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EVERY SMALL Sl-ENRICHED CATEGORY IS MORITA EQUIVALENT TO AN Sl-MONOID

Dedicated to Professor Aurelio Carboni on the occasion of his sixtieth birthday

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ABSTRACT. We show that every small category enriched over Sl—the symmetric monoidal closed category of sup-lattices and sup-preserving morphisms—is Morita equivalent to a Sl-monoid. As a corollary, we obtain a result of Borceux and Vitale [1] asserting that every separable Sl-category is Morita equivalent to a separable Sl-monoid.

1. Theorem

We use [2] as a reference for enriched category theory.

Let $Sl = (Sl_{\circ}, -\otimes -, I)$ denote the symmetric monoidal closed category of sup-lattices with sup-preserving maps.

It is well-known [3] that Sl_{\circ} is a complete, cocomplete pointed category. Moreover, (small) coproducts are biproducts; that is, if $\{x_i\}$ is a (small) family of objects of Sl_{\circ} , the unique morphism

$$\delta: \coprod x_i \longrightarrow \prod x_i$$

with components $\delta_{ii} = 1_{x_i}$ and $\delta_{ij} = 0$ otherwise is an isomorphism. For any small $\mathcal{S}l$ -category \mathcal{A} , this property transfers to the $\mathcal{S}l$ -functor category $[\mathcal{A}^{op}, \mathcal{S}l]$, since limits and colimits there are computed pointwise.

We will write $\oplus x_i$ for the biproduct of the family $\{x_i\}$.

1.1. THEOREM. Every small Sl-category is Morita equivalent to an Sl-monoid.

PROOF. We first observe that for each pair of objects c, c' of an Sl-enriched category C, the set $C_{\circ}(c, c')$ is non-empty because every sup-lattice contains at least the element 0.

Now let \mathcal{A} be a small $\mathcal{S}l$ -category. Define the $\mathcal{S}l$ -functor $P_{\mathcal{A}} : \mathcal{A}^{op} \longrightarrow \mathcal{S}l$ to be $\bigoplus_{a \in \mathcal{A}} \mathcal{A}(-, a)$. Since arbitrary (small) coproducts (=products) of representable $\mathcal{S}l$ -functors are in the Cauchy completion $\overline{\mathcal{A}}$ of \mathcal{A} (see [4]), $P_{\mathcal{A}}$ is in $\overline{\mathcal{A}}$. Thus the $\mathcal{S}l$ -functor $P_{\mathcal{A}}$ is small projective in $[\mathcal{A}^{op}, \mathcal{S}l]$.

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Next, by the observation at the beginning of the proof, no $[\mathcal{A}^{op}, \mathcal{S}l](\mathcal{A}(-, a), \mathcal{A}(-, a'))$ is empty. Moreover, the small $\mathcal{S}l$ -full subcategory determined by the representable $\mathcal{S}l$ functors is dense in $[\mathcal{A}^{op}, \mathcal{S}l]$. So that, in view of Proposition 5.22 of [2], the $\mathcal{S}l$ -full subcategory $[P_{\mathcal{A}}] \subseteq [\mathcal{A}^{op}, \mathcal{S}l]$ generated by the $\mathcal{S}l$ -functor $P_{\mathcal{A}}$ is dense, and a fortiori strongly generating, in $[\mathcal{A}^{op}, \mathcal{S}l]$. Thus $P_{\mathcal{A}} : \mathcal{A}^{op} \longrightarrow \mathcal{S}l$, besides being small projective in $[\mathcal{A}^{op}, \mathcal{S}l]$, is a strong generator for $[\mathcal{A}^{op}, \mathcal{S}l]$. It now follows from Theorem 5.26 of [2] that the $\mathcal{S}l$ -functor

$$\mathcal{N}_{\mathcal{A}}: [\mathcal{A}^{op}, \mathcal{S}l] \longrightarrow [[P_{\mathcal{A}}]^{op}, \mathcal{S}l]$$

given by

$$\mathcal{N}_{\mathcal{A}}(F) = \left[\mathcal{A}^{op}, \mathcal{S}l\right] \left(P_{\mathcal{A}}, F\right)$$

is an equivalence of $\mathcal{S}l$ -categories. Whence \mathcal{A} is Morita equivalent to the $\mathcal{S}l$ -monoid $[\mathcal{A}^{op}, \mathcal{S}l](P_{\mathcal{A}}, P_{\mathcal{A}}).$

1.2. REMARK. There is precisely an analogous result wherein Sl is replaced by the symmetric monoidal closed category of abelian groups, **Ab**, and A by an **Ab**-enriched category with finitely many objects (see, for example, [5]); and still another with Sl replaced by the symmetric monoidal closed category of commutative monoids, once again in the finitely-many-object case.

2. Application

Let $\mathcal{V} = (\mathcal{V}_{\circ}, -\otimes -, I)$ be a symmetric monoidal closed category whose underlying ordinary category \mathcal{V}_{\circ} is locally small, complete and cocomplete. Write $\mathcal{B}im(\mathcal{V})$ for the bicategory of small \mathcal{V} -categories, \mathcal{V} -bimodules and \mathcal{V} -natural transformation between them.

Recall [1] that a small \mathcal{V} -category \mathcal{A} is separable when the canonical \mathcal{V} -functor

$$\mathcal{A}(-,-):\mathcal{A}^{op}\otimes\mathcal{A}\longrightarrow\mathcal{V}$$

is small projective in the \mathcal{V} -functor category $[\mathcal{A}^{op} \otimes \mathcal{A}, \mathcal{V}]$.

2.1. PROPOSITION. Separability is invariant under Morita equivalence.

PROOF. Recall (for instance from [2]) that two small \mathcal{V} -categories \mathcal{A} and \mathcal{B} are Morita equivalent if and only if there exist bimodules $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ and $\psi : \mathcal{B} \longrightarrow \mathcal{A}$ such that $\psi \circ \phi \simeq \mathcal{A}$ and $\phi \circ \psi \simeq \mathcal{B}$. Then

$$[\mathcal{A}^{op} \otimes \mathcal{A}, \mathcal{V}] \simeq \mathcal{B}im(\mathcal{V})(\mathcal{A}, \mathcal{A}) \xrightarrow{-\circ\psi} \mathcal{B}im(\mathcal{V})(\mathcal{B}, \mathcal{A}) \xrightarrow{\phi\circ-} \mathcal{B}im(\mathcal{V})(\mathcal{B}, \mathcal{B}) \simeq [\mathcal{B}^{op} \otimes \mathcal{B}, \mathcal{V}],$$

and (up to isomorphism) this composite takes the \mathcal{V} -functor $\mathcal{A}(-,-): \mathcal{A}^{op} \otimes \mathcal{A} \longrightarrow \mathcal{V}$ to the \mathcal{V} -functor $\mathcal{B}(-,-): \mathcal{B}^{op} \otimes \mathcal{B} \longrightarrow \mathcal{V}$. The result now follows from the fact that any equivalence of cocomplete \mathcal{V} -categories preserves and reflects the property of objects of being small projective.

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It follows from Theorem 1.1 and Proposition 2.1 that

2.2. THEOREM. [1] Every separable Sl-category is Morita equivalent to a separable Sl-monoid. In particular, every Azumaya Sl-category is Morita equivalent to an Azumaya Sl-monoid.

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