph with $U_q = \{u_1, u_2\}, F_q = \emptyset$ and $d_q : D_q \to G_q$ the diagram in G_q determined by $d_{q1}(u_1) = d_{q1}(u_2) = Q_1$. Define the cone $Q_2 \lhd d_q$ in G_q by specifying that $p_1 : Q_2 \to Q_1$ and $p_2 : Q_2 \to Q_1$ be the two cone projections.

2.4. Definition. [Barr & Wells (1990)] *A* **limit sketch** $\mathcal{S} = \langle G, Q, L\rangle$ *consists of a graph* G, *a set* Q *of equations in* G *and a set* L *of cones in* G. *If all cones in* L *are product cones then* $\mathcal{S}$ *is called a* **product sketch** *and if they are all finite, then* $\mathcal{S}$ *is called a* **finite product sketch** *or an* **FP-sketch**.

Let $\mathcal{S}_q = \langle G_q, Q_q, L_q\rangle$ be the sketch with graph G_q, set of equations Q_q, containing all the equations displayed above, and set of cones $L_q = \{Q_2 \lhd d_q\}$. $\mathcal{S}_q$ is the **sketch for quasigroups**.

2.5. Definition. *Let* $\mathcal{S} = \langle G, Q, L\rangle$ *be a limit sketch and* $\mathcal{C}$ *a category. A* **model** $M : \mathcal{S} \to \mathcal{C}$ *of* $\mathcal{S}$ *in* $\mathcal{C}$ *is a graph morphism* $M : G \to C$, *where* C *is the underlying graph of* $\mathcal{C}$, *such that all equations in* Q *become commuting diagrams in* $\mathcal{C}$ *and all cones in* L *become limit cones in* $\mathcal{C}$.

Given two models M_1, M_2 *of* $\mathcal{S}$ *in* $\mathcal{C}$, *a* **model morphism** $h : M_1 \to M_2$ *is a natural transformation from* M_1 *to* M_2, *i.e., a family* $h_v : M_1(v) \to M_2(v), v \in V$, *of morphisms in* $\mathcal{C}$, *such that, for all* $e \in E$, *with* $s(e) = v_1, t(e) = v_2$, *the following rectangle commutes in* $\mathcal{C}$:

$$\begin{tikzcd}
M_1(v_1) \arrow[r, "h_{v_1}"] \arrow[d, "M_1(e)"'] & M_2(v_1) \arrow[d, "M_2(e)"] \\
M_1(v_2) \arrow[r, "h_{v_2}"'] & M_2(v_2)
\end{tikzcd}$$

Models of $\mathcal{S}$ *in* $\mathcal{C}$ *together with model morphisms form a category, which is denoted by* $\mathrm{Mod}_{\mathcal{C}}(\mathcal{S})$.

In case $\mathcal{C}$ *is a category with specified limits, a model of* $\mathcal{S}$ *in* $\mathcal{C}$ *has to carry all cones in* L *to specified limit cones and a model morphism has to preserve all limits corresponding to limit cones on the nose.*

2.6. Definition. *A* **quasigroup** $\mathbf{Q} = \langle Q, \cdot, /, \backslash\rangle$ *is a set* Q *equipped with binary operations* $x \cdot y$ *or, simply,* xy *of* **multiplication**, x/y *of* **right division** *and* $x\backslash y$ *of* **left division**, *such that the following identities hold:*

$$\begin{array}{ll} x\backslash(x \cdot y) \approx y & (x \cdot y)/y \approx x \\ x \cdot (x\backslash y) \approx y & (x/y) \cdot y \approx x \end{array}$$

Let $\mathbf{Q}$ *and* $\mathbf{P}$ *be two quasigroups. A* **quasigroup homomorphism** $h : \mathbf{Q} \to \mathbf{P}$ *is a function* $h : Q \to P$, *such that* $h(x \cdot y) = h(x) \cdot h(y)$, *for all* $x, y \in Q$.

A triple $(h_1, h_2, h_3) : \mathbf{Q} \to \mathbf{P}$ *of functions from* Q *to* P *is a* **quasigroup homotopy** *if* $h_3(x \cdot y) = h_1(x) \cdot h_2(y)$, *for all* $x, y \in Q$.

Note that, if $h : \mathbf{Q} \to \mathbf{P}$ is a quasigroup homomorphism, then, for all $x, y \in Q$, $h(x/y) = h(x)/h(y)$ and $h(x\backslash y) = h(x)\backslash h(y)$. Similarly, if $(h_1, h_2, h_3) : \mathbf{Q} \to \mathbf{P}$ is a quasigroup homotopy, then $h_1(x/y) = h_3(x)/h_2(y)$ and $h_2(x\backslash y) = h_1(x)\backslash h_3(y)$, for all $x, y \in Q$ (see [Smith (1997)]).

We denote by **Set** the category of all small sets and by $\mathbf{Set}_{\rightarrow}$ the category of all small sets with the usual specified limits. The same notation will also be used to denote the underlying graphs of these two categories for simplicity. The following proposition is a first step in relating the notions that were introduced in this section.

2.7. Proposition. *Let* $\mathbf{Q} = \langle Q, \cdot, /, \backslash \rangle$ *be a quasigroup. Define the graph morphism* $M_{\mathbf{Q}} : G_q \to \mathbf{Set}_{\rightarrow}$, *as follows:*

$$\begin{array}{ll}
M_{\mathbf{Q}}(Q_1) = Q & M_{\mathbf{Q}}(Q_2) = Q \times Q \\
M_{\mathbf{Q}}(p_1)((x,y)) = x & M_{\mathbf{Q}}(p_2)((x,y)) = y \\
M_{\mathbf{Q}}(m)((x,y)) = x \cdot y & M_{\mathbf{Q}}(l)((x,y)) = x\backslash y \\
M_{\mathbf{Q}}(r)((x,y)) = x/y & \\
M_{\mathbf{Q}}(\langle p_1, m\rangle)((x,y)) = (x, x \cdot y) & M_{\mathbf{Q}}(\langle m, p_2\rangle)((x,y)) = (x \cdot y, y) \\
M_{\mathbf{Q}}(\langle p_1, l\rangle)((x,y)) = (x, x\backslash y) & M_{\mathbf{Q}}(\langle r, p_2\rangle)((x,y)) = (x/y, y)
\end{array}$$

Then $M_{\mathbf{Q}}$ *is a model of* $\mathcal{S}_q$ *in* $\mathbf{Set}_{\rightarrow}$.

In addition, quasigroup homomorphisms give us concrete examples of model morphisms in $\mathrm{Mod}_{\mathbf{Set}_{\rightarrow}}(\mathcal{S}_q)$:

2.8. Proposition. *Let* $\mathbf{Q}, \mathbf{P}$ *be two quasigroups,* $h : \mathbf{Q} \to \mathbf{P}$ *a quasigroup homomorphism and* $M_{\mathbf{Q}}, M_{\mathbf{P}} : \mathcal{S}_q \to \mathbf{Set}_{\rightarrow}$ *the two models of* $\mathcal{S}_q$ *in* $\mathbf{Set}_{\rightarrow}$ *defined as in Proposition 2.7.* $\eta : M_{\mathbf{Q}} \to M_{\mathbf{P}}$, *defined by* $\eta_{Q_1} = h$ *and* $\eta_{Q_2} = (h, h)$ *is a model morphism of* $\mathcal{S}_q$ *in* $\mathbf{Set}_{\rightarrow}$.

3. Homomorphisms of Quasigroups

It is now shown that the only objects in $\mathrm{Mod}_{\mathbf{Set}_{\rightarrow}}(\mathcal{S}_q)$ are the ones given by Proposition 2.7 and, similarly, that the only model morphisms in this category are the ones provided by Proposition 2.8. What role do homotopies play in the context of the categorical model theory of quasigroups? The use of direct squares and, more generally, direct powers and the associate projections in the sketch that specifies a particular structure makes it impossible to accomodate homotopy-like morphisms in the form of natural transformations. To answer this question a new categorical specification of quasigroups will be introduced in the next section. The trick is to specify multiplication as a multisorted operation distinguishing between the values that can be substituted for each of its arguments (see [Gvaramiya & Plotkin (1992)]).

3.1. Proposition. *Let* $M : \mathcal{S}_q \to \mathbf{Set}_{\rightarrow}$ *be a model in* $\mathrm{Mod}_{\mathbf{Set}_{\rightarrow}}(\mathcal{S}_q)$. *Then* $\langle M(Q_1), M(m), M(r), M(l)\rangle$ *is a quasigroup.*

PROOF. Since $Q_1 \overset{p_1}{\leftarrow} Q_2 \overset{p_2}{\rightarrow} Q_1$ is carried by M to a specified product cone, we have $M(Q_2) = M(Q_1) \times M(Q_1)$, with $M(p_1), M(p_2)$ the first and second coordinate projections, respectively. Now it is clear that $M(m), M(r)$ and $M(l)$ are binary operations on $M(Q_1)$.

The pair of equations

$$p_1\langle p_1, m\rangle \approx p_1 \quad p_2\langle p_1, m\rangle \approx m,$$

interpreted in M, give, for all $(x, y) \in M(Q_1)^2$,

$$M(p_1)(M(\langle p_1, m\rangle)(x, y)) = M(p_1)(x, y) \quad \text{and}$$

$$M(p_2)(M(\langle p_1, m\rangle)(x, y)) = M(m)(x, y),$$

whence, since $M(p_i)$ is the i-th projection, $i = 1, 2$,

$$M(\langle p_1, m\rangle)(x, y) = (x, M(m)(x, y)), \quad \text{for all } x, y \in M(Q_1).$$

Similarly, one obtains that

$$M(\langle m, p_2\rangle)(x, y) = (M(m)(x, y), y),$$

$$M(\langle p_1, l\rangle)(x, y) = (x, M(l)(x, y)) \quad \text{and}$$

$$M(\langle r, p_2\rangle)(x, y) = (M(r)(x, y), y), \quad \text{for all } x, y \in M(Q_1).$$

Now from $l\langle p_1, m\rangle \approx p_2$ we get $M(l)(M(\langle p_1, m\rangle)(x, y)) = M(p_2)(x, y)$, whence

$$M(l)(x, M(m)(x, y)) = y,$$

and, similarly,

$$M(r)(M(m)(x, y), y) = x, \quad M(m)(x, M(l)(x, y)) = y, \quad \text{and}$$

$$M(m)(M(r)(x, y), y) = x.$$

Thus $\langle M(Q_1), M(m), M(r), M(l)\rangle$ is a quasigroup, as claimed. ■

A similar result is obtained next concerning the morphisms in the model category $\mathrm{Mod}_{\mathbf{Set}_{\rightarrow}}(\mathcal{S}_q)$.

3.2. PROPOSITION. *Let* $M, N : \mathcal{S}_q \rightarrow \mathbf{Set}_{\rightarrow}$ *be models in* $\mathrm{Mod}_{\mathbf{Set}_{\rightarrow}}(\mathcal{S}_q)$ *and* $\eta : M \rightarrow N$ *a morphism in* $\mathrm{Mod}_{\mathbf{Set}_{\rightarrow}}(\mathcal{S}_q)$. *Then* $\eta_{Q_1} : M(Q_1) \rightarrow N(Q_1)$ *is a quasigroup homomorphism from* $\langle M(Q_1), M(m), M(r), M(l)\rangle$ *into* $\langle N(Q_1), N(m), N(r), N(l)\rangle$.

Proof. It suffices to show that,

$$\text{for all } x, y \in M(Q_1), \eta_{Q_1}(M(m)(x,y)) = N(m)(\eta_{Q_1}(x), \eta_{Q_1}(y)),$$

i.e., that the following rectangle commutes

$$\begin{tikzcd}
M(Q_1) \times M(Q_1) \arrow[r, "\eta_{Q_1} \times \eta_{Q_1}"] \arrow[d, "M(m)"'] & N(Q_1) \times N(Q_1) \arrow[d, "N(m)"] \\
M(Q_1) \arrow[r, "\eta_{Q_1}"'] & N(Q_1)
\end{tikzcd}$$

Since $M(Q_1) \times M(Q_1) = M(Q_2)$ and $N(Q_1) \times N(Q_1) = N(Q_2)$, this would certainly be true if $\eta_{Q_1} \times \eta_{Q_1} = \eta_{Q_2}$. But, since $\eta : M \to N$ is a natural transformation, we have commutativity of

$$\begin{tikzcd}
M(Q_1) \times M(Q_1) \arrow[r, "\eta_{Q_2}"] \arrow[d, "M(p_1)"'] & N(Q_1) \times N(Q_1) \arrow[d, "N(p_1)"] & M(Q_1) \times M(Q_1) \arrow[r, "\eta_{Q_2}"] \arrow[d, "M(p_2)"'] & N(Q_1) \times N(Q_1) \arrow[d, "N(p_2)"] \\
M(Q_1) \arrow[r, "\eta_{Q_1}"'] & N(Q_1) & M(Q_1) \arrow[r, "\eta_{Q_1}"'] & N(Q_1)
\end{tikzcd}$$

whence

$$\eta_{Q_2}(x,y) = (\eta_{Q_1}(x), \eta_{Q_1}(y)), \quad \text{for all } x, y \in M(Q_1),$$

as required. ■

The results that we have obtained so far may be summarized in the following

3.3. Theorem. $\mathrm{Mod}_{\mathbf{Set}_{\rightarrow}}(\mathcal{S}_q)$ *is the category corresponding to the usual universal algebraic variety of quasigroups, i.e., the category of quasigroups with homomorphisms between them.*

Next, let $M : \mathcal{S}_q \to \mathbf{Set}$ be a model of $\mathcal{S}_q$ in **Set**. Since **Set** is not assumed to have specified limits, it is not necessarily the case that $M(Q_2) = M(Q_1) \times M(Q_1)$. One may now only conclude that $M(Q_2) \cong M(Q_1) \times M(Q_1)$. Denote by $\phi_M : M(Q_1) \times M(Q_1) \to M(Q_2)$ the isomorphism that makes the following diagram commute

$$\begin{tikzcd}
 & M(Q_1) \times M(Q_1) \arrow[dl, "\pi_1"'] \arrow[d, "\phi_M"] \arrow[dr, "\pi_2"] & \\
M(Q_1) & M(Q_2) \arrow[l, "M(p_1)"] \arrow[r, "M(p_2)"'] & M(Q_1)
\end{tikzcd}$$

where π_1, π_2 are the ordinary coordinate projections in **Set**.

The following propositions show that the model category obtained is essentially the same as before modulo the isomorphism ϕ_M.

3.4. Proposition. *Let* $M : \mathcal{S}_q \to \mathbf{Set}$ *and* $\phi_M : M(Q_1) \times M(Q_1) \to M(Q_2)$ *be as above. Then* $\langle M(Q_1), M(m)\phi_M, M(r)\phi_M, M(l)\phi_M\rangle$ *is a quasigroup.*

Proof. It is clear that $M(m)\phi_M, M(r)\phi_M$ and $M(l)\phi_M$ are binary operations on $M(Q_1)$.

$$p_1\langle p_1, m\rangle \approx p_1 \quad \text{and} \quad p_2\langle p_1, m\rangle \approx m$$

give

$$M(p_1)M(\langle p_1, m\rangle) = M(p_1) \quad \text{and} \quad M(p_2)M(\langle p_1, m\rangle) = M(m),$$

whence

$$M(p_1)M(\langle p_1, m\rangle)\phi_M = \pi_1 \quad \text{and} \quad M(p_2)M(\langle p_1, m\rangle)\phi_M = M(m)\phi_M,$$

which together imply that

$$M(\langle p_1, m\rangle)\phi_M = \phi_M\langle \pi_1, M(m)\phi_M\rangle.$$

Similarly, we obtain

$$M(\langle m, p_2\rangle)\phi_M = \phi_M\langle M(m)\phi_M, \pi_2\rangle, \quad M(\langle p_1, l\rangle)\phi_M = \phi_M\langle \pi_1, M(l)\phi_M\rangle$$

$$\text{and} \quad M(\langle r, p_2\rangle)\phi_M = \phi_M\langle M(r)\phi_M, \pi_2\rangle.$$

Now, we have $M(l)M(\langle p_1, m\rangle) = M(p_2)$, whence

$$M(l)M(\langle p_1, m\rangle)\phi_M = M(p_2)\phi_M, \quad \text{i.e.,} \quad M(l)\phi_M\langle \pi_1, M(m)\phi_M\rangle = \pi_2.$$

Similarly, one may obtain the remaining three identities for the quasigroup $\langle M(Q_1), M(m)\phi_M, M(r)\phi_M, M(l)\phi_M\rangle$. ■

Similarly, one obtains the following proposition, whose proof is omitted.

3.5. Proposition. *Let* $M, N : \mathcal{S}_q \to \mathbf{Set}$ *and* $\phi_M : M(Q_1) \times M(Q_1) \to M(Q_2)$, $\phi_N : N(Q_1) \times N(Q_1) \to N(Q_2)$ *be as above. Suppose that* $\eta : M \to N$ *is a morphism in* $\mathrm{Mod}_{\mathbf{Set}}(\mathcal{S}_q)$. *Then* $\eta_{Q_1} : M(Q_1) \to N(Q_1)$ *is a quasigroup homomorphism from the quasigroup* $\langle M(Q_1), M(m)\phi_M, M(r)\phi_M, M(l)\phi_M\rangle$ *to the quasigroup* $\langle N(Q_1), N(m)\phi_N, N(r)\phi_N, N(l)\phi_N\rangle$.

So, what modification is needed in the quasigroup sketch, so that its model category be the category **Qtp** of quasigroups with quasigroup homotopies between them? In the next section a modified sketch is introduced whose model category will be shown to be naturally equivalent to **Qtp**. We do not know, however, whether a limit sketch exists whose category of models is isomorphic to **Qtp**. Smith's semisymmetrization result in [Smith (1997)] may prove helpful in answering this question.

4. The Modified Sketch

The new graph G_t has vertex set

$$V_t = \{Q_1, Q_2, Q_3, Q_{12}, Q_{13}, Q_{32}\}$$

and its edges are given diagrammatically as follows:

$$p_1^{12} : Q_{12} \to Q_1 \qquad p_2^{12} : Q_{12} \to Q_2$$

and, similarly, for p_1^{13}, p_3^{13} and p_3^{32}, p_2^{32},

$$m : Q_{12} \to Q_3 \qquad l : Q_{13} \to Q_2 \qquad r : Q_{32} \to Q_1$$

and, finally,

$$\begin{array}{rr} \langle m, p_2^{12}\rangle : Q_{12} \to Q_{32} & \langle p_1^{12}, m\rangle : Q_{12} \to Q_{13} \\ \langle p_1^{13}, l\rangle : Q_{13} \to Q_{12} & \langle r, p_2^{32}\rangle : Q_{32} \to Q_{12}. \end{array}$$

The following is a list of equations in G_t. The set of these equations is denoted by Q_t :

$$\begin{array}{cccc} p_1^{13}\langle p_1^{12}, m\rangle \approx p_1^{12} & p_3^{13}\langle p_1^{12}, m\rangle \approx m & p_3^{32}\langle m, p_2^{12}\rangle \approx m & p_2^{32}\langle m, p_2^{12}\rangle \approx p_2^{12} \\ p_1^{12}\langle p_1^{13}, l\rangle \approx p_3^{13} & p_1^{12}\langle p_1^{13}, l\rangle \approx l & p_1^{12}\langle r, p_2^{32}\rangle \approx r & p_2^{12}\langle r, p_2^{32}\rangle \approx p_2^{32} \end{array}$$

$$\begin{array}{cc} l\langle p_1^{12}, m\rangle \approx p_2^{12} & r\langle m, p_2^{12}\rangle \approx p_1^{12} \\ m\langle p_1^{13}, l\rangle \approx p_3^{13} & m\langle r, p_2^{32}\rangle \approx p_3^{32} \end{array}$$

Compare the equations in Q_t with those in Q_q displayed in Section 2.

Finally, let L_t be the set consisting of the following cones, given in diagrammatic form

$$\begin{tikzcd}
 & Q_{12} \arrow[dl, "p_1^{12}"'] \arrow[dr, "p_2^{12}"] & & & Q_{13} \arrow[dl, "p_1^{13}"'] \arrow[dr, "p_3^{13}"] & & & Q_{32} \arrow[dl, "p_3^{32}"'] \arrow[dr, "p_2^{32}"] & \\
Q_1 & & Q_2 & Q_1 & & Q_3 & Q_3 & & Q_2
\end{tikzcd}$$

Let $\mathcal{S}_t = \langle G_t, Q_t, L_t\rangle$ be the sketch with graph G_t, set of equations Q_t and set of cones L_t, as constructed above. $\mathcal{S}_t$ is the **sketch for quasigroup homotopies**.

Now the following proposition may be easily verified.

4.1. PROPOSITION. *Let* $\mathbf{Q} = \langle Q, \cdot, /, \backslash\rangle$ *be a quasigroup. Define the graph morphism* $N_{\mathbf{Q}} : G_t \to \mathbf{Set}_{\to}$ *as follows:*

$$N_{\mathbf{Q}}(Q_1) = N_{\mathbf{Q}}(Q_2) = N_{\mathbf{Q}}(Q_3) = Q,$$

$$N_{\mathbf{Q}}(Q_{12}) = N_{\mathbf{Q}}(Q_{13}) = N_{\mathbf{Q}}(Q_{32}) = Q \times Q,$$

$$N_{\mathbf{Q}}(p_1^{12})(x, y) = N_{\mathbf{Q}}(p_1^{13})(x, y) = N_{\mathbf{Q}}(p_3^{32})(x, y) = x$$

and

$$N_{\mathbf{Q}}(p_2^{12})(x, y) = N_{\mathbf{Q}}(p_3^{13})(x, y) = N_{\mathbf{Q}}(p_2^{32})(x, y) = y,$$

$$N_{\mathbf{Q}}(m)(x,y)=x\cdot y \qquad N_{\mathbf{Q}}(l)(x,y)=x\backslash y \qquad N_{\mathbf{Q}}(r)(x,y)=x/y$$
$$N_{\mathbf{Q}}(\langle m,p_2^{12}\rangle)(x,y)=(x\cdot y,y) \qquad N_{\mathbf{Q}}(\langle p_1^{12},m\rangle)(x,y)=(x,x\cdot y)$$
$$N_{\mathbf{Q}}(\langle p_1^{13},l\rangle)(x,y)=(x,x\backslash y) \qquad N_{\mathbf{Q}}(\langle r,p_3^{13}\rangle)(x,y)=(x/y,y)$$

Then $N_{\mathbf{Q}}$ *is a model of* $\mathcal{S}_t$ *in* $\mathbf{Set}_{\rightarrow}$.

Moreover, by analogy with Proposition 2.8, we have the following:

4.2. PROPOSITION. *Let* $\mathbf{Q},\mathbf{P}$ *be two quasigroups,* $(h_1,h_2,h_3):\mathbf{Q}\rightarrow\mathbf{P}$ *a quasigroup homotopy and* $N_{\mathbf{Q}},N_{\mathbf{P}}:\mathcal{S}_t\rightarrow\mathbf{Set}_{\rightarrow}$ *the two models of* $\mathcal{S}_t$ *defined as in Proposition 4.1.* $\eta:N_{\mathbf{Q}}\rightarrow N_{\mathbf{P}}$*, defined by*

$$\eta_{Q_1}=h_1 \qquad \eta_{Q_2}=h_2 \qquad \eta_{Q_3}=h_3$$
$$\eta_{Q_{12}}=(h_1,h_2) \qquad \eta_{Q_{13}}=(h_1,h_3) \qquad \eta_{Q_{32}}=(h_3,h_2),$$

is a model morphism of $\mathcal{S}_t$ *in* $\mathbf{Set}_{\rightarrow}$.

5. Homotopies of Quasigroups

In this section, it is shown that the category of models $\mathrm{Mod}_{\mathbf{Set}_{\rightarrow}}(\mathcal{S}_t)$ essentially contains quasigroups with homotopies between them. More precisely, that $\mathrm{Mod}_{\mathbf{Set}_{\rightarrow}}(\mathcal{S}_t)$ is naturally equivalent to **Qtp**. The proof is based on the fact that the three sets in which the vertices Q_1,Q_2 and Q_3 of G_t are mapped in $\mathbf{Set}_{\rightarrow}$ by any model $M:\mathcal{S}_t\rightarrow\mathbf{Set}_{\rightarrow}$ are isomorphic. So up to isomorphism, i.e., a renaming of the elements corresponding to Q_1 and Q_2, M will be shown to define a quasigroup with universe $M(Q_3)$. Then all model morphisms in $\mathrm{Mod}_{\mathbf{Set}_{\rightarrow}}(\mathcal{S}_t)$ between two models M and N may be appropriately translated to homotopies between the quasigroups with the universes $M(Q_3)$ and $N(Q_3)$.

5.1. LEMMA. *Let* $M:\mathcal{S}_t\rightarrow\mathbf{Set}_{\rightarrow}$ *be a model in* $\mathrm{Mod}_{\mathbf{Set}_{\rightarrow}}(\mathcal{S}_t)$, $x_M\in M(Q_1)$ *and* $y_M\in M(Q_2)$. *Then* $\phi_{y_M}:M(Q_1)\rightarrow M(Q_3)$ *and* $\psi_{x_M}:M(Q_2)\rightarrow M(Q_3)$*, defined by*

$$\phi_{y_M}(x)=M(m)(x,y_M) \quad \text{and} \quad \psi_{x_M}(y)=M(m)(x_M,y),$$

for all $x\in M(Q_1),y\in M(Q_2)$*, respectively, are bijections.*

PROOF. We only show that $\phi_{y_M}:M(Q_1)\rightarrow M(Q_3)$ is a bijection. The case of ψ_{x_M} may be handled similarly.

Suppose $x_1,x_2\in M(Q_1)$, with $\phi_{y_M}(x_1)=\phi_{y_M}(x_2)$. Then

$$M(m)(x_1,y_M)=M(m)(x_2,y_M),$$

whence $M(r)(M(m)(x_1,y_M),y_M)=M(r)(M(m)(x_2,y_M),y_M)$ and, therefore, $x_1=x_2$. Thus ϕ_{y_M} is one-to-one.

Next, let $z\in M(Q_3)$. Then, for $x=M(r)(z,y_M)\in M(Q_1)$, we have

$$\phi_{y_M}(x)=M(m)(x,y_M)=M(m)(M(r)(z,y_M),y_M)=z.$$

Thus, ϕ_{y_M} is also onto. ∎

With Lemma 5.1 at hand, it may now be shown that $\mathrm{Mod}_{\mathbf{Set}_\rightarrow}(\mathcal{S}_t)$ is essentially the category **Qtp** of quasigroups with homotopies between them. Note that, because of Lemma 5.1, to make the correspondence that is established between $\mathrm{Mod}_{\mathbf{Set}_\rightarrow}(\mathcal{S}_t)$ and the category **Qtp** of quasigroups with homotopies between them natural we must fix a way of choosing the elements x_M and y_M in $M(Q_1)$ and $M(Q_2)$, respectively. Luckily enough any choice will do. To this end, an arbitrary choice function c for the class $|\mathbf{Set}_\rightarrow| - \{\emptyset\}$ will be fixed later in the section.

The work is divided again into two steps. In the first we deal with objects and in the second with morphisms in $\mathrm{Mod}_{\mathbf{Set}_\rightarrow}(\mathcal{S}_t)$.

5.2. PROPOSITION. *Let* $M : \mathcal{S}_t \to \mathbf{Set}_\rightarrow$ *be a model in* $\mathrm{Mod}_{\mathbf{Set}_\rightarrow}(\mathcal{S}_t)$, $x_M \in M(Q_1)$ *and* $y_M \in M(Q_2)$. *Then*

$$\langle M(Q_3), M(m)\langle \phi_{y_M}^{-1}, \psi_{x_M}^{-1}\rangle, \phi_{y_M} M(r)\langle i_{M(Q_3)}, \psi_{x_M}^{-1}\rangle, \psi_{x_M} M(l)\langle \phi_{y_M}^{-1}, i_{M(Q_3)}\rangle\rangle$$

is a quasigroup.

PROOF. Using Lemma 5.1, it is easy to verify that $M(m)\langle \phi_{y_M}^{-1}, \psi_{x_M}^{-1}\rangle$, $\phi_{y_M} M(r)\langle i_{M(Q_3)}, \psi_{x_M}^{-1}\rangle$ and $\psi_{x_M} M(l)\langle \phi_{y_M}^{-1}, i_{M(Q_3)}\rangle$ are all binary operations on $M(Q_3)$. So it suffices to show that they obey the quasigroup laws. We only verify one of the four laws. The proofs of the remaining three are very similar.

$$\begin{aligned}
&\psi_{x_M} M(l)\langle \phi_{y_M}^{-1}, i_{M(Q_3)}\rangle(x, M(m)\langle \phi_{y_M}^{-1}, \psi_{x_M}^{-1}\rangle(x,y)) \\
&\qquad = \psi_{x_M} M(l)(\phi_{y_M}^{-1}(x), M(m)(\phi_{y_M}^{-1}(x), \psi_{x_M}^{-1}(y))) \\
&\qquad = \psi_{x_M}(\psi_{x_M}^{-1}(y)) = y,
\end{aligned}$$

the second equality being valid because of the corresponding equation imposed by Q_s. ■

Given a model $M : \mathcal{S}_t \to \mathbf{Set}_\rightarrow$, $x_M \in M(Q_1)$ and $y_M \in M(Q_2)$, denote by $M^*_{x_M,y_M}(Q_3)$ the quasigroup

$$\langle M(Q_3), M(m)\langle \phi_{y_M}^{-1}, \psi_{x_M}^{-1}\rangle, \phi_{y_M} M(r)\langle i_{M(Q_3)}, \psi_{x_M}^{-1}\rangle, \psi_{x_M} M(l)\langle \phi_{y_M}^{-1}, i_{M(Q_3)}\rangle\rangle$$

associated with it by Proposition 5.2.

Now for the morphisms in $\mathrm{Mod}_{\mathbf{Set}_\rightarrow}(\mathcal{S}_t)$ we have the following proposition.

5.3. PROPOSITION. *Let* $M, N : \mathcal{S}_t \to \mathbf{Set}_\rightarrow$ *be two models in* $\mathrm{Mod}_{\mathbf{Set}_\rightarrow}(\mathcal{S}_t)$, $x_M \in M(Q_1), y_M \in M(Q_2), x_N \in N(Q_1)$ *and* $y_N \in N(Q_2)$. *Finally, let* $\eta : M \to N$ *be a morphism in* $\mathrm{Mod}_{\mathbf{Set}_\rightarrow}(\mathcal{S}_t)$. *Then*

$$(\phi_{y_N}\eta_{Q_1}\phi_{y_M}^{-1}, \psi_{x_N}\eta_{Q_2}\psi_{x_M}^{-1}, \eta_{Q_3}) : M(Q_3) \to N(Q_3)$$

is a quasigroup homotopy from $M^*_{x_M,y_M}(Q_3)$ *into* $N^*_{x_N,y_N}(Q_3)$.

Proof. We need to show that

$$\eta_{Q_3}(M(m)\langle\phi^{-1}_{y_M},\psi^{-1}_{x_M}\rangle(x,y)) = N(m)\langle\phi^{-1}_{y_N},\psi^{-1}_{x_N}\rangle(\phi_{y_N}\eta_{Q_1}\phi^{-1}_{y_M}(x),\psi_{x_N}\eta_{Q_2}\psi^{-1}_{x_M}(y)).$$

We have

$$\begin{aligned} N(m)\langle\phi^{-1}_{y_N},\psi^{-1}_{x_N}\rangle(\phi_{y_N}\eta_{Q_1}\phi^{-1}_{y_M}(x),\psi_{x_N}\eta_{Q_2}\psi^{-1}_{x_M}(y)) &= N(m)(\eta_{Q_1}\phi^{-1}_{y_M}(x),\eta_{Q_2}\psi^{-1}_{x_M}(y)) \\ &= N(m)(\langle\eta_{Q_1},\eta_{Q_2}\rangle(\phi^{-1}_{y_M}(x),\psi^{-1}_{x_M}(y))) \\ &= \eta_{Q_3}(M(m)\langle\phi^{-1}_{y_M},\psi^{-1}_{x_M}\rangle(x,y)), \end{aligned}$$

the last equality being valid because $\eta : M \to N$ is a natural transformation. ■

Now suppose that there is available a choice function c for the class $|\mathbf{Set}_{\to}| - \{\emptyset\}$ of all nonempty sets, i.e., for all $X \in |\mathbf{Set}_{\to}|, X \neq \emptyset$, $c(X) \in X$. Then, it can be shown that the category **Qtp** of quasigroups with homotopies between them is naturally equivalent to $\mathrm{Mod}_{\mathbf{Set}_{\to}}(\mathcal{S}_t)$.

The functor $F : \mathbf{Qtp} \to \mathrm{Mod}_{\mathbf{Set}_{\to}}(\mathcal{S}_t)$ is defined by

$$F(\mathbf{Q}) = N_{\mathbf{Q}}, \quad \text{for all } \mathbf{Q} \in |\mathbf{Qtp}|,$$

and, given $(h_1,h_2,h_3) \in \mathbf{Qtp}(\mathbf{Q},\mathbf{P})$, $F((h_1,h_2,h_3))$ is the model morphism in $\mathrm{Mod}_{\mathbf{Set}_{\to}}(\mathcal{S}_t)$ defined in Proposition 4.2.

The functor $G : \mathrm{Mod}_{\mathbf{Set}_{\to}}(\mathcal{S}_t) \to \mathbf{Qtp}$ is defined by

$$G(M) = M^*_{c(M(Q_1)),c(M(Q_2))}(Q_3), \quad \text{for all } M : \mathcal{S}_t \to \mathbf{Set}_{\to},$$

and, given $\eta \in \mathrm{Mod}_{\mathbf{Set}_{\to}}(\mathcal{S}_t)(M,N)$,

$$G(\eta) = (\phi_{c(N(Q_2))}\eta_{Q_1}\phi^{-1}_{c(M(Q_2))},\psi_{c(N(Q_1))}\eta_{Q_2}\psi^{-1}_{c(M(Q_1))},\eta_{Q_3}).$$

It is a routine calculation to check that F and G are indeed functors. So to prove the natural equivalence it suffices to exhibit natural isomorphisms $\mu : I_{\mathbf{Qtp}} \to G \circ F$ and $\nu : I_{\mathrm{Mod}_{\mathbf{Set}_{\to}}(\mathcal{S}_t)} \to F \circ G$. Note that $G(F(\langle Q,\cdot,/,\backslash\rangle))$ is the quasigroup with universe Q and multiplication, right division and left division given, respectively, by

$$\begin{gathered} (x,y) \mapsto (x/c(Q)) \cdot (c(Q)\backslash y) \\ (x,y) \mapsto (x/(c(Q)\backslash y)) \cdot c(Q) \\ (x,y) \mapsto c(Q) \cdot ((x/c(Q))\backslash y). \end{gathered}$$

So, it is natural to define

$$\mu_{\mathbf{Q}1}(x) = x \cdot c(Q), \mu_{\mathbf{Q}2}(x) = c(Q) \cdot x, \mu_{\mathbf{Q}3}(x) = x,$$

i.e.,

$$\mu_{\mathbf{Q}1} = \phi_{c(Q)}, \mu_{\mathbf{Q}2} = \psi_{c(Q)}, \mu_{\mathbf{Q}3} = i_Q.$$

$\mu_{\mathbf{Q}}$ is an isotopy and, for all $\mathbf{Q}, \mathbf{P} \in |\mathbf{Qtp}|$, and $(h_1, h_2, h_3) \in \mathbf{Qtp}(\mathbf{Q}, \mathbf{P})$, commutativity of

$$\begin{tikzcd}
\mathbf{Q} \arrow[r, "\mu_{\mathbf{Q}}"] \arrow[d, "{(h_1,h_2,h_3)}"'] & G(F(\mathbf{Q})) \arrow[d, "{G(F((h_1,h_2,h_3)))}"] \\
\mathbf{P} \arrow[r, "\mu_{\mathbf{P}}"'] & G(F(\mathbf{P}))
\end{tikzcd}$$

is easy to verify. For instance, for the first component, diagram chasing gives

$$(\phi_{c(P)} h_1 \phi_{c(Q)}^{-1})\phi_{c(Q)} = \phi_{c(P)} h_1.$$

Next, given $M : \mathcal{S}_t \to \mathbf{Set}_{\to}$, the model $F(G(M))$ has

$$F(G(M))(Q_1) = F(G(M))(Q_2) = F(G(M))(Q_3) = M(Q_3)$$

and, moreover,

$$\begin{aligned}
F(G(M))(m) &= M(m)\langle \phi_{c(M(Q_2))}^{-1}, \psi_{c(M(Q_1))}^{-1}\rangle \\
F(G(M))(r) &= \phi_{c(M(Q_2))} M(r)\langle i_{M(Q_3)}, \psi_{c(M(Q_1))}^{-1}\rangle \\
F(G(M))(l) &= \psi_{c(M(Q_1))} M(l)\langle \phi_{c(M(Q_2))}^{-1}, i_{M(Q_3)}\rangle.
\end{aligned}$$

So, now, we define

$$\nu_M(Q_1) = \phi_{c(M(Q_2))}, \nu_M(Q_2) = \psi_{c(M(Q_1))}, \nu_M(Q_3) = i_{M(Q_3)}.$$

Clearly, ν_M is also an isotopy and, for all $M, N : \mathcal{S}_t \to \mathbf{Set}_{\to}$ and $\eta : M \to N$ in $\mathrm{Mod}_{\mathbf{Set}_{\to}}(\mathcal{S}_t)$, commutativity of

$$\begin{tikzcd}
M \arrow[r, "\nu_M"] \arrow[d, "\eta"'] & F(G(M)) \arrow[d, "F(G(\eta))"] \\
N \arrow[r, "\nu_N"'] & F(G(N))
\end{tikzcd}$$

may be verified as follows:

$$\begin{aligned}
(F(G(\eta)) \circ \nu_M)_{Q_1} &= F(G(\eta))_{Q_1} \nu_M(Q_1) \\
&= \phi_{c(N(Q_2))} \eta_{Q_1} \phi_{c(M(Q_2))}^{-1} \phi_{c(M(Q_2))} \\
&= \phi_{c(N(Q_2))} \eta_{Q_1} \\
&= \nu_N(Q_1) \eta_{Q_1} \\
&= (\nu_N \circ \eta)_{Q_1}
\end{aligned}$$

and, similarly, for Q_2, Q_3, Q_{12}, Q_{13} and Q_{32}.

We have, thus, shown the following

5.4. THEOREM. *The category* **Qtp** *of quasigroups with homotopies between them and the model category* $\mathrm{Mod}_{\mathbf{Set}_{\rightarrow}}(\mathcal{S}_t)$ *of the product sketch* $\mathcal{S}_t$ *in* $\mathbf{Set}_{\rightarrow}$ *are naturally isomorphic categories.*

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