Theory and Applications of Categories, Vol. 17, No. 2, 2006, pp. 17–29.

COHERENCE OF THE DOUBLE INVOLUTION ON $*$-AUTONOMOUS CATEGORIES

J.R.B. COCKETT, M. HASEGAWA AND R.A.G. SEELY

ABSTRACT. We show that any free $*$-autonomous category is equivalent (in a strict sense) to a free $*$-autonomous category in which the double-involution $(-)^{**}$ is the identity functor and the canonical isomorphism $A \simeq A^{**}$ is an identity arrow for all A.

1. Introduction

Many formulations of proof nets and sequent calculi for *Classical Linear Logic (CLL)* [9, 10] take it for granted that a type A is *identical* to its double negation $A^{\perp\perp}$. On the other hand, since Seely [16], it has been assumed that $*$-*autonomous categories* [1, 2] are the appropriate semantic models of (the multiplicative fragment of) CLL. However, in general, in a $*$-autonomous category an object A is only *canonically isomorphic* to its double involution A^{**}. For instance, in the category of finite dimensional vector spaces and linear maps, a vector space V is only isomorphic to its double dual V^{**}. This raises the questions whether $*$-autonomous categories do not, after all, provide an accurate semantic model for these proof nets and whether there could be semantically non-identical proofs (or morphisms), which must be identified in any system which assumes a type is identical to its double negation. Whether this can happen is not completely obvious even when one examines purely syntactic descriptions of proofs with the isomorphism between A and $A^{\perp\perp}$ present such as [14, 11] or the alternative proof net systems of [5] which are faithful to the categorical semantics.

Fortunately, there is no such semantic gap: in this paper we provide a *coherence theorem* for the double involution on $*$-autonomous categories, which tells us that there is no difference between the up-to-identity approach and the up-to-isomorphism approach, as far as this double-negation problem is concerned. This remains true under the presence of linear exponential comonads and finite products (the semantic counterpart of exponentials and additives respectively). Our proof is fairly short and simple, and we suspect that this is folklore among specialists (at least everyone would expect such a result), though we are not aware of an explicit treatment of this issue in the literature.

This result should be compared with the classical coherence theorem for monoidal categories, as found *e.g.* in [15, 13]. In fact, we follow the proof strategy by Joyal and Street in [13]. We first show a weaker form of the coherence theorem which turns a $*$-autonomous category into an equivalent one with "strict involution" (where A^{**} is identical

Received by the editors 2003-03-10 and, in revised form, 2004-08-04.
Published on 2006-12-16 in the volume *Chu spaces: theory and applications*.
2000 Mathematics Subject Classification: 03F52,18D10,18D15.

to A), for which we make use of (a simplified version of) a construction of Cockett and Seely [8]. We then strengthen it to a form of "all diagrams commute" result by some additional fairly standard arguments on the structure-preserving functors. In this way, this work also demonstrates the applicability of the Joyal-Street argument (which actually can be seen as an instance of a general flexibility result on free algebras of 2-monads developed by Blackwell, Kelly and Power [4]) to other sorts of coherence problems.

We should warn the reader that there is no particularly novel technique in this short note, and the result itself is probably unsurprising. The reader who merely wants the essence of the story should read the definitions and theorems, with the assurance that the standard proofs that spring to mind do actually work. In other words, our point is just that the expected approach to this matter does actually work to give the expected result. Why would one bother? Primarily to guard against the seduction of the obvious: when it comes to coherence, it is easy to assume "obvious" conclusions are in fact true, whereas in some cases unpleasant surprises may occur. By taking a little care in the presentation, we hope to convince the reader that this is not such a case. If as a side-effect we encourage the reader to investigate the notion of a linear functor [8], we will not be displeased at that.

This work grew out of discussions during the CTCS'02 conference held at Ottawa on August 2002. The authors are grateful to the organisers of this fruitful meeting.

2. Preliminaries

Let us first fix our terminology.

2.1. Definition. ($*$-autonomous categories [[1, 2]].) *A* $*$-autonomous category *is a symmetric monoidal closed category $\mathcal{C} = (\mathcal{C}, I, \otimes, \multimap)$ with a contravariant functor (an* involution*) $(-)^*\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ given by a "dualising object" $\bot$: $A^* = A \multimap \bot$, for which the canonical morphism $A \to A^{**}$ is an isomorphism.*

As usual, below we write $A \mathbin{⅋} B$ for $(A^* \otimes B^*)^*$; note that we may suppose $\bot = I^*$.

Next we introduce the class of $*$-autonomous categories which supports the "double-negation identification".

2.2. Definition. ($*$-autonomous categories with strict involution.) *A* $*$-autonomous category with strict involution *is a $*$-autonomous category in which the functor $(-)^{**}$ is the identity functor and the canonical isomorphism $A \simeq A^{**}$ is the identity for all A.*

To discuss the precise relationship between $*$-autonomous categories, we introduce two notions of structure-preserving functors: strong (up-to-iso) and strict ones. We will also need the notion of isomorphisms between these functors.

2.3. Definition. (strong/strict $*$-autonomous functors.)

- *A* strong $*$-autonomous functor *between $*$-autonomous categories $\mathcal{C}$ and $\mathcal{D}$ is a strong symmetric monoidal functor $F\colon\mathcal{C}\to\mathcal{D}$ equipped with a natural isomorphism $\theta_A\colon(FA)^*\xrightarrow{\simeq}F(A^*)$ such that $(F(A^*))^*\xrightarrow{\theta_{A^*}}F(A^{**})\xrightarrow{\simeq}FA$ agrees with $(F(A^*))^*\xrightarrow{\theta_A^*}(FA)^{**}\xrightarrow{\simeq}FA$.*
- *A* strict $*$-autonomous functor *is a strong $*$-autonomous functor which is strict symmetric monoidal with θ given by the identity natural transformation (so that $(FA)^*=F(A^*)$).*

2.4. Definition. (isomorphisms between strong $*$-autonomous functors.) *An* isomorphism *between strong $*$-autonomous functors $F,G\colon\mathcal{C}\to\mathcal{D}$ is a monoidal natural isomorphism $\tau\colon F\to G$ such that $\theta_A^F\circ\tau_A^*\circ\theta_A^{G-1}$ is the inverse of τ_{A^*}.*

3. A Weak Coherence Theorem

Our first task, given a $*$-autonomous category, is to construct an equivalent $*$-autonomous category with strict involution. For this purpose, it turns out that the bi-adjunction between the 2-category of linearly distributive categories and that of $*$-autonomous categories in [8] is helpful: any $*$-autonomous category constructed from a linearly distributive category as the "category of complemented objects" does have a strict involution. Since a $*$-autonomous category is of course a linearly distributive category, we can apply this construction to $*$-autonomous categories. Below we recall the construction in a slightly simplified form. The essential idea is that, to realise a strict involution, for each object we explicitly specify its "complement". (See also [6] for a further sophistication of this construction, called CMap$(-)$, in the context of linear bicategories.)

3.1. Definition. (categories of complemented objects [[8]].) *Let $\mathcal{C}$ be a $*$-autonomous category. The category $\mathbf{C}(\mathcal{C})$ of* complemented objects *is defined as follows. $\mathbf{C}(\mathcal{C})$'s objects are triples $\mathbf{A}=(A,A',\tau_A)$ such that $\tau_A\colon A'\xrightarrow{\simeq}A^*$ in $\mathcal{C}$. An arrow from $\mathbf{A}=(A,A',\tau_A)$ to $\mathbf{B}=(B,B',\tau_B)$ in $\mathbf{C}(\mathcal{C})$ is just an arrow from A to B in $\mathcal{C}$.*

Let us define

$$\begin{array}{rcl} I & = & (I,\perp,id_\perp)\\ \mathbf{A}\otimes\mathbf{B} & = & \big(A\otimes B, A'\mathbin{⅋} B', A'\mathbin{⅋} B'\xrightarrow{\tau_A ⅋ \tau_B}A^*\mathbin{⅋} B^*\xrightarrow{\simeq}(A\otimes B)^*\big)\end{array}$$

With the obvious action on arrows, this determines a symmetric monoidal structure on $\mathbf{C}(\mathcal{C})$. Moreover, we have an obvious strict involution on $\mathbf{C}(\mathcal{C})$ by

$$\begin{array}{rcl} \mathbf{A}^* & = & (A',A,A\xrightarrow{\simeq}A^{**}\xrightarrow{\tau_A^*}A'^*)\\ (f\colon\mathbf{A}\to\mathbf{B})^* & = & B'\xrightarrow{\tau_B}B^*\xrightarrow{f^*}A^*\xrightarrow{\tau_A^{-1}}A'\end{array}$$

3.1.1. The Chu construction and $\mathbf{C}(\mathcal{C})$. Another way to view the category of complemented objects $\mathbf{C}(\mathcal{C})$ is as a full subcategory of the Chu construction $\mathbf{Chu}(\mathcal{C}, \perp)$ determined by complements and their complementation map $(A, B, \gamma: A \otimes B \to \perp)$. We can "thin" this even further, by considering pairs (A, A^*), the complementation map $\gamma: A \otimes A^* \to \perp$ being implicit. The construction of $\mathbf{Chu}(\mathcal{C}, \perp)$ requires pullbacks, and the structure of the tensor and par in $\mathbf{Chu}(\mathcal{C}, \perp)$ makes essential use of those pullbacks. However, when restricted to the complemented objects, the pullbacks are in fact all along isomorphisms, and so exist even if $\mathcal{C}$ does not have pullbacks in general. It is easy to verify that this essentially gives us the multiplicative structure described above. So from this perspective it is clear that we have constructed a $*$-autonomous category (the "diagonal elements" of $\mathbf{Chu}(\mathcal{C}, \perp)$, if the latter exists) which is strongly $*$-autonomous equivalent to $\mathcal{C}$ and has a strict negation.

3.2. Proposition. $\mathbf{C}(\mathcal{C})$ *is a $*$-autonomous category with strict involution.*

The equivalence $F: \mathcal{C} \to \mathbf{C}(\mathcal{C})$ and $G: \mathbf{C}(\mathcal{C}) \to \mathcal{C}$ is given by $F(A) = (A, A^*, id_{A^*})$ and $G(A, A', \tau) = A$. Obviously $G \circ F$ is the identity functor on $\mathcal{C}$, and since any A' in (A, A', τ) is isomorphic to A^*, we can see that $F \circ G \simeq Id_{\mathbf{C}(\mathcal{C})}$. Also it is immediate to see that F is strong monoidal and G is strict monoidal. Furthermore, the obvious natural transformations from $(FA)^*$ to $F(A^*)$ (realised by the identity arrow id_{A^*}) and $(G\mathbf{A})^*$ to $G(\mathbf{A}^*)$ (realised by τ_A^{-1}) satisfy the requirement for strong $*$-autonomous functors. Finally, we shall note that F is fully faithful (this will be important for showing the coherence theorem later).

3.3. Theorem.(the weak coherence theorem.) *Every $*$-autonomous category is strongly $*$-autonomous equivalent to a $*$-autonomous category with strict involution.*

Note, however, that this equivalence preserves the $*$-autonomous structure only up to isomorphism (*i.e.* not "strictly" but only "strongly") — in particular, it does not strictly preserve the involution: $F(A^*) = (A^*, A^{**}, A^{**} \xrightarrow{id_{A^{**}}} A^{**})$ while $(FA)^* = (A^*, A, A \xrightarrow{\simeq} A^{**})$. (This remark also applies to the "diagonal elements of the Chu construction" approach above.) We would like to strengthen this result to obtain the usual coherence theorem of the form "every diagram (of certain type) commutes". There are several arguments which can accomplish this; we shall present a standard one along the lines of [13] in the next section.

3.3.1. Linearly distributive categories and $\mathbf{C}(\mathcal{C})$. If we take *(symmetric) linearly distributive categories with negation* as the starting point instead of $*$-autonomous categories and re-examine the construction given here, we can reformulate this result, and actually obtain a stronger coherence result. Recall that a linearly distributive category is a category with two distinguished symmetric monoidal structures $(\otimes, I)$ and $(\parr, \perp)$ with certain coherence morphisms expressing linear distributive laws [7]. It has been shown in [7] that the notion of linearly distributive category with negation and that of $*$-autonomous category coincide. However, unlike most systems for linear logic, in linearly distributive categories with negation, objects $A \parr B$ and $(A^* \otimes B^*)^*$, and similarly $\perp$ and

I^*, are only canonically isomorphic, and not identified in general. Thus here we need some more delicacy than just talking about the $*$-autonomous structure. We shall consider this in the next section.

However, for the moment we shall merely comment that the construction of the category of complemented objects can be presented [8] so as to apply to a linearly distributive category $\mathcal{C}$, and constructs (freely, in an appropriate sense) a $*$-autonomous category $\mathbf{C}(\mathcal{C})$ (with a strict negation) generated by $\mathcal{C}$. If $\mathcal{C}$ is in fact $*$-autonomous, then $\mathbf{C}(\mathcal{C})$ is equivalent to it.

3.4. Strictifying other isomorphisms. With the weak coherence theorem, we seem only to have dealt with the strictness of the double involution, but we ought to remark that in fact much more is possible. All the isomorphisms we might "expect" to be strict can be made strict in the sense that there is an equivalent category in which they are strict. While we cannot expect the commutativity of tensors to be strict, one can "strictify" the associative and unit isomorphisms. For tensor alone, this result has been known for decades, although the first published proof seems to be in [13]. However, in our context, there is a block to applying this result, in that we must "strictify" both tensor and par if we are to have a strict tensor in $\mathbf{C}(\mathcal{C})$ as well. To do that requires a somewhat more involved construction than the ones we've seen so far.

3.5. Proposition. *Given a linearly distributive category $\mathcal{C}$, there is a linearly distributive category $\mathbf{W}(\mathcal{C})$ with strict multiplicatives, and satisfies the following.*

1. *There is a linear equivalence $w\colon \mathcal{C} \to \mathbf{W}(\mathcal{C})$.*
2. *If $\mathcal{C}$ is $*$-autonomous, then so is $\mathbf{W}(\mathcal{C})$.*
3. *If $\mathcal{C}$ is strictly $*$-autonomous, then so is $\mathbf{W}(\mathcal{C})$.*

Proof. (Sketch) Note that by a "linearly equivalence" we mean that the equivalence is a linear functor [8]. The objects of $\mathbf{W}(\mathcal{C})$ are "words" in the free algebra on the objects of $\mathcal{C}$ on two associative operations (with units) having no interaction. Of course, the two operations are to be interpreted as tensor and par. We may represent the words of this algebra by a normal form for terms consisting of objects of $\mathcal{C}$ and of alternating non-singleton lists of such objects, such as

$$\{A_1, [A_2, \{A_3, A_4\}, A_5], [\,], [\{\}, \{A_6, A_7\}]\}$$

where we may interpret (for example) $[A, B]$ as the tensor of A, B, and $\{A, B\}$ as their par. (For longer lists, we may associate to the right.) If $\mathcal{C}$ has negation, this is easily extended to $\mathbf{W}(\mathcal{C})$ by interchanging the two types of brackets and negating the objects, so the negation of the list above would be

$$[A_1^*, \{A_2^*, [A_3^*, A_4^*], A_5^*\}, \{\}, \{[\,], [A_6^*, A_7^*]\}]$$

This representation is a normal form with respect to a rewriting system which associates to the right (and makes $[\,]$ and $\{\}$ the units). (Such a rewriting system is easily shown to be

Church-Rosser.) Tensor and par for $\mathbf{W}(\mathcal{C})$ are defined in the obvious way, as the normal forms of the induced lists. This essentially means doing the obvious "concatenation", and then rewriting to obtain the appropriate normal form.

That $\mathbf{W}(\mathcal{C})$ is closed under tensor and par is straightforward, although a little care is needed to show these operations are functorial (here Mac Lane's coherence theorem is needed in showing that no matter how one normalizes, one obtains the same morphism). The linearly distributive structure lifts from $\mathcal{C}$, and so one may show that $\mathbf{W}(\mathcal{C})$ is linearly equivalent to $\mathcal{C}$, and has strictly associative and strictly unitary tensor and par, as claimed.

∎

Since the construction of $\mathbf{C}(\mathcal{C})$ clearly preserves the strictness of the associative and unit isomorphisms for tensor and par, in order to have a strict involution as well as strict associativity and unit isomorphisms, we merely need to strictify the latter first, *via* $\mathbf{W}(\mathcal{C})$, then construct $\mathbf{C}(\mathbf{W}(\mathcal{C}))$ from that. In addition, if we start with a $*$-autonomous category $\mathcal{C}$, we have also strictified the de Morgan isomorphisms, since we define the other connectives in terms of those as equations. Both $\mathbf{C}(\mathbf{W}(\mathcal{C}))$ and $\mathbf{W}(\mathbf{C}(\mathcal{C}))$ are strict $*$-autonomous with strict multiplicatives.

3.6. Corollary. *Any $*$-autonomous category is strongly $*$-autonomous equivalent to a $*$-autonomous category with strict associativity and unit isomorphisms as well as strict involution and strict de Morgan isomorphisms.*

Moreover, looking at this construction shows that we can similarly turn a linearly distributive category with negation into one in which not only are A and A^{**} identified, but also $A \mathbin{⅋} B$ and $(A^* \otimes B^*)^*$ are identified, and $\bot$ and I^* are identified. Thus, in terms of linearly distributive categories with negation, we can conclude the following.

3.7. Corollary. *Any linearly distributive category with negation is strongly equivalent to a linearly distributive category with strict associativity and unit isomorphisms as well as strict involution and strict de Morgan isomorphisms.*

Since the notion of linearly distributive category with strict negation and that of $*$-autonomous category with strict involution coincide, this also says that any linearly distributive category with negation is linearly equivalent to a $*$-autonomous category (viewed as a linearly distributive category) with strict involution. Since the equivalence is a linear functor, it preserves linear structure [8], such as the additives and exponentials, as outlined in the final section of this paper.

In fact, the proof of the coherence theorem in the following section also applies to this refined setting, thus any free linearly distributive category with negation is equivalent (in a strict way) to a free linearly distributive category with strict negation. (So, from mathematical point of view, the title of this paper could be "coherence of the double negation on linearly distributive categories".)

3.8. Remark. (other examples) We should note what these results do *not* say. Not all monoidal categories are strict; a famous example of Isbell's is that in the skeleton of Sets, as a category with finite products, the associativity isomorphisms for the product cannot be strict. Isbell's example works because there is an object A with $A \otimes A \cong A$, together with a strongly epic map $!\colon A \to I$ (in the sense that $! \otimes id_X$ is epic for any object X). Such an object is present in any $*$-autonomous category which is sufficiently cocomplete, as one can use $\coprod_\omega I$. Abelian groups form another example (with the usual tensor product). Neither of these categories is free, so the strong coherence theorem in the next section does not apply; however each can be "strictified", in that an equivalent category may be constructed which is strict. Of course, the equivalence will only preserve the product up to isomorphism. Similar remarks apply to the strictness of involutions. For example, one can construct simple examples of $*$-autonomous categories whose involution cannot be strict; one such, a preorder, is illustrated below, being inspired by the four-element Boolean algebra with a fifth element isomorphic to one of the objects as shown. There are two objects R, R' with the same negation L, but L has only one, R, as its negation. Clearly this negation cannot be strictified.

$$
\begin{tikzcd}
& \top & & \\
L \arrow[ur] & & R \arrow[ul] \arrow[r, phantom, "\cong"] & R' \arrow[ull] \\
& \bot \arrow[ul] \arrow[ur] \arrow[urr] & &
\end{tikzcd}
$$

Finally, we note that although a coherence theorem of the sort in this paper cannot be expected for the symmetry isomorphisms, there certainly are $*$-autonomous categories with strict symmetries. A trivial, one-object example is the category with one object, let us call it k, whose morphisms are all linear endomorphisms of some field k. This is essentially the sub-$*$-autonomous category of finite dimensional vector spaces generated by the unit; it is clearly strict in "every" sense: the unit, symmetry, associativity, and double negation isomorphisms are strict. Taking the product of this with any Boolean algebra (for example) gives non-trivial examples. If we take the product of k with the preorder given above, then we lose the double negation strictness, while preserving the rest.

4. The Coherence Theorem

Our weak coherence theorem had the unsatisfactory property that the equivalence did not preserve the relevant structure "on the nose". This is a familiar situation, and one for which well-known standard arguments exist to strengthen the result. There are two familiar ways to present such an argument: either syntactically or semantically. While categorical logicians might prefer the former, we shall use the (essentially equivalent) semantic approach, primarily because it is simpler to present in a self-contained concise manner which requires no informal argument about the nature of isomorphisms and equalities; to take sufficient care about these matters with a syntactic presentation of the argument would require a longer development than what follows.

In order to express the stronger coherence theorem, we first need to introduce the notions of free $*$-autonomous categories as well as free $*$-autonomous categories with strict involution.

Let $\mathcal{FC}$ denote the *free $*$-autonomous category on the category $\mathcal{C}$*, with $i: \mathcal{C} \to \mathcal{FC}$ the unit (*i.e.* the inclusion of generators); thus for any $*$-autonomous category $\mathcal{V}$ with a functor $H: \mathcal{C} \to \mathcal{V}$ there is a unique strict $*$-autonomous functor $H': \mathcal{FC} \to \mathcal{V}$ such that

$$\begin{tikzcd} \mathcal{C} \arrow[r, "i"] \arrow[dr, "H"'] & \mathcal{FC} \arrow[d, dotted, "H'"] \\ & \mathcal{V} \end{tikzcd}$$

commutes. Also let $\mathcal{F}_s\mathcal{C}$ be the *free $*$-autonomous category with strict involution on the category $\mathcal{C}$*. The functor $\Gamma: \mathcal{FC} \to \mathcal{F}_s\mathcal{C}$ given by the freeness of $\mathcal{FC}$ is strict $*$-autonomous, and sends the canonical isomorphism $A \xrightarrow{\simeq} A^{**}$ to the identity $A \xrightarrow{id} A$. We claim:

4.1. Theorem. (the coherence theorem.) *For every category $\mathcal{C}$, the (strict $*$-autonomous) functor $\Gamma: \mathcal{FC} \to \mathcal{F}_s\mathcal{C}$ is an equivalence.*

Intuitively, applying Γ amounts to throwing away all the information on the canonical isomorphisms $A \xrightarrow{\simeq} A^{**}$. Nevertheless, this theorem (most importantly the faithfulness of Γ) tells us that there is nothing lost! In the rest of this section, we shall show this result by adapting the construction in [13].

4.2. Definition. *Given functors $S, T: \mathcal{A} \to \mathcal{B}$, the category of "iso-inserters" $\mathbf{Eq}(S,T)$ has objects (A,h) consisting of an object A of $\mathcal{A}$ and an isomorphism $h: SA \simeq TA$ in $\mathcal{B}$, and an arrow $f: (A,h) \to (A',h')$ is an arrow $f: A \to A'$ in $\mathcal{A}$ such that $Tf \circ h = h' \circ Sf$ holds.*

There is a projection functor $P: \mathbf{Eq}(S,T) \to \mathcal{A}$ sending (A,h) to A, and then we have a natural isomorphism $\sigma: S \circ P \xrightarrow{\simeq} T \circ P$ whose (A,h)-component is h.

$$\begin{tikzcd} & \mathbf{Eq}(S,T) \arrow[dl, "P"'] \arrow[dr, "P"] \arrow[dd, phantom, "\overset{\sigma}{\simeq}"] & \\ \mathcal{A} \arrow[dr, "S"'] & & \mathcal{A} \arrow[dl, "T"] \\ & \mathcal{B} & \end{tikzcd}$$

We are interested in the case where S and T are strong $*$-autonomous functors:

4.3. Lemma. *If $S, T: \mathcal{A} \to \mathcal{B}$ are strong $*$-autonomous functors, then $\mathbf{Eq}(S,T)$ supports a unique $*$-autonomous structure such that P becomes a strict $*$-autonomous functor and σ becomes an isomorphism of strong $*$-autonomous functors.*

PROOF. This result is quite straightforward, although the application to the current situation requires some obvious extensions of the original discussion in [13]. Explicitly, this $*$-autonomous structure on $\mathbf{Eq}(S,T)$ is described as follows.

$$\begin{aligned}
I &= (I, SI \xrightarrow{\simeq} I \xrightarrow{\simeq} TI)\\
(A,h)\otimes(B,k) &= (A\otimes B, S(A\otimes B) \xrightarrow{\simeq} SA\otimes SB \xrightarrow{h\otimes k} TA\otimes TB \xrightarrow{\simeq} T(A\otimes B))\\
(A,h)^* &= (A^*, S(A^*) \xrightarrow{\theta_A^{S\,-1}} (SA)^* \xrightarrow{(h^{-1})^*} (TA)^* \xrightarrow{\theta_A^T} T(A^*))
\end{aligned}$$

where θ^S and θ^T are the natural isomorphisms associated with S and T respectively. ■

4.4. PROPOSITION. (FLEXIBILITY.) *Every strong $*$-autonomous functor $T\colon \mathcal{FC} \to \mathcal{V}$ is isomorphic to a strict $*$-autonomous functor $S\colon \mathcal{FC} \to \mathcal{V}$.*

PROOF. By freeness, there is a unique strict $*$-autonomous functor $S\colon \mathcal{FC} \to \mathcal{V}$ such that $S\circ i = T\circ i\colon \mathcal{C} \to \mathcal{V}$ holds. Also the functor $H\colon \mathcal{C} \to \mathbf{Eq}(S,T)$ given by $H(C) = (i(C), id)$ is the unique H such that $P\circ H = i$ and σH is an identity. So we have a unique strict $*$-autonomous functor $H'\colon \mathcal{FC} \to \mathbf{Eq}(S,T)$ with $H'\circ i = H$.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "i"] \arrow[dr, "T\circ i"'] & \mathcal{FC} \arrow[d, dotted, "S"] \\
 & \mathcal{V}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[r, "i"] \arrow[dr, "H"] \arrow[ddr, "i"'] & \mathcal{FC} \arrow[d, dotted, "H'"] \\
 & \mathbf{Eq}(S,T) \arrow[d, "P"] \\
 & \mathcal{FC}
\end{tikzcd}$$

By freeness of $\mathcal{FC}$, the strictness of $P\circ H'$, and the equality

$$P\circ H'\circ i = P\circ H = i = Id_{\mathcal{FC}}\circ i$$

we obtain $P\circ H' = Id_{\mathcal{FC}}$. Hence we have $\sigma H'\colon S = S\circ P\circ H' \xrightarrow{\simeq} T\circ P\circ H' = T$, *i.e.*, an isomorphism from S to T.

■

PROOF. (OF THE COHERENCE THEOREM.) Since Γ is surjective on objects and also full, it remains to see its faithfulness. The weak coherence theorem gives a faithful strong $*$-autonomous functor $F\colon \mathcal{FC} \to \mathbf{C}(\mathcal{FC})$. By the flexibility result, we have an isomorphism $S \simeq F$ with $S\colon \mathcal{FC} \to \mathbf{C}(\mathcal{FC})$ strict $*$-autonomous. By the universal property of $\mathcal{F}_s\mathcal{C}$ and $\mathcal{FC}$, there is a unique strict $*$-autonomous functor $R\colon \mathcal{F}_s\mathcal{C} \to \mathbf{C}(\mathcal{FC})$ such that $R\circ\Gamma = S$. But S is faithful because it is isomorphic to F which is faithful. Then $R\circ\Gamma = S$ implies that Γ is faithful too.

$$\begin{tikzcd}
\mathcal{FC} \arrow[d, "\Gamma"'] \arrow[dr, dotted, "S"'] \arrow[dr, bend left, "F"] \arrow[dr, phantom, bend left=20, "\simeq"] & \\
\mathcal{F}_s\mathcal{C} \arrow[r, dotted, "R"'] & \mathbf{C}(\mathcal{FC})
\end{tikzcd}$$

■

In view of the comments we made following Corollaries 3.6, 3.7, we have the following extension of the strong coherence theorem.

4.5. Corollary. *Any free $*$-autonomous category is $*$-autonomous equivalent,* via *a strict canonical $*$-autonomous functor, to a free $*$-autonomous category with strict associativity and unit isomorphisms as well as strict involution and strict de Morgan isomorphisms.*

4.6. Remark. More abstractly, Proposition 4.4 follows from the fact that $*$-autonomous categories are algebras of a 2-monad on the 2-category $\mathbf{Cat}_g$ of (small) categories, functors and natural isomorphisms (see Remark 5.7 in [4] on the flexibility of free algebras). This is also the case for the extensions discussed below.

5. Exponentials and Additives

The results above all smoothly extend to the cases with exponentials and additives (hence full propositional Classical Linear Logic). Below we recall the needed notions and outline the constructions used in the proofs.

5.1. Definition. (linear exponential comonads [[12]].) *A symmetric monoidal comonad $! = (!, \varepsilon, \delta, m_{A,B}, m_I)$ on a symmetric monoidal category $\mathcal{C}$ is called a* linear exponential comonad *when the category of its coalgebras is a category of commutative comonoids.*

In other words (*cf.* [16, 3]):

- there are specified monoidal natural transformations $e_A\colon !A \to I$ and $d_A\colon !A \to !A \otimes !A$ which form a commutative comonoid $(!A, e_A, d_A)$ in $\mathcal{C}$ and there also are coalgebra morphisms from $(!A, \delta_A)$ to (I, m_I) and $(!A \otimes !A, m_{!A,!A} \circ (\delta_A \otimes \delta_A))$ respectively, and
- any coalgebra morphism from $(!A, \delta_A)$ to $(!B, \delta_B)$ is also a comonoid morphism from $(!A, e_A, d_A)$ to $(!B, e_B, d_B)$.

The notion of strong functors between $*$-autonomous categories with linear exponential comonads $\mathcal{C}$ and $\mathcal{D}$ is defined as strong $*$-autonomous functors $F\colon \mathcal{C} \to \mathcal{D}$ equipped with a natural isomorphism $\kappa\colon !F \xrightarrow{\simeq} F!$ which is a distributive law and also respects the comonoid structure. The strict functors between $*$-autonomous categories with linear exponential comonads are those preserving the structure on the nose.

First, it is easily seen that the category of complemented objects $\mathbf{C}(\mathcal{C})$ has a linear exponential comonad if $\mathcal{C}$ does:

$$!(A, A', \tau_A) \quad = \quad (!A, ?A', ?A' \xrightarrow{?\tau_A} ?A^* \xrightarrow{\simeq} (!A)^*)$$

where we write $?A$ for $(!A^*)^*$. Therefore Prop. 3.2 remains true under the presence of linear exponential comonads: if $\mathcal{C}$ is a $*$-autonomous category with a linear exponential comonad, then $\mathbf{C}(\mathcal{C})$ is one with strict involution. The equivalence between $\mathcal{C}$ and $\mathbf{C}(\mathcal{C})$ is obviously seen to be strong, hence we have the weak coherence theorem for this extension.

The same consideration applies to the case with finite products, where the construction on the category of complemented objects is

$$\begin{array}{rcl} \top & = & (\top, 0, id_0) \\ \mathbf{A} \,\&\, \mathbf{B} & = & \big(A \,\&\, B, A' \oplus B', A' \oplus B' \xrightarrow{\tau_A \oplus \tau_B} A^* \oplus B^* \xrightarrow{\simeq} (A \,\&\, B)^*\big) \end{array}$$

where $0 = \top^*$ and $A \oplus B = (A^* \,\&\, B^*)^*$. (Note: here we use $\top$ for the terminal object and $\&$ for binary products, while 0 and $\oplus$ are the initial object and coproducts respectively.)

5.2. Theorem. (the weak coherence theorem.) *Every $*$-autonomous category with linear exponential comonad and/or finite products is strongly equivalent to one with strict involution.*

To derive the coherence theorem, we need to identify the required structure on the category $\mathbf{Eq}(S, T)$ for strong functors S and T. This is routinely done as

$$\begin{array}{rcl} !(A, h) & = & (!A, S(!A) \xrightarrow{\kappa_A^{S\,-1}} !(SA) \xrightarrow{!h} !(TA) \xrightarrow{\kappa_A^T} T(!A)) \\ \top & = & (\top, S\top \xrightarrow{\simeq} \top \xrightarrow{\simeq} T\top) \\ (A, h) \,\&\, (B, k) & = & (A \,\&\, B, S(A \,\&\, B) \xrightarrow{\simeq} SA \,\&\, SB \xrightarrow{h \,\&\, k} TA \,\&\, TB \xrightarrow{\simeq} T(A \,\&\, B)) \end{array}$$

By repeating the argument in the last section, now we obtain the coherence theorem for these extensions. Let $\mathcal{F}\mathcal{C}$ and $\mathcal{F}_s\mathcal{C}$ denote the free $*$-autonomous category with linear exponential comonad and/or finite products on the category $\mathcal{C}$ and that with strict involution respectively.

5.3. Theorem. (the coherence theorem.) *For every category $\mathcal{C}$, the strict functor $\Gamma\colon \mathcal{F}\mathcal{C} \to \mathcal{F}_s\mathcal{C}$ is an equivalence.*

References

[1] Barr, M. (1979) *$*$-Autonomous Categories.* Springer Lecture Notes in Math. **752**.

[2] Barr, M. (1991) $*$-autonomous categories and linear logic. *Math. Struct. Comp. Sci.* **1**, 159–178.

[3] Bierman, G.M. (1995) What is a categorical model of intuitionistic linear logic? In *Proc. Typed Lambda Calculi and Applications (TLCA'95)*, Springer Lecture Notes in Comput. Sci. **902**, pp. 78–93.

[4] Blackwell, R., Kelly, G.M. and Power, A.J. (1989) Two-dimensional monad theory. *J. Pure All. Algebra* **59**, 1–41.

[5] Blute, R.F., Cockett, J.R.B., Seely, R.A.G. and Trimble, T.H. (1996) Natural deduction and coherence for weakly distributive categories. *J. Pure Appl. Algebra* **113**(3), 229–296.

[6] Cockett, J.R.B., Koslowski, J. and Seely, R.A.G. (2000) Introduction to linear bicategories. *Math. Structures Comput. Sci.* **10**(2), 165–203.

[7] Cockett, J.R.B. and Seely, R.A.G. (1991) Weakly distributive categories. In *Applications of Categories in Computer Science*, London Mathematical Society Lecture Note Series **177**, pp. 45–65.

[8] Cockett, J.R.B. and Seely, R.A.G. (1999) Linearly distributive functors. *J. Pure Appl. Algebra* **143**, 155–203.

[9] Girard, J.-Y. (1987) Linear logic. *Theoret. Comp. Sci.* **50**, 1–102.

[10] Girard, J.-Y. (1995) Linear logic: its syntax and semantics. In *Advances in Linear Logic*, London Mathematical Society Lecture Note Series **222**, pp. 1–42.

[11] Hasegawa, M. (2002) Classical linear logic of implications. In *Proc. Computer Science Logic (CSL'02)*, Springer Lecture Notes in Comp. Sci. **2471**, pp. 458–472.

[12] Hyland, M. and Schalk, A. (2003) Glueing and orthogonality for models of linear logic. *Theoret. Comp. Sci.* **294**(1/2), 183–231.

[13] Joyal, A. and Street, R. (1993) Braided tensor categories. *Adv. Math.* **102**, 20–78.

[14] Koh, T.W. and Ong, C.-H.L. (1999) Explicit substitution internal languages for autonomous and $*$-autonomous categories. In *Proc. Category Theory and Computer Science (CTCS'99)*, Electron. Notes Theor. Comput. Sci. **29**.

[15] Mac Lane, S. (1971) *Categories for the Working Mathematician.* Graduate Texts in Mathematics **5**, Springer-Verlag.

[16] Seely, R.A.G. (1989) Linear logic, $*$-autonomous categories and cofree coalgebras. In *Categories in Computer Science*, AMS Contemporary Mathematics **92**, pp. 371–389.

Department of Computer Science
University of Calgary
2500 University Drive N.W.
Calgary, Alberta
Canada T2N 1N4

Research Institute for Mathematical Sciences
Kyoto University
Kyoto, 606-8502 Japan
and
Information and Systems
PRESTO, Japan Science and Technology Agency

Department of Mathematics
McGill University
805 Sherbrooke St. W.
Montreal, Quebec
Canada H3A 2K6

Email: `robin@cpsc.ucalgary.ca`
`hassei@kurims.kyoto-u.ac.jp`
`rags@math.mcgill.ca`

This article may be accessed via WWW at `http://www.tac.mta.ca/tac/` or by anonymous ftp at `ftp://ftp.tac.mta.ca/pub/tac/html/volumes/17/2/17-02.{dvi,ps,pdf}`

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.
Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.
The method of distribution of the journal is via the Internet tools WWW/ftp. The journal is archived electronically and in printed paper format.

SUBSCRIPTION INFORMATION. Individual subscribers receive (by e-mail) abstracts of articles as they are published. Full text of published articles is available in .dvi, Postscript and PDF. Details will be e-mailed to new subscribers. To subscribe, send e-mail to tac@mta.ca including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, rrosebrugh@mta.ca.

INFORMATION FOR AUTHORS. The typesetting language of the journal is TEX, and LATEX2e is the preferred flavour. TEX source of articles for publication should be submitted by e-mail directly to an appropriate Editor. They are listed below. Please obtain detailed information on submission format and style files from the journal's WWW server at http://www.tac.mta.ca/tac/. You may also write to tac@mta.ca to receive details by e-mail.