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# QUOTIENTS OF UNITAL $A_\infty$-CATEGORIES

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Abstract. Assuming that $\mathcal{B}$ is a full $A_\infty$-subcategory of a unital $A_\infty$-category $\mathcal{C}$ we construct the quotient unital $A_\infty$-category $\mathcal{D}$ =‘$\mathcal{C}/\mathcal{B}$’. It represents the $A^u_\infty$-2-functor $\mathcal{A} \mapsto A^u_\infty(\mathcal{C}, \mathcal{A})_{\operatorname{mod}\mathcal{B}}$, which associates with a given unital $A_\infty$-category $\mathcal{A}$ the $A_\infty$-category of unital $A_\infty$-functors $\mathcal{C} \to \mathcal{A}$, whose restriction to $\mathcal{B}$ is contractible. Namely, there is a unital $A_\infty$-functor $e : \mathcal{C} \to \mathcal{D}$ such that the composition $\mathcal{B} \hookrightarrow \mathcal{C} \xrightarrow{e} \mathcal{D}$ is contractible, and for an arbitrary unital $A_\infty$-category $\mathcal{A}$ the restriction $A_\infty$-functor $(e \boxtimes 1)M : A^u_\infty(\mathcal{D}, \mathcal{A}) \to A^u_\infty(\mathcal{C}, \mathcal{A})_{\operatorname{mod}\mathcal{B}}$ is an equivalence.

Let $\underline{\mathsf{C}}_\Bbbk$ be the differential graded category of differential graded $\Bbbk$-modules. We prove that the Yoneda $A_\infty$-functor $Y : \mathcal{A} \to A^u_\infty(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_\Bbbk)$ is a full embedding for an arbitrary unital $A_\infty$-category $\mathcal{A}$. In particular, such $\mathcal{A}$ is $A_\infty$-equivalent to a differential graded category with the same set of objects.

## Contents



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Let $\mathcal{A}$ be an Abelian category. The question: what is the quotient

$$\{\text{category of complexes in } \mathcal{A}\}/\{\text{category of acyclic complexes}\}?$$

admits several answers. The first answer – the derived category of $\mathcal{A}$ – was given by Grothendieck and Verdier [Ver77].

The second answer – a differential graded category $\mathcal{D}$ – is given by Drinfeld [Dri04]. His article is based on the work of Bondal and Kapranov [BK90] and of Keller [Kel99]. The derived category $D(\mathcal{A})$ can be obtained as $H^0(\mathcal{D}^{\text{pre-tr}})$.

The third answer – an $A_\infty$-category of bar-construction type – is given by Lyubashenko and Ovsienko [LO06]. This $A_\infty$-category is especially useful when the basic ring $\Bbbk$ is a field. It is an $A_\infty$-version of one of the constructions of Drinfeld [Dri04].

The fourth answer – an $A_\infty$-category freely generated over the category of complexes in $\mathcal{A}$ – is given in this article. It is $A_\infty$-equivalent to the third answer and enjoys a certain universal property of the quotient. Thus, it passes this universal property also to the third answer.

# 1. Introduction

Since $A_\infty$-algebras were introduced by Stasheff [Sta63, II] there existed a possibility to consider $A_\infty$-generalizations of categories. It did not happen until $A_\infty$-categories were encountered in studies of mirror symmetry by Fukaya [Fuk93] and Kontsevich [Kon95]. $A_\infty$-categories may be viewed as generalizations of differential graded categories for which the binary composition is associative only up to a homotopy. The possibility to define $A_\infty$-functors was mentioned by Smirnov [Smi89], who reformulated one of his results in the language of $A_\infty$-functors between differential graded categories. The definition of $A_\infty$-functors between $A_\infty$-categories was published by Keller [Kel01], who studied their applications to homological algebra. Homomorphisms of $A_\infty$-algebras (e.g. [Kad82]) are particular cases of $A_\infty$-functors.

$A_\infty$-transformations between $A_\infty$-functors are certain coderivations. Given two $A_\infty$-categories $\mathcal{A}$ and $\mathcal{B}$, one can construct a third $A_\infty$-category $A_\infty(\mathcal{A},\mathcal{B})$, whose objects are $A_\infty$-functors $f:\mathcal{A}\to\mathcal{B}$, and morphisms are $A_\infty$-transformations (Fukaya [Fuk02], Kontsevich and Soibelman [KS06, KS07], Lefèvre-Hasegawa [LH03], as well as [Lyu03]). For an $A_\infty$-category $\mathcal{C}$ there is a homotopy invariant notion of unit elements (identity morphisms) [Lyu03]. They are cycles ${}_X\mathbf{i}_0^{\mathcal{C}}\in s\mathcal{C}(X,X)$ of degree $-1$ such that the maps $(1\otimes\mathbf{i}_0^{\mathcal{C}})b_2, -(\mathbf{i}_0^{\mathcal{C}}\otimes 1)b_2 : s\mathcal{C}(X,Y)\to s\mathcal{C}(X,Y)$ are homotopic to the identity map. This allows us to define the 2-category $\overline{A_\infty^u}$, whose objects are unital $A_\infty$-categories (those which have units), 1-morphisms are unital $A_\infty$-functors (their first components preserve the units up to a boundary) and 2-morphisms are equivalence classes of natural $A_\infty$-transformations [Lyu03]. We continue to study this 2-category. Notations and terminology follow [Lyu03], complemented by [LO06] and [LM06].

Unital $A_\infty$-categories and unital $A_\infty$-functors can be considered as strong homotopy generalizations of differential graded categories and functors. Let us illustrate the notion of $A_\infty$-transformations in a familiar context.

1.1. Differential for $A_\infty$-transformations compared with the Hochschild differential. Let $\mathcal{A}$, $\mathcal{B}$ be ordinary $\Bbbk$-linear categories. We consider $\mathcal{A}(\_,\_)$ and $\mathcal{B}(\_,\_)$ as complexes of $\Bbbk$-modules concentrated in degree 0. This turns $\mathcal{A}$ and $\mathcal{B}$ into differential graded categories and, thereby, into unital $A_\infty$-categories. An $A_\infty$-functor between $\mathcal{A}$ and $\mathcal{B}$ is necessarily strict, for $(s\mathcal{A})^{\otimes k} = \mathcal{A}^{\otimes k}[k]$ and $s\mathcal{B} = \mathcal{B}[1]$ are concentrated in different degrees if $k > 1$. Thus, a unital $A_\infty$-functor $f : \mathcal{A} \to \mathcal{B}$ is the same as an ordinary $\Bbbk$-linear functor $f$. Let $f, g : \mathcal{A} \to \mathcal{B}$ be $\Bbbk$-linear functors. All complexes $\underline{\mathsf{C}}_\Bbbk((s\mathcal{A})^{\otimes k}(X,Y), s\mathcal{B}(Xf,Yg))$ are concentrated in degree $k-1$. Their direct product

$$\Psi_k = \prod_{X,Y\in\operatorname{Ob}\mathcal{A}} \underline{\mathsf{C}}_\Bbbk((s\mathcal{A})^{\otimes k}(X,Y), s\mathcal{B}(Xf,Yg))$$

is the same, whether taken in the category of $\Bbbk$-modules or graded $\Bbbk$-modules or complexes of $\Bbbk$-modules. It is the module of $k$-th components of $A_\infty$-transformations. The graded $\Bbbk$-module of $A_\infty$-transformations $sA_\infty(\mathcal{A},\mathcal{B})(f,g)$ is isomorphic to the direct product $\prod_{k=0}^\infty \Psi_k$ taken in the category of graded $\Bbbk$-modules [Lyu03, Section 2.7]. That is, $[sA_\infty(\mathcal{A},\mathcal{B})(f,g)]^n = \prod_{k=0}^\infty \Psi_k^n$, where $\Psi_k^n$ is the degree $n$ part of $\Psi_k$. Therefore, in our case it simply coincides with the graded $\Bbbk$-module $\Psi[1] : \mathbb{Z} \ni n \mapsto \Psi_{n+1} \in \Bbbk\text{-mod}$. The graded $\Bbbk$-module $sA_\infty(\mathcal{A},\mathcal{B})(f,g)$ is equipped with the differential $B_1$, $rB_1 = rb^{\mathcal{B}} - (-)^r b^{\mathcal{A}} r$ [Lyu03, Proposition 5.1]. Since the only non-vanishing component of $b$ (resp. $f$) is $b_2$ (resp. $f_1$), the explicit formula for components of $rB_1$ is the following:

$$(rB_1)_{k+1} = (f_1 \otimes r_k)b_2 + (r_k \otimes g_1)b_2 - (-)^{r_k} \sum_{a+c=k-1} (1^{\otimes a} \otimes b_2 \otimes 1^{\otimes c}) r_k.$$

Recalling that $\deg r_k = k-1$, we get the differential in $\Psi[1]$ also denoted $B_1$:

$$r_k B_1 = (f_1 \otimes r_k)b_2 + (r_k \otimes g_1)b_2 + (-)^k \sum_{a+c=k-1} (1^{\otimes a} \otimes b_2 \otimes 1^{\otimes c}) r_k,$$

where $r_k \in \Psi_k$, $r_k B_1 \in \Psi_{k+1}$. We consider an isomorphism of graded $\Bbbk$-modules $\Psi \to \Psi' : \mathbb{Z} \ni k \mapsto \Psi'_k$ given by

$$\Psi_k \ni r_k \mapsto (s \otimes \cdots \otimes s) r_k s^{-1} = s^{\otimes k} r_k s^{-1} \in \Psi'_k \overset{\text{def}}{=} \prod_{X,Y\in\operatorname{Ob}\mathcal{A}} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}^{\otimes k}(X,Y), \mathcal{B}(Xf,Yg)).$$

Its inverse is $\Psi'_k \ni t_k \mapsto (s^{\otimes k})^{-1} t_k s \in \Psi_k$. This isomorphism induces the differential

$$d : \Psi'_k \to \Psi'_{k+1}, \qquad t_k d = s^{\otimes k+1} \cdot [(s^{\otimes k})^{-1} t_k s] B_1 \cdot s^{-1}.$$

The explicit formula for $d$ is

$$t_k d = (f \otimes t_k) m_2 + \sum_{a+c=k-1} (-1)^{a+1} (1^{\otimes a} \otimes m_2 \otimes 1^{\otimes c}) t_k + (-1)^{k+1} (t_k \otimes g) m_2.$$

Up to an overall sign this coincides with the differential in the Hochschild cochain complex $C^\bullet(\mathcal{A}, {}_f\mathcal{B}_g)$ (cf. [Mac63, Section X.3]). The $\mathcal{A}$-bimodule ${}_f\mathcal{B}_g$ acquires its left $\mathcal{A}$-module structure via $f$ and its right $\mathcal{A}$-module structure via $g$. Therefore, in our situation $A_\infty$-transformations are nothing else but Hochschild cochains. Natural $A_\infty$-transformations $r : f \to g : \mathcal{A} \to \mathcal{B}$ (such that $\deg r = -1$ and $rB_1 = 0$) are identified with the Hochschild cocycles of degree 0, that is, with natural transformations $t = rs^{-1} : f \to g : \mathcal{A} \to \mathcal{B}$ in the ordinary sense.

When $\mathcal{A}$, $\mathcal{B}$ are differential graded categories and $f, g : \mathcal{A} \to \mathcal{B}$ are differential graded functors, we may still interpret the complex $(sA_\infty(\mathcal{A}, \mathcal{B})(f, g), B_1)$ as the complex of Hochschild cochains $C^\bullet(\mathcal{A}, {}_f\mathcal{B}_g)$ for the differential graded category $\mathcal{A}$ and the differential graded bimodule ${}_f\mathcal{B}_g$. Indeed, for a homogeneous element $r \in sA_\infty(\mathcal{A}, \mathcal{B})(f, g)$ the components of $rB_1$ are

$$
\begin{aligned}
(rB_1)_k = r_k b_1 &+ (f_1 \otimes r_{k-1})b_2 + (r_{k-1} \otimes g_1)b_2 \\
&- (-)^r \sum_{a+1+c=k} (1^{\otimes a} \otimes b_1 \otimes 1^{\otimes c}) r_k - (-)^r \sum_{a+2+c=k} (1^{\otimes a} \otimes b_2 \otimes 1^{\otimes c}) r_{k-1}.
\end{aligned}
$$

For $A_\infty$-functors between differential graded categories or $A_\infty$-categories the differential $B_1$ is not interpreted as Hochschild differential any more. But we may view the complex of $A_\infty$-transformations as a generalization of the Hochschild cochain complex.

1.2. Main result. By Definition 6.4 of [LO06] an $A_\infty$-functor $g : \mathcal{B} \to \mathcal{A}$ from a unital $A_\infty$-category $\mathcal{B}$ is *contractible* if for all objects $X$, $Y$ of $\mathcal{B}$ the chain map $g_1 : s\mathcal{B}(X, Y) \to s\mathcal{A}(Xg, Yg)$ is null-homotopic. If $g : \mathcal{B} \to \mathcal{A}$ is a unital $A_\infty$-functor, then it is contractible if and only if for any $X \in \operatorname{Ob}\mathcal{B}$ and any $V \in \operatorname{Ob}\mathcal{A}$ the complexes $s\mathcal{A}(Xg, V)$ and $s\mathcal{A}(V, Xg)$ are contractible. Equivalently, $g\mathbf{i}^{\mathcal{A}} \equiv 0 : g \to g : \mathcal{B} \to \mathcal{A}$ [LO06, Proposition 6.1(C5)]. Other equivalent conditions are listed in Propositions 6.1–6.3 of [LO06].

Let $\mathcal{B}$ be a full $A_\infty$-subcategory of a unital $A_\infty$-category $\mathcal{C}$. Let $\mathcal{A}$ be an arbitrary unital $A_\infty$-category. Denote by $A^u_\infty(\mathcal{C}, \mathcal{A})_{\operatorname{mod}\mathcal{B}}$ the full $A_\infty$-subcategory of $A^u_\infty(\mathcal{C}, \mathcal{A})$, whose objects are unital $A_\infty$-functors $\mathcal{C} \to \mathcal{A}$, whose restriction to $\mathcal{B}$ is contractible. We allow consideration of $A_\infty$-categories with the empty set of objects.

1.3. Main Theorem. *In the above assumptions there exists a unital $A_\infty$-category $\mathcal{D} = \mathsf{q}(\mathcal{C}|\mathcal{B})$ and a unital $A_\infty$-functor $e : \mathcal{C} \to \mathcal{D}$ such that*

1) *the composition $\mathcal{B} \hookrightarrow \mathcal{C} \xrightarrow{e} \mathcal{D}$ is contractible;*
2) *the strict $A_\infty$-functor given by composition with $e$*

$$(e \boxtimes 1)M : A^u_\infty(\mathcal{D}, \mathcal{A}) \to A^u_\infty(\mathcal{C}, \mathcal{A})_{\operatorname{mod}\mathcal{B}}, \qquad f \mapsto ef,$$

*is an $A_\infty$-equivalence for an arbitrary unital $A_\infty$-category $\mathcal{A}$.*

Proof. Let us prove the statement first in a particular case, for a full subcategory $\widetilde{\mathcal{B}}$ of a strictly unital $A_\infty$-category $\widetilde{\mathcal{C}}$. Then the representing $A_\infty$-category $\mathcal{D} = \mathsf{Q}(\widetilde{\mathcal{C}}|\widetilde{\mathcal{B}})$ is constructed in Section 5.2 as an $A_\infty$-category, freely generated over $\widetilde{\mathcal{C}}$ by application of contracting homotopies $H$ to morphisms, whose source or target is in $\widetilde{\mathcal{B}}$. The strict $A_\infty$-functor $\widetilde{e} : \widetilde{\mathcal{C}} \to \mathcal{D}$ is identity on objects and $\widetilde{e}_1$ is an embedding. Theorem 6.5 asserts unitality of $\mathcal{D} = \mathsf{Q}(\widetilde{\mathcal{C}}|\widetilde{\mathcal{B}})$. By construction, the $A_\infty$-functor $\widetilde{e} : \widetilde{\mathcal{C}} \to \mathcal{D}$ is unital and $\widetilde{\mathcal{B}} \hookrightarrow \widetilde{\mathcal{C}} \xrightarrow{\widetilde{e}} \mathcal{D}$ is contractible. By Theorem 5.13 the restriction strict $A_\infty$-functor restr : $A^u_\infty(\mathcal{D},\mathcal{A}) \to A^u_\infty(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}}$ is an $A_\infty$-equivalence. Thus, $\mathcal{D}$ and $\widetilde{e} : \widetilde{\mathcal{C}} \to \mathcal{D}$ represent the $A^u_\infty$-2-functor $\mathcal{A} \mapsto A^u_\infty(\widetilde{\mathcal{C}},\mathcal{A})_{\operatorname{mod}\widetilde{\mathcal{B}}}$ in the sense of 1), 2), as claimed.

Let now $\mathcal{B}$ be a full $A_\infty$-subcategory of a unital $A_\infty$-category $\mathcal{C}$. There exists a differential graded category $\widetilde{\mathcal{C}}$ with $\operatorname{Ob}\widetilde{\mathcal{C}} = \operatorname{Ob}\mathcal{C}$, and quasi-inverse to each other $A_\infty$-functors $\widetilde{Y} : \mathcal{C} \to \widetilde{\mathcal{C}}$, $\Psi : \widetilde{\mathcal{C}} \to \mathcal{C}$ such that $\operatorname{Ob}\widetilde{Y} = \operatorname{Ob}\Psi = \mathrm{id}_{\operatorname{Ob}\mathcal{C}}$ (by Remark A.9 this follows from the $A_\infty$-version of Yoneda Lemma – Theorem A.7). Let $\widetilde{\mathcal{B}} \subset \widetilde{\mathcal{C}}$ be the full differential graded subcategory with $\operatorname{Ob}\widetilde{\mathcal{B}} = \operatorname{Ob}\mathcal{B}$. By the previous case there is a unital $A_\infty$-category $\mathcal{D}$ and a unital $A_\infty$-functor $\widetilde{e} : \widetilde{\mathcal{C}} \to \mathcal{D}$ representing the $A^u_\infty$-2-functor $\mathcal{A} \mapsto A^u_\infty(\widetilde{\mathcal{C}},\mathcal{A})_{\operatorname{mod}\widetilde{\mathcal{B}}}$ in the sense of 1), 2). By considerations in Appendix B (Corollaries B.10, B.11) the pair $\bigl(\mathcal{D}, e = (\mathcal{C} \xrightarrow{\widetilde{Y}} \widetilde{\mathcal{C}} \xrightarrow{\widetilde{e}} \mathcal{D})\bigr)$ represents $A^u_\infty(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}}$. Indeed,

$$(e \boxtimes 1)M = \bigl(A^u_\infty(\mathcal{D},\mathcal{A}) \xrightarrow{(\widetilde{e}\boxtimes 1)M} A^u_\infty(\widetilde{\mathcal{C}},\mathcal{A})_{\operatorname{mod}\widetilde{\mathcal{B}}} \xrightarrow{(\widetilde{Y}\boxtimes 1)M} A^u_\infty(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}}\bigr)$$

is a composition of two $A_\infty$-equivalences. ∎

Notation for our quotient constructions is the following. The construction of Section 5.2 is denoted $\mathsf{Q}(\_|\_)$. When it is combined with the Yoneda $A_\infty$-equivalence of Remark A.9 we denote it $\mathsf{q}(\_|\_)$.

The 2-category $\overline{A^u_\infty} = H^0 A^u_\infty$ has unital $A_\infty$-categories as objects, unital $A_\infty$-functors as 1-morphisms and equivalence classes of natural $A_\infty$-transformations as 2-morphisms [Lyu03]. Thus, $\overline{A^u_\infty}(\mathcal{C},\mathcal{A})(f,g) = [H^0 A^u_\infty(\mathcal{C},\mathcal{A})](f,g) = H^0(A^u_\infty(\mathcal{C},\mathcal{A})(f,g), m_1)$.

A zero object of a category $\mathcal{E}$ is an object $Z$, which is simultaneously initial and terminal. For a linear (*Ab*-enriched, not necessarily additive) category $\mathcal{E}$ this can be formulated as follows: $\mathcal{E}(Z,X) = 0$ and $\mathcal{E}(X,Z) = 0$ for any object $X$ of $\mathcal{E}$. This condition is equivalent to the equation $1_Z = 0 \in \mathcal{E}(Z,Z)$.

1.4. Corollary. *The unital $A_\infty$-functor $e : \mathcal{C} \to \mathcal{D}$ from the main theorem has the following property: composition with e in the sequence of functors*

$$\overline{A^u_\infty}(\mathcal{D},\mathcal{A}) \xrightarrow{e\bullet} \overline{A^u_\infty}(\mathcal{C},\mathcal{A}) \longrightarrow \overline{A^u_\infty}(\mathcal{B},\mathcal{A})$$

*is an equivalence of the category $\overline{A^u_\infty}(\mathcal{D},\mathcal{A})$ with the full subcategory*

$$\operatorname{Ker}(\overline{A^u_\infty}(\mathcal{C},\mathcal{A}) \to \overline{A^u_\infty}(\mathcal{B},\mathcal{A})) = H^0(A^u_\infty(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}}, m_1) \subset \overline{A^u_\infty}(\mathcal{C},\mathcal{A}),$$

*consisting of unital $A_\infty$-functors $f : \mathcal{C} \to \mathcal{A}$ such that the restriction $f\big|_{\mathcal{B}} : \mathcal{B} \to \mathcal{A}$ is a zero object of $\overline{A^u_\infty}(\mathcal{B},\mathcal{A})$.*

PROOF. A unital $A_\infty$-functor $g : \mathcal{B} \to \mathcal{A}$ is contractible if and only if $g\mathbf{i}^{\mathcal{A}} \equiv 0 : g \to g : \mathcal{B} \to \mathcal{A}$, that is, $1_g = 0 \in \overline{A^u_\infty}(\mathcal{B},\mathcal{A})(g,g)$. Thus, for unital $g$ contractibility is equivalent to $g$ being a zero object of $\overline{A^u_\infty}(\mathcal{B},\mathcal{A})$. ∎

The main theorem asserts that the chain map $e_\bullet = s(e\boxtimes 1)M_{01}s^{-1} : A^u_\infty(\mathcal{D},\mathcal{A})(f,g) \to A^u_\infty(\mathcal{C},\mathcal{A})(ef,eg)$ is a homotopy isomorphism, while Corollary 1.4 claims only that it induces isomorphism in 0-th homology.

1.5. UNIQUENESS OF THE REPRESENTING $A_\infty$-CATEGORY. With each strict $A^u_\infty$-2-functor $F : A^u_\infty \to A^u_\infty$ is associated an ordinary strict 2-functor $\overline{F} : \overline{A^u_\infty} \to \overline{A^u_\infty}$, $\overline{F}\mathcal{A} = F\mathcal{A}$ [LM06, Section 3.2]. With a strict $A^u_\infty$-2-transformation $\lambda = (\lambda_{\mathcal{A}}) : F \to G : A^u_\infty \to A^u_\infty$ is associated an ordinary strict 2-transformation $\overline{\lambda} = (\overline{\lambda_{\mathcal{A}}}) : \overline{F} \to \overline{G} : \overline{A^u_\infty} \to \overline{A^u_\infty}$ in cohomology [*ibid*]. Assume that $\lambda$ is a natural $A^u_\infty$-2-equivalence. Since $\lambda_{\mathcal{A}} : F\mathcal{A} \to G\mathcal{A}$ are $A_\infty$-equivalences, the 1-morphisms $\overline{\lambda_{\mathcal{A}}} : \overline{F}\mathcal{A} \to \overline{G}\mathcal{A}$ are equivalences in the 2-category $\overline{A^u_\infty}$. Composing $\overline{\lambda}$ with the 0-th cohomology 2-functor $H^0 : \overline{A^u_\infty} \to \mathcal{C}at$, we get a 2-natural equivalence $H^0\overline{\lambda} : H^0\overline{F} \to H^0\overline{G} : \overline{A^u_\infty} \to \mathcal{C}at$, which consists of equivalences of ordinary categories $H^0(\lambda_{\mathcal{A}}) : H^0(F\mathcal{A}) \to H^0(G\mathcal{A})$. In particular, if $F = A^u_\infty(\mathcal{D},\_)$ for some unital $A_\infty$-category $\mathcal{D}$, then

$$H^0\overline{F} = H^0\overline{A^u_\infty(\mathcal{D},\_)} = \overline{A^u_\infty}(\mathcal{D},\_).$$

Indeed, both the categories $H^0\overline{A^u_\infty(\mathcal{D},\mathcal{A})}$ and $\overline{A^u_\infty}(\mathcal{D},\mathcal{A})$ have unital $A_\infty$-functors $\mathcal{D} \to \mathcal{A}$ as objects and equivalence classes of natural $A_\infty$-transformations as morphisms. If $\lambda : A^u_\infty(\mathcal{D},\_) \to G : A^u_\infty \to A^u_\infty$ is a natural $A^u_\infty$-2-equivalence ($G$ is *unitally representable* by $\mathcal{D}$), then $H^0(\lambda_{\mathcal{A}}) : \overline{A^u_\infty}(\mathcal{D},\mathcal{A}) \to H^0(G\mathcal{A}) : \overline{A^u_\infty} \to \mathcal{C}at$ is a 2-natural equivalence. Thus, $H^0\overline{G}$ is represented by $\mathcal{D}$ in the 2-category sense and $\mathcal{D}$ is unique up to an equivalence by Section C.18.

In particular, if $G = A^u_\infty(\mathcal{C},\_)_{\operatorname{mod}\mathcal{B}}$, then with each object $e$ of $G\mathcal{D} = A^u_\infty(\mathcal{C},\mathcal{D})_{\operatorname{mod}\mathcal{B}}$ is associated a strict $A^u_\infty$-2-transformation

$$\lambda = (e\boxtimes 1)M : A^u_\infty(\mathcal{D},\mathcal{A}) \to A^u_\infty(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}} : A^u_\infty \to A^u_\infty.$$

We have identified $H^0\overline{G}(\mathcal{A})$ with $\operatorname{Ker}(\overline{A^u_\infty}(\mathcal{C},\mathcal{A}) \to \overline{A^u_\infty}(\mathcal{B},\mathcal{A}))$ in Corollary 1.4. The strict 2-natural equivalence $H^0\overline{\lambda} : H^0\overline{F} \to H^0\overline{G} : \overline{A^u_\infty} \to \mathcal{C}at$ identifies with the strict 2-transformation

$${}^{H^0\overline{G}}\lambda^{\mathcal{D},e} : \overline{A^u_\infty}(\mathcal{D},\_) \to H^0\overline{G} : \overline{A^u_\infty} \to \mathcal{C}at$$

from Proposition C.12, since

$$(f : \mathcal{D} \to \mathcal{A}) \xmapsto{(e\boxtimes 1)M} ef = (e)(H^0\overline{G}(f)) \overset{\text{def}}{=}: (f)^{H^0\overline{G}}\lambda^{\mathcal{D},e},$$
$$(r : f \to g : \mathcal{D} \to \mathcal{A}) \xmapsto{(e\boxtimes 1)M} er = (e)(H^0\overline{G}(r)) \overset{\text{def}}{=}: (r)^{H^0\overline{G}}\lambda^{\mathcal{D},e}.$$

Therefore, the pair $(\mathcal{D},e)$ represents the strict 2-functor $H^0\overline{G} : \overline{A^u_\infty} \to \mathcal{C}at$ in the sense of Definition C.17.

1.6. Corollary. *The pair $(\mathcal{D}, e : \mathcal{C} \to \mathcal{D})$ is unique up to an equivalence, that is, for any other quotient $(\mathcal{D}', e' : \mathcal{C} \to \mathcal{D}')$ there exists an $A_\infty$-equivalence $\phi : \mathcal{D} \to \mathcal{D}'$ such that $e\phi$ is isomorphic to $e'$.*

The proof immediately follows from results of Section C.18.

The unital $A_\infty$-category $\mathcal{D}$ obtained in the main theorem can be replaced with a differential graded category by the $A_\infty$-version of Yoneda Lemma (Theorem A.7). We may restrict Corollary 1.4 to differential graded categories $\mathcal{B}$, $\mathcal{C}$, $\mathcal{A}$ for the same reason. Then it becomes parallel to the second half of main Theorem 1.6.2 of Drinfeld's work [Dri04], which asserts exactness of the sequence of categories

$$T(\mathcal{D}, \mathcal{A}) \to T(\mathcal{C}, \mathcal{A}) \to T(\mathcal{B}, \mathcal{A})$$

in the same sense as in Corollary 1.4. Here $T$ is a certain 2-category whose objects are differential graded categories. It is not known in general whether categories $T(\mathcal{C}, \mathcal{A})$ and $\overline{A^u_\infty}(\mathcal{C}, \mathcal{A})$ are equivalent. If $\Bbbk$ is a field, then, as B. Keller explained to us, equivalence of $\overline{A^u_\infty}(\mathcal{C}, \mathcal{A})$ and $T(\mathcal{C}, \mathcal{A})$ can be deduced from results of Lefèvre-Hasegawa [LH03, Section 8.2].

1.7. Basic properties of the main construction. The proof of universality of $\mathcal{D} = \mathsf{Q}(\mathcal{C}|\mathcal{B})$ is based on the fact that $\mathcal{D}$ is *relatively free* over $\mathcal{C}$, that is, it admits a filtration

$$\mathcal{C} = \mathcal{D}_0 \subset \mathcal{Q}_1 \subset \mathcal{D}_1 \subset \mathcal{Q}_2 \subset \mathcal{D}_2 \subset \mathcal{Q}_3 \subset \dots \subset \mathcal{D}$$

by $A_\infty$-subcategories $\mathcal{D}_j$ and differential graded subquivers $\mathcal{Q}_j$, such that the graded subquiver $\mathcal{D}_j \subset \mathcal{Q}_{j+1}$ has a direct complement $\mathcal{N}_{j+1}$ (a graded subquiver of $\mathcal{Q}_{j+1}$), and such that $\mathcal{D}_{j+1}$ is generated by $\mathcal{N}_{j+1}$ over $\mathcal{D}_j$. The precise conditions are given in Definition 5.1 (see also Proposition 2.2). This filtration allows to write down a sequence of restriction $A_\infty$-functors

$$\begin{aligned} A^u_\infty(\mathcal{C}, \mathcal{A})_{\operatorname{mod}\mathcal{B}} \longleftarrow A^{\psi u}_{\infty 1}(\mathcal{D}_0, \mathcal{Q}_1; \mathcal{A}) \longleftarrow A^{\psi u}_\infty(\mathcal{D}_1, \mathcal{A}) \\ \longleftarrow A^{\psi u}_{\infty 1}(\mathcal{D}_1, \mathcal{Q}_2; \mathcal{A}) \longleftarrow A^{\psi u}_\infty(\mathcal{D}_2, \mathcal{A}) \longleftarrow A^{\psi u}_{\infty 1}(\mathcal{D}_2, \mathcal{Q}_3; \mathcal{A}) \longleftarrow \dots \end{aligned} \tag{41}$$

and to prove that each of these $A_\infty$-functors is an equivalence, surjective on objects (Theorem 4.7, Propositions 5.7, 5.10). The category $A^{\psi u}_{\infty 1}(\mathcal{D}_j, \mathcal{Q}_{j+1}; \mathcal{A})$ is defined via pullback square (14)

$$\begin{tikzcd} A^{\psi u}_{\infty 1}(\mathcal{D}_j, \mathcal{Q}_{j+1}; \mathcal{A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A_1(\mathcal{Q}_{j+1}, \mathcal{A}) \arrow[d] \\ A^{\psi u}_\infty(\mathcal{D}_j, \mathcal{A}) \arrow[r] & A_1(\mathcal{D}_j, \mathcal{A}) \end{tikzcd}$$

The $A_\infty$-categories $\mathcal{D}_j$ are not unital, but only pseudounital – there are distinguished cycles $\mathbf{i}^{\mathcal{C}}_0 \in (s\mathcal{D}_j)^{-1}$, which are not unit elements of $\mathcal{D}_j$ if $j > 0$. The index $\psi u$ in $A^{\psi u}_\infty$ indicates that we consider pseudounital $A_\infty$-functors – a generalization of unital ones (Definition 4.1). Their first components preserve the distinguished cycles up to a boundary. The $A_\infty$-equivalence $A_\infty^{\psi u}(\mathcal{D},\mathcal{A}) \to A_\infty^u(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}}$ is the limit case of (41) (Theorem 5.13).

The proof of unitality of $\mathcal{D} = \mathsf{Q}(\mathcal{C}|\mathcal{B})$ for strictly unital $\mathcal{C}$ is based on the study of the multicategory of $A_\infty$-operations and contracting homotopies operating in $\mathcal{D}$ (Theorem 6.5).

1.8. Description of various results. The proof of Theorem 4.7 is based on description of chain maps $P \to sA_\infty(\mathcal{F}\mathcal{Q},\mathcal{A})(\phi,\psi)$ to the complex of $(\phi,\psi)$-coderivations (Proposition 2.7), where $\mathcal{F}\mathcal{Q}$ is the free $A_\infty$-category, generated by a differential graded quiver $\mathcal{Q}$. A similar result for the quotient $\mathcal{F}\mathcal{Q}/s^{-1}I$ over an $A_\infty$-ideal $I$ is given in Proposition 2.10. Of course, any $A_\infty$-category is a quotient of a free one (Proposition 3.2). We also describe homotopies between chain maps $P \to sA_\infty(\mathcal{F}\mathcal{Q},\mathcal{A})(\phi,\psi)$ (Corollary 2.8), and generalize the result to quotients $\mathcal{F}\mathcal{Q}/s^{-1}I$ (Corollary 2.11).

In Section 8 we consider the example of differential graded category $\mathcal{C} = \underline{\mathsf{C}}(\mathcal{A})$ of complexes in a $\Bbbk$-linear Abelian category $\mathcal{A}$, and the full subcategory $\mathcal{B} \subset \mathcal{C}$ of acyclic complexes. The functor $H^0e$ factors through a functor $g : D(\mathcal{A}) \to H^0(\mathsf{Q}(\mathcal{C}|\mathcal{B}))$ by Corollary 8.2. It is an equivalence, when $\Bbbk$ is a field.

In Appendix A we define, following Fukaya [Fuk02, Lemma 9.8], the Yoneda $A_\infty$-functor $Y : \mathcal{A}^{\mathrm{op}} \to A_\infty(\mathcal{A},\underline{\mathsf{C}}_\Bbbk)$, where $\underline{\mathsf{C}}_\Bbbk$ is the differential graded category of complexes of $\Bbbk$-modules. We prove for an arbitrary unital $A_\infty$-category $\mathcal{A}$ that the Yoneda $A_\infty$-functor $Y$ is an equivalence of $\mathcal{A}^{\mathrm{op}}$ with its image – full differential graded subcategory of $A_\infty(\mathcal{A},\underline{\mathsf{C}}_\Bbbk)$ (Theorem A.7). This is already proven by Fukaya in the case of strictly unital $A_\infty$-category $\mathcal{A}$ [Fuk02, Theorem 9.1]. As a corollary we deduce that any $\mathscr{U}$-small unital $A_\infty$-category $\mathcal{A}$ is $A_\infty$-equivalent to a $\mathscr{U}$-small differential graded category (Corollary A.8).

In Appendix C we lift the classical Yoneda Lemma one dimension up – to strict 2-categories, weak 2-functors and weak 2-transformations. In the completely strict set-up such lifting can be obtained via enriched category theory, namely, that of $\mathcal{C}at$-categories, see Street and Walters [SW78], Kelly [Kel82]. The present weak generalization admits a direct proof.

An important result from another paper is recalled in simplified form, in which it is used in the present paper:

1.9. Corollary. [to Theorem 8.8 [Lyu03]] *Let $\mathcal{C}$ be an $A_\infty$-category and let $\mathcal{B}$ be a unital $A_\infty$-category. Let $\phi : \mathcal{C} \to \mathcal{B}$ be an $A_\infty$-functor such that for all objects $X$, $Y$ of $\mathcal{C}$ the chain map $\phi_1 : (s\mathcal{C}(X,Y),b_1) \to (s\mathcal{B}(X\phi,Y\phi),b_1)$ is homotopy invertible. If $\operatorname{Ob}\phi : \operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{B}$ is surjective, then $\mathcal{C}$ is unital and $\phi$ is an $A_\infty$-equivalence.*

Proof. Let $h : \operatorname{Ob}\mathcal{B} \to \operatorname{Ob}\mathcal{C}$ be an arbitrary mapping such that $h \cdot \operatorname{Ob}\phi = \mathrm{id}_{\operatorname{Ob}\mathcal{B}}$. The remaining data required in Theorem 8.8 of [Lyu03] can be chosen as ${}_Ur_0 = {}_Up_0 = {}_U\mathbf{i}_0^{\mathcal{B}} : \Bbbk \to (s\mathcal{B})^{-1}(U,U)$ for all objects $U$ of $\mathcal{B}$. We conclude by this theorem that there exists an $A_\infty$-functor $\psi : \mathcal{B} \to \mathcal{C}$ with $\operatorname{Ob}\psi = h$, quasi-inverse to $\phi$. ∎

Logical dependence of sections is the following. Appendices A and C do not depend on other sections. Appendix B depends on Appendix A. Sections 2–8 depend on appendices and on sections with smaller number. Dependence on the first section means dependence on Corollary 1.9 and on overall notations and conventions. Being a summary, the first section depends on all the rest of the article.

1.10. Conventions and preliminaries. We keep the notations and conventions of [Lyu03, LO06, LM06], sometimes without explicit mentioning. Some of the conventions are recalled here.

We assume that most quivers, $A_\infty$-categories, etc. are small with respect to some universe $\mathscr{U}$. It means that the set of objects and the set of morphisms are $\mathscr{U}$-small, that is, isomorphic as sets to an element of $\mathscr{U}$ [GV73, Section 1.0]. The universe $\mathscr{U}$ is supposed to be an element of a universe $\mathscr{U}'$, which in its turn is an element of a universe $\mathscr{U}''$, and so on. All sets are supposed to be in bijection with some elements of some of the universes. Some differential graded categories in this paper will be $\mathscr{U}'$-small $\mathscr{U}$-categories. A category $\mathcal{C}$ is a $\mathscr{U}$-category if all its sets of morphisms $\mathcal{C}(X,Y)$ are $\mathscr{U}$-small [GV73, Definition 1.1].

The $\mathscr{U}$-small ground ring $\Bbbk$ is a unital associative commutative ring. A $\Bbbk$-module means a $\mathscr{U}$-small $\Bbbk$-module.

We use the right operators: the composition of two maps (or morphisms) $f : X \to Y$ and $g : Y \to Z$ is denoted $fg : X \to Z$; a map is written on elements as $f : x \mapsto xf = (x)f$. However, these conventions are not used systematically, and $f(x)$ might be used instead.

The set of non-negative integers is denoted $\mathbb{N} = \mathbb{Z}_{\geqslant 0}$.

We consider only such $A_\infty$-categories $\mathcal{C}$ that the differential $b : Ts\mathcal{C} \to Ts\mathcal{C}$ vanishes on $T^0s\mathcal{C}$, that is, $b_0 = 0$. We consider only those $A_\infty$-functors $f : \mathcal{A} \to \mathcal{B}$, whose 0-th component $f_0$ vanishes.

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## 2. Quotients of free $A_\infty$-categories

2.1. Definition. *Let $\mathcal{C}$ be an $A_\infty$-category, and let $I \subset s\mathcal{C}$ be a graded subquiver with $\operatorname{Ob} I = \operatorname{Ob} \mathcal{C}$. The subquiver $I$ is called an $A_\infty$-*ideal *of $\mathcal{C}$ if*

$$\operatorname{Im}\bigl(b_{\alpha+1+\beta} : (s\mathcal{C})^{\otimes\alpha} \otimes I \otimes (s\mathcal{C})^{\otimes\beta} \to s\mathcal{C}\bigr) \subset I$$

*for all* $\alpha$, $\beta \geqslant 0$.

If $I \subset s\mathcal{C}$ is an $A_\infty$-ideal of an $A_\infty$-category $\mathcal{C}$, then the quotient graded quiver $\mathcal{E} = \mathcal{C}/s^{-1}I$ with $\operatorname{Ob}\mathcal{E} = \operatorname{Ob}\mathcal{C}$, $\mathcal{E}(X,Y) = \mathcal{C}(X,Y)/s^{-1}I(X,Y)$ has a unique $A_\infty$-category structure such that the natural projection $\pi_1 : s\mathcal{C} \to s\mathcal{E} = s\mathcal{C}/I$ determines a strict $A_\infty$-functor $\pi : \mathcal{C} \to \mathcal{E}$. Multiplications $b_k^{\mathcal{E}}$ in $\mathcal{E}$ are well-defined for $k \geqslant 1$ by the equation

$$\begin{aligned}&\bigl[(s\mathcal{C})^{\otimes k} \xrightarrow{b_k^{\mathcal{C}}} s\mathcal{C} \xrightarrow{\pi_1} s\mathcal{C}/I\bigr]\\ &\qquad = \Bigl[(s\mathcal{C})^{\otimes k} \longrightarrow (s\mathcal{C})^{\otimes k}\Big/ \sum_{\alpha+1+\beta=k} (s\mathcal{C})^{\otimes\alpha} \otimes I \otimes (s\mathcal{C})^{\otimes\beta} \simeq (s\mathcal{C}/I)^{\otimes k} \xrightarrow{b_k^{\mathcal{E}}} s\mathcal{C}/I\Bigr].\end{aligned}$$

Let $\mathcal{Q}$ be a differential graded $\Bbbk$-quiver. There is a free $A_\infty$-category $\mathcal{FQ}$, generated by $\mathcal{Q}$ [LM06, Section 2.1]. Let $R \subset s\mathcal{FQ}$ be a graded subquiver. Denote by $I = (R) \subset s\mathcal{FQ}$ the graded subquiver spanned by multiplying elements of $R$ with some elements of $s\mathcal{FQ}$ via several operations $b_k^{\mathcal{FQ}}$, $k > 1$. It can be described as follows. Let $t \in \mathcal{T}^n_{\geqslant 2}$ be a plane rooted tree with $i(t) = n$ input leaves. Each decomposition

$$(t, \leqslant) = (1^{\sqcup\alpha_1} \sqcup \mathfrak{t}_{k_1} \sqcup 1^{\sqcup\beta_1}) \cdot (1^{\sqcup\alpha_2} \sqcup \mathfrak{t}_{k_2} \sqcup 1^{\sqcup\beta_2}) \cdot \dots \cdot \mathfrak{t}_{k_N}, \qquad (1)$$

of $t$ into the product of elementary forests gives a linear ordering $\leqslant$ of $t$. Here $\alpha_1+k_1+\beta_1 = n$ and $N = |t|$ is the number of internal vertices of $t$. An operation

$$b^{\mathcal{FQ}}_{(t,\leqslant)} = \bigl(s\mathcal{FQ}^{\otimes n} \xrightarrow{1^{\otimes\alpha_1}\otimes b^{\mathcal{FQ}}_{k_1}\otimes 1^{\otimes\beta_1}} s\mathcal{FQ}^{\otimes\alpha_1+1+\beta_1} \xrightarrow{1^{\otimes\alpha_2}\otimes b^{\mathcal{FQ}}_{k_2}\otimes 1^{\otimes\beta_2}} \dots \xrightarrow{b^{\mathcal{FQ}}_{k_N}} s\mathcal{FQ}\bigr) \qquad (2)$$

is associated with the linearly ordered tree $(t, \leqslant)$. Different choices of the ordering of $t$ change only the sign of the above map. In particular, one may consider the canonical linear ordering $t_<$ of $t$ [LM06, Section 1.7] and the corresponding map $b^{\mathcal{FQ}}_{t_<}$. So the subquiver $I \subset s\mathcal{FQ}$ is defined as

$$I = (R) = \sum_{t\in\mathcal{T}^{\alpha+1+\beta}_{\geqslant 2}} \operatorname{Im}(b^{\mathcal{FQ}}_{t_<} : s\mathcal{Q}^{\otimes\alpha} \otimes R \otimes s\mathcal{Q}^{\otimes\beta} \to s\mathcal{FQ}),$$

where the summation goes over all $\alpha, \beta \in \mathbb{Z}_{\geqslant 0}$ and all trees with $\alpha + 1 + \beta$ input leaves.

2.2. Proposition. *Let $R \subset s\mathcal{FQ}$ be a graded subquiver such that $Rb_1^{\mathcal{FQ}} \subset (R) = I$. Then $Ib_1^{\mathcal{FQ}} \subset I$, $I$ is an $A_\infty$-ideal of $\mathcal{FQ}$, and $\mathcal{E} = \mathcal{FQ}/s^{-1}I$ is an $A_\infty$-category.*

Proof. Clearly, $I$ is closed under multiplications $b_k$, $k > 1$, with elements of $s\mathcal{FQ}$.

Let us prove that for all $t \in \mathcal{T}_{\geqslant 2}$

$$\operatorname{Im}(b^{\mathcal{FQ}}_{t_<} : s\mathcal{Q}^{\otimes\alpha} \otimes R \otimes s\mathcal{Q}^{\otimes\beta} \to s\mathcal{FQ})b_1^{\mathcal{FQ}} \subset I \qquad (3)$$

using induction on $|t|$. This holds for $|t| = 0$, $t = \big|$ by assumption. Let $|t| = N > 0$ and assume that (3) holds for all $t' \in \mathcal{T}_{\geqslant 2}$ with $|t'| < N$. The tree $t$ can be presented as $t = (t_1 \sqcup \dots \sqcup t_k)\mathfrak{t}_k$ for some $k > 1$. We have $b^{\mathcal{FQ}}_{t_<} = \pm(b^{\mathcal{FQ}}_{t_1} \otimes \dots \otimes b^{\mathcal{FQ}}_{t_k})b^{\mathcal{FQ}}_k$, $|t_i| < N$ and

$$b_k^{\mathcal{FQ}} b_1^{\mathcal{FQ}} = -\sum_{a+q+c=k}^{a+c>0} (1^{\otimes a} \otimes b_q^{\mathcal{FQ}} \otimes 1^{\otimes c}) b^{\mathcal{FQ}}_{a+1+c}.$$

One of the $a+1+c$ factors of

$$(s\mathcal{Q}^{\otimes\alpha}\otimes R\otimes s\mathcal{Q}^{\otimes\beta})(b^{\mathcal{FQ}}_{t_1}\otimes\dots\otimes b^{\mathcal{FQ}}_{t_k})(1^{\otimes a}\otimes b^{\mathcal{FQ}}_q\otimes 1^{\otimes c}) \tag{4}$$

is contained in $I$ (for $q=1$ this is the induction assumption). Hence,

$$(s\mathcal{Q}^{\otimes\alpha}\otimes R\otimes s\mathcal{Q}^{\otimes\beta})(b^{\mathcal{FQ}}_{t_1}\otimes\dots\otimes b^{\mathcal{FQ}}_{t_k})(1^{\otimes a}\otimes b^{\mathcal{FQ}}_q\otimes 1^{\otimes c})b^{\mathcal{FQ}}_{a+1+c}\subset I,$$

and the inclusion $Ib^{\mathcal{FQ}}_1\subset I$ follows by induction.

Therefore, $I$ is stable under all $b^{\mathcal{FQ}}_k$, $k\geqslant 1$, so it is an $A_\infty$-ideal, and $\mathcal{E}=\mathcal{FQ}/s^{-1}I$ is an $A_\infty$-category. ■

2.3. $A_\infty$-FUNCTORS FROM A QUOTIENT OF A FREE $A_\infty$-CATEGORY. The following statement is Proposition 2.3 from [LM06].

2.4. PROPOSITION. *Let $\mathcal{Q}$ be a differential graded quiver, and let $\mathcal{A}$ be an $A_\infty$-category. $A_\infty$-functors $f:\mathcal{FQ}\to\mathcal{A}$ are in bijection with sequences $(f'_1,f_k)_{k>1}$, where $f'_1:s\mathcal{Q}\to(s\mathcal{A},b_1)$ is a chain morphism of differential graded quivers with the underlying mapping of objects $\operatorname{Ob}f:\operatorname{Ob}\mathcal{Q}\to\operatorname{Ob}\mathcal{A}$ and $f_k:T^ks\mathcal{FQ}\to s\mathcal{A}$ are $\Bbbk$-quiver morphisms of degree 0 with the same underlying map $\operatorname{Ob}f$ for all $k>1$. The morphisms $f_k$ are components of $f$ for $k>1$. The component $f_1:s\mathcal{FQ}\to s\mathcal{A}$ is an extension of $f'_1$.*

In fact, it is shown in [LM06, Proposition 2.3] that such a sequence $(f'_1,f_k)_{k>1}$ extends to a sequence of components of an $A_\infty$-functor $(f_1,f_k)_{k>1}$ in a unique way.

We are going to extend this description to quotients of free $A_\infty$-categories. Let a graded subquiver $R\subset s\mathcal{FQ}$ satisfy the assumption $Rb^{\mathcal{FQ}}_1\subset(R)=I$. Denote by $\pi$ : $\mathcal{FQ}\longrightarrow\mathcal{E}=\mathcal{FQ}/s^{-1}I$ the natural projection strict $A_\infty$-functor.

2.5. PROPOSITION. *An $A_\infty$-functor $f:\mathcal{FQ}\to\mathcal{A}$ factorizes as $f=\bigl(\mathcal{FQ}\xrightarrow{\pi}\mathcal{E}\xrightarrow{\tilde f}\mathcal{A}\bigr)$ for some (unique) $A_\infty$-functor $\tilde f:\mathcal{E}\to\mathcal{A}$ if and only if the following two conditions are satisfied:*

1. *$Rf_1=0$;*
2. *$(s\mathcal{FQ}^{\otimes\alpha}\otimes I\otimes s\mathcal{FQ}^{\otimes\beta})f_{\alpha+1+\beta}=0$ for all $\alpha,\beta\in\mathbb{Z}_{\geqslant 0}$ such that $\alpha+\beta>0$.*

PROOF. If $f=\pi\tilde f$, then $f_k=\pi_1^{\otimes k}\tilde f_k$ and the above conditions are necessary.

Assume that the both conditions are satisfied. Let us prove that $If_1=0$. We are going to prove that for all $t\in\mathcal{T}_{\geqslant 2}$

$$\bigl(s\mathcal{Q}^{\otimes\alpha}\otimes R\otimes s\mathcal{Q}^{\otimes\beta}\xrightarrow{b^{\mathcal{FQ}}_{t_<}}s\mathcal{FQ}\xrightarrow{f_1}s\mathcal{A}\bigr)=0 \tag{5}$$

by induction on $|t|$. This holds for $|t|=0$, $t=\bigm|$ by assumption. Let $|t|=N>0$ and assume that (5) holds for all $t'\in\mathcal{T}_{\geqslant 2}$ with $|t'|<N$. The tree $t$ can be presented as $t=(t_1\sqcup\dots\sqcup t_k)\mathfrak{t}_k$ for some $k>1$. We have $b^{\mathcal{FQ}}_{t_<}=\pm(b^{\mathcal{FQ}}_{t_1}\otimes\dots\otimes b^{\mathcal{FQ}}_{t_k})b^{\mathcal{FQ}}_k$, $|t_i|<N$ and

$$b^{\mathcal{FQ}}_kf_1=\sum_{i_1+\dots+i_l=k}(f_{i_1}\otimes\dots\otimes f_{i_l})b^{\mathcal{A}}_l-\sum_{a+q+c=n}^{a+c>0}(1^{\otimes a}\otimes b^{\mathcal{FQ}}_q\otimes 1^{\otimes c})f_{a+1+c}. \tag{6}$$

One of the $a+1+c$ factors of (4) is contained in $I$ and also one of the $l$ factors of

$$(s\mathcal{Q}^{\otimes\alpha}\otimes R\otimes s\mathcal{Q}^{\otimes\beta})(b^{\mathcal{FQ}}_{t_1}\otimes\dots\otimes b^{\mathcal{FQ}}_{t_k})(f_{i_1}\otimes\dots\otimes f_{i_l})$$

is contained in $I$ (for $i_j=1$ this is the induction assumption). Hence, the right hand side of (6) is contained in $I$, and $If_1\subset I$ follows by induction. Therefore,

$$(s\mathcal{FQ}^{\otimes\alpha}\otimes I\otimes s\mathcal{FQ}^{\otimes\beta})f\subset\sum_{i_1+\dots+i_l=\alpha+1+\beta}(s\mathcal{FQ}^{\otimes\alpha}\otimes I\otimes s\mathcal{FQ}^{\otimes\beta})(f_{i_1}\otimes\dots\otimes f_{i_l})=0$$

for all $\alpha,\beta\geqslant 0$. Clearly, $f$ factorizes as $f=\pi\tilde{f}$. ■

2.6. Transformations from a free $A_\infty$-category. The following statement is Proposition 2.8 from [LM06].

2.7. Proposition. *Let $\phi,\psi:\mathcal{FQ}\to\mathcal{A}$ be $A_\infty$-functors. For an arbitrary complex $P$ of $\Bbbk$-modules chain maps $u:P\to sA_\infty(\mathcal{FQ},\mathcal{A})(\phi,\psi)$ are in bijection with the following data: $(u',u_k)_{k>1}$*

1. *a chain map $u':P\to sA_1(\mathcal{Q},\mathcal{A})(\phi,\psi)$,*
2. *$\Bbbk$-linear maps*

$$u_k:P\to\prod_{X,Y\in\operatorname{Ob}\mathcal{Q}}\underline{\mathsf{C}}_\Bbbk\big((s\mathcal{FQ})^{\otimes k}(X,Y),s\mathcal{A}(X\phi,Y\psi)\big)$$

*of degree 0 for all $k>1$.*

*The bijection maps $u$ to $u_k=u\cdot\mathrm{pr}_k$,*

$$u'=\big(P\xrightarrow{u}sA_\infty(\mathcal{FQ},\mathcal{A})(\phi,\psi)\xrightarrow{\mathrm{restr}_{\leqslant 1}}sA_1(\mathcal{FQ},\mathcal{A})(\phi,\psi)\xrightarrow{\mathrm{restr}}sA_1(\mathcal{Q},\mathcal{A})(\phi,\psi)\big).\qquad(7)$$

*The inverse bijection can be recovered from the recurrent formula*

$$\begin{aligned}(-)^p b^{\mathcal{FQ}}_k(pu_1)=-(pd)u_k+\sum_{a+q+c=k}^{\alpha,\beta}(\phi_{a\alpha}\otimes pu_q\otimes\psi_{c\beta})b^{\mathcal{A}}_{\alpha+1+\beta}\\ -(-)^p\sum_{\alpha+q+\beta=k}^{\alpha+\beta>0}(1^{\otimes\alpha}\otimes b^{\mathcal{FQ}}_q\otimes 1^{\otimes\beta})(pu_{\alpha+1+\beta}):(s\mathcal{FQ})^{\otimes k}\to s\mathcal{A},\end{aligned}\qquad(8)$$

*where $k>1$, $p\in P$, and $\phi_{a\alpha}$, $\psi_{c\beta}$ are matrix elements of $\phi$, $\psi$.*

The following statement is Corollary 2.10 from [LM06].

2.8. Corollary. *Let $\phi, \psi : \mathcal{FQ} \to \mathcal{A}$ be $A_\infty$-functors. Let $P$ be a complex of $\Bbbk$-modules. Let $w : P \to sA_\infty(\mathcal{FQ}, \mathcal{A})(\phi, \psi)$ be a chain map. The set (possibly empty) of homotopies $h : P \to sA_\infty(\mathcal{FQ}, \mathcal{A})(\phi, \psi)$, $\deg h = -1$, such that $w = dh + hB_1$ is in bijection with the set of data $(h', h_k)_{k>1}$, consisting of*

1. *a homotopy $h' : P \to sA_1(\mathcal{Q}, \mathcal{A})(\phi, \psi)$, $\deg h' = -1$, such that $dh' + h'B_1 = w'$, where*

$$w' = \bigl(P \xrightarrow{w} sA_\infty(\mathcal{FQ}, \mathcal{A})(\phi, \psi) \xrightarrow{\mathrm{restr}_{\leqslant 1}} sA_1(\mathcal{FQ}, \mathcal{A})(\phi, \psi) \xrightarrow{\mathrm{restr}} sA_1(\mathcal{Q}, \mathcal{A})(\phi, \psi)\bigr);$$

2. *$\Bbbk$-linear maps*

$$h_k : P \to \prod_{X,Y \in \operatorname{Ob}\mathcal{Q}} \underline{\mathsf{C}}_\Bbbk\bigl((s\mathcal{FQ})^{\otimes k}(X, Y), s\mathcal{A}(X\phi, Y\psi)\bigr)$$

*of degree $-1$ for all $k > 1$.*

*The bijection maps $h$ to $h_k = h \cdot \mathrm{pr}_k$,*

$$h' = \bigl(P \xrightarrow{h} sA_\infty(\mathcal{FQ}, \mathcal{A})(\phi, \psi) \xrightarrow{\mathrm{restr}_{\leqslant 1}} sA_1(\mathcal{FQ}, \mathcal{A})(\phi, \psi) \xrightarrow{\mathrm{restr}} sA_1(\mathcal{Q}, \mathcal{A})(\phi, \psi)\bigr). \quad (9)$$

*The inverse bijection can be recovered from the recurrent formula*

$$(-)^p b_k(ph_1) = pw_k - (pd)h_k - \sum_{a+q+c=k}^{\alpha,\beta} (\phi_{a\alpha} \otimes ph_q \otimes \psi_{c\beta}) b_{\alpha+1+\beta} - (-)^p \sum_{a+q+c=k}^{a+c>0} (1^{\otimes a} \otimes b_q \otimes 1^{\otimes c})(ph_{a+1+c}) : (s\mathcal{FQ})^{\otimes k} \to s\mathcal{A}, \quad (10)$$

*where $k > 1$, $p \in P$, and $\phi_{a\alpha}$, $\psi_{c\beta}$ are matrix elements of $\phi$, $\psi$.*

2.9. Transformations from a quotient of a free $A_\infty$-category. We are going to extend the above description to quotients of free $A_\infty$-categories. Assume that a graded subquiver $R \subset s\mathcal{FQ}$ satisfies $Rb_1^{\mathcal{FQ}} \subset (R) = I$. Let $\mathcal{A}$ be an $A_\infty$-category. Composition with the projection $A_\infty$-functor $\pi : \mathcal{FQ} \to \mathcal{E} = \mathcal{FQ}/s^{-1}I$ gives a strict $A_\infty$-functor $L^\pi = (\pi \boxtimes 1)M : A_\infty(\mathcal{E}, \mathcal{A}) \to A_\infty(\mathcal{FQ}, \mathcal{A})$. It is injective on objects and morphisms, that is, both maps $\operatorname{Ob} L^\pi : \phi \mapsto \pi\phi$ and $L_1^\pi : sA_\infty(\mathcal{E}, \mathcal{A})(\phi, \psi) \to sA_\infty(\mathcal{FQ}, \mathcal{A})(\pi\phi, \pi\psi)$, $r \mapsto \pi r$ are injective. We are going to characterize the subcomplex $sA_\infty(\mathcal{E}, \mathcal{A})(\phi, \psi) \hookrightarrow sA_\infty(\mathcal{FQ}, \mathcal{A})(\pi\phi, \pi\psi)$.

2.10. Proposition. *Let $P$ be a complex of $\Bbbk$-modules, and let $u : P \to sA_\infty(\mathcal{FQ}, \mathcal{A})(\pi\phi, \pi\psi)$ be a chain map. Denote*

$$u_k = u \cdot \mathrm{pr}_k : P \to \prod_{X,Y \in \operatorname{Ob}\mathcal{Q}} \underline{\mathsf{C}}_\Bbbk\bigl((s\mathcal{FQ})^{\otimes k}(X, Y), s\mathcal{A}(X\phi, Y\psi)\bigr), \qquad k \geqslant 0.$$

*Then the image of $u$ is contained in the subcomplex $sA_\infty(\mathcal{E}, \mathcal{A})(\phi, \psi)$ if and only if the following two conditions are satisfied:*

1. *$R(pu_1) = \operatorname{Im}(pu_1 : R(X,Y) \to s\mathcal{A}(X\phi, Y\psi)) = 0$ for all $p \in P$;*
2. *$(s\mathcal{F}\mathcal{Q}^{\otimes\alpha} \otimes I \otimes s\mathcal{F}\mathcal{Q}^{\otimes\beta})(pu_k) = 0$ for all $p \in P$ and all $\alpha, \beta \geqslant 0$ such that $\alpha + 1 + \beta = k > 1$.*

Proof. The conditions are obviously necessary. Let us prove that they are sufficient. First of all, we are going to show that $I(pu_1) = 0$. Namely, we claim that

$$(s\mathcal{Q}^{\otimes\alpha} \otimes R \otimes s\mathcal{Q}^{\otimes\beta}) b_{t_<}^{\mathcal{F}\mathcal{Q}}(pu_1) = 0 \tag{11}$$

for all trees $t \in \mathcal{T}_{\geqslant 2}^{\alpha+1+\beta}$. We prove it by induction on $|t|$. For $|t| = 0$, $t = \big|$ this holds by assumption 1. Let $t \in \mathcal{T}_{\geqslant 2}$ be a tree with $|t| = N > 0$ internal vertices. Assume that (11) holds for all $t' \in \mathcal{T}_{\geqslant 2}$ with $|t'| < N$. The tree $t$ can be presented as $t = (t_1 \sqcup \dots \sqcup t_k)\mathfrak{t}_k$ for some $k > 1$, so $b_{t_<}^{\mathcal{F}\mathcal{Q}} = \pm(b_{t_1}^{\mathcal{F}\mathcal{Q}} \otimes \dots \otimes b_{t_k}^{\mathcal{F}\mathcal{Q}}) b_k^{\mathcal{F}\mathcal{Q}}$, and $|t_i| < N$. Formula (8) in the form

$$\begin{aligned}(-)^p b_k^{\mathcal{F}\mathcal{Q}}(pu_1) = -(pd)u_k &+ \sum_{a+q+c=k}^{m,n} (\pi_1^{\otimes a}\phi_{am} \otimes pu_q \otimes \pi_1^{\otimes c}\psi_{cn}) b_{m+1+n}^{\mathcal{A}} \\ &- (-)^p \sum_{a+q+c=k}^{a+c>0} (1^{\otimes a} \otimes b_q^{\mathcal{F}\mathcal{Q}} \otimes 1^{\otimes c})(pu_{a+1+c}) : s\mathcal{F}_{t_1}\mathcal{Q} \otimes \dots \otimes s\mathcal{F}_{t_k}\mathcal{Q} \to s\mathcal{A}\end{aligned} \tag{12}$$

implies that

$$\begin{aligned}(s\mathcal{Q}^{\otimes\alpha} \otimes R \otimes s\mathcal{Q}^{\otimes\beta}) b_{t_<}^{\mathcal{F}\mathcal{Q}}(pu_1) &= (s\mathcal{Q}^{\otimes\alpha} \otimes R \otimes s\mathcal{Q}^{\otimes\beta})(b_{t_1}^{\mathcal{F}\mathcal{Q}} \otimes \dots \otimes b_{t_k}^{\mathcal{F}\mathcal{Q}}) b_k^{\mathcal{F}\mathcal{Q}}(pu_1) \\ &\subset (s\mathcal{F}\mathcal{Q}^{\otimes\gamma} \otimes I \otimes s\mathcal{F}\mathcal{Q}^{\otimes\delta})[(pd)u_k] \\ &+ \sum (s\mathcal{Q}^{\otimes\alpha} \otimes R \otimes s\mathcal{Q}^{\otimes\beta})(b_{t_1}^{\mathcal{F}\mathcal{Q}} \otimes \dots \otimes b_{t_k}^{\mathcal{F}\mathcal{Q}})(\pi_1^{\otimes a}\phi_{am} \otimes pu_q \otimes \pi_1^{\otimes c}\psi_{cn}) b_{m+1+n}^{\mathcal{A}} \\ &+ \sum_{\lambda+\mu>0} (s\mathcal{F}\mathcal{Q}^{\otimes\lambda} \otimes I \otimes s\mathcal{F}\mathcal{Q}^{\otimes\mu})(pu_{\lambda+1+\mu}) = 0.\end{aligned}$$

Indeed, summands with $q = 0$ vanish due to $I\pi_1 = 0$, summands with $q = 1$ vanish due to induction assumption (11), and other summands vanish due to condition 2.

Thus condition 2 holds not only for $k > 1$ but for $k = 1$ as well. At last

$$(s\mathcal{F}\mathcal{Q}^{\otimes\alpha} \otimes I \otimes s\mathcal{F}\mathcal{Q}^{\otimes\beta})(pu) \subset \sum_{a+q+c=\alpha+1+\beta}^{m,n} (s\mathcal{F}\mathcal{Q}^{\otimes\alpha} \otimes I \otimes s\mathcal{F}\mathcal{Q}^{\otimes\beta})(\pi_1^{\otimes a}\phi_{am} \otimes pu_q \otimes \pi_1^{\otimes c}\psi_{cn}) = 0,$$

and the proof is finished. ∎

Let us extend the description of homotopies between chain maps to the case of quotient of a free $A_\infty$-category. We keep the assumptions of Section 2.9.

2.11. Corollary. *Let $P$ be a complex of $\Bbbk$-modules, and let $v : P \to sA_\infty(\mathcal{E},\mathcal{A})(\phi,\psi)$ be a chain map. Denote*

$$w = \bigl(P \xrightarrow{v} sA_\infty(\mathcal{E},\mathcal{A})(\phi,\psi) \hookrightarrow sA_\infty(\mathcal{FQ},\mathcal{A})(\pi\phi,\pi\psi)\bigr).$$

*Let $h : P \to sA_\infty(\mathcal{FQ},\mathcal{A})(\pi\phi,\pi\psi)$ be a homotopy, $\deg h = -1$, $w = dh + hB_1$. Then the image of $h$ is contained in the subcomplex $sA_\infty(\mathcal{E},\mathcal{A})(\phi,\psi)$ if and only if it factorizes as*

$$h = \bigl(P \xrightarrow{\eta} sA_\infty(\mathcal{E},\mathcal{A})(\phi,\psi) \hookrightarrow sA_\infty(\mathcal{FQ},\mathcal{A})(\pi\phi,\pi\psi)\bigr),$$

*where $\deg\eta = -1$, $v = d\eta + \eta B_1$, or if and only if the following two conditions are satisfied:*

1. *$R(ph_1) = \mathrm{Im}(ph_1 : R(X,Y) \to s\mathcal{A}(X\phi,Y\psi)) = 0$ for all $p \in P$;*
2. *$(s\mathcal{FQ}^{\otimes\alpha} \otimes I \otimes s\mathcal{FQ}^{\otimes\beta})(ph_k) = 0$ for all $p \in P$ and all $\alpha,\beta \geqslant 0$ such that $\alpha+1+\beta = k > 1$.*

Proof. Given pair $(w,h)$ defines a chain map $\overline{w} : \mathrm{Cone}(\mathrm{id}_P) \to sA_\infty(\mathcal{FQ},\mathcal{A})(\pi\phi,\pi\psi)$, $(q,ps) \mapsto qw + ph$, such that

$$w = \bigl(P \xrightarrow{\mathrm{in}_1} P \oplus P[1] = \mathrm{Cone}(\mathrm{id}_P) \xrightarrow{\overline{w}} sA_\infty(\mathcal{FQ},\mathcal{A})(\pi\phi,\pi\psi)\bigr).$$

The image of $h$ is contained in $sA_\infty(\mathcal{E},\mathcal{A})(\phi,\psi)$ if and only if the image of $\overline{w}$ is contained in $sA_\infty(\mathcal{E},\mathcal{A})(\phi,\psi)$. By Proposition 2.10 this is equivalent to conditions:

| | |
|---|---|
| $1'$. $R(qw_1) = 0$; | $1''$. $R(ph_1) = 0$; |
| $2'$. $(s\mathcal{FQ}^{\otimes\alpha} \otimes I \otimes s\mathcal{FQ}^{\otimes\beta})(qw_k) = 0$; | $2''$. $(s\mathcal{FQ}^{\otimes\alpha} \otimes I \otimes s\mathcal{FQ}^{\otimes\beta})(ph_k) = 0$ |

for all $q,p \in P$ and all $\alpha,\beta \geqslant 0$ such that $\alpha+1+\beta = k > 1$. However, the conditions $1'$ and $2'$ are satisfied automatically by Proposition 2.10 applied to $w$ and $v$. ∎

## 3. A simple example

We want to consider an example, which is almost tautological. The non-trivial part of it is the concrete choice of a system of relations $R$ generating an ideal. Let $\mathcal{D}$ be an $A_\infty$-category. We view it as a differential graded quiver and construct the free $A_\infty$-category $\mathcal{FD}$ out of it. We choose the following subquiver of relations: $R_\mathcal{D} = \sum_{n\geqslant 2} \mathrm{Im}(b_n^{\mathcal{FD}} - b_n^{\mathcal{D}} : s\mathcal{D}^{\otimes n} \to s\mathcal{FD})$. More precisely, a map $\delta_n : (s\mathcal{D})^{\otimes n} \to s\mathcal{FD}$, $n \geqslant 2$, is defined as the difference

$$\delta_n = \bigl((s\mathcal{D})^{\otimes n} = (s\mathcal{F}_{|}\mathcal{D})^{\otimes n} \xrightarrow[s^{-1}]{b_n^{\mathcal{FD}}} s\mathcal{F}_{\mathfrak{t}_n}\mathcal{D} \hookrightarrow s\mathcal{FD}\bigr) - \bigl((s\mathcal{D})^{\otimes n} \xrightarrow{b_n^{\mathcal{D}}} s\mathcal{D} = s\mathcal{F}_{|}\mathcal{D} \hookrightarrow s\mathcal{FD}\bigr),$$

and $R_\mathcal{D} = \sum_{n\geqslant 2} \mathrm{Im}(\delta_n)$.

3.1. Lemma. *The subquiver $I = (R_\mathcal{D}) \subset s\mathcal{FD}$ is an $A_\infty$-ideal.*

Proof. According to Proposition 2.2 it suffices to check that $R_{\mathcal{D}} b_1^{\mathcal{FD}} \subset (R_{\mathcal{D}})$. As

$$(b_n^{\mathcal{FD}} - b_n^{\mathcal{D}}) b_1^{\mathcal{FD}} = -b_n^{\mathcal{D}} b_1^{\mathcal{FD}} - \sum_{a+k+c=n}^{a+c>0} (1^{\otimes a} \otimes b_k^{\mathcal{FD}} \otimes 1^{\otimes c}) b_{a+1+c}^{\mathcal{FD}},$$

we have

$$\begin{aligned}\operatorname{Im}(b_n^{\mathcal{FD}} - b_n^{\mathcal{D}}) b_1^{\mathcal{FD}} &\subset -\operatorname{Im}\Big(b_n^{\mathcal{D}} b_1^{\mathcal{FD}} + \sum_{a+k+c=n}^{a+c>0} (1^{\otimes a} \otimes b_k^{\mathcal{D}} \otimes 1^{\otimes c}) b_{a+1+c}^{\mathcal{FD}}\Big) + (R_{\mathcal{D}}) \\ &\subset -\operatorname{Im}\Big(b_n^{\mathcal{D}} b_1^{\mathcal{D}} + \sum_{a+k+c=n}^{a+c>0} (1^{\otimes a} \otimes b_k^{\mathcal{D}} \otimes 1^{\otimes c}) b_{a+1+c}^{\mathcal{D}}\Big) + (R_{\mathcal{D}}) = (R_{\mathcal{D}}),\end{aligned}$$

and the lemma is proven. ∎

Let us describe the ideal $I = (R_{\mathcal{D}}) \subset s\mathcal{FD}$. Let $t' \in \mathcal{T}$ be a tree, and let $t = (1^{\sqcup\alpha} \sqcup \mathfrak{t}_k \sqcup 1^{\sqcup\beta}) \cdot t'_<$. Define a map $\delta_{\alpha,k,t'}$ as the difference

$$\begin{aligned}\delta_{\alpha,k,t'} = &\Big((s\mathcal{D})^{\otimes n} \xrightarrow{1^{\otimes\alpha}\otimes b_k^{\mathcal{FD}}\otimes 1^{\otimes\beta}} (s\mathcal{F}_|\mathcal{D})^{\otimes\alpha} \otimes s\mathcal{F}_{\mathfrak{t}_k}\mathcal{D} \otimes (s\mathcal{F}_|\mathcal{D})^{\otimes\beta} \xrightarrow{b_{t'_<}^{\mathcal{FD}}} s\mathcal{F}_t\mathcal{D} \longrightarrow s\mathcal{FD}\Big) \\ &- \Big((s\mathcal{D})^{\otimes n} \xrightarrow{1^{\otimes\alpha}\otimes b_k^{\mathcal{D}}\otimes 1^{\otimes\beta}} (s\mathcal{D})^{\otimes\alpha+1+\beta} = (s\mathcal{F}_|\mathcal{D})^{\otimes\alpha+1+\beta} \xrightarrow{b_{t'_<}^{\mathcal{FD}}} s\mathcal{F}_{t'}\mathcal{D} \longrightarrow s\mathcal{FD}\Big)\end{aligned}$$

for $k > 1$, $\alpha+k+\beta = n$. Clearly, $\sum \operatorname{Im}(\delta_{\alpha,k,t'}) = I$. These compositions can be simplified. Let $h$ be the height of the distinguished vertex $\mathfrak{t}_k$ in $t_<$. Then the above difference equals

$$\begin{aligned}\delta_{\alpha,k,t'} = &\Big((s\mathcal{D})^{\otimes\alpha+k+\beta} = (s\mathcal{F}_|\mathcal{D})^{\otimes n} \xrightarrow{(-)^{|t|-h}s^{-|t|}} s\mathcal{F}_t\mathcal{D} \longrightarrow s\mathcal{FD}\Big) \\ &- \Big((s\mathcal{D})^{\otimes\alpha+k+\beta} \xrightarrow{1^{\otimes\alpha}\otimes b_k^{\mathcal{D}}\otimes 1^{\otimes\beta}} (s\mathcal{D})^{\otimes\alpha+1+\beta} = (s\mathcal{F}_|\mathcal{D})^{\otimes\alpha+1+\beta} \xrightarrow{s^{-|t'|}} s\mathcal{F}_{t'}\mathcal{D} \longrightarrow s\mathcal{FD}\Big).\end{aligned}$$

With the identity map $\mathrm{id} : \mathcal{D} \to \mathcal{D}$ is associated a strict $A_\infty$-functor $\widehat{\mathrm{id}} : \mathcal{FD} \to \mathcal{D}$ [LM06, Section 2.6]. Its first component equals

$$\widehat{\mathrm{id}}_1 = \Big(s\mathcal{F}_t\mathcal{D} \xrightarrow{[b_{(t,\leqslant)}^{\mathcal{FD}}]^{-1}} T^{i(t)}s\mathcal{D} \xrightarrow{b_{(t,\leqslant)}^{\mathcal{D}}} s\mathcal{D}\Big) \tag{13}$$

for each linear ordering $(t,\leqslant)$ of a plane rooted tree $t \in \mathcal{T}_{\geqslant 2}^n$ with $i(t) = n$ input leaves. Note that in the above formula $b_{(t,\leqslant)}^{\mathcal{FD}} = \pm s^{-|t|}$. In particular, for the canonical linear ordering $t_<$ the formula becomes

$$\widehat{\mathrm{id}}_1 = \Big(s\mathcal{F}_t\mathcal{D} \xrightarrow{s^{|t|}} T^{i(t)}s\mathcal{D} \xrightarrow{b_{t_>}^{\mathcal{D}}} s\mathcal{D}\Big).$$

Here

$$b_{(t,\leqslant)}^{\mathcal{D}} = \Big(s\mathcal{D}^{\otimes n} \xrightarrow{1^{\otimes\alpha_1}\otimes b_{k_1}^{\mathcal{D}}\otimes 1^{\otimes\beta_1}} s\mathcal{D}^{\otimes\alpha_1+1+\beta_1} \xrightarrow{1^{\otimes\alpha_2}\otimes b_{k_2}^{\mathcal{D}}\otimes 1^{\otimes\beta_2}} \dots \xrightarrow{b_{k_N}^{\mathcal{D}}} s\mathcal{D}\Big)$$

corresponds to the ordered decomposition

$$(t,\leqslant) = (1^{\sqcup\alpha_1} \sqcup \mathfrak{t}_{k_1} \sqcup 1^{\sqcup\beta_1}) \cdot (1^{\sqcup\alpha_2} \sqcup \mathfrak{t}_{k_2} \sqcup 1^{\sqcup\beta_2}) \cdot \dots \cdot \mathfrak{t}_{k_N},$$

of $(t,\leqslant)$ into the product of forests, $\alpha_1 + k_1 + \beta_1 = n$, and $b_{(t,\leqslant)}^{\mathcal{FD}}$ has a similar meaning.

3.2. Proposition. *The $A_\infty$-category $\mathcal{FD}/s^{-1}(R_\mathcal{D})$ is isomorphic to $\mathcal{D}$.*

Proof. Let $\mathcal{E} = \mathcal{FD}/s^{-1}(R_\mathcal{D})$ be the quotient category. The projection map $\pi_1 : s\mathcal{FD} \to s\mathcal{E}$ with the underlying map of objects $\operatorname{Ob}\pi = \mathrm{id}_{\operatorname{Ob}\mathcal{D}}$ determines a strict $A_\infty$-functor $\pi : \mathcal{FD} \to \mathcal{E}$. The embedding $\iota_1 = \bigl(s\mathcal{D} \hookrightarrow s\mathcal{FD} \xrightarrow{\pi_1} s\mathcal{E}\bigr)$ with the underlying identity map of objects $\operatorname{Ob}\iota = \mathrm{id}_{\operatorname{Ob}\mathcal{D}}$ determines a strict $A_\infty$-functor $\iota : \mathcal{D} \to \mathcal{E}$. Indeed, $\iota_1^{\otimes n} b_n^\mathcal{E} = b_n^\mathcal{D}\iota_1 : s\mathcal{D}^{\otimes n} \to s\mathcal{E}$, for $\operatorname{Im}(b_n^{\mathcal{FD}} - b_n^\mathcal{D}) = \operatorname{Im}\delta_n \subset I \subset s\mathcal{FD}$.

We claim that the $A_\infty$-functor $\widehat{\mathrm{id}} : \mathcal{FD} \to \mathcal{D}$ factorizes as $\widehat{\mathrm{id}} = \bigl(\mathcal{FD} \xrightarrow{\pi} \mathcal{E} \xrightarrow{\widetilde{\mathrm{id}}} \mathcal{D}\bigr)$ for some $A_\infty$-functor $\widetilde{\mathrm{id}} : \mathcal{E} \to \mathcal{D}$. Indeed, both conditions of Proposition 2.5 are satisfied. The second is obvious since $\widehat{\mathrm{id}}$ is strict. The first condition $R_\mathcal{D}\widehat{\mathrm{id}}_1 = 0$ follows from the computation $(b_n^{\mathcal{FD}} - b_n^\mathcal{D})\widehat{\mathrm{id}}_1 = \mathrm{id}_1^{\otimes n} b_n^\mathcal{D} - b_n^\mathcal{D}\,\mathrm{id}_1 = 0 : s\mathcal{D}^{\otimes n} \to s\mathcal{D}$. Thus, $\widetilde{\mathrm{id}} : \mathcal{E} \to \mathcal{D}$ is a strict $A_\infty$-functor.

Both $\iota$ and $\widetilde{\mathrm{id}}$ induce the identity map on objects. Clearly, $\iota \cdot \widetilde{\mathrm{id}} = \mathrm{id}_\mathcal{D}$. Furthermore, $\iota$ is surjective on morphisms because every element of $\mathcal{F}_t\mathcal{D}$ reduces to an element of $\mathcal{D}$ modulo $(R_\mathcal{D})$. Therefore, $\iota$ is invertible and $A_\infty$-functors $\iota$ and $\widetilde{\mathrm{id}}$ are strictly inverse to each other. ∎

3.3. Corollary. $(R_\mathcal{D})\widehat{\mathrm{id}}_1 = 0$.

## 4. $A_\infty$-categories and quivers

4.1. Definition. [Pseudounital $A_\infty$-categories] *A* pseudounital *structure of an $A_\infty$-category $\mathcal{D}$ is a choice of an element $[\iota_X] \in H^{-1}(s\mathcal{D}(X,X), b_1)$ for each object $X$ of $\mathcal{D}$. By $\iota_X$ we mean a representative of the chosen cohomology class,* $\deg\iota_X = -1$. *An $A_\infty$-functor $f : \mathcal{D} \to \mathcal{A}$ between two pseudounital $A_\infty$-categories is called* pseudounital *if it preserves the distinguished cohomology classes, that is, $\iota^\mathcal{A}_{Xf} - \iota^\mathcal{D}_X f_1 \in \operatorname{Im} b_1$ for all objects $X$ of $\mathcal{D}$.*

The composition of pseudounital $A_\infty$-functors is pseudounital as well. The full subcategory of pseudounital $A_\infty$-functors is denoted $A_\infty^{\psi u}(\mathcal{D},\mathcal{A}) \subset A_\infty(\mathcal{D},\mathcal{A})$.

A unital $A_\infty$-category $\mathcal{A}$ has a canonical pseudounital structure: $\iota^\mathcal{A}_X = {}_X\mathbf{i}_0^\mathcal{A}$. An $A_\infty$-functor $f : \mathcal{D} \to \mathcal{A}$ between unital $A_\infty$-categories is unital if and only if it is pseudounital for the canonical pseudounital structures of $\mathcal{D}$ and $\mathcal{A}$ [Lyu03, Definition 8.1].

Let $\mathcal{D}$ be a pseudounital $A_\infty$-category, let $\mathcal{Q}$ be a differential graded quiver with $\operatorname{Ob}\mathcal{Q} = \operatorname{Ob}\mathcal{D}$, and let $\mathrm{in}^\mathcal{D} : \mathcal{D} \to \mathcal{Q}$ be an $A_1$-functor, such that $\operatorname{Ob}\mathrm{in}^\mathcal{D} = \mathrm{id}_{\operatorname{Ob}\mathcal{D}}$ and $\mathrm{in}_1^\mathcal{D} : s\mathcal{D} \hookrightarrow s\mathcal{Q}$ is an embedding. Let $\mathcal{A}$ be a pseudounital $A_\infty$-category.

4.2. $A_{\infty1}$-functors and transformations. We define $A_\infty$-category $A_{\infty1}^{\psi u}(\mathcal{D},\mathcal{Q};\mathcal{A})$ via pull-back square

$$
\begin{tikzcd}
A_{\infty1}^{\psi u}(\mathcal{D},\mathcal{Q};\mathcal{A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A_1(\mathcal{Q},\mathcal{A}) \arrow[d, "A_1(\mathrm{in}^\mathcal{D},\mathcal{A})"] \\
A_\infty^{\psi u}(\mathcal{D},\mathcal{A}) \arrow[r, "\mathrm{restr}_{\leqslant 1}"] & A_1(\mathcal{D},\mathcal{A})
\end{tikzcd}
\tag{14}
$$

In details, the objects of $A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ are pairs $(f,f')$, where $f:\mathcal{D}\to\mathcal{A}$ is a pseudounital $A_\infty$-functor, $f':\mathcal{Q}\to\mathcal{A}$ is an $A_1$-functor such that $\operatorname{Ob} f=\operatorname{Ob} f'$ and $f_1=f'_1\big|_{s\mathcal{D}}$. Morphisms of $A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ from $(f,f')$ to $(g,g')$ are pairs $(r,r')$, where $r\in A_\infty(\mathcal{D},\mathcal{A})(f,g)$, $r'\in A_1(\mathcal{Q},\mathcal{A})(f',g')$ are such that $\deg r=\deg r'$, $r_0=r'_0$ and $r_1=r'_1\big|_{s\mathcal{D}}$. For any $n$-tuple of composable morphisms $(r^j,p^j)\in sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})((f^{j-1},g^{j-1}),(f^j,g^j))$, $1\leqslant j\leqslant n$, their $n$-th product is defined as

$$[(r^1,p^1)\otimes\dots\otimes(r^n,p^n)]B_n\overset{\text{def}}{=}\bigl((r^1\otimes\dots\otimes r^n)B_n,(p^1\otimes\dots\otimes p^n)B_n\bigr).$$

It is well-defined, because formulas for $B_k$ agree for all $A_N$-categories, $1\leqslant N\leqslant\infty$. The identity $B^2=0$ for $A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ follows from that for $A^{\psi u}_{\infty}(\mathcal{D},\mathcal{A})$ and $A_1(\mathcal{Q},\mathcal{A})$. Thus, $A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ is an $A_\infty$-category.

4.3. Proposition. *If $\mathcal{A}$ is unital, then the $A_\infty$-category $A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ is unital as well.*

Proof. Let us denote by $(1\boxtimes\mathbf{i}^{\mathcal{A}})M$ the $A_\infty$-transformation $\mathrm{id}\to\mathrm{id}:A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\to A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$, whose $n$-th component is

$$(r^1,p^1)\otimes\dots\otimes(r^n,p^n)\mapsto\bigl((r^1\otimes\dots\otimes r^n\boxtimes\mathbf{i}^{\mathcal{A}})M_{n1},(p^1\otimes\dots\otimes p^n\boxtimes\mathbf{i}^{\mathcal{A}})M_{n1}\bigr).$$

It is well-defined, since formulas for multiplication $M$ agree for all $A_N$-categories, $1\leqslant N\leqslant\infty$. In a sense, $(1\boxtimes\mathbf{i}^{\mathcal{A}})M$ for $A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ is determined by a pair of $A_\infty$-transformations: $(1\boxtimes\mathbf{i}^{\mathcal{A}})M$ for $A^{\psi u}_{\infty}(\mathcal{D},\mathcal{A})$ and for $A_1(\mathcal{Q},\mathcal{A})$. Since the latter two $A_\infty$-transformations are natural and satisfy

$$[(1\boxtimes\mathbf{i}^{\mathcal{A}})M\otimes(1\boxtimes\mathbf{i}^{\mathcal{A}})M]B_2\equiv(1\boxtimes\mathbf{i}^{\mathcal{A}})M,$$

the $A_\infty$-transformation $(1\boxtimes\mathbf{i}^{\mathcal{A}})M$ for $A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ has the same properties.

We claim that $(1\boxtimes\mathbf{i}^{\mathcal{A}})M$ is the unit transformation of $A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ as defined in [Lyu03, Definition 7.6]. Indeed, it remains to prove that chain endomorphisms

$$a\overset{\text{def}}{=}\bigl(1\otimes{}_{(g,g')}[(1\boxtimes\mathbf{i}^{\mathcal{A}})M]_0\bigr)B_2,\quad c\overset{\text{def}}{=}\bigl({}_{(f,f')}[(1\boxtimes\mathbf{i}^{\mathcal{A}})M]_0\otimes1\bigr)B_2:$$
$$\Phi\overset{\text{def}}{=}\bigl(sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})((f,f'),(g,g')),B_1\bigr)\to\Phi$$

are homotopy invertible for all pairs $(f,f')$, $(g,g')$ of objects of $A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$. We have

$$(r,r')a=\bigl((r\otimes g\mathbf{i}^{\mathcal{A}})B_2,(r'\otimes g'\mathbf{i}^{\mathcal{A}})B_2\bigr),$$
$$(r,r')c=\bigl(r(f\mathbf{i}^{\mathcal{A}}\otimes1)B_2,r'(f'\mathbf{i}^{\mathcal{A}}\otimes1)B_2\bigr).$$

As a graded $\Bbbk$-module $\Phi=\prod_{n=0}^\infty V_n$, where

$$V_0=\prod_{X\in\operatorname{Ob}\mathcal{D}}s\mathcal{A}(Xf,Xg),$$
$$V_1=\prod_{X,Y\in\operatorname{Ob}\mathcal{D}}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{Q}(X,Y),s\mathcal{A}(Xf,Yg)),$$
$$V_n=\prod_{X,Y\in\operatorname{Ob}\mathcal{D}}\underline{\mathsf{C}}_{\Bbbk}((s\mathcal{D})^{\otimes n}(X,Y),s\mathcal{A}(Xf,Yg))\qquad\text{for}\qquad n\geqslant2. \tag{15}$$

Consider the decreasing filtration $\Phi = \Phi_0 \supset \Phi_1 \supset \dots \supset \Phi_n \supset \Phi_{n+1} \supset \dots$ of the complex $\Phi$, defined by $\Phi_n = 0 \times \dots \times 0 \times \prod_{m=n}^\infty V_m$. We may write

$$\Phi_1 = \{(r, r') \in \Phi \mid r_0 = 0\},$$
$$\Phi_n = \{(r, 0) \in \Phi \mid \forall l < n \quad r_l = 0\} \qquad \text{for} \qquad n \geqslant 2.$$

The differential $B_1$ preserves the submodules $\Phi_n$. It induces the differential

$$d = \underline{\mathsf{C}}_{\Bbbk}(1, b_1) - \sum_{\alpha+1+\beta=n} \underline{\mathsf{C}}_{\Bbbk}(1^{\otimes\alpha} \otimes b_1 \otimes 1^{\otimes\beta}, 1)$$

in the quotient $V_n = \Phi_n/\Phi_{n+1}$. Due to

$$[(r \otimes g\mathbf{i}^{\mathcal{A}})B_2]_k = \sum_l (r \otimes g\mathbf{i}^{\mathcal{A}})\theta_{kl} b_l = \sum_{a+p+c+q+e=k}^{\alpha,\gamma,\varepsilon} (f_{a\alpha} \otimes r_p \otimes g_{c\gamma} \otimes (g\mathbf{i}^{\mathcal{A}})_q \otimes g_{e\varepsilon}) b_{\alpha+\gamma+\varepsilon+2}$$

and similar formulas, the endomorphisms $a, c : \Phi \to \Phi$ preserve the subcomplexes $\Phi_n$. They induce the endomorphisms $\operatorname{gr} a, \operatorname{gr} c : V_n \to V_n$ in the quotient complex $V_n$:

$$(r'_1) \operatorname{gr} a = \prod_{X,Y \in \operatorname{Ob} \mathcal{D}} ({}_{X,Y}r'_1 \otimes {}_{Yg}\mathbf{i}_0^{\mathcal{A}}) b_2^{\mathcal{A}}, \qquad r'_1 \in V_1,$$
$$(r'_1) \operatorname{gr} c = \prod_{X,Y \in \operatorname{Ob} \mathcal{D}} {}_{X,Y}r'_1 ({}_{Xf}\mathbf{i}_0^{\mathcal{A}} \otimes 1) b_2^{\mathcal{A}},$$
$$(r_n) \operatorname{gr} a = \prod_{X,Y \in \operatorname{Ob} \mathcal{D}} ({}_{X,Y}r_n \otimes {}_{Yg}\mathbf{i}_0^{\mathcal{A}}) b_2^{\mathcal{A}}, \qquad r_n \in V_n, \quad n = 0 \text{ or } n \geqslant 2.$$
$$(r_n) \operatorname{gr} c = \prod_{X,Y \in \operatorname{Ob} \mathcal{D}} {}_{X,Y}r_n ({}_{Xf}\mathbf{i}_0^{\mathcal{A}} \otimes 1) b_2^{\mathcal{A}}.$$

Due to unitality of $\mathcal{A}$, for each pair $X$, $Y$ of objects of $\mathcal{D}$ there exist $\Bbbk$-linear maps ${}_{X,Y}h, {}_{X,Y}h' : s\mathcal{A}(Xf, Yg) \to s\mathcal{A}(Xf, Yg)$ of degree $-1$ such that

$$(1 \otimes {}_{Yg}\mathbf{i}_0^{\mathcal{A}}) b_2^{\mathcal{A}} = 1 + {}_{X,Y}h \cdot d + d \cdot {}_{X,Y}h, \tag{16}$$
$$({}_{Xf}\mathbf{i}_0^{\mathcal{A}} \otimes 1) b_2^{\mathcal{A}} = -1 + {}_{X,Y}h' \cdot d + d \cdot {}_{X,Y}h'. \tag{17}$$

We equip the $\Bbbk$-modules $\Phi^d = \prod_{n=0}^\infty V_n^d$ with the topology of the product of discrete Abelian groups $V_n^d$. Thus the $\Bbbk$-submodules $\Phi_m^d = 0^{m-1} \times \prod_{n=m}^\infty V_n^d$ form a basis of neighborhoods of 0 in $\Phi^d$. Continuous maps $A : V^d \to V^{d+p}$ are identified with $\mathbb{N} \times \mathbb{N}$-matrices of linear maps $A_{nm} : V_n^d \to V_m^{d+p}$ with finite number of non-vanishing elements in each column.

In particular, the maps $B_1 : \Phi^d \to \Phi^{d+1}$ are continuous for all $d \in \mathbb{Z}$. Let us introduce continuous $\Bbbk$-linear maps $H, H' : \prod_{n=0}^\infty V_n^d \to \prod_{n=0}^\infty V_n^{d-1}$ by diagonal matrices ${}_{X,Y}r_n \mapsto {}_{X,Y}r_n {}_{X,Y}h$, ${}_{X,Y}r_n \mapsto {}_{X,Y}r_n {}_{X,Y}h'$. We may view $\operatorname{gr} a$, $\operatorname{gr} c$ as diagonal matrices and as the corresponding continuous endomorphisms of $\prod_{n=0}^{\infty} V_n^d$. Equations (16), (17) can be written as

$$\operatorname{gr} a = 1 + Hd + dH, \qquad \operatorname{gr} c = -1 + H'd + dH'.$$

The continuous chain maps

$$N = a - HB_1 - B_1H - 1, \quad N' = c - H'B_1 - B_1H' + 1 : \prod_{n=0}^{\infty} V_n^d \to \prod_{n=0}^{\infty} V_n^d$$

have strictly upper triangular matrices. Therefore, the endomorphisms $1 + N, -1 + N'$ : $\prod_{n=0}^{\infty} V_n^d \to \prod_{n=0}^{\infty} V_n^d$ are invertible. Their inverse maps correspond to well-defined $\mathbb{N}\times\mathbb{N}$-matrices $\sum_{i=0}^{\infty}(-N)^i$ and $-\sum_{i=0}^{\infty}(N')^i$. Since $a$, $c$ are homotopic to invertible maps, they are homotopy invertible and the proposition is proven. ■

4.4. A STRICT $A_\infty^u$-2-FUNCTOR. Let us describe a strict $A_\infty^u$-2-functor $F : A_\infty^u \to A_\infty^u$. It maps a unital $A_\infty$-category $\mathcal{A}$ to the unital $A_\infty$-category $A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; \mathcal{A})$. The strict unital $A_\infty$-functor

$$F_{\mathcal{A},\mathcal{B}} = A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; \_) : A_\infty^u(\mathcal{A}, \mathcal{B}) \to A_\infty^u\bigl(A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; \mathcal{A}), A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; \mathcal{B})\bigr)$$

is specified by the following data. It maps an object $f : \mathcal{A} \to \mathcal{B}$ (a unital $A_\infty$-functor) to the object

$$A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; f) = (A_\infty^{\psi u}(\mathcal{D}, f), A_1(\mathcal{Q}, f)) = ((1 \boxtimes f)M, (1 \boxtimes f)M),$$

a unital $A_\infty$-functor $A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; \mathcal{A}) \to A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; \mathcal{B})$, which sends $(g, g') \in \operatorname{Ob} A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; \mathcal{A})$ to $(gf, g'f) \in \operatorname{Ob} A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; \mathcal{B})$. Its $n$-th component is

$$[A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; f)]_n : (r^1, p^1) \otimes \cdots \otimes (r^n, p^n) \mapsto \bigl((r^1 \otimes \cdots \otimes r^n \boxtimes f)M_{n0}, (p^1 \otimes \cdots \otimes p^n \boxtimes f)M_{n0}\bigr).$$

An $A_\infty$-transformation $q : f \to g : \mathcal{A} \to \mathcal{B}$ is mapped by $[A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; \_)]_1$ to the $A_\infty$-transformation

$$A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; q) = (A_\infty^{\psi u}(\mathcal{D}, q), A_1(\mathcal{Q}, q)) = ((1 \boxtimes q)M, (1 \boxtimes q)M),$$

whose $n$-th component is

$$[A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; q)]_n : (r^1, p^1) \otimes \cdots \otimes (r^n, p^n) \mapsto \bigl((r^1 \otimes \cdots \otimes r^n \boxtimes q)M_{n1}, (p^1 \otimes \cdots \otimes p^n \boxtimes q)M_{n1}\bigr).$$

Thus, a strict $A_\infty$-functor $F_{\mathcal{A},\mathcal{B}} = A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; \_)$ is constructed. It is unital, because the unit element $f\mathbf{i}^{\mathcal{B}}$ of $f \in \operatorname{Ob} A_\infty^u(\mathcal{A}, \mathcal{B})$ is mapped to the unit element $A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; f\mathbf{i}^{\mathcal{B}}) = ((1 \boxtimes f\mathbf{i}^{\mathcal{B}})M, (1 \boxtimes f\mathbf{i}^{\mathcal{B}})M)$ of $A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; f) \in \operatorname{Ob} A_\infty^u(F\mathcal{A}, F\mathcal{B})$.

Necessary equation, given by diagram (3.1.1) of [LM06] follows from the same equation written for $A_\infty^{\psi u}(\mathcal{D}, \_)$ and for $A_1(\mathcal{Q}, \_)$. Therefore, the strict $A_\infty^u$-2-functor $F$ is constructed.

4.5. An $A_\infty$-category freely generated over an $A_\infty$-category. Let $\mathcal{D}$ be a pseudounital $A_\infty$-category, let $\mathcal{Q}$ be a differential graded quiver with $\operatorname{Ob}\mathcal{Q} = \operatorname{Ob}\mathcal{D}$, equipped with a chain embedding $s\mathcal{D} \hookrightarrow s\mathcal{Q}$, identity on objects. Assume that there exists a graded $\Bbbk$-subquiver $\mathcal{N} \subset \mathcal{Q}$, $\operatorname{Ob}\mathcal{N} = \operatorname{Ob}\mathcal{D}$, which is a direct complement of $\mathcal{D}$. Thus, $\mathcal{D} \oplus \mathcal{N} = \mathcal{Q}$ is an isomorphism of graded $\Bbbk$-linear quivers. Then there exists a differential in the graded quiver $s\mathcal{M} = \mathcal{N}$ and a chain map $\alpha : s\mathcal{M} \to s\mathcal{D}$ such that $\operatorname{Ob}\alpha = \mathrm{id}_{\operatorname{Ob}\mathcal{D}}$ and $\mathcal{Q} = \operatorname{Cone}\alpha$. Indeed, the embedding $\mathrm{in}^{\mathcal{D}}$ in the exact sequence

$$0 \to s\mathcal{D} \xrightarrow{\mathrm{in}^{\mathcal{D}}} s\mathcal{Q} \xrightarrow{\mathrm{pr}^{\mathcal{N}}} \mathcal{M}[2] \to 0$$

is a chain map. Thus the graded $\Bbbk$-quiver $\mathcal{M}[2] = s\mathcal{N} = \operatorname{Coker}(\mathrm{in}^{\mathcal{D}})$ acquires a differential $d^{\mathcal{M}[2]}$, such that $\mathrm{pr}^{\mathcal{N}}$ is a chain map. The differential in $s\mathcal{Q}$ has the form

$$(t, ms)b_1^{\mathcal{Q}} = (tb_1^{\mathcal{D}} + m\alpha, msd^{\mathcal{M}[2]}) = (tb_1^{\mathcal{D}} + m\alpha, -md^{\mathcal{M}[1]}s) \tag{18}$$

for $t \in s\mathcal{D}(X,Y)$, $m \in \mathcal{M}[1](X,Y)$, where $\alpha : s\mathcal{M}(X,Y) \to s\mathcal{D}(X,Y)$ are $\Bbbk$-linear maps of degree 0. The condition $(b_1^{\mathcal{Q}})^2 = 0$ is equivalent to $\alpha$ being a chain map. Therefore, $s\mathcal{Q} = \operatorname{Cone}(\alpha : s\mathcal{M} \to s\mathcal{D})$.

Define a pseudounital $A_\infty$-category $\mathcal{E} = \mathcal{F}\mathcal{Q}/s^{-1}(R_{\mathcal{D}})$, where $R_{\mathcal{D}} = \sum_{n\geqslant 2} \operatorname{Im}(\delta_n)$ for

$$\delta_n = \big((s\mathcal{D})^{\otimes n} \hookrightarrow (s\mathcal{F}\mathcal{Q})^{\otimes n} \xrightarrow{b_n^{\mathcal{F}\mathcal{Q}}} s\mathcal{F}\mathcal{Q}\big) - \big((s\mathcal{D})^{\otimes n} \xrightarrow{b_n^{\mathcal{D}}} s\mathcal{D} \hookrightarrow s\mathcal{F}\mathcal{Q}\big).$$

Repeating word by word the proof of Lemma 3.1 we deduce that $J = (R_{\mathcal{D}}) \subset s\mathcal{F}\mathcal{Q}$ is an $A_\infty$-ideal. The distinguished elements $\iota_X \in (s\mathcal{E})^{-1}(X,X)$ are those of $\mathcal{D} \subset \mathcal{E}$. Let $\mathcal{A}$ be a pseudounital $A_\infty$-category. There is the restriction strict $A_\infty$-functor

$$\mathrm{restr} : A_\infty^{\psi u}(\mathcal{E},\mathcal{A}) \to A_{\infty 1}^{\psi u}(\mathcal{D},\mathcal{Q};\mathcal{A}), \qquad x \mapsto (x\big|_{\mathcal{D}}, x\big|_{\mathcal{Q}}).$$

If $\mathcal{A}$ is unital, then the above $A_\infty$-functor is unital, because the unit element $f\mathbf{i}^{\mathcal{A}}$ of $f \in \operatorname{Ob} A_\infty^{\psi u}(\mathcal{E},\mathcal{A})$ is mapped to the unit element $\big((f\mathbf{i}^{\mathcal{A}})\big|_{\mathcal{D}}, (f\mathbf{i}^{\mathcal{A}})\big|_{\mathcal{Q}}\big) = \big(f\big|_{\mathcal{D}}\mathbf{i}^{\mathcal{A}}, f\big|_{\mathcal{Q}}\mathbf{i}^{\mathcal{A}}\big)$ of $(f\big|_{\mathcal{D}}, f\big|_{\mathcal{Q}}) \in \operatorname{Ob} A_{\infty 1}^{\psi u}(\mathcal{D},\mathcal{Q};\mathcal{A})$.

4.6. Proposition. *The map* $\operatorname{Ob}\mathrm{restr} : \operatorname{Ob} A_\infty^{\psi u}(\mathcal{E},\mathcal{A}) \to \operatorname{Ob} A_{\infty 1}^{\psi u}(\mathcal{D},\mathcal{Q};\mathcal{A})$ *is surjective. An object* $(f, f')$ *of* $A_{\infty 1}^{\psi u}(\mathcal{D},\mathcal{Q};\mathcal{A})$ *is the restriction of a unique pseudounital $A_\infty$-functor* $\widetilde{f} : \mathcal{E} \to \mathcal{A}$ *such that* $\operatorname{Ob}\widetilde{f} = \operatorname{Ob} f$, $\widetilde{f}_1\big|_{s\mathcal{Q}} = f_1'$, $\widetilde{f}_k\big|_{s\mathcal{D}^{\otimes k}} = f_k$, *and* $\widetilde{f}_k$ *vanishes on all summands of* $T^k s\mathcal{E}$ *containing the factor* $s\mathcal{N}$ *for* $k > 1$.

Proof. Let us define an $A_\infty$-functor $\hat{f} : \mathcal{F}\mathcal{Q} \to \mathcal{A}$ via Proposition 2.4 by the following data. On objects it is $\operatorname{Ob}\hat{f} = \operatorname{Ob} f = \operatorname{Ob} f'$, the restriction to $s\mathcal{Q}$ of the first component is $\hat{f}_1\big|_{s\mathcal{Q}} = f_1'$. On each direct summand of $T^k s\mathcal{F}\mathcal{Q}$, $k > 1$, containing the factor $s\mathcal{N}$ we set $\hat{f}_k = 0$. On the direct summand $T^k s\mathcal{F}\mathcal{D}$ of $T^k s\mathcal{F}\mathcal{Q}$ we define

$$\hat{f}_k = \big(T^k s\mathcal{F}\mathcal{D} \xrightarrow{\widehat{\mathrm{id}}_1^{\otimes k}} T^k s\mathcal{D} \xrightarrow{f_k} s\mathcal{A}\big)$$

for $k > 1$, where $\widehat{\mathrm{id}}_1$ is defined by (13). These requirements specify $\hat{f}$ completely. It is pseudounital, since $\hat{f}_1\big|_{s\mathcal{D}} = f_1$ and $f$ is pseudounital.

Let us prove that the $A_\infty$-functor $\hat{f}$ factors as $\hat{f} = \bigl(\mathcal{FQ} \xrightarrow{\pi} \mathcal{E} \xrightarrow{\widetilde{f}} \mathcal{A}\bigr)$ for some unique $A_\infty$-functor $\widetilde{f}$. Denote $J = (R_{\mathcal{D}}) \subset s\mathcal{FQ}$. We have to check conditions of Proposition 2.5. The second condition, $(s\mathcal{FQ}^{\otimes\alpha} \otimes J \otimes s\mathcal{FQ}^{\otimes\beta})\hat{f}_{\alpha+1+\beta} = 0$ if $\alpha + \beta > 0$, holds on direct summands of $s\mathcal{FQ}^{\otimes\alpha} \otimes J \otimes s\mathcal{FQ}^{\otimes\beta}$, which contain a factor $s\mathcal{N}$ in some of $\mathcal{FQ}$. It holds also on summands of $J$ of the form $\operatorname{Im}(b_t^{\mathcal{FQ}} : \cdots \otimes s\mathcal{N} \otimes \cdots \otimes R_{\mathcal{D}} \otimes \cdots \to s\mathcal{FQ})$ or $\operatorname{Im}(b_t^{\mathcal{FQ}} : \cdots \otimes R_{\mathcal{D}} \otimes \cdots \otimes s\mathcal{N} \otimes \cdots \to s\mathcal{FQ})$. We have to verify that $(s\mathcal{FD}^{\otimes\alpha} \otimes I \otimes s\mathcal{FD}^{\otimes\beta})\hat{f}_{\alpha+1+\beta} = 0$ if $\alpha + \beta > 0$, where $I \subset s\mathcal{FD}$ is the ideal described in Lemma 3.1. This equation holds true because

$$(s\mathcal{FD}^{\otimes\alpha} \otimes I \otimes s\mathcal{FD}^{\otimes\beta})\hat{f}_{\alpha+1+\beta} = (s\mathcal{FD}^{\otimes\alpha} \otimes I \otimes s\mathcal{FD}^{\otimes\beta})\widehat{\mathrm{id}}_1^{\,\alpha+1+\beta} f_{\alpha+1+\beta} = 0$$

due to equation $I\widehat{\mathrm{id}}_1 = 0$, obtained in Corollary 3.3. Therefore, the second condition of Proposition 2.5 is verified.

Let us observe that the restriction of $\hat{f}$ to $\mathcal{FD}$ coincides with $\mathcal{FD} \xrightarrow{\widehat{\mathrm{id}}} \mathcal{D} \xrightarrow{f} \mathcal{A}$. Indeed, their components are equal, $\hat{f}_k\big|_{\mathcal{FD}} = \widehat{\mathrm{id}}_1^{\otimes k} \cdot f_k$ for $k > 1$, and $\hat{f}_1\big|_{s\mathcal{F}_{|}\mathcal{D}} = f_1 = \widehat{\mathrm{id}}_1 \cdot f_1 : s\mathcal{D} \to s\mathcal{A}$. Hence, $\hat{f}\big|_{\mathcal{FD}} = \widehat{\mathrm{id}} \cdot f$ by Proposition 2.4.

In particular, $\hat{f}_1\big|_{s\mathcal{FD}} = \bigl(s\mathcal{FD} \xrightarrow{\widehat{\mathrm{id}}_1} s\mathcal{D} \xrightarrow{f_1} s\mathcal{A}\bigr)$. Hence, $R_{\mathcal{D}}\hat{f}_1 = R_{\mathcal{D}}\widehat{\mathrm{id}}_1 f_1 = 0$ due to Corollary 3.3. Therefore, the first condition of Proposition 2.5 is also verified and $\widetilde{f}$ exists. Uniqueness of such $\widetilde{f}$ is obvious. ∎

The projection map $\pi_1 : s\mathcal{FQ} \to s\mathcal{E}$ with the underlying map of objects $\operatorname{Ob}\pi = \mathrm{id}_{\operatorname{Ob}\mathcal{D}}$ determines a strict $A_\infty$-functor $\pi : \mathcal{FQ} \to \mathcal{E}$. The embedding $\iota_1 = \bigl(s\mathcal{D} \xhookrightarrow{i} s\mathcal{FQ} \xrightarrow{\pi_1} s\mathcal{E}\bigr)$ with the underlying identity map of objects $\operatorname{Ob}\iota = \mathrm{id}_{\operatorname{Ob}\mathcal{D}}$ determines a strict $A_\infty$-functor $\iota : \mathcal{D} \to \mathcal{E}$. Indeed, $\iota_1^{\otimes n} b_n^{\mathcal{E}} = b_n^{\mathcal{D}} \iota_1 : s\mathcal{D}^{\otimes n} \to s\mathcal{E}$, for $\operatorname{Im}(b_n^{\mathcal{FD}} - b_n^{\mathcal{D}}) = \operatorname{Im}\delta_n \subset J \subset s\mathcal{FQ}$. These $A_\infty$-functors produce other strict $A_\infty$-functors for an arbitrary $A_\infty$-category $\mathcal{A}$. For instance, the functor

$$(\pi \boxtimes 1)M : A_\infty(\mathcal{E}, \mathcal{A}) \to A_\infty(\mathcal{FQ}, \mathcal{A}), \qquad x \mapsto \pi x,$$

injective on objects and morphisms, and the restriction $A_\infty$-functor

$$\mathrm{restr} = (\iota \boxtimes 1)M : A_\infty(\mathcal{E}, \mathcal{A}) \to A_\infty(\mathcal{D}, \mathcal{A}), \qquad y \mapsto \iota y = \overline{y}.$$

4.7. Theorem. *Let $\mathcal{E} = \mathcal{FQ}/s^{-1}(R_{\mathcal{D}})$, where $\mathcal{D}$, $\mathcal{Q}$ satisfy assumptions of Section 4.5. Then the restriction $A_\infty$-functor*

$$\mathrm{restr} : A_\infty^{\psi u}(\mathcal{E}, \mathcal{A}) \to A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; \mathcal{A}) \tag{19}$$

*is an $A_\infty$-equivalence, surjective on objects. The chain surjections* $\mathrm{restr}_1$ *admit a chain splitting.*

Proof. Let us prove that the chain map

$$\mathrm{restr}_1 : sA^{\psi u}_\infty(\mathcal{E},\mathcal{A})(f,g) \to sA^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\big((f|_\mathcal{D}, f|_\mathcal{Q}),(g|_\mathcal{D}, g|_\mathcal{Q})\big) \tag{20}$$

is homotopy invertible for all pairs of pseudounital $A_\infty$-functors $f,g:\mathcal{E}\to\mathcal{A}$. This will be achieved in a sequence of Lemmata.

The graded $\Bbbk$-quiver decomposition $\mathcal{Q}=\mathcal{D}\oplus\mathcal{N}$ implies that the graded $\Bbbk$-quiver $\mathcal{FD}$ is a direct summand of $\mathcal{FQ}$. The projection $\mathrm{pr}^{\mathcal{FD}} : s\mathcal{FQ}\to s\mathcal{FD}$ annihilates all summands with factors $s\mathcal{N}$. Define a degree 0 map

$$\varpi = \big(s\mathcal{FQ} \xrightarrow{\mathrm{pr}^{\mathcal{FD}}} s\mathcal{FD} \xrightarrow{\widehat{\mathrm{id}}_1} s\mathcal{D}\big).$$

4.8. Lemma. *There exists a unique chain map*

$$u : sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\big((f|_\mathcal{D}, f|_\mathcal{Q}),(g|_\mathcal{D}, g|_\mathcal{Q})\big) \to sA_\infty(\mathcal{FQ},\mathcal{A})(\pi f,\pi g)$$

*such that $u'$ obtained from $u$ via formula* (7) *equals*

$$u' = \mathrm{restr}_\mathcal{Q} : sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\big((f|_\mathcal{D}, f|_\mathcal{Q}),(g|_\mathcal{D}, g|_\mathcal{Q})\big) \to sA_1(\mathcal{Q},\mathcal{A})(f|_\mathcal{Q},g|_\mathcal{Q}), \quad (p,p')\mapsto p',$$

*and for $k>1$ the maps $u_k$ are*

$$u_k = \Big[sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\big((f|_\mathcal{D}, f|_\mathcal{Q}),(g|_\mathcal{D}, g|_\mathcal{Q})\big) \xrightarrow{\mathrm{pr}_k} \prod_{X,Y\in\mathrm{Ob}\,\mathcal{D}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{D}^{\otimes k}(X,Y), s\mathcal{A}(Xf,Yg))$$
$$\xrightarrow{\prod \underline{\mathsf{C}}_\Bbbk(\varpi^{\otimes k},1)} \prod_{X,Y\in\mathrm{Ob}\,\mathcal{D}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{FQ}^{\otimes k}(X,Y), s\mathcal{A}(Xf,Yg))\Big], \qquad (p,p')\mapsto p_k \mapsto \varpi^{\otimes k}p_k.$$

Proof. Apply Proposition 2.7 to $P = sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\big((f|_\mathcal{D}, f|_\mathcal{Q}),(g|_\mathcal{D}, g|_\mathcal{Q})\big)$. ∎

4.9. Lemma. *The map $u$ from Lemma 4.8 takes values in*

$$sA_\infty(\mathcal{E},\mathcal{A})(f,g) \subset sA_\infty(\mathcal{FQ},\mathcal{A})(\pi f,\pi g).$$

Proof. Let us verify conditions of Proposition 2.10. We have $J\varpi=0$. Indeed, $\varpi$ vanishes on summands of $J=(R_\mathcal{D})$ of the form $\mathrm{Im}(b^{\mathcal{FQ}}_t : \dots\otimes s\mathcal{N}\otimes\dots\otimes R_\mathcal{D}\otimes\dots\to s\mathcal{FQ})$ or $\mathrm{Im}(b^{\mathcal{FQ}}_t : \dots\otimes R_\mathcal{D}\otimes\dots\otimes s\mathcal{N}\otimes\dots\to s\mathcal{FQ})$. Looking at $I=J\cap\mathcal{FD}$ we find that $J\varpi = I\varpi = I\widehat{\mathrm{id}}_1 = 0$ by Corollary 3.3. Therefore,

$$(s\mathcal{FQ}^{\otimes\alpha}\otimes J\otimes s\mathcal{FQ}^{\otimes\beta})((p,p')u_k) = (s\mathcal{FQ}^{\otimes\alpha}\otimes J\otimes s\mathcal{FQ}^{\otimes\beta})\varpi^{\otimes k}p_k = 0, \tag{21}$$

and the second condition of Proposition 2.10 is verified.

Let us check now that $R_\mathcal{D}((p,p')u_1)=0$ for any element

$$(p,p')\in sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\big((f|_\mathcal{D}, f|_\mathcal{Q}),(g|_\mathcal{D}, g|_\mathcal{Q})\big).$$

That is,

$$(-)^p(i^{\otimes k}b_k^{\mathcal{FQ}} - b_k^{\mathcal{D}}i)((p,p')u_1) = 0 : s\mathcal{D}^{\otimes k}(X,Y) \to s\mathcal{A}(Xf,Yg) \tag{22}$$

for $k>1$, where $i : s\mathcal{D} \hookrightarrow s\mathcal{FQ}$ is the embedding of differential graded $\Bbbk$-quivers. By definition (8)

$$\begin{aligned}(-)^p i^{\otimes k}b_k^{\mathcal{FQ}}((p,p')u_1) = {} & -i^{\otimes k}[(pB_1,p'B_1)u_k] \\ & + \sum_{a+q+c=k}^{m,n} (i^{\otimes a}\pi_1^{\otimes a}f_{am} \otimes i^{\otimes q}((p,p')u_q) \otimes i^{\otimes c}\pi_1^{\otimes c}g_{cn})b^{\mathcal{A}}_{m+1+n} \\ & - (-)^p i^{\otimes k}\sum_{a+q+c=k}^{a+c>0} (1^{\otimes a} \otimes b_q^{\mathcal{FQ}} \otimes 1^{\otimes c})((p,p')u_{a+1+c}) : s\mathcal{D}^{\otimes k}(X,Y) \to s\mathcal{A}(Xf,Yg).\end{aligned}$$

For an arbitrary $(t,t') \in sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\big((f|_{\mathcal{D}},f|_{\mathcal{Q}}),(g|_{\mathcal{D}},g|_{\mathcal{Q}})\big)$, in particular for $(p,p')$ or $(pB_1,p'B_1)$, we have for $k>1$

$$i^{\otimes k}((t,t')u_k) = (t,t')u_k\underline{\mathsf{C}}_{\Bbbk}(i^{\otimes k},1) = t_k\underline{\mathsf{C}}_{\Bbbk}(\varpi^{\otimes k},1)\underline{\mathsf{C}}_{\Bbbk}(i^{\otimes k},1) = t_k\underline{\mathsf{C}}_{\Bbbk}(i^{\otimes k}\varpi^{\otimes k},1) = t_k \tag{23}$$

due to relation $i\varpi = \mathrm{id}_{s\mathcal{D}}$. For $k=0$ or $1$ we also have $i^{\otimes k}((t,t')u_k) = t_k$. Indeed,

$$(t,t')u_0 = (t,t')u'\,\mathrm{pr}_0 = t'_0 = t_0,$$
$$i((t,t')u_1) = \mathrm{in}^{\mathcal{D}}((t,t')u')\,\mathrm{pr}_1 = \mathrm{in}^{\mathcal{D}}\,t'_1 = t_1.$$

Notice also that $\big(s\mathcal{D} \overset{i}{\hookrightarrow} s\mathcal{FQ} \xrightarrow{\pi_1} s\mathcal{E}\big) = \iota_1$, hence, $i^{\otimes a}\pi_1^{\otimes a}f_{am} = \iota_1^{\otimes a}f_{am} = \overline{f}_{am} \overset{\text{def}}{=} (f|_{\mathcal{D}})_{am}$. Due to already proven property (21) we may replace $i^{\otimes q}b_q^{\mathcal{FQ}}$ in the last sum with $b_q^{\mathcal{D}}i$. Therefore,

$$\begin{aligned}(-)^p i^{\otimes k}b_k^{\mathcal{FQ}}((p,p')u_1) &= -(pB_1)_k + \sum_{a+q+c=k}^{m,n} (\overline{f}_{am} \otimes p_q \otimes \overline{g}_{cn})b^{\mathcal{A}}_{m+1+n} \\ &\qquad - (-)^p \sum_{a+q+c=k}^{a+c>0} (1^{\otimes a} \otimes b_q^{\mathcal{D}} \otimes 1^{\otimes c})i^{\otimes a+1+c}((p,p')u_{a+1+c}) \\ &= -(pB_1)_k + (pb^{\mathcal{A}})_k - (-)^p \sum_{a+q+c=k} (1^{\otimes a} \otimes b_q^{\mathcal{D}} \otimes 1^{\otimes c})p_{a+1+c} + (-)^p b_k^{\mathcal{D}}p_1 \\ &= -\big(pb^{\mathcal{A}} - (-)^p b^{\mathcal{D}}p\big)_k + (pb^{\mathcal{A}})_k - (-)^p(b^{\mathcal{D}}p)_k + (-)^p b_k^{\mathcal{D}}i((p,p')u_1) \\ &= (-)^p b_k^{\mathcal{D}}i((p,p')u_1) : s\mathcal{D}^{\otimes k}(X,Y) \to s\mathcal{A}(Xf,Yg)\end{aligned}$$

and (22) is proven. We conclude by Proposition 2.10 that there is a chain map $\Phi$ such that

$$\begin{aligned}u = \big(sA^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\big((f|_{\mathcal{D}},f|_{\mathcal{Q}}),(g|_{\mathcal{D}},g|_{\mathcal{Q}})\big) &\xrightarrow{\Phi} sA^{\psi u}_{\infty}(\mathcal{E},\mathcal{A})(f,g) \\ &\xhookrightarrow{(\pi\boxtimes 1)M_{01}} sA_{\infty}(\mathcal{FQ},\mathcal{A})(\pi f,\pi g)\big),\end{aligned} \tag{24}$$

so the lemma is proven. ∎

4.10. Lemma. *The map $\Phi$ from (24) is a one-sided inverse to* $\mathrm{restr}_1$*:*

$$\Bigl(sA^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f|_{\mathcal{D}},f|_{\mathcal{Q}}),(g|_{\mathcal{D}},g|_{\mathcal{Q}})\bigr) \xrightarrow{\Phi} sA^{\psi u}_{\infty}(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{\mathrm{restr}_1} sA^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f|_{\mathcal{D}},f|_{\mathcal{Q}}),(g|_{\mathcal{D}},g|_{\mathcal{Q}})\bigr)\Bigr) = \mathrm{id}\,.$$

Proof. Recall that $\mathrm{restr}_1$ is the componentwise map

$$\begin{aligned}\bigl((\iota\boxtimes 1)M_{01},(j\boxtimes 1)M_{01}\bigr) : sA^{\psi u}_{\infty}(\mathcal{E},\mathcal{A})(f,g) &\to sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f|_{\mathcal{D}},f|_{\mathcal{Q}}),(g|_{\mathcal{D}},g|_{\mathcal{Q}})\bigr)\\ r=(r_k)_k &\mapsto (\iota r, jr) = \bigl((\iota_1^{\otimes k}r_k)_{k\geqslant 0},(j_1^{\otimes k}r_k)_{k=0,1}\bigr)\end{aligned}$$

Also $(\pi\boxtimes 1)M_{01} : sA^{\psi u}_{\infty}(\mathcal{E},\mathcal{A})(f,g) \hookrightarrow sA_{\infty}(\mathcal{F}\mathcal{Q},\mathcal{A})(\pi f,\pi g)$, $r=(r_k)_k \mapsto \pi r = (\pi_1^{\otimes k}r_k)_k$ is componentwise. Introduce another componentwise map of degree 0

$$\begin{aligned}L^i : sA_{\infty}(\mathcal{F}\mathcal{Q},\mathcal{A})(\pi f,\pi g) &\to sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f|_{\mathcal{D}},f|_{\mathcal{Q}}),(g|_{\mathcal{D}},g|_{\mathcal{Q}})\bigr),\\ q=(q_k)_k &\mapsto \bigl((i^{\otimes k}q_k)_{k\geqslant 0},(\mathrm{in}^{\mathcal{Q}\otimes k}\, q_k)_{k=0,1}\bigr).\end{aligned}$$

As $\iota_1 = \bigl(s\mathcal{D} \xhookrightarrow{i} s\mathcal{F}\mathcal{Q} \xrightarrow{\pi_1} s\mathcal{E}\bigr)$ and $j_1 = \bigl(s\mathcal{Q} \xhookrightarrow{\mathrm{in}^{\mathcal{Q}}} s\mathcal{F}\mathcal{Q} \xrightarrow{\pi_1} s\mathcal{E}\bigr)$, the lower triangle in the following diagram commutes:

$$\begin{tikzcd}
sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f|_{\mathcal{D}},f|_{\mathcal{Q}}),(g|_{\mathcal{D}},g|_{\mathcal{Q}})\bigr) \arrow[dd, "\Phi\cdot\mathrm{restr}_1"'] \arrow[dr, "\Phi"] \arrow[rr, phantom, ""] & & \phantom{X} \arrow[dd, phantom, "="] \\
\arrow[r, phantom, "="] & sA^{\psi u}_{\infty}(\mathcal{E},\mathcal{A})(f,g) \arrow[dl, "\mathrm{restr}_1"'] \arrow[dr, hook, "(\pi\boxtimes 1)M_{01}"] \arrow[d, phantom, "="] & \\
sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f|_{\mathcal{D}},f|_{\mathcal{Q}}),(g|_{\mathcal{D}},g|_{\mathcal{Q}})\bigr) & \phantom{X} & sA_{\infty}(\mathcal{F}\mathcal{Q},\mathcal{A})(\pi f,\pi g) \arrow[ll, "L^i"']
\arrow[from=1-1, to=3-3, rounded corners, to path={-| (\tikztotarget)}, "u"]
\end{tikzcd} \tag{25}$$

Thus the whole diagram is commutative and $\Phi\cdot\mathrm{restr}_1 = uL^i$. We have proved in (23) and (4.5) that for all $(p,p') \in sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f|_{\mathcal{D}},f|_{\mathcal{Q}}),(g|_{\mathcal{D}},g|_{\mathcal{Q}})\bigr)$ and all $k\in\mathbb{Z}_{\geqslant 0}$ we have $i^{\otimes k}((p,p')u_k) = p_k$. Similarly,

$$\mathrm{in}^{\mathcal{Q}}[(p,p')u_1] = (p,p')u'\,\mathrm{pr}_1 = p'_1.$$

Therefore,

$$(p,p')\Phi\,\mathrm{restr}_1 = \bigl((i^{\otimes k}(p,p')u_k)_{k\geqslant 0},(\mathrm{in}^{\mathcal{Q}\otimes k}(p,p')u_k)_{k=0,1}\bigr) = \bigl((p_k)_{k\geqslant 0},(p'_k)_{k=0,1}\bigr) = (p,p'),$$

and the equation $\Phi\cdot\mathrm{restr}_1 = \mathrm{id}_{sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})((f|_{\mathcal{D}},f|_{\mathcal{Q}}),(g|_{\mathcal{D}},g|_{\mathcal{Q}}))}$ is proven. ∎

4.11. LEMMA. *Denote by $v$ the chain map*

$$v = \mathrm{id} - \mathrm{restr}_1 \cdot \Phi : sA_\infty(\mathcal{E},\mathcal{A})(f,g) \to sA_\infty(\mathcal{E},\mathcal{A})(f,g).$$

*Denote by $w$ the chain map*

$$w = \bigl[sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{v} sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xhookrightarrow{(\pi\boxtimes1)M_{01}} sA_\infty(\mathcal{F}\mathcal{Q},\mathcal{A})(\pi f,\pi g)\bigr].$$

*There exists a unique homotopy $h : sA_\infty(\mathcal{E},\mathcal{A})(f,g) \to sA_\infty(\mathcal{F}\mathcal{Q},\mathcal{A})(\pi f,\pi g)$ of degree $-1$ such that $w = B_1 h + hB_1$,*

$$h' = \bigl(sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{h} sA_\infty(\mathcal{F}\mathcal{Q},\mathcal{A})(\pi f,\pi g) \xrightarrow{\mathrm{restr}_{\leqslant1}} sA_1(\mathcal{F}\mathcal{Q},\mathcal{A})(\pi f,\pi g) \xrightarrow{\mathrm{restr}} sA_1(\mathcal{Q},\mathcal{A})(jf,jg)\bigr) = 0,$$

$$h_k = 0 : sA_\infty(\mathcal{E},\mathcal{A})(f,g) \to \prod_{X,Y\in\mathrm{Ob}\,\mathcal{D}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{F}\mathcal{Q}^{\otimes k}(X,Y), s\mathcal{A}(Xf,Yg)), \qquad \textit{for } k>1.$$

PROOF. We use Corollary 2.8, setting $P = sA_\infty(\mathcal{E},\mathcal{A})(f,g)$. We have

$$\begin{aligned} w = {} & \bigl[sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xhookrightarrow{(\pi\boxtimes1)M_{01}} sA_\infty(\mathcal{F}\mathcal{Q},\mathcal{A})(\pi f,\pi g)\bigr] \\ & -\bigl[sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{\mathrm{restr}_1} sA_{\infty1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f|_\mathcal{D},f|_\mathcal{Q}),(g|_\mathcal{D},g|_\mathcal{Q})\bigr) \xrightarrow{u} sA_\infty(\mathcal{F}\mathcal{Q},\mathcal{A})(\pi f,\pi g)\bigr] \end{aligned}$$

due to (24). Due to (7), $w'$ defined in condition 1 of Corollary 2.8 is

$$\begin{aligned} w' = {} & \bigl[sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xhookrightarrow{(\pi\boxtimes1)M_{01}} sA_\infty(\mathcal{F}\mathcal{Q},\mathcal{A})(\pi f,\pi g) \\ & \qquad \xrightarrow{\mathrm{restr}_{\leqslant1}} sA_1(\mathcal{F}\mathcal{Q},\mathcal{A})(\pi f,\pi g) \xrightarrow{\mathrm{restr}} sA_1(\mathcal{Q},\mathcal{A})(jf,jg)\bigr] \\ & -\bigl[sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{\mathrm{restr}_1} sA_{\infty1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f|_\mathcal{D},f|_\mathcal{Q}),(g|_\mathcal{D},g|_\mathcal{Q})\bigr) \xrightarrow[\mathrm{restr}_\mathcal{Q}]{u'} sA_1(\mathcal{Q},\mathcal{A})(jf,jg)\bigr], \end{aligned}$$

where $j : \mathcal{Q} \hookrightarrow \mathcal{E}$ is the embedding $A_1$-functor, $j_1 = \bigl(s\mathcal{Q} \hookrightarrow s\mathcal{F}\mathcal{Q} \xrightarrow{\pi_1} s\mathcal{E}\bigr)$. We get

$$\begin{aligned} w' = {} & \bigl[sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{\mathrm{restr}_{\leqslant1}} sA_1(\mathcal{E},\mathcal{A})(f,g) \\ & \qquad \xhookrightarrow{(\pi\boxtimes1)M_{01}} sA_1(\mathcal{F}\mathcal{Q},\mathcal{A})(\pi f,\pi g) \xrightarrow{(\mathrm{in}^\mathcal{Q}\boxtimes1)M_{01}} sA_1(\mathcal{Q},\mathcal{A})(jf,jg)\bigr] \\ & - \bigl[sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{((\iota\boxtimes1)M_{01},(j\boxtimes1)M_{01})} sA_{\infty1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f|_\mathcal{D},f|_\mathcal{Q}),(g|_\mathcal{D},g|_\mathcal{Q})\bigr) \\ & \qquad \xrightarrow{\mathrm{restr}_\mathcal{Q}} sA_1(\mathcal{Q},\mathcal{A})(jf,jg)\bigr] \\ = {} & \bigl[sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{\mathrm{restr}_{\leqslant1}} sA_1(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{(j\boxtimes1)M_{01}} sA_1(\mathcal{Q},\mathcal{A})(jf,jg)\bigr] \\ & - \bigl[sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{\mathrm{restr}_{\leqslant1}} sA_1(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{(j\boxtimes1)M_{01}} sA_1(\mathcal{Q},\mathcal{A})(jf,jg)\bigr] = 0. \end{aligned}$$

Therefore, $h' = 0$ satisfies $B_1h' + h'B_1 = 0 = w'$. Hence, the unique homotopy $h$ is constructed by Corollary 2.8. ■

4.12. Remark. In the case of Lemma 4.11 $h \cdot \mathrm{pr}_k = h_k = 0$ if $k > 1$ or if $k = 0$. Indeed, (9) together with $h' = 0$ implies that $h_0 = h\,\mathrm{pr}_0 = h'\,\mathrm{pr}_0 = 0$, moreover,

$$rh_1|_{s\mathcal{Q}} = rh'\,\mathrm{pr}_1 = 0 : s\mathcal{Q}(X,Y) = s\mathcal{F}_|\mathcal{Q}(X,Y) \to s\mathcal{A}(Xf,Yg), \tag{26}$$

where $r \in sA_\infty(\mathcal{E},\mathcal{A})(f,g)$. Therefore, recurrent formula (10) simplifies here to

$$(-)^r b_k^{\mathcal{FQ}}(rh_1) = rw_k - \sum_{a+1+c=k}^{m,n} (\pi_1^{\otimes a} f_{am} \otimes rh_1 \otimes \pi_1^{\otimes c} g_{cn}) b^{\mathcal{A}}_{m+1+n} : \\ (s\mathcal{FQ})^{\otimes k}(X,Y) \to s\mathcal{A}(Xf,Yg). \tag{27}$$

4.13. Lemma. *The homotopy $h$ constructed in Lemma 4.11 factorizes as*

$$h = \bigl(sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{\eta} sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{(\pi\boxtimes 1)M_{01}} sA_\infty(\mathcal{FQ},\mathcal{A})(\pi f,\pi g)\bigr)$$

*for a unique homotopy $\eta$ of degree $-1$ such that $v = B_1\eta + \eta B_1$.*

Proof. Let us show that $h$ satisfies conditions of Corollary 2.11. The second is obvious. The first is $R_{\mathcal{D}}(rh_1) = 0$ for any $r \in sA_\infty(\mathcal{E},\mathcal{A})(f,g)$, that is,

$$(-)^r(i^{\otimes k} b_k^{\mathcal{FQ}} - b_k^{\mathcal{D}} i)(rh_1) = 0 : s\mathcal{D}^{\otimes k}(X,Y) \to s\mathcal{A}(Xf,Yg). \tag{28}$$

From (27) we find the formula for $k > 1$

$$(-)^r i^{\otimes k} b_k^{\mathcal{FQ}}(rh_1) = i^{\otimes k}(rw_k) - \sum_{a+1+c=k}^{m,n} (\iota_1^{\otimes a} f_{am} \otimes i(rh_1) \otimes \iota_1^{\otimes c} g_{cn}) b^{\mathcal{A}}_{m+1+n}$$

$$= \iota_1^{\otimes k} r_k - i^{\otimes k}[(\iota r, jr)u_k] - \sum_{a+1+c=k}^{m,n} (\overline{f}_{am} \otimes i(rh_1) \otimes \overline{g}_{cn}) b^{\mathcal{A}}_{m+1+n} : s\mathcal{D}^{\otimes k}(X,Y) \to s\mathcal{A}(Xf,Yg).$$

A particular case of (26) is

$$i(rh_1) = [\mathrm{in}^{\mathcal{D}}(rh')]\,\mathrm{pr}_1 = 0, \tag{29}$$

due to $h' = 0$, where the $A_1$-functor $\mathrm{in}^{\mathcal{D}} : \mathcal{D} \hookrightarrow \mathcal{Q}$ is the natural embedding. For our concrete choice of $u_k$ we get

$$(-)^r i^{\otimes k} b_k^{\mathcal{FQ}}(rh_1) = \iota_1^{\otimes k} r_k - i^{\otimes k} \varpi^{\otimes k} \iota_1^{\otimes k} r_k = 0 : s\mathcal{D}^{\otimes k}(X,Y) \to s\mathcal{A}(Xf,Yg),$$

since $\bigl(s\mathcal{D} \xrightarrow{i} s\mathcal{FQ} \xrightarrow{\varpi} s\mathcal{D}\bigr) = \mathrm{id}$. Therefore, $i^{\otimes k} b_k^{\mathcal{FQ}}(rh_1) = 0$ and $b_k^{\mathcal{D}} i(rh_1) = 0$ due to (29). We conclude that (28) is satisfied, and by Corollary 2.11 there exists a homotopy

$$\eta : sA_\infty(\mathcal{E},\mathcal{A})(f,g) \to sA_\infty(\mathcal{E},\mathcal{A})(f,g),$$

such that $\deg \eta = -1$, $h = \eta \cdot [(\pi \boxtimes 1)M_{01}]$ and $v = B_1\eta + \eta B_1$. ■

Lemmata 4.10 and 4.13 show that the maps $\mathrm{restr}_1$ and $\Phi$ given by (20) and (24) are homotopy inverse to each other.

The $A_\infty$-functor restr is surjective on objects by Proposition 4.6, and its first component is a homotopy isomorphism. Therefore, it is an $A_\infty$-equivalence by Corollary 1.9, and Theorem 4.7 is proven. ■

4.14. Corollary. *The collection of $A_\infty$-functors* (19) *is a natural $A^u_\infty$-2-equivalence.*

Proof. The restriction $A_\infty$-functors $A^{\psi u}_\infty(\mathcal{E},\mathcal{A}) \to A^{\psi u}_\infty(\mathcal{D},\mathcal{A})$ and $A^{\psi u}_\infty(\mathcal{E},\mathcal{A}) \to A_1(\mathcal{Q},\mathcal{A})$ are strict $A^u_\infty$-2-transformations. By (14) the restriction $A_\infty$-functor $A^{\psi u}_\infty(\mathcal{E},\mathcal{A}) \to A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ is also a strict $A^u_\infty$-2-transformation. It is an $A_\infty$-equivalence by Theorem 4.7. ■

4.15. Remark. The maps $\Phi$, $\eta$ constructed in the proof of Theorem 4.7 satisfy

$$\Phi\cdot\eta = 0 : sA^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\big((f|_{\mathcal{D}},f|_{\mathcal{Q}}),(g|_{\mathcal{D}},g|_{\mathcal{Q}})\big) \to sA^{\psi u}_\infty(\mathcal{E},\mathcal{A})(f,g).$$

Indeed, $\Phi\cdot\eta$ composed with an embedding,

$$\Phi\cdot\eta\cdot(\pi\boxtimes 1)M_{01} = \Phi\cdot h : sA^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\big((f|_{\mathcal{D}},f|_{\mathcal{Q}}),(g|_{\mathcal{D}},g|_{\mathcal{Q}})\big) \to sA_\infty(\mathcal{F}\mathcal{Q},\mathcal{A})(\pi f,\pi g),$$

is a degree $-1$ homotopy such that

$$\begin{aligned} B_1(\Phi h) + (\Phi h)B_1 &= \Phi(B_1 h + hB_1) = \Phi w \\ &= \Phi(\mathrm{id} - \mathrm{restr}_1\Phi)(\pi\boxtimes 1)M_{01} = (\mathrm{id} - \Phi\,\mathrm{restr}_1)\Phi(\pi\boxtimes 1)M_{01} = 0. \end{aligned}$$

We have $\Phi h\,\mathrm{pr}_k = \Phi\cdot h_k = 0$ for $k>1$ and

$$(\Phi h)' = \Phi h\,\mathrm{restr}_{\leqslant 1}\mathrm{restr} = \Phi\cdot h' = 0.$$

The 0 homotopy for 0 chain map also has these properties, and by Corollary 2.8 we conclude that $\Phi h = 0$.

4.16. Remark. The equation

$$\eta\cdot\mathrm{restr}_1 = 0 : sA^{\psi u}_\infty(\mathcal{E},\mathcal{A})(f,g) \to sA^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\big((f|_{\mathcal{D}},f|_{\mathcal{Q}}),(g|_{\mathcal{D}},g|_{\mathcal{Q}})\big)$$

also holds. Indeed, the decomposition $\mathrm{restr}_1 = (\pi\boxtimes 1)M_{01}\cdot L^i$ from diagram (25) implies that

$$\eta\cdot\mathrm{restr}_1 = \eta\cdot(\pi\boxtimes 1)M_{01}\cdot L^i = h\cdot L^i.$$

For any $r \in sA^{\psi u}_\infty(\mathcal{E},\mathcal{A})(f,g)$ all components of the element

$$rhL^i = \big((i^{\otimes k}(rh_k))_{k\geqslant 0}, (\mathrm{in}^{\mathcal{Q}\otimes k}(rh_k))_{k=0,1}\big) \in sA^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\big((f|_{\mathcal{D}},f|_{\mathcal{Q}}),(g|_{\mathcal{D}},g|_{\mathcal{Q}})\big)$$

vanish except, possibly, those indexed by $k=1$ by Remark 4.12. However, $i(rh_1)=0$ by (29), and, moreover, $\mathrm{in}^{\mathcal{Q}}(rh_1)=0$ by (26), thus, all the components of $rhL^i$ vanish.

4.17. REMARK. We have not used in the proof of Theorem 4.7 the assumption of pseudounitality of $\mathcal{D}$ and $\mathcal{E}$. Its assertion holds without this property. If $\mathcal{A}$ is unital, then the restriction $A_\infty$-functor

$$\mathrm{restr} : A_\infty(\mathcal{E}, \mathcal{A}) \to A_{\infty 1}(\mathcal{D}, \mathcal{Q}; \mathcal{A})$$

is an $A_\infty$-equivalence, surjective on objects. Its first component maps admit a chain splitting. In the particular case $\mathcal{D}(X,Y) = 0$ for all $X, Y \in \operatorname{Ob}\mathcal{Q}$ we get Theorem 2.12 of [LM06]: the $A_\infty$-functor

$$\mathrm{restr} : A_\infty(\mathcal{F}\mathcal{Q}, \mathcal{A}) \to A_1(\mathcal{Q}, \mathcal{A})$$

is an $A_\infty$-equivalence.

## 5. Relatively free $A_\infty$-categories

Hinich [Hin97] defines standard cofibrations of differential graded algebras. This notion is generalized by Drinfeld to semi-free differential graded categories [Dri04]. We give a definition in the spirit of these two definitions in the framework of $A_\infty$-categories.

5.1. DEFINITION. *Let $e : \mathcal{C} \to \mathcal{D}$ be a strict $A_\infty$-functor such that $\operatorname{Ob} e$ is an isomorphism, and $e_1 : s\mathcal{C} \to s\mathcal{D}$ is an embedding. The $A_\infty$-category $\mathcal{D}$ is* relatively free *over $\mathcal{C}$, if it can be represented as the union of an increasing sequence of its $A_\infty$-subcategories $\mathcal{D}_j$ and differential graded subquivers $\mathcal{Q}_j$*

$$\mathcal{D}_0 \subset \mathcal{Q}_1 \subset \mathcal{D}_1 \subset \mathcal{Q}_2 \subset \mathcal{D}_2 \subset \mathcal{Q}_3 \subset \cdots \subset \mathcal{D} \tag{30}$$

*with the same set of objects $\operatorname{Ob}\mathcal{D}$, such that*

1. *$s\mathcal{D}_0 = (s\mathcal{C})e_1$;*
2. *for each $j \geqslant 0$ the embedding of graded quivers $\mathcal{D}_j \hookrightarrow \mathcal{Q}_{j+1}$ admits a splitting map $\mathcal{Q}_{j+1} \longrightarrow \mathcal{D}_j$ of degree 0;*
3. *for each $j > 0$ the unique strict $A_\infty$-functor $\mathcal{F}\mathcal{Q}_j \to \mathcal{D}_j$ extending the embedding $\mathcal{Q}_j \hookrightarrow \mathcal{D}_j$ factors into the natural projection and an isomorphism*

$$\mathcal{F}\mathcal{Q}_j \longrightarrow \mathcal{F}\mathcal{Q}_j/s^{-1}(R_j) \xrightarrow{\sim} \mathcal{D}_j,$$

*where the system of relations $R_j = R_{\mathcal{D}_{j-1}} \subset s\mathcal{F}\mathcal{Q}_j$ is $R_j = \sum_{n\geqslant 2} \operatorname{Im}(\delta_n)$ for*

$$\delta_n = \big((s\mathcal{D}_{j-1})^{\otimes n} \hookrightarrow (s\mathcal{F}\mathcal{Q}_j)^{\otimes n} \xrightarrow{b_n^{\mathcal{F}\mathcal{Q}_j}} s\mathcal{F}\mathcal{Q}_j\big) - \big((s\mathcal{D}_{j-1})^{\otimes n} \xrightarrow{b_n^{\mathcal{D}_{j-1}}} s\mathcal{D}_{j-1} \hookrightarrow s\mathcal{F}\mathcal{Q}_j\big).$$

When all differential graded quivers $\mathcal{N}_j = \operatorname{Coker}(\mathcal{D}_{j-1} \hookrightarrow \mathcal{Q}_j)$ have zero differential and the $\Bbbk$-modules $\mathcal{N}_j^k(X,Y)$ are free for all $j \geqslant 1$, $k \in \mathbb{Z}$, we say that $\mathcal{D}$ is *semi-free* over $\mathcal{C}$ in accordance with terminology of Drinfeld. In fact, if in Definition 5.1 one replaces $A_\infty$-categories with differential graded categories and adds the above assumption on $\mathcal{N}_j$, then one recovers Definition 13.4 from [Dri04] of semi-free differential graded categories.

The system of relations $R_j$ is the minimal one that ensures that the natural embedding $s\mathcal{D}_{j-1} \hookrightarrow s\mathcal{F}\mathcal{Q}_j/(R_j) = s\mathcal{D}_j$ is the first component of a strict $A_\infty$-functor. In semi-free case we may say that $\mathcal{D}_j$ is freely generated by $\mathcal{N}_j$ over $\mathcal{D}_{j-1}$.

5.2. THE MAIN CONSTRUCTION. Let $\mathcal{C}$ be a unital $A_\infty$-category, and let $\mathcal{B} \subset \mathcal{C}$ be its full subcategory. The unit $\mathbf{i}_0^{\mathcal{C}}$ is abbreviated to $\mathbf{i}_0$.

A vertex of a tree is $k$-ary if it is adjacent to $k+1$ edges. A unary vertex is a 1-ary one.

Define a *labeled tree* $t = (t; X_0, X_1, \dots, X_n)$ as a non-empty (non-reduced) plane rooted tree $t$ with $n$ leaves, such that unary vertices are not joined by an edge, equipped with a sequence $(X_0, X_1, \dots, X_n)$ of objects of $\mathcal{C}$.

Let $e$ be an edge of $t$. If $i$ is the smallest number such that $i$-th leaf is above $e$, the *domain* of $e$ is defined as $\operatorname{dom}(e) = X_{i-1}$. If $k$ is the biggest number such that $k$-th leaf is above $e$, the *codomain* of $e$ is defined as $\operatorname{codom}(e) = X_k$. An *admissible tree* is a labeled tree $(t; X_0, X_1, \dots, X_n)$ such that for each edge $e$ adjacent to a unary vertex $\operatorname{dom}(e) \in \operatorname{Ob}\mathcal{B}$ or $\operatorname{codom}(e) \in \operatorname{Ob}\mathcal{B}$ (or both).

The set of vertices $V(t)$ of a rooted tree $t$ has a canonical ordering: $x \preccurlyeq y$ iff the minimal path connecting the root with $y$ contains $x$. A $\mathcal{C}$-*admissible tree* is an admissible tree $(t; X_0, X_1, \dots, X_n)$ such that top (maximal with respect to $\preccurlyeq$) internal vertices are unary.

Define a graded quiver $\mathcal{E}$ with the set of objects $\operatorname{Ob}\mathcal{E} = \operatorname{Ob}\mathcal{C}$. The $\mathbb{Z}$-graded $\Bbbk$-module of morphisms between $X, Y \in \operatorname{Ob}\mathcal{E}$ is defined as

$$s\mathcal{E}(X,Y) = \bigoplus_{n \geqslant 1} \bigoplus_{\text{admissible } (t; X_0, X_1, \dots, X_n)}^{X_0 = X,\, X_n = Y} s\mathcal{E}(t)(X,Y), \tag{31}$$

$$s\mathcal{E}(t)(X_0, X_n) = s\mathcal{E}(t) = s\mathcal{C}(X_0, X_1) \otimes \dots \otimes s\mathcal{C}(X_{n-1}, X_n)\bigl[|t|_1 - |t|_>\bigr],$$

where $|t|_1$ is the number of unary internal vertices of $t$, and $|t|_>$ is the number of internal vertices of arity $> 1$.

The vertices of arity $k > 1$ are interpreted as $k$-ary multiplications of degree 1. Unary vertices are interpreted as contracting homotopies $H$ of degree $-1$. Define an $A_\infty$-structure on $\mathcal{E}$ in which operations $b_k$, $k > 1$, are given by grafting. So for $k > 1$ the operation $b_k$ is a direct sum of maps

$$b_k = s^{|t_1|} \otimes \dots \otimes s^{|t_{k-1}|} \otimes s^{|t_k| - |t|} : s\mathcal{E}(t_1)(Y_0, Y_1) \otimes \dots \otimes s\mathcal{E}(t_k)(Y_{k-1}, Y_k) \to s\mathcal{E}(t)(Y_0, Y_k),$$

where $|t| = |t|_> - |t|_1$ and $t = (t_1 \sqcup \dots \sqcup t_k) \cdot \mathfrak{t}_k$. In particular, $|t| = |t_1| + \dots + |t_k| + 1$.

Let $t = (t; X_0, X_1, \dots, X_n)$ be an admissible tree, whose lowest internal vertex is not unary. In particular, $t$ might be the trivial tree $t = (|; X_0, X_1)$. Assume that $X_0 \in \operatorname{Ob}\mathcal{B}$ or $X_n \in \operatorname{Ob}\mathcal{B}$ (or both). Denote by $H$ the $\Bbbk$-linear map

$$\begin{aligned} H = s : s\mathcal{E}(t)(X_0, X_n) &= s\mathcal{C}(X_0, X_1) \otimes \dots \otimes s\mathcal{C}(X_{n-1}, X_n)\bigl[|t|_1 - |t|_>\bigr] \\ &\to s\mathcal{C}(X_0, X_1) \otimes \dots \otimes s\mathcal{C}(X_{n-1}, X_n)\bigl[1 + |t|_1 - |t|_>\bigr] = s\mathcal{E}(t \cdot \mathfrak{t}_1)(X_0, X_n) \end{aligned}$$

of degree $-1$. Here $\mathfrak{t}_1 = (\,\dot{|}\,; X_0, X_n)$ is the unary corolla.

The operation $b_1$ is determined by the given differential $b_1 : s\mathcal{C} \to s\mathcal{C}$ and by the recursive substitutions

$$b_k b_1 := -\sum_{\alpha+p+\beta=k}^{\alpha+\beta>0} (1^{\otimes\alpha} \otimes b_p \otimes 1^{\otimes\beta}) b_{\alpha+1+\beta}, \qquad k > 1, \tag{32}$$

$$H b_1 := 1 - b_1 H, \tag{33}$$

where $H$ stands for a unary vertex. Identity (32) is satisfied for $k > 1$ by definition of $b_1$. We have to prove that $b_1^2 = 0$. Assume that $k > 1$, then

$$\begin{aligned}
b_k b_1^2 &= -\sum_{\alpha+p+\beta=k}^{\alpha+\beta>0} (1^{\otimes\alpha} \otimes b_p \otimes 1^{\otimes\beta}) b_{\alpha+1+\beta} b_1 \\
&= \sum_{\substack{\alpha+p+\beta=k \\ \gamma+q+\delta=\alpha+1+\beta}}^{\gamma+\delta>0} (1^{\otimes\alpha} \otimes b_p \otimes 1^{\otimes\beta})(1^{\otimes\gamma} \otimes b_q \otimes 1^{\otimes\delta}) b_{\gamma+1+\delta} \\
&= \sum_{\alpha+p+\varepsilon+q+\delta=k} (1^{\otimes\alpha} \otimes b_p \otimes 1^{\otimes\varepsilon} \otimes b_q \otimes 1^{\otimes\delta}) b_{\alpha+\varepsilon+\delta+2} \\
&\quad - \sum_{\gamma+q+\eta+p+\beta=k} (1^{\otimes\gamma} \otimes b_q \otimes 1^{\otimes\eta} \otimes b_p \otimes 1^{\otimes\beta}) b_{\gamma+\eta+\beta+2} \\
&\quad + \sum_{\gamma+r+\delta=k}^{\gamma+\delta>0} \Bigl\{ 1^{\otimes\gamma} \otimes \Bigl[ \sum_{\kappa+p+\lambda=r} (1^{\otimes\kappa} \otimes b_p \otimes 1^{\otimes\lambda}) b_{\kappa+1+\lambda} \Bigr] \otimes 1^{\otimes\delta} \Bigr\} b_{\gamma+1+\delta} \\
&= \sum_{\gamma+1+\delta=k} (1^{\otimes\gamma} \otimes b_1^2 \otimes 1^{\otimes\delta}) b_k,
\end{aligned}$$

because the sum in square brackets vanishes for $r > 1$ by (32). We also have

$$H b_1^2 = (1 - b_1 H) b_1 = b_1 - b_1(1 - b_1 H) = b_1^2 H.$$

By induction the equation $b_1^2\big|_{s\mathcal{C}} = 0$ implies that $b_1^2 = 0$ on $s\mathcal{E}$. Therefore, $\mathcal{E}$ is an $A_\infty$-category by the same argument as in the proof of Proposition 2.2 of [LM06].

It has an ideal $(R_\mathcal{C})_+$, generated by the $\Bbbk$-subquiver $R_\mathcal{C} = \sum_{n\geqslant 2} \operatorname{Im}(\delta_n)$ for

$$\delta_n = \bigl((s\mathcal{C})^{\otimes n} \hookrightarrow (s\mathcal{E})^{\otimes n} \xrightarrow{b_n^\mathcal{E}} s\mathcal{E}\bigr) - \bigl((s\mathcal{C})^{\otimes n} \xrightarrow{b_n^\mathcal{C}} s\mathcal{C} \hookrightarrow s\mathcal{E}\bigr)$$

by application of operations $1^{\otimes\alpha} \otimes H \otimes 1^{\otimes\beta}$, $1^{\otimes\alpha} \otimes b_p \otimes 1^{\otimes\beta}$ for $p \geqslant 2$. By Lemma 3.1 $R_\mathcal{C} b_1^\mathcal{E} \subset (R_\mathcal{C}) \subset (R_\mathcal{C})_+$, where $(R_\mathcal{C})$ denotes the ideal generated by application of $1^{\otimes\alpha} \otimes b_p \otimes 1^{\otimes\beta}$ ($p \geqslant 2$) only. Similarly to Proposition 2.2 this implies that $(R_\mathcal{C})_+ b_1^\mathcal{E} \subset (R_\mathcal{C})_+$. Indeed, let $t = (t; X_0, X_1, \dots, X_n)$ be an admissible tree, whose lowest internal vertex is not unary. Assume that $X_0 \in \operatorname{Ob} \mathcal{B}$ or $X_n \in \operatorname{Ob} \mathcal{B}$, so that $t \cdot \mathfrak{t}_1$ is admissible. For an arbitrary $z \in (R_\mathcal{C})_+(t)(X_0, X_n)$ there exists $zH \in (R_\mathcal{C})_+(t \cdot \mathfrak{t}_1)(X_0, X_n)$, and $z b_1 \in (R_\mathcal{C})_+(X_0, X_n)$

by induction. Due to (33) $zHb_1 = z - zb_1H \in (R_{\mathcal{C}})_+$, which proves the claim. Therefore, the ideal $(R_{\mathcal{C}})_+$ is stable with respect to all $A_\infty$-operations, including $b_1$.

Denote by $\mathcal{D} = \mathcal{E}/s^{-1}(R_{\mathcal{C}})_+ = \mathsf{Q}(\mathcal{C}|\mathcal{B})$ the quotient $A_\infty$-category. It has a direct sum decomposition similar to that of $\mathcal{E}$

$$s\mathcal{D}(X,Y) = \bigoplus_{n\geqslant 1} \bigoplus_{\mathcal{C}\text{-admissible } (t;X_0,X_1,\dots,X_n)}^{X_0=X,\quad X_n=Y} s\mathcal{D}(t)(X,Y),$$

$$s\mathcal{D}(t)(X_0,X_n) = s\mathcal{E}(t)(X_0,X_n) = s\mathcal{C}(X_0,X_1) \otimes \dots \otimes s\mathcal{C}(X_{n-1},X_n)\bigl[|t|_1 - |t|_>\bigr],$$

with the only difference that the sum is taken over $\mathcal{C}$-admissible trees $t$. We can view $\mathcal{D}$ as a graded $\Bbbk$-subquiver of $\mathcal{E}$.

The category $\mathcal{C}$ is embedded in $\mathcal{D}$ (via a strict $A_\infty$-functor) as

$$s\mathcal{D}_0(X,Y) = s\mathcal{D}(|)(X,Y) = s\mathcal{C}(X,Y)$$

for the trivial tree $t = (|;X,Y)$. Let us show that $\mathcal{D}$ is relatively free over $\mathcal{C}$.

Let us define for $j \geqslant 0$ the $A_\infty$-subcategories $\mathcal{D}_j$ and differential graded subquivers $\mathcal{Q}_{j+1}$ of $\mathcal{D}$ so that embeddings (30) hold. Each leaf $\ell$ and the root of a tree can be connected by the unique minimal path. We say that internal vertices occurring at this path are between the root and the leaf $\ell$. Define for $j \geqslant 0$ the $A_\infty$-subcategory $\mathcal{D}_j = \oplus_t \mathcal{D}(t)$ of $\mathcal{D}$, where the summation goes over all

$\mathcal{C}$-admissible trees $t$ with no more than $j$ unary internal vertices between the root and any leaf. (C1)

Define for $j \geqslant 1$ the graded subquiver $\mathcal{N}_j = \oplus_t \mathcal{D}(t)$ of $\mathcal{D}$, where the summation goes over all trees $t$ satisfying (C1) and such that

there exists a leaf $\ell$ of $t$ with $j$ unary internal vertices between the root and $\ell$; the lowest internal vertex (adjacent to the root) is unary. (C2)

One can easily see that for $j \geqslant 1$

$$s\mathcal{Q}_j = s\mathcal{D}_{j-1} \oplus s\mathcal{N}_j$$

is a differential graded subquiver of $\mathcal{D}_j \subset \mathcal{D}$. For example, $\mathcal{D}_0 = \mathcal{D}(|) = \mathcal{C}$, $\mathcal{N}_1 = \mathcal{D}\bigl(\dot{|}\bigr)$, $\mathcal{Q}_1 = \mathcal{D}(|) \oplus \mathcal{D}\bigl(\dot{|}\bigr)$, and $\mathcal{D}_1 = \mathcal{D}(|) \oplus \oplus_t \mathcal{D}(t)$, where $t$ runs over admissible trees with the only unary internal vertex $v$, such that all other internal vertices lie on the minimal path between the root and $v$.

The inclusion map of differential graded quivers $i : s\mathcal{Q}_j \hookrightarrow s\mathcal{D}_j$ induces a unique strict $A_\infty$-functor $\hat{\imath} : \mathcal{F}\mathcal{Q}_j \to \mathcal{D}_j$ [LM06, Corollary 2.4].

5.3. Proposition. *The map $\hat{\imath}_1 : s\mathcal{F}\mathcal{Q}_j \to s\mathcal{D}_j$ is surjective and its kernel is $(R_{\mathcal{D}_{j-1}})$. Thus it induces an isomorphism $\iota_1 : s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}}) \to s\mathcal{D}_j$.*

PROOF. The strict $A_\infty$-functor $\hat{\imath} : \mathcal{FQ}_j \to \mathcal{D}_j$ is described in [LM06, Section 2.6] as follows. Let $t$ be a reduced labeled tree, with $n$ input leaves, and let $\leqslant$ be a linear order on $\mathrm{Vert}(t)$, such that $x \preccurlyeq y$ implies $x \leqslant y$ for all $x, y \in \mathrm{Vert}(t)$. The choice of $\leqslant$ is equivalent to the choice of decomposition into product of elementary forests (1). The linearly ordered tree $(t, \leqslant)$ determines the map $b^{\mathcal{FQ}_j}_{(t,\leqslant)} : (s\mathcal{FQ}_j)^{\otimes n} \to s\mathcal{FQ}_j$ given by (2) and a similar map $b^{\mathcal{D}_j}_{(t,\leqslant)} : (s\mathcal{D}_j)^{\otimes n} \to s\mathcal{D}_j$. In the commutative diagram from [LM06, Section 2.6]

$$
\begin{tikzcd}
(s\mathcal{Q}_j)^{\otimes n} \arrow[r, "b^{\mathcal{FQ}_j}_{(t,\leqslant)}", "\pm s^{-|t|}"'] \arrow[d, "i_1^{\otimes n}"'] & s\mathcal{F}_t\mathcal{Q}_j \arrow[d, "\hat{\imath}_1"] \\
(s\mathcal{D}_j)^{\otimes n} \arrow[r, "b^{\mathcal{D}_j}_{(t,\leqslant)}"] & s\mathcal{D}_j
\end{tikzcd}
\tag{34}
$$

the top map is invertible, so $\hat{\imath}_1$ is uniquely determined by this diagram.

Being the first component of a strict $A_\infty$-functor $\hat{\imath}_1$ satisfies, in particular, the equation

$$
\bigl((s\mathcal{D}_{j-1})^{\otimes n} \hookrightarrow (s\mathcal{F}_{|}\mathcal{Q}_j)^{\otimes n} \xrightarrow{b_n^{\mathcal{FQ}_j}} s\mathcal{FQ}_j \xrightarrow{\hat{\imath}_1} s\mathcal{D}_j\bigr) = \bigl((s\mathcal{D}_{j-1})^{\otimes n} \xhookrightarrow{i_1^{\otimes n}} (s\mathcal{D}_j)^{\otimes n} \xrightarrow{b_n^{\mathcal{D}_j}} s\mathcal{D}_j\bigr)
$$
$$
= \bigl((s\mathcal{D}_{j-1})^{\otimes n} \xrightarrow{b_n^{\mathcal{D}_{j-1}}} s\mathcal{D}_{j-1} \xhookrightarrow{i_1} s\mathcal{D}_j\bigr) = \bigl((s\mathcal{D}_{j-1})^{\otimes n} \xrightarrow{b_n^{\mathcal{D}_{j-1}}} s\mathcal{D}_{j-1} \hookrightarrow s\mathcal{F}_{|}\mathcal{Q}_j \xrightarrow{\hat{\imath}_1} s\mathcal{D}_j\bigr).
$$

It implies $R_{\mathcal{D}_{j-1}}\hat{\imath}_1 = 0$. Since $\hat{\imath}$ is strict we have also $(R_{\mathcal{D}_{j-1}})\hat{\imath}_1 = 0$. Thus there is a strict $A_\infty$-functor $\iota$ with the first component $\iota_1 : s\mathcal{FQ}_j/(R_{\mathcal{D}_{j-1}}) \to s\mathcal{D}_j$, identity on objects.

Let us construct a degree 0 map $\phi : s\mathcal{D}_j \to s\mathcal{FQ}_j$ for $j \geqslant 1$. Let $t$ be a tree that satisfies (C1). Denote by $\mathrm{UV}(t) \subset \mathrm{Vert}(t)$ the subset of unary internal vertices. Let $\mathrm{minUV}(t)$ be the subset of partially ordered set $(\mathrm{UV}(t), \preccurlyeq)$ consisting of minimal elements. Let $L \subset \mathrm{Leaf}(t)$ be the subset of leaves $\ell$ such that between $\ell$ and the root there are no unary vertices. Let $\overline{L}$ be the set of leaf vertices above leaves from $L$. Using the canonical linear ordering $t_< = (t, \leqslant)$ of $\overline{\mathrm{Vert}}(t)$ [LM06, Section 1.7] we can write the set $\overline{L} \sqcup \mathrm{minUV}(t)$ as $\{u_1 < \dots < u_k\}$. For any $1 \leqslant p \leqslant k$ denote by $t_p$ the $\mathcal{C}$-admissible subtree of $t$ with

$$
\overline{\mathrm{Vert}}(t_p) = \{y \in \overline{\mathrm{Vert}}(t) \mid y \succcurlyeq u_p\} \sqcup \{\text{new root vertex } r_p\}.
$$

Edges of $t_p$ are all edges of $t$ above $u_p$ plus a new root edge between $u_p$ and $r_p$. In particular, if $u_p \in \overline{L}$, then $t_p = |$ is the trivial tree. Denote by $t'$ the reduced labeled tree, which is $t$ with all vertices and edges above $\mathrm{minUV}(t)$ removed. It has precisely $k$ leaves. Thus $t$ is the concatenation of a forest and $t'$:

$$
t = (t_1 \sqcup t_2 \sqcup \dots \sqcup t_k) \cdot t'. \tag{35}
$$

We have

$$
\mathrm{Vert}(t) = \mathrm{Vert}(t') \sqcup \bigsqcup_{p=1}^{k} \mathrm{Vert}(t_p), \qquad \mathrm{Leaf}(t) = \bigsqcup_{p=1}^{k} \mathrm{Leaf}(t_p), \qquad \mathrm{Leaf}(t') = \bigsqcup_{p=1}^{k} \mathrm{Out}(t_p).
$$

Correspondingly, the labels of the $p$-th leaf of $t'$ are those of $\operatorname{Out}(t_p)$.

Being simply a shift, the $\Bbbk$-linear map $b^{\mathcal{D}}_{t'_<} : s\mathcal{D}(t_1)\otimes\cdots\otimes s\mathcal{D}(t_k) \to s\mathcal{D}(t)$ is invertible. Therefore, for any element $x \in s\mathcal{D}(t)$ there exists a unique tensor $\sum_i z^i_1 \otimes \cdots \otimes z^i_k \in s\mathcal{D}(t_1)\otimes\cdots\otimes s\mathcal{D}(t_k)$ such that $x = \sum_i (z^i_1 \otimes \cdots \otimes z^i_k) b^{\mathcal{D}}_{t'_<}$. We have $z^i_p \in s\mathcal{D}(t_p) \subset s\mathcal{Q}_j$, in particular, $z^i_p \in s\mathcal{D}(|) = s\mathcal{C}$, if $u_p \in \overline{L}$. Define

$$x\phi = \sum_i (z^i_1 \otimes \cdots \otimes z^i_k) b^{\mathcal{F}\mathcal{Q}_j}_{t'_<} \in s\mathcal{F}_{t'}\mathcal{Q}_j.$$

Commutative diagram (34) implies that $x\phi\hat{\imath}_1 = \sum_i (z^i_1 \otimes \cdots \otimes z^i_k) b^{\mathcal{D}_j}_{t'_<} = x$. Therefore,

$$\big[s\mathcal{D}_j \xrightarrow{\phi} s\mathcal{F}\mathcal{Q}_j \xrightarrow{\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}}) \xrightarrow{\iota_1} s\mathcal{D}_j\big] = \mathrm{id}\,. \tag{36}$$

Let us prove that $\phi\pi_1$ preserves the operations $b_n$ for $n > 1$. Indeed,

$$b^{\mathcal{D}_j}_n(\phi\pi_1) - (\phi\pi_1)^{\otimes n} b_n = \big[(s\mathcal{D}_j)^{\otimes n} \xrightarrow{b^{\mathcal{D}_j}_n \phi - \phi^{\otimes n} b^{\mathcal{F}\mathcal{Q}_j}_n} s\mathcal{F}\mathcal{Q}_j \xrightarrow{\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big].$$

Consider trees $\tau_1$, $\tau_2$, $\dots$, $\tau_n$ satisfying condition (C1), labeled so that the operation $b^{\mathcal{D}_j}_n : s\mathcal{D}(\tau_1)\otimes\cdots\otimes s\mathcal{D}(\tau_n) \to s\mathcal{D}_j$ makes sense. The quiver $(s\mathcal{D}_j)^{\otimes n}$ is a direct sum of such $s\mathcal{D}(\tau_1)\otimes\cdots\otimes s\mathcal{D}(\tau_n)$. If some of trees $\tau_p$ are not trivial, then $b^{\mathcal{D}_j}_n \phi = \phi^{\otimes n} b^{\mathcal{F}\mathcal{Q}_j}_n$, because constructing $\phi$ for $\tau = (\tau_1 \sqcup \tau_2 \sqcup \cdots \sqcup \tau_n)\cdot \mathfrak{t}_n$ is equivalent to decomposing each $\tau_p$ as in (35), collecting the upper parts, and gluing the lowest parts $\tau'_p$ into $\tau' = (\tau'_1 \sqcup \tau'_2 \sqcup \cdots \sqcup \tau'_n)\cdot \mathfrak{t}_n$. If all trees $\tau_p$ are trivial, then $\mathcal{D}(\tau_p) = \mathcal{D}(|) = \mathcal{C}$ and

$$(b^{\mathcal{C}}_n\phi - \phi^{\otimes n} b^{\mathcal{F}\mathcal{Q}_j}_n)\pi_1 = (b^{\mathcal{C}}_n - b^{\mathcal{F}\mathcal{Q}_j}_n)\pi_1 = 0 : (s\mathcal{C})^{\otimes n} \to s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}}),$$

due to $R_{\mathcal{C}}\pi_1 \subset R_{\mathcal{D}_{j-1}}\pi_1 = 0$.

We claim that

$$\big[s\mathcal{F}\mathcal{Q}_j \xrightarrow{\hat{\imath}_1} s\mathcal{D}_j \xrightarrow{\phi} s\mathcal{F}\mathcal{Q}_j \xrightarrow{\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big] = \big[s\mathcal{F}\mathcal{Q}_j \xrightarrow{\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big]. \tag{37}$$

First of all, the restriction of this equation to $s\mathcal{Q}_j = s\mathcal{F}_|\mathcal{Q}_j$ holds true:

$$\big[s\mathcal{Q}_j \xrightarrow{i_1} s\mathcal{D}_j \xrightarrow{\phi} s\mathcal{F}\mathcal{Q}_j \xrightarrow{\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big] = \big[s\mathcal{Q}_j \hookrightarrow s\mathcal{F}\mathcal{Q}_j \xrightarrow{\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big]. \tag{38}$$

Indeed, $s\mathcal{Q}_j = s\mathcal{D}_{j-1} \oplus s\mathcal{N}_j$. On the first summand we get for $x \in s\mathcal{D}_{j-1}(t)$

$$\begin{aligned} x \xmapsto{i_1} x = \sum_i (z^i_1 \otimes \cdots \otimes z^i_k) b^{\mathcal{D}_j}_{t'_<} &\xmapsto{\phi} \sum_i (z^i_1 \otimes \cdots \otimes z^i_k) b^{\mathcal{F}\mathcal{Q}_j}_{t'_<} \\ &\xmapsto{\pi_1} \sum_i (z^i_1 \otimes \cdots \otimes z^i_k) b^{\mathcal{F}\mathcal{Q}_j}_{t'_<} + (R_{\mathcal{D}_{j-1}}) = \sum_i (z^i_1 \otimes \cdots \otimes z^i_k) b^{\mathcal{D}_{j-1}}_{t'_<} + (R_{\mathcal{D}_{j-1}}) = x\pi_1 \end{aligned}$$

by Proposition 3.2 because $z^i_p \in s\mathcal{D}_{j-1}$. On the second summand we get for $x \in s\mathcal{N}_j(t)$

$$x \xmapsto{i_1} x \xmapsto{\phi} x \xmapsto{\pi_1} x\pi_1$$

since $t' = |$. Thus, (38) is verified.

Now we prove (37) on the generic summand $s\mathcal{F}_\tau\mathcal{Q}_j$ of $s\mathcal{F}\mathcal{Q}_j$, where $\tau$ is a reduced labeled tree with $n$ leaves. The first map below is an isomorphism:

$$\begin{aligned}
&\big[(s\mathcal{Q}_j)^{\otimes n} \xrightarrow{b_{\tau_<}^{\mathcal{F}\mathcal{Q}_j}} s\mathcal{F}_\tau\mathcal{Q}_j \xrightarrow{\hat\imath_1} s\mathcal{D}_j \xrightarrow{\phi\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big]\\
&= \big[(s\mathcal{Q}_j)^{\otimes n} \xhookrightarrow{i_1^{\otimes n}} (s\mathcal{D}_j)^{\otimes n} \xrightarrow{b_{\tau_<}^{\mathcal{D}_j}} s\mathcal{D}_j \xrightarrow{\phi\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big]\\
&= \big[(s\mathcal{Q}_j)^{\otimes n} \xhookrightarrow{i_1^{\otimes n}} (s\mathcal{D}_j)^{\otimes n} \xrightarrow{\phi\pi_1} (s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}}))^{\otimes n} \xrightarrow{b_{\tau_<}} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big]\\
&= \big[(s\mathcal{Q}_j)^{\otimes n} \hookrightarrow (s\mathcal{F}\mathcal{Q}_j)^{\otimes n} \xrightarrow{\pi_1^{\otimes n}} (s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}}))^{\otimes n} \xrightarrow{b_{\tau_<}} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big]\\
&= \big[(s\mathcal{Q}_j)^{\otimes n} \xrightarrow{b_{\tau_<}^{\mathcal{F}\mathcal{Q}_j}} s\mathcal{F}_\tau\mathcal{Q}_j \xrightarrow{\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big]
\end{aligned}$$

by (38) and by the fact that the considered maps $\hat\imath_1$, $\phi\pi_1$ and $\pi_1$ commute with $b_{\tau_<}$.

Rewriting (37) in the form

$$\begin{aligned}
\big[s\mathcal{F}\mathcal{Q}_j \xrightarrow{\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}}) \xrightarrow{\iota_1} s\mathcal{D}_j \xrightarrow{\phi} s\mathcal{F}\mathcal{Q}_j \xrightarrow{\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big]&\\
= \big[s\mathcal{F}\mathcal{Q}_j \xrightarrow{\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big]&,
\end{aligned}$$

we find by surjectivity of $\pi_1$ that

$$\big[s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}}) \xrightarrow{\iota_1} s\mathcal{D}_j \xrightarrow{\phi} s\mathcal{F}\mathcal{Q}_j \xrightarrow{\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big] = \mathrm{id}\,.$$

Together with (36) this proves that $\phi\pi_1$ is an inverse to $\iota_1$. ∎

5.4. Corollary. *The $A_\infty$-category $\mathcal{D} = \mathsf{Q}(\mathcal{C}|\mathcal{B})$ is relatively free over $\mathcal{C}$.*

5.5. The first equivalence. Let $\mathcal{A}$ be pseudounital, then the restriction functor

$$A^{\psi u}_{\infty 1}(\mathcal{D}_0, \mathcal{Q}_1; \mathcal{A}) \to A^{\psi u}_\infty(\mathcal{C}, \mathcal{A}), \qquad (f, f') \mapsto f$$

takes values in the full subcategory $A^{\psi u}_\infty(\mathcal{C}, \mathcal{A})_{\mathrm{mod}\,\mathcal{B}}$. Indeed, let $(f, f')$ be an object of $A^{\psi u}_{\infty 1}(\mathcal{D}_0, \mathcal{Q}_1; \mathcal{A})$. For arbitrary objects $X$, $Y$ of $\mathcal{B}$ we have

$$\begin{aligned}
f_1 &= \big[s\mathcal{C}(X,Y) \hookrightarrow s\mathcal{Q}_1(X,Y) \xrightarrow{f'_1} s\mathcal{A}(Xf, Yf)\big]\\
&= \big[s\mathcal{D}(|)(X,Y) \xrightarrow{b_1H + Hb_1} s\mathcal{Q}_1(X,Y) \xrightarrow{f'_1} s\mathcal{A}(Xf, Yf)\big]\\
&= \big[s\mathcal{D}(|)(X,Y) \xrightarrow{b_1(Hf'_1) + (Hf'_1)b_1} s\mathcal{A}(Xf, Yf)\big],
\end{aligned}$$

where $H$ is the map $H : s\mathcal{D}(|) \xrightarrow{s} s\mathcal{D}(\mathfrak{t}_1) \hookrightarrow s\mathcal{Q}_1$. Hence, the above $f_1$ is null-homotopic. By Definition 6.4 of [LO06] the $A_\infty$-functor $f\big|_{\mathcal{B}}$ is contractible.

A short exact sequence of chain maps of complexes is *semisplit* (resp. *semisplittable*) if it is split (resp. splittable) as a sequence of degree 0 maps of graded $\Bbbk$-modules.

5.6. Lemma. *Let* $0 \to C \xrightarrow{\alpha} A \xrightarrow{\beta} B \to 0$ *be a semisplittable exact sequence of complexes of* $\Bbbk$*-modules. If* $C$ *is contractible, then this sequence is splittable, and the splitting chain map* $\nu : B \to A$ *can be chosen so that* $\nu$ *is homotopy inverse to* $\beta$*.*

Proof. Let $\phi : A \to C$ be a map of degree 0, such that $\alpha\phi = 1_C$. Assume that $1_C = Hd + dH$ for a homotopy $H : C \to C$ of degree $-1$. Then $\psi = (\phi H)d = \phi Hd + d\phi H : A \to C$ is a chain map, and $\alpha\psi = \alpha\phi Hd + d\alpha\phi H = Hd + dH = 1_C$. Denote by $\nu : B \to A$ the unique $\Bbbk$-linear map such that $\nu\psi = 0$, $\nu\beta = 1$. The splitting injection $\nu = \ker\psi$ is a chain map. The sequence looks as follows

$$0 \to C \underset{\psi}{\overset{\alpha}{\rightleftarrows}} A \underset{\nu}{\overset{\beta}{\rightleftarrows}} B \to 0.$$

Since $A$ is a direct sum $C \oplus B$, we have

$$\mathrm{id}_A - \beta\nu = \psi\alpha = (\phi Hd + d\phi H)\alpha = (\phi H\alpha)d + d(\phi H\alpha) = \gamma d + d\gamma,$$

where $\gamma = \bigl(A \xrightarrow{\phi} C \xrightarrow{H} C \xrightarrow{\alpha} A\bigr)$ is a homotopy. ■

5.7. Proposition. *Let* $\mathcal{A}$ *be unital, then the restriction strict* $A_\infty$*-functor*

$$\mathrm{restr} : A^{\psi u}_{\infty 1}(\mathcal{D}_0, \mathcal{Q}_1; \mathcal{A}) \to A^u_\infty(\mathcal{C}, \mathcal{A})_{\mathrm{mod}\,\mathcal{B}} \tag{39}$$

*is an* $A_\infty$*-equivalence, surjective on objects. The chain surjections* $\mathrm{restr}_1$ *admit a chain splitting.*

Proof. First of all, restr is surjective on objects. Indeed, assume that $f : \mathcal{C} \to \mathcal{A}$ is unital and $\mathcal{B} \hookrightarrow \mathcal{C} \xrightarrow{f} \mathcal{A}$ is contractible. We have to extend the chain maps $f_1 : s\mathcal{C}(X,Y) \to s\mathcal{A}(Xf,Yf)$ to chain maps $f'_1 : s\mathcal{Q}_1(X,Y) \to s\mathcal{A}(Xf,Yf)$. If $X,Y \notin \operatorname{Ob}\mathcal{B}$, then $s\mathcal{Q}_1(X,Y) = s\mathcal{C}(X,Y)$ and $f'_1 = f_1$. If $X \in \operatorname{Ob}\mathcal{B}$ or $Y \in \operatorname{Ob}\mathcal{B}$, then $s\mathcal{Q}_1(X,Y) = s\mathcal{C}(X,Y) \oplus s\mathcal{D}(\mathsf{t}_1)(X,Y)$ as a graded quiver, and the complex $\mathcal{A}(Xf,Yf)$ is contractible by Proposition 6.1(C1), (C2) of [LO06]. Let $\chi_{XY} : s\mathcal{A}(Xf,Yf) \to s\mathcal{A}(Xf,Yf)$ be a contracting homotopy for $\mathcal{A}(Xf,Yf)$. Define

$$f'_1 = \bigl(s\mathcal{D}(\mathsf{t}_1)(X,Y) \xrightarrow{s^{-1}} s\mathcal{C}(X,Y) \xrightarrow{f_1} s\mathcal{A}(Xf,Yf) \xrightarrow{\chi_{XY}} s\mathcal{A}(Xf,Yf)\bigr).$$

Then $Hf'_1 = f_1\chi_{XY}$. Since $H = s : s\mathcal{C} \to s\mathcal{D}(\mathsf{t}_1)$ is invertible, the equation

$$Hf'_1b_1 - Hb_1f'_1 = Hf'_1b_1 + b_1Hf'_1 - f'_1 = f_1\chi_{XY}b_1 + b_1f_1\chi_{XY} - f_1 = 0 : \\ s\mathcal{C}(X,Y) \to s\mathcal{A}(Xf,Yf)$$

implies that $f'_1$ is a chain map.

Let us prove that the restriction chain map

$$\mathrm{restr}_1 : sA^{\psi u}_{\infty 1}(\mathcal{C}, \mathcal{Q}_1; \mathcal{A})\bigl((f,f'),(g,g')\bigr) \to sA_\infty(\mathcal{C},\mathcal{A})(f,g), \qquad (r,r') \mapsto r$$

is homotopy invertible. This map is a product over $n \in \mathbb{Z}_{\geqslant 0}$ of the restriction maps $\rho_n : V_n \to V'_n$ of the graded $\Bbbk$-modules of $n$-th components (compare (15) with analogous decomposition of $sA_\infty(\mathcal{C},\mathcal{A})(f,g)$). Clearly, the maps $\rho_n = \mathrm{id}$ for $n = 0$ or for $n > 1$. On the other hand, for $n = 1$

$$\rho_1 = \prod \underline{\mathsf{C}}_\Bbbk(\mathrm{in}^{\mathcal{C}}, 1) : \prod_{X,Y\in \mathrm{Ob}\,\mathcal{C}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{Q}_1(X,Y), s\mathcal{A}(Xf,Yg)) \to \prod_{X,Y\in \mathrm{Ob}\,\mathcal{C}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{C}(X,Y), s\mathcal{A}(Xf,Yg))$$

is surjective with the kernel $\operatorname{Ker}\rho_1 = \prod_{X,Y\in\mathrm{Ob}\,\mathcal{C}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{D}(\mathfrak{t}_1)(X,Y), s\mathcal{A}(Xf,Yg))$, because the sequence $0 \to s\mathcal{C} \to s\mathcal{Q}_1 \to s\mathcal{D}(\mathfrak{t}_1) \to 0$ is semisplit. Since we may write the kernel as

$$\operatorname{Ker}\rho_1 = \prod_{\substack{X\in\mathrm{Ob}\,\mathcal{B}, Y\in\mathrm{Ob}\,\mathcal{C}\\ \text{or } X\in\mathrm{Ob}\,\mathcal{C}, Y\in\mathrm{Ob}\,\mathcal{B}}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{D}(\mathfrak{t}_1)(X,Y), s\mathcal{A}(Xf,Yg)),$$

it is contractible, because contractibility of $f\big|_{\mathcal{B}}, g\big|_{\mathcal{B}} : \mathcal{B} \to \mathcal{A}$ implies contractibility of complexes $\mathcal{A}(Xf,Yg)$ by Proposition 6.1(C1), (C2) of [LO06].

Summing up, the first term of the semisplit exact sequence

$$0 \to \operatorname{Ker}\mathrm{restr}_1 \longrightarrow sA_{\infty 1}(\mathcal{C},\mathcal{Q}_1;\mathcal{A})\big((f,f'),(g,g')\big) \xrightarrow{\mathrm{restr}_1} sA_\infty(\mathcal{C},\mathcal{A})(f,g) \to 0$$

is contractible. By Lemma 5.6 this sequence admits a splitting chain map

$$\nu : sA_\infty(\mathcal{C},\mathcal{A})(f,g) \to sA_{\infty 1}(\mathcal{C},\mathcal{Q};\mathcal{A})\big((f,f'),(g,g')\big),$$

and $\nu$ is homotopy inverse to $\mathrm{restr}_1$. Applying Corollary 1.9 we conclude that (39) is an $A_\infty$-equivalence. ■

5.8. Corollary. *$A^u_\infty$-2-transformation* (39) *is a natural $A^u_\infty$-2-equivalence.*

5.9. Proposition. *Let $\mathcal{A}$ be a unital $A_\infty$-category, let $\mathcal{D}$ be a pseudounital $A_\infty$-category with distinguished cycles $\iota^{\mathcal{D}}_X$, let $\mathcal{Q}$ be a differential graded quiver and let $\mathcal{N}$ be a graded quiver such that $\mathrm{Ob}\,\mathcal{D} = \mathrm{Ob}\,\mathcal{Q} = \mathrm{Ob}\,\mathcal{N}$ and $\mathcal{Q} = \mathcal{D}\oplus\mathcal{N}$. Suppose that $\mathcal{N}(X,Y) \neq 0$ implies that $\iota^{\mathcal{D}}_X \in \operatorname{Im} b_1$ or $\iota^{\mathcal{D}}_Y \in \operatorname{Im} b_1$. Then an arbitrary pseudounital $A_\infty$-functor $f : \mathcal{D} \to \mathcal{A}$ extends to an object $(f,f') \in \mathrm{Ob}\,A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ for some $f'$.*

Proof. Let $s\mathcal{M} = \mathcal{N}$ be the differential graded quiver and let $\alpha : s\mathcal{M} \to s\mathcal{D}$ be the chain map defined in Section 4.5. There exists a homotopy $\overline{f} : s\mathcal{M}(X,Y) \to s\mathcal{A}(Xf,Yf)$ of degree $-1$ such that

$$\alpha f_1 = \overline{f} b_1 + d^{\mathcal{M}[1]}\overline{f} : s\mathcal{M}(X,Y) \longrightarrow s\mathcal{A}(Xf,Yf).$$

Indeed, the case of $\mathcal{M}(X,Y) = 0$ being obvious, we may assume that $\iota_X \in \operatorname{Im} b_1$ or $\iota_Y \in \operatorname{Im} b_1$. Then ${}_{Xf}\mathbf{i}^{\mathcal{A}}_0 \in \iota_X f_1 + \operatorname{Im} b_1 \subset \operatorname{Im} b_1$ or ${}_{Yf}\mathbf{i}^{\mathcal{A}}_0 \in \iota_Y f_1 + \operatorname{Im} b_1 \subset \operatorname{Im} b_1$. Since $\mathcal{A}$ is unital, the complex $s\mathcal{A}(Xf,Yf)$ is contractible with some contracting homotopy $\overline{h}$. We may take $\overline{f} = \alpha f_1 \overline{h}$.

Define a degree 0 map

$$f_1' = \bigl(s\mathcal{Q}(X,Y) = s\mathcal{D}(X,Y) \oplus s\mathcal{N}(X,Y) \xrightarrow{(f_1, s^{-1}\overline{f})} s\mathcal{A}(Xf,Yf)\bigr).$$

For arbitrary $p \in s\mathcal{D}(X,Y)$, $m \in s\mathcal{M}(X,Y) = \mathcal{N}(X,Y)$ we have

$$\begin{aligned}(p,ms)(f_1'b_1^{\mathcal{A}} - b_1^{\mathcal{Q}}f_1') &= pf_1b_1^{\mathcal{A}} + m\overline{f}b_1^{\mathcal{A}} - (pb_1^{\mathcal{D}} + m\alpha)f_1 + (md^{\mathcal{M}[1]}s)s^{-1}\overline{f} \\ &= p(f_1b_1^{\mathcal{A}} - b_1^{\mathcal{D}}f_1) + m(\overline{f}b_1^{\mathcal{A}} + d^{\mathcal{M}[1]}\overline{f} - \alpha f_1) = 0\end{aligned}$$

by (18). Therefore, $f_1'$ is a chain map and $f_1'\big|_{s\mathcal{D}} = f_1$. ■

5.10. PROPOSITION. *In assumptions of Proposition 5.9 the restriction strict $A_\infty$-functor*

$$\mathrm{restr} : A_{\infty1}^{\psi u}(\mathcal{D},\mathcal{Q};\mathcal{A}) \to A_\infty^{\psi u}(\mathcal{D},\mathcal{A}), \qquad (x,x') \mapsto x \tag{40}$$

*is an $A_\infty$-equivalence, surjective on objects. The chain surjections* $\mathrm{restr}_1$ *admit a chain splitting.*

PROOF. Let us prove that for an arbitrary pair of objects $(f,f')$, $(g,g')$ of $A_{\infty1}^{\psi u}(\mathcal{D},\mathcal{Q};\mathcal{A})$ the restriction chain map

$$sA_{\infty1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f,f'),(g,g')\bigr) \to sA_\infty(\mathcal{D},\mathcal{A})(f,g)$$

is homotopy invertible. This map is a product over $n \in \mathbb{Z}_{\geqslant 0}$ of the restriction maps $\rho_n : V_n \to V_n'$ of the graded $\Bbbk$-modules of $n$-th components (compare (15) with analogous decomposition of $sA_\infty(\mathcal{D},\mathcal{A})(f,g)$). Clearly, the maps $\rho_n = \mathrm{id}$ for $n=0$ or for $n>1$. On the other hand, for $n=1$

$$\rho_1 = \prod \underline{\mathsf{C}}_\Bbbk(\mathrm{in}^{\mathcal{D}},1) : \\ \prod_{X,Y\in\operatorname{Ob}\mathcal{D}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{Q}(X,Y), s\mathcal{A}(Xf,Yg)) \to \prod_{X,Y\in\operatorname{Ob}\mathcal{D}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{D}(X,Y), s\mathcal{A}(Xf,Yg))$$

is surjective with the kernel $\operatorname{Ker}\rho_1 = \prod_{X,Y\in\operatorname{Ob}\mathcal{D}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{N}(X,Y), s\mathcal{A}(Xf,Yg))$. As in proof of Proposition 5.9 $\mathcal{N}(X,Y) \neq 0$ implies that ${}_{Xf}\mathbf{i}_0^{\mathcal{A}} \in \iota_X f_1 + \operatorname{Im} b_1 \subset \operatorname{Im} b_1$ or ${}_{Yg}\mathbf{i}_0^{\mathcal{A}} \in \iota_Y g_1 + \operatorname{Im} b_1 \subset \operatorname{Im} b_1$, hence, $s\mathcal{A}(Xf,Yg)$ is contractible. Therefore, for all objects $X$, $Y$ of $\mathcal{D}$ the complex $\underline{\mathsf{C}}_\Bbbk(s\mathcal{N}(X,Y), s\mathcal{A}(Xf,Yg))$ is contractible. Thus, $\operatorname{Ker}\mathrm{restr}_1 = \operatorname{Ker}\rho_1$ is contractible.

Summing up, the first term of the semisplit exact sequence

$$0 \to \operatorname{Ker}\mathrm{restr}_1 \longrightarrow sA_{\infty1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f,f'),(g,g')\bigr) \xrightarrow{\mathrm{restr}_1} sA_\infty(\mathcal{D},\mathcal{A})(f,g) \to 0$$

is contractible. By Lemma 5.6 this sequence admits a splitting chain map

$$\nu : sA_\infty(\mathcal{D},\mathcal{A})(f,g) \to sA_{\infty1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f,f'),(g,g')\bigr),$$

and $\nu$ is homotopy inverse to $\operatorname{restr}_1$.

By Proposition 5.9 the surjection

$$\operatorname{Ob} A^{\psi u}_{\infty 1}(\mathcal{D}, \mathcal{Q}; \mathcal{A}) \ni (f, f') \mapsto f \in \operatorname{Ob} A^{\psi u}_{\infty}(\mathcal{D}, \mathcal{A})$$

admits a splitting $f \mapsto \widetilde{f} = (f, f')$. Applying Corollary 1.9 we conclude that (40) is an $A_\infty$-equivalence. ∎

5.11. Corollary. *$A^u_\infty$-2-transformation* (40) *is a natural $A^u_\infty$-2-equivalence.*

An easy converse to Lemma 5.6 is given by

5.12. Lemma. *Let $\beta : A \to B$, $\nu : B \to A$ be chain maps of complexes of $\Bbbk$-modules, such that $\nu\beta = \mathrm{id}_B$. Denote $C = \operatorname{Ker} \nu$, then $A \simeq C \oplus B$. If $\beta$ is a homotopy isomorphism, then the chain complex $C$ is contractible.*

Proof. Being one-sided inverse to homotopy isomorphism $\beta$, the map $\nu$ is homotopy inverse to $\beta$. Therefore, $\mathrm{id}_A - \beta\nu = hd + dh$ for some homotopy $h : A \to A$ of degree $-1$. Since $C$ is the image of the idempotent $\mathrm{id}_A - \beta\nu = \mathrm{pr}^C \cdot \mathrm{in}^C$, we have

$$\mathrm{id}_C = \mathrm{in}^C \mathrm{pr}^C \mathrm{in}^C \mathrm{pr}^C = \mathrm{in}^C (\mathrm{id}_A - \beta\nu) \mathrm{pr}^C = (\mathrm{in}^C h \, \mathrm{pr}^C) d + d(\mathrm{in}^C h \, \mathrm{pr}^C).$$

Thus, $\mathrm{id}_C = Hd + dH$ for $H = \mathrm{in}^C h \, \mathrm{pr}^C : C \to C$, and $C$ is contractible. ∎

5.13. Theorem. *Let $\mathcal{A}$ be a unital $A_\infty$-category, and let $\mathcal{D} = \cup_{j \geqslant 0} \mathcal{D}_j = \varinjlim_j \mathcal{D}_j = \mathsf{Q}(\mathcal{C}|\mathcal{B})$ be as in Section 5.2. Then the restriction strict $A_\infty$-functor*

$$\operatorname{restr} : A^{\psi u}_\infty(\mathcal{D}, \mathcal{A}) \to A^u_\infty(\mathcal{C}, \mathcal{A})_{\operatorname{mod} \mathcal{B}}$$

*is an $A_\infty$-equivalence, 2-natural in $\mathcal{A}$, surjective on objects. The chain surjections* $\operatorname{restr}_1$ *admit a chain splitting.*

Proof. All restriction strict $A_\infty$-functors in the sequence

$$\begin{aligned} A^u_\infty(\mathcal{C}, \mathcal{A})_{\operatorname{mod} \mathcal{B}} \longleftarrow A^{\psi u}_{\infty 1}(\mathcal{D}_0, \mathcal{Q}_1; \mathcal{A}) \longleftarrow A^{\psi u}_\infty(\mathcal{D}_1, \mathcal{A}) \\ \longleftarrow A^{\psi u}_{\infty 1}(\mathcal{D}_1, \mathcal{Q}_2; \mathcal{A}) \longleftarrow A^{\psi u}_\infty(\mathcal{D}_2, \mathcal{A}) \longleftarrow A^{\psi u}_{\infty 1}(\mathcal{D}_2, \mathcal{Q}_3; \mathcal{A}) \longleftarrow \dots \end{aligned} \tag{41}$$

are $A_\infty$-equivalences (and natural $A^u_\infty$-2-equivalences). They are surjective on objects. The first components are surjective and admit a chain splitting. For the first functor it follows from Proposition 5.7. For other odd-numbered functors it follows from Proposition 5.10. Indeed, if $X \in \operatorname{Ob} \mathcal{B}$, then ${}_X\mathbf{i}^{\mathcal{C}}_0 = {}_X\mathbf{i}^{\mathcal{C}}_0 H b_1 \in \operatorname{Im}(b_1 : s\mathcal{D}_1(X, X) \to s\mathcal{D}_1(X, X))$. For even-numbered functors it follows from Theorem 4.7.

Let us show that $A^{\psi u}_\infty(\mathcal{D}, \mathcal{A})$ is the inverse limit of (41) on objects and on morphisms. There are restriction strict $A_\infty$-functors

$$\begin{aligned} &\operatorname{restr} : A^{\psi u}_\infty(\mathcal{D}, \mathcal{A}) \to A^u_\infty(\mathcal{C}, \mathcal{A})_{\operatorname{mod} \mathcal{B}} \subset A^u_\infty(\mathcal{C}, \mathcal{A}), && f \mapsto f|_{\mathcal{C}}, \\ &\operatorname{restr} : A^{\psi u}_\infty(\mathcal{D}, \mathcal{A}) \to A^{\psi u}_{\infty 1}(\mathcal{D}_j, \mathcal{Q}_{j+1}; \mathcal{A}), && f \mapsto (f|_{\mathcal{D}_j}, f|_{\mathcal{Q}_{j+1}}), \quad j \geqslant 0, \\ &\operatorname{restr} : A^{\psi u}_\infty(\mathcal{D}, \mathcal{A}) \to A^{\psi u}_\infty(\mathcal{D}_j, \mathcal{A}), && f \mapsto f|_{\mathcal{D}_j}, \quad j \geqslant 1. \end{aligned}$$

They agree with the functors restr from (41) in the sense $\mathrm{restr}\cdot\mathrm{restr} = \mathrm{restr}$.

Since $\mathcal{D} = \cup_{j\geqslant 0}\mathcal{D}_j$, pseudounital $A_\infty$-functors $f : \mathcal{D} \to \mathcal{A}$ are in bijection with sequences $(f^j)_j$ of pseudounital $A_\infty$-functors $f^j : \mathcal{D}_j \to \mathcal{A}$ such that $f^{j+1}\big|_{\mathcal{D}_j} = f^j$. In other words, $\operatorname{Ob} A_\infty^{\psi u}(\mathcal{D},\mathcal{A})$ is the inverse limit of the sequence of surjections

$$\begin{aligned}\operatorname{Ob} A^u_\infty(\mathcal{C},\mathcal{A})_{\mathrm{mod}\,\mathcal{B}} \twoheadleftarrow \operatorname{Ob} A^{\psi u}_{\infty 1}(\mathcal{D}_0,\mathcal{Q}_1;\mathcal{A}) \twoheadleftarrow \operatorname{Ob} A^{\psi u}_\infty(\mathcal{D}_1,\mathcal{A})\\ \twoheadleftarrow \operatorname{Ob} A^{\psi u}_{\infty 1}(\mathcal{D}_1,\mathcal{Q}_2;\mathcal{A}) \twoheadleftarrow \operatorname{Ob} A^{\psi u}_\infty(\mathcal{D}_2,\mathcal{A}) \twoheadleftarrow \operatorname{Ob} A^{\psi u}_{\infty 1}(\mathcal{D}_2,\mathcal{Q}_3;\mathcal{A}) \twoheadleftarrow \dots\end{aligned}$$

In particular, the map

$$\operatorname{Ob}\mathrm{restr} : \operatorname{Ob} A^{\psi u}_\infty(\mathcal{D},\mathcal{A}) \to \operatorname{Ob} A^u_\infty(\mathcal{C},\mathcal{A})_{\mathrm{mod}\,\mathcal{B}}$$

is surjective.

Let $f, g : \mathcal{D} \to \mathcal{A}$ be pseudounital $A_\infty$-functors. Since $\mathcal{D} = \bigl(\cup_{j\geqslant 0}\mathcal{D}_j\bigr)\bigcup\bigl(\cup_{j\geqslant 1}\mathcal{Q}_j\bigr)$, $A_\infty$-transformations $p : f \to g : \mathcal{D} \to \mathcal{A}$ are in bijection with sequences

$$(p^0, p'^1, p^1, p'^2, p^2, p'^3, \dots)$$

of $A_\infty$-transformations $p^j : f|_{\mathcal{D}_j} \to g|_{\mathcal{D}_j} : \mathcal{D}_j \to \mathcal{A}$, $j \geqslant 0$, and $A_1$-transformations $p'^j : f|_{\mathcal{Q}_j} \to g|_{\mathcal{Q}_j} : \mathcal{Q}_j \to \mathcal{A}$, $j \geqslant 1$, such that $p^{j+1}\big|_{\mathcal{D}_j} = p^j$, $p^j\big|_{\mathcal{Q}_j} = p'^j$. In other words, $A_\infty(\mathcal{D},\mathcal{A})(f,g)$ is the inverse limit of the sequence of splittable chain surjections

$$\begin{aligned}A_\infty(\mathcal{C},\mathcal{A})(f|_{\mathcal{C}}, g|_{\mathcal{C}}) \twoheadleftarrow A_{\infty 1}(\mathcal{D}_0,\mathcal{Q}_1;\mathcal{A})\bigl((f|_{\mathcal{D}_0}, f|_{\mathcal{Q}_1}),(g|_{\mathcal{D}_0}, g|_{\mathcal{Q}_1})\bigr) \twoheadleftarrow \dots\\ \twoheadleftarrow A_\infty(\mathcal{D}_j,\mathcal{A})(f|_{\mathcal{D}_j}, g|_{\mathcal{D}_j}) \twoheadleftarrow A_{\infty 1}(\mathcal{D}_j,\mathcal{Q}_{j+1};\mathcal{A})\bigl((f|_{\mathcal{D}_j}, f|_{\mathcal{Q}_{j+1}}),(g|_{\mathcal{D}_j}, g|_{\mathcal{Q}_{j+1}})\bigr) \twoheadleftarrow \dots\end{aligned}$$

Since these surjections are splittable, the above sequence is isomorphic to the sequence of natural projections

$$C_0 \twoheadleftarrow C_0\times C_1 \twoheadleftarrow C_0\times C_1\times C_2 \twoheadleftarrow \dots \twoheadleftarrow \prod_{m=0}^{n} C_m \twoheadleftarrow \prod_{m=0}^{n+1} C_m \twoheadleftarrow \dots$$

for some complexes $C_m$ of $\Bbbk$-modules. Its inverse limit is $\prod_{m=0}^\infty C_m \simeq A_\infty(\mathcal{D},\mathcal{A})(f,g)$. By Lemma 5.12 all $C_m$ are contractible for $m > 0$. Therefore, $\prod_{m=1}^\infty C_m$ is contractible. We obtain a split exact sequence

$$0 \to \prod_{m=1}^{\infty} C_m \longrightarrow A_\infty(\mathcal{D},\mathcal{A})(f,g) \xrightarrow{\ \beta\ } A_\infty(\mathcal{C},\mathcal{A})(f|_{\mathcal{C}}, g|_{\mathcal{C}}) \to 0$$

with contractible first term. By Lemma 5.6 $\beta = s\,\mathrm{restr}_1\,s^{-1}$ is a homotopy isomorphism.

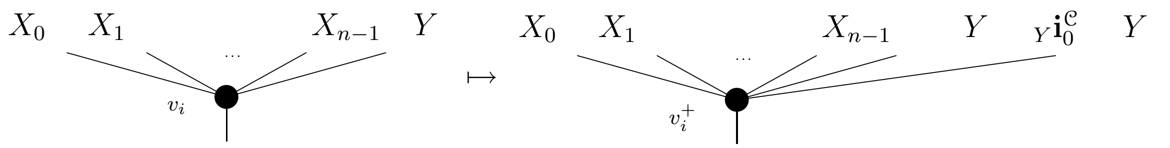
Using Corollary 1.9 we prove the theorem. ■

## 6. Unitality of $\mathcal{D}$

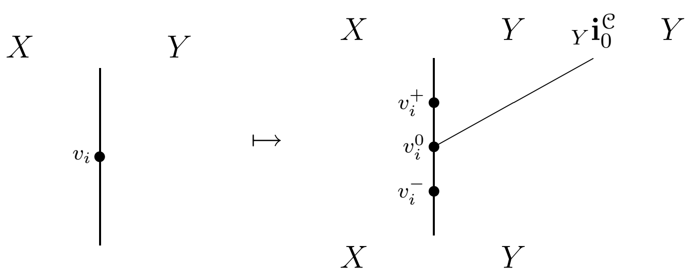
We are going to prove that if $\mathcal{C}$ is strictly unital, then the $A_\infty$-category $\mathcal{D}$ constructed in Section 5.2 is not only pseudounital, but unital with the unit elements ${}_X\mathbf{i}_0^{\mathcal{C}} \in s\mathcal{D}(|)(X,X) \subset s\mathcal{D}(X,X)$. Let us describe $\Bbbk$-linear maps $h : s\mathcal{D}(X,Y) \to s\mathcal{D}(X,Y)$ of degree $-1$ such that

$$1 - (1 \otimes \mathbf{i}_0^{\mathcal{C}})b_2 = b_1 h + h b_1 : s\mathcal{D}(X,Y) \to s\mathcal{D}(X,Y). \tag{42}$$

The homotopy $h$ is called the *right unit homotopy*. Let $t$ be a $\mathcal{C}$-admissible tree. Let $y$ be its rightmost leaf vertex. Let $v_0 \prec v_1 \prec \dots \prec v_p \prec v_{p+1} = y$ be the directed path connecting the root $v_0$ with $y$, $p \geqslant 0$. Vertices $v_i$ and $v_{i+1}$ are connected by an edge for all $0 \leqslant i \leqslant p$. Let $t_i^+$, $1 \leqslant i \leqslant p$ be the tree $t$ with an extra leaf attached on the right to the vertex $v_i$ if $v_i$ is $n$-ary, $n > 1$ as in




If $v_i$ is unary, we attach an extra leaf on the right to $v_i$ and add two more unary vertices above and below it as in




The obtained trees $t_i^+$ are $\mathcal{C}$-admissible.

Let $x$ be a homogeneous element of $s\mathcal{D}(t)(Z,Y)$. Define $x_i^+ = x \otimes_Y \mathbf{i}_0^{\mathcal{C}} \in s\mathcal{D}(t_i^+)(Z,Y) = s\mathcal{D}(t)(Z,Y) \otimes s\mathcal{C}(Y,Y)$. We claim that if $\mathcal{C}$ is strictly unital, then the map

$$h : x \mapsto \sum_{i=1}^{p} \pm x_i^+ \tag{43}$$

with an appropriate choice of signs satisfies (42). To describe the signs and to prove the claim we study the set of operations acting in $\mathcal{D}$.

### 6.1. A multicategory operating in $\mathcal{D}$.

Let $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$ be the free graded $\Bbbk$-linear (non-symmetric) operad, generated by

— a 0-ary operation $\mathbf{i}_0 \in \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}(0)$ of degree $-1$,

— a unary operation $H \in \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}(1)$ of degree $-1$,

— an $n$-ary operation $b_n \in \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}(n)$ of degree 1 for each $n > 1$.

The construction of free non-symmetric operads uses plane trees instead of abstract trees. Otherwise it is similar to the case of symmetric operads, see e.g. [MSS02]. The operad $\mathscr{A}_\infty$ of operations in $A_\infty$-algebras (e.g. [Mar00]) is a suboperad of $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$.

Actually we need a multicategory [Lam69, Lam89] rather than an operad. A multicategory is a many object version of an operad, a (non-symmetric) operad is a one-object multicategory. So we define a graded $\Bbbk$-linear multicategory $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$, whose objects are pairs of objects of $\mathcal{C}$, thus $\operatorname{Ob}\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}} = \operatorname{Ob}\mathcal{C} \times \operatorname{Ob}\mathcal{C}$. For $n > 0$ the graded $\Bbbk$-module of morphisms

$$\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}\big((X_1', X_1), (X_2', X_2), \dots, (X_n', X_n); (Y', Y)\big)$$

is 0 unless $X_1' = Y'$, $X_n = Y$ and $X_i = X_{i+1}'$ for all $1 \leqslant i < n$. For $n = 0$ the graded $\Bbbk$-module of morphisms $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}(;(Y', Y))$ is 0 unless $Y' = Y$. The morphisms of $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$ are freely generated by

— 0-ary operations ${}_X\mathbf{i}_0 \in \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}(;(X, X))$, $X \in \operatorname{Ob}\mathcal{C}$ of degree $\deg {}_X\mathbf{i}_0 = -1$,

— unary operations $H = H_{X,Y} \in \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}((X, Y); (X, Y))$, $X, Y \in \operatorname{Ob}\mathcal{C}$, where $X \in \operatorname{Ob}\mathcal{B}$ or $Y \in \operatorname{Ob}\mathcal{B}$, of degree $\deg H = -1$,

— $n$-ary operations $b_n \in \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}\big((X_0, X_1), (X_1, X_2), \dots, (X_{n-1}, X_n); (X_0, X_n)\big)$, of degree $\deg b_n = 1$ for $X_0, \dots, X_n \in \operatorname{Ob}\mathcal{C}$, $n > 1$.

We shall not insist on distinguishing between the operad $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$ and its refinement – the multicategory $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$, leaving the choice of context to the reader.

Similarly to the operad $\mathscr{A}_\infty$ [Mar00], governing $A_\infty$-algebras, the multicategory $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$ has a differential $d$ – a derivation of degree 1, such that $d^2 = 0$. In general, a derivation $d$ of degree $p$ of a graded $\Bbbk$-linear multicategory $\mathscr{M}$ is a collection of $\Bbbk$-linear endomorphisms $d$ of $\mathscr{M}(Z_1, \dots, Z_n; Z)$ of degree $p$, such that all compositions (which are of degree 0)

$$\mu_i : \mathscr{M}(Y_1, \dots, Y_k; Z_i) \otimes \mathscr{M}(Z_1, \dots, Z_n; Z) \to \mathscr{M}(Z_1, \dots, Z_{i-1}, Y_1, \dots, Y_k, Z_{i+1}, \dots, Z_n; Z) \tag{44}$$

satisfy the equation

$$\mu_i d = (1 \otimes d + d \otimes 1)\mu_i$$

with the sign conventions of this article. If $d^2 = 0$, we may say that $\mu_i$ are chain maps.

Since $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$ is free, its derivations are uniquely determined by their values on generators. In particular, the derivation $d$ of degree 1 is determined by these values:

$$\begin{aligned}
{}_X\mathbf{i}_0 d &= 0, \\
H_{X,Y} d &= 1_{(X,Y)} \qquad \text{(the unit element of } \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}\big((X, Y); (X, Y)\big)\text{)}, \\
b_n d &= -\sum_{a+p+c=n}^{p>1,\, a+c>0} (1^{\otimes a} \otimes b_p \otimes 1^{\otimes c}) b_{a+1+c}, \qquad n > 1.
\end{aligned}$$

For $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$ we use the notation $(1^{\otimes a}\otimes b_p\otimes 1^{\otimes c})b_{a+1+c}$ referring to the action of $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$ in $A_\infty$-algebras as a synonym of the usual operadic notation $(b_p\otimes b_{a+1+c})\mu_{a+1} = b_p\circ_{a+1}b_{a+1+c}$. Since the derivation $d$ has odd degree, the $\Bbbk$-linear map $d^2$ is also a derivation. Its value on all generators is 0 (for $b_n$ it follows from a similar result for $\mathscr{A}_\infty$, see e.g. [Mar00, MSS02]). Therefore, $d^2=0$, $d$ is a differential, and $\mathscr{A}_\infty \hookrightarrow \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$ is a chain embedding.

The so defined differential $d$ is distinguished by the following property. The action maps

$$\begin{aligned}\alpha : s\mathcal{E}(X_0,X_1)\otimes s\mathcal{E}(X_1,X_2)\otimes\cdots\otimes s\mathcal{E}(X_{n-1},X_n)&\\ \otimes \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}\big((X_0,X_1),(X_1,X_2),\dots,(X_{n-1},X_n);(X_0,X_n)\big) &\to s\mathcal{E}(X_0,X_n) \qquad (45)\end{aligned}$$

are chain maps, where $s\mathcal{E}(\_,\_)$ is equipped with the differential $b_1$. It suffices to check this on generators of $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$. For them the property follows from (32), (33) and the equation ${}_X\mathbf{i}_0^{\mathcal{C}}b_1=0$.

Let $\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}$ be a submulticategory of $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$, generated by $H$ and $b_n$, $n\geqslant 2$, with the same set of objects $\operatorname{Ob}\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}} = \operatorname{Ob}\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$. It is a differential graded submulticategory without 0-ary operations: $\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}(;(X,Y))=0$ for all $X,Y\in\operatorname{Ob}\mathcal{C}$. As a $\Bbbk$-linear graded multigraph $\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}$ has the following description. For $n\geqslant 1$

$$\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}\big((X_0,X_1),(X_1,X_2),\dots,(X_{n-1},X_n);(X_0,X_n)\big) = \bigoplus_{\text{admissible }(t;X_0,X_1,\dots,X_n)} \Bbbk[|t|_1-|t|_>], \qquad (46)$$

other $\Bbbk$-modules $\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}\big((X_1',X_1),(X_2',X_2),\dots,(X_n',X_n);(Y',Y)\big)$ vanish.

The embedding of $\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}$ is denoted

$$\iota : \underline{\mathscr{M}} \overset{\text{def}}{=} \underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}} \hookrightarrow \mathscr{M} \overset{\text{def}}{=} \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}. \qquad (47)$$

The following general results are valid for an arbitrary embedding $\iota:\underline{\mathscr{M}}\hookrightarrow\mathscr{M}$ of differential graded $\Bbbk$-linear multicategories, such that $\operatorname{Ob}\iota=\mathrm{id}_{\operatorname{Ob}\mathscr{M}}$, and $\underline{\mathscr{M}}(;Z)=0$ for all $Z\in\operatorname{Ob}\underline{\mathscr{M}}=\operatorname{Ob}\mathscr{M}$.

6.2. Definition. *A* right derivation $\delta$ *of degree* $p$ *of the embedding* $\iota:\underline{\mathscr{M}}\hookrightarrow\mathscr{M}$ *is a collection of* $\Bbbk$*-linear maps*

$$\delta : \underline{\mathscr{M}}(Z_1,\dots,Z_n;Z)\to\mathscr{M}(Z_1,\dots,Z_n;Z)$$

*of degree* $p$ *for all* $Z_1,\dots,Z_n,Z\in\operatorname{Ob}\mathscr{M}$*, such that compositions* $\mu_i$ *from* (44) *satisfy*

$$\begin{tikzcd} \underline{\mathscr{M}}\otimes\underline{\mathscr{M}} \arrow[r, "\mu_n"] \arrow[d, "\iota\otimes\delta+\delta\otimes\iota"'] \arrow[dr, phantom, "="] & \underline{\mathscr{M}} \arrow[d, "\delta"] \\ \mathscr{M}\otimes\mathscr{M} \arrow[r, "\mu_n"] & \mathscr{M} \end{tikzcd}, \qquad \begin{tikzcd} \underline{\mathscr{M}}\otimes\underline{\mathscr{M}} \arrow[r, "\mu_i"] \arrow[d, "\iota\otimes\delta"'] \arrow[dr, phantom, "="] & \underline{\mathscr{M}} \arrow[d, "\delta"] \\ \mathscr{M}\otimes\mathscr{M} \arrow[r, "\mu_i"] & \mathscr{M} \end{tikzcd}$$

*if* $1\leqslant i<n$*.*

Clearly, for $n = 0$ the map $\delta : \underline{\mathscr{M}}(;Z) = 0 \to \mathscr{M}(;Z)$ is 0.

An example of a right derivation is given by an *inner right derivation*. Let $\lambda_Z \in \mathscr{M}(Z;Z)$ be a family of morphisms of degree $p$. For $n > 0$ define

$$\begin{aligned} \operatorname{ad}_\lambda : \underline{\mathscr{M}}(Z_1, \dots, Z_n; Z) &\to \mathscr{M}(Z_1, \dots, Z_n; Z) \\ f &\mapsto (f \otimes \lambda_Z)\mu_1 - (-)^{f\lambda}(\lambda_{Z_n} \otimes f)\mu_n = f \circ_1 \lambda_Z - (-)^{fp}\lambda_{Z_n} \circ_n f. \end{aligned}$$

One verifies easily that $\operatorname{ad}_\lambda$ is a right derivation, which we call *inner*.

One can show that if $\delta$ is a right derivation of $\iota$, and $d$ is a derivation of $\mathscr{M}$ such that $\underline{\mathscr{M}}d \subset \underline{\mathscr{M}}$, then their commutator $[\delta, d] = \delta d - (-)^{\delta \cdot d} d\delta$ is a right derivation of $\iota$ as well. In particular, it applies to a differential $d$ of degree 1. If $\delta = \operatorname{ad}_\lambda$ is inner, then $[\delta, d] = \operatorname{ad}_{\lambda d}$ is also inner.

6.3. Examples of right derivations. Consider embedding (47) and take the family of morphisms

$$\lambda_{(X,Y)} = (1 \otimes {}_Y\mathbf{i}_0)b_2 = {}_Y\mathbf{i}_0 \circ_2 b_2 \in \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}\big((X,Y);(X,Y)\big).$$

Since $\lambda_{(X,Y)}d = 0$, we have $[\operatorname{ad}_\lambda, d] = \operatorname{ad}_{\lambda d} = 0$. Thus, $\delta = \operatorname{ad}_{(1\otimes\mathbf{i}_0)b_2}$ commutes with $d$.

Let us show that $\operatorname{ad}_\lambda = [\eta, d]$ for some right derivation $\eta : \underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}} \to \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$ of $\iota$. Since $\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}$ is a free multicategory, any such right derivation is uniquely determined by its value on the generators $H$ and $b_n$, $n > 1$. We define a right derivation $\eta$ of $\iota$ of degree $-1$ by the following assignment:

$$\begin{aligned} H_{X,Y}\eta &= H_{X,Y}(1 \otimes {}_Y\mathbf{i}_0)b_2 H_{X,Y}, \\ b_n\eta &= (1^{\otimes n} \otimes {}_{X_n}\mathbf{i}_0)b_{n+1}. \end{aligned} \tag{48}$$

Any operation $f \in \underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}$ can be presented as

$$f = (g \otimes 1)(1^{\otimes a_1} \otimes e_{p_1})(1^{\otimes a_2} \otimes e_{p_2}) \dots (1^{\otimes a_{k-1}} \otimes e_{p_{k-1}})e_{p_k}$$

for some $g \in \underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}$, where $e_1 = H$ and $e_p = b_p$ if $p > 1$. Then

$$f\eta = \sum_{i=1}^{k} (-)^{k-i}(g \otimes 1)(1^{\otimes a_1} \otimes e_{p_1}) \dots (1^{\otimes a_i} \otimes e_{p_i}\eta) \dots (1^{\otimes a_{k-1}} \otimes e_{p_{k-1}})e_{p_k}. \tag{49}$$

Let us prove that

$$[\eta, d] = \eta d + d\eta = \operatorname{ad}_{(1\otimes\mathbf{i}_0)b_2}. \tag{50}$$

Since the difference of the both sides is a right derivation of $\iota$, it suffices to prove (50) on generators. First we find that

$$\begin{aligned} H_{X,Y}(\eta d + d\eta) &= [H_{X,Y}(1 \otimes {}_Y\mathbf{i}_0)b_2 H_{X,Y}]d + 1_{(X,Y)}\eta \\ &= H_{X,Y}(1 \otimes {}_Y\mathbf{i}_0)b_2 - (1 \otimes {}_Y\mathbf{i}_0)b_2 H_{X,Y} = H_{X,Y} \operatorname{ad}_{(1\otimes\mathbf{i}_0)b_2}. \end{aligned}$$

For $n \geqslant 2$ we have

$$
\begin{aligned}
b_n(\eta d + d\eta) &= (1^{\otimes n} \otimes \mathbf{i}_0)(b_{n+1}d) - \sum_{a+p+c=n}^{p>1,\, c>0} (1^{\otimes a} \otimes b_p \otimes 1^{\otimes c})(b_{a+1+c}\eta) \\
&\quad - \sum_{a+p=n}^{p>1,\, a>0} [(1^{\otimes a} \otimes b_p)b_{a+1}]\eta \\
&= - \sum_{a+p+c=n+1}^{p>1,\, a+c>0} (1^{\otimes n} \otimes \mathbf{i}_0)(1^{\otimes a} \otimes b_p \otimes 1^{\otimes c})b_{a+1+c} \\
&\quad - \sum_{a+p+c=n}^{p>1,\, c>0} (1^{\otimes a} \otimes b_p \otimes 1^{\otimes c})(1^{\otimes a+1+c} \otimes \mathbf{i}_0)b_{a+c+2} \\
&\quad - \sum_{a+p=n}^{p>1,\, a>0} (1^{\otimes a} \otimes b_p)(1^{\otimes a+1} \otimes \mathbf{i}_0)b_{a+2} + \sum_{a+p=n}^{p>1,\, a>0} [1^{\otimes a} \otimes (1^{\otimes p} \otimes \mathbf{i}_0)b_{p+1}]b_{a+1} \\
&= - \sum_{a+p+c=n+1}^{p>1,\, a+c>0} (1^{\otimes n} \otimes \mathbf{i}_0)(1^{\otimes a} \otimes b_p \otimes 1^{\otimes c})b_{a+1+c} \\
&\quad + \sum_{a+p+e=n+1}^{p>1,\, e>1} (1^{\otimes n} \otimes \mathbf{i}_0)(1^{\otimes a} \otimes b_p \otimes 1^{\otimes e})b_{a+1+e} \\
&\quad + \sum_{a+p=n}^{p>1,\, a>0} (1^{\otimes n} \otimes \mathbf{i}_0)(1^{\otimes a} \otimes b_p \otimes 1)b_{a+2} + \sum_{a+q=n+1}^{q>2,\, a>0} (1^{\otimes n} \otimes \mathbf{i}_0)(1^{\otimes a} \otimes b_q)b_{a+1} \\
&= - \sum_{0+p+1=n+1} (1^{\otimes n} \otimes \mathbf{i}_0)(b_p \otimes 1)b_2 - \sum_{a+2+0=n+1} (1^{\otimes n} \otimes \mathbf{i}_0)(1^{\otimes a} \otimes b_2)b_{a+1} \\
&= b_n(1 \otimes \mathbf{i}_0)b_2 - [1^{\otimes n-1} \otimes (1 \otimes \mathbf{i}_0)b_2]b_n \\
&= b_n \operatorname{ad}_{(1\otimes\mathbf{i}_0)b_2} .
\end{aligned}
$$

Thus equation (50) is verified.

Notice that the graded quiver $s\mathcal{E}$ defined by (31) is a free $\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}$-algebra, generated by the graded quiver $s\mathcal{C}$. Indeed, (31) can be written as

$$
\begin{aligned}
s\mathcal{E}(X,Y) = \bigoplus_{n\geqslant 1} \bigoplus_{X_1,\dots,X_{n-1}\in \operatorname{Ob}\mathcal{C}}^{X_0=X,\, X_n=Y} & s\mathcal{C}(X_0,X_1) \otimes \cdots \otimes s\mathcal{C}(X_{n-1},X_n) \otimes \\
& \otimes \underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}\big((X_0,X_1),(X_1,X_2),\dots,(X_{n-1},X_n);(X_0,X_n)\big)
\end{aligned}
$$

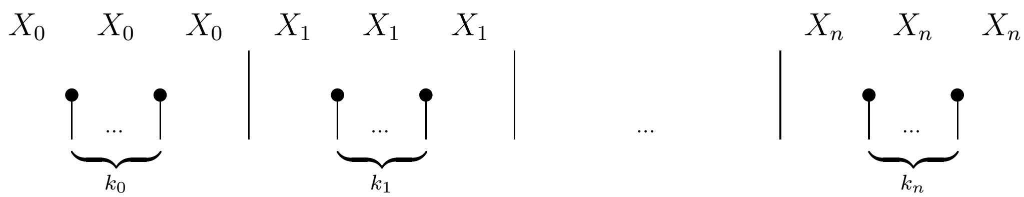
due to (46). Compare with the usual free algebras over an operad, e.g. [MSS02]. The operations $H$, $b_n$ for $n \geqslant 2$ act in $s\mathcal{E}$ via multicategory compositions in $\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}$.

The multigraph $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$ is expressed via $\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}$ as follows:

$$\begin{aligned}
&\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}\bigl((X_0,X_1),(X_1,X_2),\dots,(X_{n-1},X_n);(X_0,X_n)\bigr) \\
&\qquad = \bigoplus_{k_0,\dots,k_n\in\mathbb{Z}_{\geqslant 0}} \underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}\bigl((Y_0,Y_1),(Y_1,Y_2),\dots,(Y_{p-1},Y_p);(X_0,X_n)\bigr)[k_0+\dots+k_n], \qquad (51)
\end{aligned}$$

$$(Y_0,Y_1,\dots,Y_p) = (\underbrace{X_0,\dots,X_0}_{k_0+1},\underbrace{X_1,\dots,X_1}_{k_1+1},\dots,\underbrace{X_n,\dots,X_n}_{k_n+1}),$$

where $p = k_0+\dots+k_n+n$. Indeed, 0-ary operations can be performed first. The summand of (51) corresponds to insertion of $k_0$ symbols ${}_{X_0}\mathbf{i}_0$, $k_1$ symbols ${}_{X_1}\mathbf{i}_0$, and so on. In terms of trees such summand is described by concatenation of the forest




with an admissible tree $(t;Y_0,Y_1,\dots,Y_p)$.

The action (45) of $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$ in $s\mathcal{E}$ is described as follows. An element

$$({}_{X_0}\mathbf{i}_0^{\otimes k_0}\otimes 1\otimes {}_{X_1}\mathbf{i}_0^{\otimes k_1}\otimes 1\otimes\dots\otimes 1\otimes {}_{X_n}\mathbf{i}_0^{\otimes k_n})\cdot f \in \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}\bigl((X_0,X_1),\dots,(X_{n-1},X_n);(X_0,X_n)\bigr),$$

where $f\in \underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}\bigl((Y_0,Y_1),(Y_1,Y_2),\dots,(Y_{p-1},Y_p);(X_0,X_n)\bigr)$, acts by the map

$$\begin{aligned}
&s\mathcal{E}(X_0,X_1)\otimes\dots\otimes s\mathcal{E}(X_{n-1},X_n) \xrightarrow{{}^{\mathcal{C}}_{X_0}\mathbf{i}_0^{\otimes k_0}\otimes 1\otimes {}^{\mathcal{C}}_{X_1}\mathbf{i}_0^{\otimes k_1}\otimes 1\otimes\dots\otimes 1\otimes {}^{\mathcal{C}}_{X_n}\mathbf{i}_0^{\otimes k_n}} \\
&\ s\mathcal{C}(X_0,X_0)^{\otimes k_0}\otimes s\mathcal{E}(X_0,X_1)\otimes s\mathcal{C}(X_1,X_1)^{\otimes k_1}\otimes\dots\otimes s\mathcal{E}(X_{n-1},X_n)\otimes s\mathcal{C}(X_n,X_n)^{\otimes k_n}
\end{aligned}$$

followed by the action of $f$ via multiplication in multicategory $\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}$, that is, via grafting of trees.

Define a $\Bbbk$-linear map $\bar{h}: s\mathcal{E}(X,Y)\to s\mathcal{E}(X,Y)$ of degree $-1$ for all objects $X$, $Y$ of

$\mathcal{C}$ as follows:

$$\begin{array}{c}
s\mathcal{E}(X,Y) \\
\Big\| \\
\bigoplus_{X_1,\dots,X_{n-1}\in\operatorname{Ob}\mathcal{C}}^{n,\,X_0=X,\,X_n=Y} s\mathcal{C}(X_0,X_1)\otimes\cdots\otimes s\mathcal{C}(X_{n-1},X_n)\otimes\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}\big((X_0,X_1),\dots,(X_{n-1},X_n);(X_0,X_n)\big) \\
\Big\downarrow{\scriptstyle -1^{\otimes n}\otimes\eta} \\
\bigoplus_{X_1,\dots,X_{n-1}\in\operatorname{Ob}\mathcal{C}}^{n,\,X_0=X,\,X_n=Y} s\mathcal{C}(X_0,X_1)\otimes\cdots\otimes s\mathcal{C}(X_{n-1},X_n)\otimes\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}\big((X_0,X_1),\dots,(X_{n-1},X_n);(X_0,X_n)\big) \\
\Big\downarrow{\scriptstyle \alpha} \\
s\mathcal{E}(X,Y)
\end{array}$$

The concrete choice (48) of value of $\eta$ on generators shows that on the summand

$$s\mathcal{C}(X_0,X_1)\otimes\cdots\otimes s\mathcal{C}(X_{n-1},X_n)\otimes\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}\big((X_0,X_1),\dots,(X_{n-1},X_n);(X_0,X_n)\big)$$

the map $\bar{h}$ takes values in

$$s\mathcal{C}(X_0,X_1)\otimes\cdots\otimes s\mathcal{C}(X_{n-1},X_n)\otimes s\mathcal{C}(X_n,X_n)\otimes\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}\big(\dots,(X_{n-1},X_n),(X_n,X_n);(X_0,X_n)\big)$$

Due to (49) the explicit formula for $\bar{h}$ has the form (43) with the concrete choice of signs.

Let us compute the commutator

$$\begin{aligned}
-\bar{h}b_1^{\mathcal{E}} - b_1^{\mathcal{E}}\bar{h} &= (1^{\otimes n}\otimes\eta)\alpha b_1^{\mathcal{E}} + b_1^{\mathcal{E}}(1^{\otimes n}\otimes\eta)\alpha \\
&= (1^{\otimes n}\otimes\eta)\Big(\sum_{a+1+c=n}(1^{\otimes a}\otimes b_1\otimes 1^{\otimes c})\otimes 1 + 1^{\otimes n}\otimes d\Big)\alpha \\
&\quad + \Big(\sum_{a+1+c=n}(1^{\otimes a}\otimes b_1\otimes 1^{\otimes c})\otimes 1 + 1^{\otimes n}\otimes d\Big)(1^{\otimes n}\otimes\eta)\alpha \\
&= [1^{\otimes n}\otimes(\eta d + d\eta)]\alpha = [1^{\otimes n}\otimes \operatorname{ad}_{(1\otimes\mathbf{i}_0)b_2}]\alpha :
\end{aligned}$$

$$s\mathcal{C}(X_0,X_1)\otimes\cdots\otimes s\mathcal{C}(X_{n-1},X_n)\otimes\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}\big((X_0,X_1),\dots,(X_{n-1},X_n);(X_0,X_n)\big)\to$$
$$s\mathcal{C}(X_0,X_1)\otimes\cdots\otimes s\mathcal{C}(X_{n-1},X_n)\otimes s\mathcal{C}(X_n,X_n)\otimes\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}\big(\dots,(X_{n-1},X_n),(X_n,X_n);(X_0,X_n)\big).$$

We write an element $z_1\otimes\cdots\otimes z_n\otimes f$ of the source, which is a direct summand of $s\mathcal{E}(X_0,X_n)$, in the form $(z_1\otimes\cdots\otimes z_n)f$, meaning that $z_i\in s\mathcal{C}(X_{i-1},X_i)$ and $f$ is a composition of expressions $1^{\otimes a}\otimes b_p\otimes 1^{\otimes c}$ and $1^{\otimes a}\otimes H\otimes 1^{\otimes c}$ ending in $b_p$ or $H$. Then

$$\begin{aligned}
-[(z_1\otimes\cdots\otimes z_n)f](\bar{h}b_1^{\mathcal{E}} + b_1^{\mathcal{E}}\bar{h}) &= [(z_1\otimes\cdots\otimes z_n)(f.\operatorname{ad}_{(1\otimes\mathbf{i}_0)b_2})]\alpha \\
&= \big\{[(z_1\otimes\cdots\otimes z_n)f]\otimes{}_{X_n}\mathbf{i}_0^{\mathcal{C}}\big\}b_2^{\mathcal{E}} - (z_1\otimes\cdots\otimes z_n\otimes{}_{X_n}\mathbf{i}_0^{\mathcal{C}})(1^{\otimes n-1}\otimes b_2^{\mathcal{E}})f.
\end{aligned}\tag{52}$$

6.4. Proposition. *Assume that $\mathcal{C}$ is a unital $A_\infty$-category with the unit elements satisfying equations*

$$(1 \otimes \mathbf{i}_0^{\mathcal{C}})b_2 = 1,$$
$$(1^{\otimes n} \otimes \mathbf{i}_0^{\mathcal{C}})b_{n+1} = 0, \qquad \text{if } n > 1.$$

*Then there exists a map $h : s\mathcal{D} \to s\mathcal{D}$ such that*

$$\big(s\mathcal{E} \xrightarrow{\bar{h}} s\mathcal{E} \xrightarrow{\pi} s\mathcal{D}\big) = \big(s\mathcal{E} \xrightarrow{\pi} s\mathcal{D} \xrightarrow{h} s\mathcal{D}\big).$$

*The map $h : s\mathcal{D} \to s\mathcal{D}$ is a right unit homotopy for $\mathcal{D}$.*

Proof. We have to prove that $(R)_+\bar{h} \subset (R)_+$. The left hand side is the sum of images of maps

$$[(1^{\otimes a} \otimes b_n^{\mathcal{E}} \otimes 1^{\otimes c})f]\eta - (1^{\otimes a} \otimes b_n^{\mathcal{C}} \otimes 1^{\otimes c})(f\eta) : s\mathcal{C}^{\otimes k} \to s\mathcal{E}. \tag{53}$$

If $c > 0$, this map equals

$$(1^{\otimes a} \otimes b_n^{\mathcal{E}} \otimes 1^{\otimes c})(f\eta) - (1^{\otimes a} \otimes b_n^{\mathcal{C}} \otimes 1^{\otimes c})(f\eta),$$

and the claim holds. If $c = 0$, expression (53) is

$$\begin{aligned}(1^{\otimes a} \otimes b_n^{\mathcal{E}})(f\eta) &- (1^{\otimes a} \otimes b_n^{\mathcal{C}})(f\eta) + (-)^f(1^{\otimes a} \otimes (1^{\otimes n} \otimes \mathbf{i}_0^{\mathcal{C}})b_{n+1}^{\mathcal{E}})f \\ &= (1^{\otimes a} \otimes b_n^{\mathcal{E}})(f\eta) - (1^{\otimes a} \otimes b_n^{\mathcal{C}})(f\eta) \\ &\quad + (-)^f(1^{\otimes a+n} \otimes \mathbf{i}_0^{\mathcal{C}})(1^{\otimes a} \otimes b_{n+1}^{\mathcal{E}})f - (-)^f(1^{\otimes a+n} \otimes \mathbf{i}_0^{\mathcal{C}})(1^{\otimes a} \otimes b_{n+1}^{\mathcal{C}})f \\ &\quad + (-)^f(1^{\otimes a} \otimes (1^{\otimes n} \otimes \mathbf{i}_0^{\mathcal{C}})b_{n+1}^{\mathcal{C}})f : s\mathcal{C}^{\otimes k} \to s\mathcal{E}\end{aligned}$$

and the last summand equals 0. So the claim holds, and $h$ exists.

Property (52) turns into

$$hb_1^{\mathcal{D}} + b_1^{\mathcal{D}}h = 1 - (1 \otimes \mathbf{i}_0^{\mathcal{C}})b_2 : s\mathcal{D}(X, Y) \to s\mathcal{D}(X, Y).$$

Therefore, $h$ is a right unit homotopy for $\mathcal{D}$. ∎

6.5. Theorem. *Assume that $\mathcal{C}$ is a unital $A_\infty$-category with the unit elements satisfying equations*

$$(1 \otimes \mathbf{i}_0^{\mathcal{C}})b_2 = 1, \qquad (\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2 = -1,$$
$$(1^{\otimes n} \otimes \mathbf{i}_0^{\mathcal{C}})b_{n+1} = 0, \qquad (\mathbf{i}_0^{\mathcal{C}} \otimes 1^{\otimes n})b_{n+1} = 0, \qquad \text{if } n > 1.$$

*Then the $A_\infty$-category $\mathcal{D}$ is unital.*

Proof. Besides constructing $\mathcal{D}$ from the pair $(\mathcal{C}, \mathcal{B})$, we may apply the construction to the pair $(\mathcal{C}^{\mathrm{op}}, \mathcal{B}^{\mathrm{op}})$, and we get an $A_\infty$-category isomorphic to $\mathcal{D}^{\mathrm{op}}$. The opposite $A_\infty$-category $\mathcal{A}^{\mathrm{op}}$ to an $A_\infty$-category $\mathcal{A}$ is the opposite quiver, equipped with operations $b_k^{\mathrm{op}}$, see Definition A.4. In particular, $b_1^{\mathrm{op}} = b_1$ and $(x \otimes \mathbf{i}_0)b_2^{\mathrm{op}} = -x(\mathbf{i}_0 \otimes 1)b_2$. Thus we may use $h_{\mathrm{op}} = h$ for $\mathcal{D}^{\mathrm{op}}$ in place of $h'$ for $\mathcal{D}$. Thus, $A_\infty$-category $\mathcal{D}$ is unital. ∎

6.6. Corollary. *If $\mathcal{C}$ is strictly unital, then the $A_\infty$-category $\mathcal{D}$ is unital.*

6.7. Corollary. *The $A^u_\infty$-2-functor $\mathcal{A} \mapsto A^u_\infty(\mathcal{C},\mathcal{A})_{\mathrm{mod}\,\mathcal{B}}$ is unitally representable for an arbitrary unital $A_\infty$-category $\mathcal{C}$ by*

$$\big(\mathcal{C}, e : \mathcal{C} \to \mathsf{q}(\mathcal{C}|\mathcal{B})\big) \overset{def}{=} \big(\mathcal{C}, \mathcal{C} \xrightarrow{\widetilde{Y}} \widetilde{\mathcal{C}} \xrightarrow{\widetilde{e}} \mathsf{Q}(\widetilde{\mathcal{C}}|\widetilde{\mathcal{B}})\big),$$

*where $\widetilde{Y} : \mathcal{C} \to \widetilde{\mathcal{C}}$ is the Yoneda $A_\infty$-equivalence identity on objects from Remark A.9.*

Proof. By Corollary B.11 it suffices to prove unital representability for differential graded categories $\widetilde{\mathcal{C}}$ in place of $\mathcal{C}$. In this case the representing unital $A_\infty$-category $\mathsf{Q}(\widetilde{\mathcal{C}}|\widetilde{\mathcal{B}})$ exists by Corollary 6.6. ∎

## 7. Equivalence of two quotients of $A_\infty$-categories

Let $\mathcal{B}$ be a full $A_\infty$-subcategory of a unital $A_\infty$-category $\mathcal{C}$. By Remark A.9 there exists a differential graded category $\widetilde{\mathcal{C}}$ with $\operatorname{Ob}\widetilde{\mathcal{C}} = \operatorname{Ob}\mathcal{C}$, and quasi-inverse to each other $A_\infty$-functors $\widetilde{Y} : \mathcal{C} \to \widetilde{\mathcal{C}}$, $\Psi : \widetilde{\mathcal{C}} \to \mathcal{C}$ such that $\operatorname{Ob}\widetilde{Y} = \operatorname{Ob}\Psi = \mathrm{id}_{\operatorname{Ob}\mathcal{C}}$. Let $\widetilde{\mathcal{B}} \subset \widetilde{\mathcal{C}}$ be the full differential graded subcategory with $\operatorname{Ob}\widetilde{\mathcal{B}} = \operatorname{Ob}\mathcal{B}$. The quotient $A_\infty$-category $\mathsf{Q}(\widetilde{\mathcal{C}}|\widetilde{\mathcal{B}})$ and the quotient $A_\infty$-functor $\widetilde{e} : \widetilde{\mathcal{C}} \to \mathcal{D}$ is constructed in Theorem 5.13. The same $\mathsf{Q}(\widetilde{\mathcal{C}}|\widetilde{\mathcal{B}})$ denoted also $\mathsf{q}(\mathcal{C}|\mathcal{B})$ with the quotient $A_\infty$-functor $e = \big(\mathcal{C} \xrightarrow{\widetilde{Y}} \widetilde{\mathcal{C}} \xrightarrow{\widetilde{e}} \mathcal{D}\big)$ represents the $A^u_\infty$-2-functor $\mathcal{A} \mapsto A^u_\infty(\mathcal{C},\mathcal{A})_{\mathrm{mod}\,\mathcal{B}}$.

There is also a construction of [LO06] which gives a unital $A_\infty$-category $\mathsf{D}(\mathcal{C}|\mathcal{B})$ and, in particular, a differential graded category $\mathsf{D}(\widetilde{\mathcal{C}}|\widetilde{\mathcal{B}})$. These are smaller than $\mathsf{Q}(\widetilde{\mathcal{C}}|\widetilde{\mathcal{B}})$, however, we are going to prove that all these three $A_\infty$-categories are equivalent. Thus a simpler construction $\mathsf{D}(\mathcal{C}|\mathcal{B})$ enjoys the same universal properties as $\mathsf{q}(\mathcal{C}|\mathcal{B})$ does. As a graded $\Bbbk$-quiver $\mathcal{A} = \mathsf{D}(\mathcal{C}|\mathcal{B})$ has the set of objects $\operatorname{Ob}\mathcal{A} = \operatorname{Ob}\mathcal{C}$, the morphisms for $X, Y \in \operatorname{Ob}\mathcal{A}$ are

$$s\mathcal{A}(X,Y) = \oplus_{C_1,\dots,C_{n-1}\in\mathcal{B}} s\mathcal{C}(X,C_1) \otimes s\mathcal{C}(C_1,C_2) \otimes \cdots \otimes s\mathcal{C}(C_{n-2},C_{n-1}) \otimes s\mathcal{C}(C_{n-1},Y),$$

where the summation extends over all sequences of objects $(C_1,\dots,C_{n-1})$ of $\mathcal{B}$. To the empty sequence ($n = 1$) corresponds the summand $s\mathcal{C}(X,Y)$. The operations $\bar{b}_n : s\mathcal{A}^{\otimes n} \to s\mathcal{A}$ are restrictions of maps $\bar{b}_0 = 0$, $\bar{b}_1 = b$ and for $n \geqslant 2$

$$\bar{b}_n = \mu^{(n)} \sum_{m;q<k;t<l} 1^{\otimes q} \otimes b_m \otimes 1^{\otimes t} : T^k s\mathcal{C} \otimes (T^{\geqslant 1}s\mathcal{C})^{\otimes n-2} \otimes T^l s\mathcal{C} \to T^{\geqslant 1}s\mathcal{C}.$$

via the natural embedding $s\mathcal{A} \subset T^{\geqslant 1}s\mathcal{C}$ of graded $\Bbbk$-quivers [LO06, Proposition 2.2]. Here $\mu^{(k)} : T^k T^{\geqslant 1}s\mathcal{C} \to T^{\geqslant 1}s\mathcal{C}$, $k \geqslant 1$, is the multiplication in the tensor algebra.

Denote by $\mathsf{Q}(\mathcal{C}|\mathcal{B}) = \mathcal{D}$ the quotient $A_\infty$-category, constructed in Theorem 5.13.

7.1. Lemma. *There is a chain quiver map $\psi : s\mathsf{D}(\mathcal{C}|\mathcal{B}) \to s\mathsf{Q}(\mathcal{C}|\mathcal{B})$, whose summands*

$$\psi_n : s\mathcal{C}(X, C_1) \otimes s\mathcal{C}(C_1, C_2) \otimes \cdots \otimes s\mathcal{C}(C_{n-2}, C_{n-1}) \otimes s\mathcal{C}(C_{n-1}, Y) \to s\mathsf{Q}(\mathcal{C}|\mathcal{B})(X, Y)$$

*for $C_i \in \operatorname{Ob}\mathcal{B}$ are defined by recurrent relation: $\psi_1 = e_1 : s\mathcal{C}(X,Y) \hookrightarrow s\mathsf{Q}(\mathcal{C}|\mathcal{B})(X,Y)$ is the natural embedding, and for $n > 1$*

$$\psi_n = -\sum_{k=1}^{n-1} (e_1^{\otimes k} \otimes \psi_{n-k}H)b_{k+1}. \tag{54}$$

For example, $\psi_2 = -(e_1 \otimes e_1 H)b_2^{\mathcal{D}}$,

$$\psi_3 = (e_1 \otimes e_1 \otimes e_1 H)(1 \otimes b_2^{\mathcal{D}} H)b_2^{\mathcal{D}} - (e_1 \otimes e_1 \otimes e_1 H)b_3^{\mathcal{D}}.$$

In general, expansion of (54) contains $2^{n-2}$ summands.

Proof. We have to prove equation

$$\psi_n b_1 = b\psi : s\mathcal{C}(X, C_1) \otimes s\mathcal{C}(C_1, C_2) \otimes \cdots \otimes s\mathcal{C}(C_{n-1}, Y) \to s\mathsf{Q}(\mathcal{C}|\mathcal{B})(X, Y)$$

for all $n \geqslant 1$. It is obvious for $n = 1$. Let us prove it by induction. Assume that it holds for number of factors smaller than $n$. Then

$$\psi_n b_1 = -\sum_{k=1}^{n-1} (e_1^{\otimes k} \otimes \psi_{n-k}H)b_{k+1}b_1 = \sum_{\substack{0<k<n \\ k+1=a+p+c}}^{a+c>0} (e_1^{\otimes k} \otimes \psi_{n-k}H)(1^{\otimes a} \otimes b_p \otimes 1^{\otimes c})b_{a+1+c}$$

$$= -\sum_{\substack{0<k<n \\ k+1=a+p+c}}^{a\geqslant 0,\, c>0} (1^{\otimes a} \otimes b_p \otimes 1^{\otimes n-a-p})(e_1^{\otimes a+c} \otimes \psi_{n-k}H)b_{a+c+1}$$

$$+ \sum_{\substack{0<k<n \\ k+1=a+p}}^{a>0,\, p>1} (e_1^{\otimes k} \otimes \psi_{n-k}H)(1^{\otimes a} \otimes b_p)b_{a+1} + \sum_{0<a<n} (e_1^{\otimes a} \otimes \psi_{n-a})b_{a+1} - \sum_{0<k<n} (e_1^{\otimes k} \otimes \psi_{n-k}b_1 H)b_{k+1}.$$

The second sum in the right hand side can be presented as

$$\sum_{0<a<n-1} \Big[ e_1^{\otimes a} \otimes \sum_{0<k-a<n-a} (e_1^{\otimes k-a} \otimes \psi_{n-a-(k-a)}H)b_{k-a+1} \Big] b_{a+1}.$$

Thus, it nearly cancels with the third sum except for one summand, corresponding to $a = n - 1$. In the fourth sum in the right hand side we replace $\psi_{n-k}b_1$ with $b\psi$ by induction assumption, and we get

$$\psi_n b_1 = -\sum_{\substack{0<k<n \\ k+1=a+p+c}}^{a\geqslant 0,\, c>0} (1^{\otimes a} \otimes b_p \otimes 1^{\otimes n-a-p})(e_1^{\otimes a+c} \otimes \psi_{n-k}H)b_{a+c+1} + (e_1^{\otimes n-1} \otimes \psi_1)b_n$$

$$- \sum_{\substack{0<k<n \\ k+\alpha+\beta+\gamma=n}} (1^{\otimes k+\alpha} \otimes b_\beta \otimes 1^{\otimes \gamma})(e_1^{\otimes k} \otimes \psi_{\alpha+1+\gamma}H)b_{k+1} = b\psi,$$

since $(e_1^{\otimes n-1} \otimes \psi_1)b_n = e_1^{\otimes n} b_n = b_n e_1 = b_n \psi_1$. ∎

According to Theorem 5.13 for any unital $A_\infty$-category $\mathcal{A}$ the map

$$\begin{aligned}\mathrm{restr} : A^u_\infty(\mathcal{D},\mathcal{A}) &\to A^u_\infty(\mathcal{C},\mathcal{A})_{\mathrm{mod}\,\mathcal{B}} \\ &= \big\{ f \in A^u_\infty(\mathcal{C},\mathcal{A}) \mid f|_{\mathcal{B}} = \big(\mathcal{B} \hookrightarrow \mathcal{C} \xrightarrow{f} \mathcal{A}\big) \text{ is contractible } \big\}\end{aligned}$$

is surjective. A splitting of this surjection is defined recurrently in Propositions 5.7, 4.6, 5.9. Let us describe another splitting map which differs from the mentioned one and is more suitable for our purposes.

7.2. Proposition. *Let $f : \mathcal{C} \to \mathcal{A}$ be a unital $A_\infty$-functor such that $f|_{\mathcal{B}}$ is contractible. For each pair $X, Y$ of objects of $\mathcal{C}$ such that $X \in \operatorname{Ob}\mathcal{B}$ or $Y \in \operatorname{Ob}\mathcal{B}$ choose a contracting homotopy $\chi_{XY} : s\mathcal{A}(Xf,Yf) \to s\mathcal{A}(Xf,Yf)$, thus, $\chi b_1 + b_1\chi = 1$. Let $\tilde{f}_k : T^k s\mathcal{D} \to s\mathcal{A}$, $k > 1$, be $\Bbbk$-quiver morphisms of degree 0 which extend the components $f_k : T^k s\mathcal{C} \to s\mathcal{A}$. Then there exists a unique extension of $f_1 : s\mathcal{C} \to s\mathcal{A}$ to a quiver morphism $\tilde{f}_1 : s\mathcal{D} \to s\mathcal{A}$ such that $(\tilde{f}_1, \tilde{f}_2, \dots)$ are components of a unital $A_\infty$-functor $\tilde{f} : \mathcal{D} \to \mathcal{A}$ and*

$$H\tilde{f}_1 = \tilde{f}_1\chi : s\mathcal{D}(X,Y) \to s\mathcal{A}(Xf,Yf), \tag{55}$$

*whenever $X \in \operatorname{Ob}\mathcal{B}$ or $Y \in \operatorname{Ob}\mathcal{B}$.*

*Warning:* Extensions $\tilde{f} : \mathcal{D} \to \mathcal{A}$ of $f : \mathcal{C} \to \mathcal{A}$ constructed in Sections 4, 5 do not, in general, satisfy condition (55).

Proof. Let us extend $f$ to an $A_\infty$-functor $\hat{f} : \mathcal{E} \to \mathcal{A}$ such that $\hat{f}_k = \big(s\mathcal{E}^{\otimes k} \longrightarrow s\mathcal{D}^{\otimes k} \xrightarrow{\tilde{f}_k} \mathcal{A}\big)$ and $H\hat{f}_1 = \hat{f}_1\chi$, whenever $X \in \operatorname{Ob}\mathcal{B}$ or $Y \in \operatorname{Ob}\mathcal{B}$. Suppose that $t_1, \dots, t_n$ are trees, $n \geqslant 1$, and $\hat{f}_1 : s\mathcal{E}(t_i) \to s\mathcal{A}$ is already defined for all $1 \leqslant i \leqslant n$. Then there is only one way to extend $\hat{f}$ on $\mathcal{E}(t)$ for $t = (t_1 \sqcup \cdots \sqcup t_n) \cdot \mathfrak{t}_n$, where $\mathfrak{t}_n$ is the corolla with $n$ leaves. Since

$$\begin{aligned} b_n = s^{|t_1|} \otimes \cdots \otimes s^{|t_{n-1}|} \otimes s^{|t_n|-|t|} : s\mathcal{E}(t_1) \otimes \cdots \otimes s\mathcal{E}(t_n) \to s\mathcal{E}(t), &\qquad \text{for } n > 1, \\ H = s : s\mathcal{E}(t_1) \to s\mathcal{E}(t), &\qquad \text{for } n = 1 \end{aligned}$$

is invertible, we find that, respectively,

$$\begin{aligned} \hat{f}_1 &= \big(s\mathcal{E}(t) \xrightarrow{b_n^{-1}} s\mathcal{E}(t_1) \otimes \cdots \otimes s\mathcal{E}(t_n) \xrightarrow{\sum(\hat{f}_{i_1}\otimes\cdots\otimes\hat{f}_{i_l})b_l - \sum_{\alpha+k+\beta=n}^{\alpha+\beta>0}(1^{\otimes\alpha}\otimes b_k\otimes 1^{\otimes\beta})\hat{f}_{\alpha+1+\beta}} s\mathcal{A}\big), \\ \hat{f}_1 &= \big(s\mathcal{E}(t) \xrightarrow{H^{-1}} s\mathcal{E}(t_1) \xrightarrow{\hat{f}_1} s\mathcal{A}(Xf,Yf) \xrightarrow{\chi_{XY}} s\mathcal{A}(Xf,Yf)\big) \qquad \text{for } n = 1. \end{aligned}$$

Let us prove that coalgebra homomorphism $\hat{f} : \mathcal{E} \to \mathcal{A}$ with recursively defined components $(\hat{f}_1, \hat{f}_2, \dots)$ is an $A_\infty$-functor. Equation

$$b_n\hat{f}_1 = \sum_{i_1+\cdots+i_l=n} (\hat{f}_{i_1} \otimes \cdots \otimes \hat{f}_{i_l})b_l - \sum_{\alpha+k+\beta=n}^{\alpha+\beta>0} (1^{\otimes\alpha} \otimes b_k \otimes 1^{\otimes\beta})\hat{f}_{\alpha+1+\beta} : T^n s\mathcal{E} \to s\mathcal{A} \tag{56}$$

is satisfied for all $n > 1$ by construction of $\hat{f}_1$. So it remains to prove that $\hat{f}_1$ is a chain map. To prove by induction on the number of vertices of $t$ that $\hat{f}_1 b_1 = b_1 \hat{f}_1 : s\mathcal{E}(t) \to s\mathcal{A}$, it suffices to show that $b_n \hat{f}_1 b_1 = b_n b_1 \hat{f}_1$ for all $n > 1$ and that $H\hat{f}_1 b_1 = H b_1 \hat{f}_1$ due to invertibility of $b_n$ and $H$. The first assertion is proven in [LO06, Proposition 2.3]. The second follows from the computation

$$\begin{aligned} H(\hat{f}_1 b_1 - b_1 \hat{f}_1) = H\hat{f}_1 b_1 - H b_1 \hat{f}_1 = \hat{f}_1 \chi b_1 - \hat{f}_1 + b_1 H \hat{f}_1 \\ = \hat{f}_1 - \hat{f}_1 b_1 \chi - \hat{f}_1 + b_1 \hat{f}_1 \chi = -(\hat{f}_1 b_1 - b_1 \hat{f}_1)\chi, \end{aligned}$$

which vanishes by induction assumption.

Using (55) and (56) one can prove that the ideal $(R_{\mathcal{C}})_+$ is mapped by $\hat{f}_1$ to 0. Therefore, $\hat{f}$ factorizes as $\mathcal{E} \longrightarrow \mathcal{D} \xrightarrow{\tilde{f}} \mathcal{A}$ for a unique $A_\infty$-functor $\tilde{f}$. It is unital since the unit elements of $\mathcal{C}$ are also the unit elements of $\mathcal{D}$. ∎

7.3. Lemma. *Let $f : \mathcal{A} \to \mathcal{B}$ be an $A_\infty$-equivalence. Let objects $Xf$, $Yf$ of $\mathcal{B}$ be isomorphic via inverse to each other isomorphisms $r \in s\mathcal{B}(Xf, Yf)$, $p \in s\mathcal{B}(Yf, Xf)$ (that is, $[rs^{-1}] \in H^0\mathcal{B}(Xf, Yf)$, $[ps^{-1}] \in H^0\mathcal{B}(Yf, Xf)$ are inverse to each other in the ordianry category $H^0\mathcal{B}$). Then the objects $X$, $Y$ of $\mathcal{A}$ are isomorphic via inverse to each other isomorphisms $q \in s\mathcal{A}(X, Y)$, $t \in s\mathcal{A}(Y, X)$ such that $qf_1 - r \in \operatorname{Im} b_1$, $tf_1 - p \in \operatorname{Im} b_1$.*

Proof. Let chain maps $g_{X,Y} : s\mathcal{B}(Xf, Yf) \to s\mathcal{A}(X, Y)$, $g_{Y,X} : s\mathcal{B}(Yf, Xf) \to s\mathcal{A}(Y, X)$ be homotopy inverse to maps $f_1 : s\mathcal{A}(X, Y) \to s\mathcal{B}(Xf, Yf)$, $f_1 : s\mathcal{A}(Y, X) \to s\mathcal{B}(Yf, Xf)$. Define $q = rg_{X,Y}$, $t = pg_{Y,X}$. Then

$$\begin{aligned} [(q \otimes t)b_2 - {}_X\mathbf{i}_0^{\mathcal{A}}]f_1 = (rg \otimes pg)b_2 f_1 - {}_X\mathbf{i}_0^{\mathcal{A}} f_1 \equiv (rg \otimes pg)(f_1 \otimes f_1)b_2 - {}_{Xf}\mathbf{i}_0^{\mathcal{B}} \\ = [(r + vb_1) \otimes (p + wb_1)]b_2 - {}_{Xf}\mathbf{i}_0^{\mathcal{B}} \equiv (r \otimes p)b_2 - {}_{Xf}\mathbf{i}_0^{\mathcal{B}} \equiv 0 \pmod{\operatorname{Im} b_1}. \end{aligned}$$

Hence,

$$(q \otimes t)b_2 - {}_X\mathbf{i}_0^{\mathcal{A}} \equiv [(q \otimes t)b_2 - {}_X\mathbf{i}_0^{\mathcal{A}}]f_1 g_{X,X} \equiv 0 \pmod{\operatorname{Im} b_1}.$$

By symmetry, $(t \otimes q)b_2 - {}_Y\mathbf{i}_0^{\mathcal{A}} \in \operatorname{Im} b_1$. Other properties are easy to verify. ∎

We are going to apply Proposition 7.2 to the unital quotient $A_\infty$-functor $\bar{\jmath} : \mathcal{C} \to \mathsf{D}(\mathcal{C}|\mathcal{B})$, constructed in [LO06]. When restricted to $\mathcal{B}$ the $A_\infty$-functor $\bar{\jmath}$ becomes contractible, therefore, there exists a unital $A_\infty$-functor $f : \mathsf{q}(\mathcal{C}|\mathcal{B}) \to \mathsf{D}(\mathcal{C}|\mathcal{B})$ (unique up to an isomorphism) such that $\bar{\jmath}$ is isomorphic to the composition $\mathcal{C} \xrightarrow{e} \mathsf{q}(\mathcal{C}|\mathcal{B}) \xrightarrow{f} \mathsf{D}(\mathcal{C}|\mathcal{B})$.

7.4. Proposition. *The $A_\infty$-functor $f : \mathsf{q}(\mathcal{C}|\mathcal{B}) \to \mathsf{D}(\mathcal{C}|\mathcal{B})$ (defined up to an isomorphism) is an equivalence.*

Proof. In the following diagram the top and the bottom rows compose to contractible $A_\infty$-functors.

$$
\begin{tikzcd}
\mathcal{B} \arrow[r, hook] \arrow[dd, "\widetilde{Y}"'] \arrow[rdd, phantom, "="] & \mathcal{C} \arrow[rr, "\bar\jmath^{\mathcal{C}}"] \arrow[dd, "\widetilde{Y}"'] \arrow[dr, "e"] \arrow[d, phantom, shift left=6, "="] & \arrow[d, phantom, "\Downarrow\alpha"] & \mathsf{D}(\mathcal{C}|\mathcal{B}) \arrow[dd, "\mathsf{D}(\widetilde{Y})"] \\
 & {} & \mathsf{Q}(\widetilde{\mathcal{C}}|\widetilde{\mathcal{B}}) \arrow[ur, "f"] \arrow[dr, "g"] \arrow[d, phantom, "="] \arrow[r, phantom, "\overset{\beta}{\Uparrow}"] & {} \\
\widetilde{\mathcal{B}} \arrow[r, hook] & \widetilde{\mathcal{C}} \arrow[ur, "\widetilde{e}"] \arrow[rr, "\bar\jmath^{\widetilde{\mathcal{C}}}"'] & {} & \mathsf{D}(\widetilde{\mathcal{C}}|\widetilde{\mathcal{B}})
\end{tikzcd}
$$

Here the $A_\infty$-functor $f$ and an isomorphism $\alpha$ exist due to $e$ being quotient $A_\infty$-functor. The existence of $A_\infty$-functor $g$ such that $\widetilde{e}g = \bar\jmath^{\widetilde{\mathcal{C}}}$ follows from Theorem 5.13. The isomorphism

$$\alpha\mathsf{D}(\widetilde{Y}) : eg = \bar\jmath^{\mathcal{C}}\mathsf{D}(\widetilde{Y}) \to ef\mathsf{D}(\widetilde{Y}) : \mathcal{C} \to \mathsf{D}(\widetilde{\mathcal{C}}|\widetilde{\mathcal{B}})$$

is equivalent by Lemma 7.3 to $e\beta$ for some isomorphism

$$\beta : g \to f\mathsf{D}(\widetilde{Y}) : \mathsf{Q}(\widetilde{\mathcal{C}}|\widetilde{\mathcal{B}}) \to \mathsf{D}(\widetilde{\mathcal{C}}|\widetilde{\mathcal{B}}).$$

The $A_\infty$-functor $\mathsf{D}(\widetilde{Y})$ is an equivalence by corollary 4.9 and section 5 of [LO06]. Therefore, if we prove that $g$ is an $A_\infty$-equivalence, then $f$ is an $A_\infty$-equivalence as well. So in the following we consider only the case of strictly unital $A_\infty$-category $\mathcal{C}$, and we are proving that the natural $A_\infty$-functor $f : \mathsf{Q}(\mathcal{C}|\mathcal{B}) \to \mathsf{D}(\mathcal{C}|\mathcal{B})$ (defined up to an isomorphism) is an equivalence.

Let $\mathcal{E}$ be an arbitrary unital $A_\infty$-category. Let $\mathcal{F} \subset \mathcal{E}$ be its full contractible subcategory, that is, complexes $(s\mathcal{E}(X,X), b_1)$ are contractible for all objects $X$ of $\mathcal{F}$. Let $e : \mathcal{C} \to \mathcal{E}$ be a unital $A_\infty$-functor such that $e(\operatorname{Ob}\mathcal{B}) \subset \operatorname{Ob}\mathcal{F}$. Then there is a unital $A_\infty$-functor $\mathsf{D}(e) : \mathsf{D}(\mathcal{C}|\mathcal{B}) \to \mathsf{D}(\mathcal{E}|\mathcal{F})$ [LO06, Corollary 5.6]. There is a unital $A_\infty$-functor $\pi^{\mathcal{E}} : \mathsf{D}(\mathcal{E}|\mathcal{F}) \to \mathcal{E}$, quasi-inverse to the canonical strict embedding $\bar\jmath^{\mathcal{E}} : \mathcal{E} \to \mathsf{D}(\mathcal{E}|\mathcal{F})$ and such that $\bar\jmath^{\mathcal{E}}\pi^{\mathcal{E}} = \mathrm{id}_{\mathcal{E}}$ [LO06, Proposition 7.4]. In particular, $\operatorname{Ob}\pi^{\mathcal{E}} = \mathrm{id}_{\mathcal{E}}$. The diagram

$$
\begin{tikzcd}
\mathcal{B} \arrow[r, hook] \arrow[d, "e"'] & \mathcal{C} \arrow[r, "\bar\jmath^{\mathcal{C}}"] \arrow[d, "e"] & \mathsf{D}(\mathcal{C}|\mathcal{B}) \arrow[d, "\mathsf{D}(e)"] & \\
\mathcal{F} \arrow[r, hook] & \mathcal{E} \arrow[r, "\bar\jmath^{\mathcal{E}}"] & \mathsf{D}(\mathcal{E}|\mathcal{F}) \arrow[r, "\pi^{\mathcal{E}}"] & \mathcal{E}
\end{tikzcd}
$$

is commutative due to [*ibid*, Corollary 3.2]. Thus the composition $h = \mathsf{D}(e)\pi^{\mathcal{E}} : \mathsf{D}(\mathcal{C}|\mathcal{B}) \to \mathcal{E}$ is a unital $A_\infty$-functor such that $\operatorname{Ob}h = \operatorname{Ob}e$ and $\bar\jmath^{\mathcal{C}}h = \bar\jmath^{\mathcal{C}}\mathsf{D}(e)\pi^{\mathcal{E}} = e\bar\jmath^{\mathcal{E}}\pi^{\mathcal{E}} = e$.

Now we apply these considerations to the quotient functor $e : \mathcal{C} \to \mathcal{E} = \mathcal{D} = \mathsf{Q}(\mathcal{C}|\mathcal{B})$. Define $\mathcal{F} \subset \mathsf{Q}(\mathcal{C}|\mathcal{B})$ as a full $A_\infty$-subcategory with $\operatorname{Ob}\mathcal{F} = \operatorname{Ob}\mathcal{B}$. It is contractible. In the commutative diagram

$$\begin{tikzcd}
\mathcal{B} \arrow[r, hook] \arrow[d, "e"'] & \mathcal{C} \arrow[r, "\bar{\jmath}^{\mathcal{C}}"] \arrow[d, "e"'] & \mathsf{D}(\mathcal{C}|\mathcal{B}) \arrow[d, "\mathsf{D}(e)"'] \arrow[dr, "h"] & \\
\mathcal{F} \arrow[r, hook] & \mathcal{D} \arrow[r, "\bar{\jmath}^{\mathcal{D}}"] & \mathsf{D}(\mathcal{D}|\mathcal{F}) \arrow[r, "\pi^{\mathcal{D}}"] & \mathcal{D}
\end{tikzcd}$$

the composition $h = \mathsf{D}(e)\pi^{\mathcal{D}} : \mathsf{D}(\mathcal{C}|\mathcal{B}) \to \mathcal{D}$ satisfies equation $\bar{\jmath}^{\mathcal{C}} h = e$.

The restriction of the strict $A_\infty$-functor $\bar{\jmath}^{\mathcal{C}} : \mathcal{C} \to \mathsf{D}(\mathcal{C}|\mathcal{B})$ to $\mathcal{B}$ is contractible by [LO06, Example 6.6]. Choose the maps

$$\chi = -{}_X\mathbf{i}_0^{\mathcal{C}} \otimes 1 = [(\mathbf{i}_0^{\mathcal{C}} \otimes \mathbf{i}_0^{\mathcal{C}}) \otimes 1]\bar{b}_2 : s\mathsf{D}(\mathcal{C}|\mathcal{B})(X,Y) \to s\mathsf{D}(\mathcal{C}|\mathcal{B})(X,Y)$$

as contracting homotopies if $X \in \operatorname{Ob}\mathcal{B}$. Indeed, $(\mathbf{i}_0^{\mathcal{C}} \otimes \mathbf{i}_0^{\mathcal{C}})\bar{b}_1 = \mathbf{i}_0^{\mathcal{C}}$ and

$$[(\mathbf{i}_0^{\mathcal{C}} \otimes \mathbf{i}_0^{\mathcal{C}}) \otimes 1]\bar{b}_2\bar{b}_1 + \bar{b}_1[(\mathbf{i}_0^{\mathcal{C}} \otimes \mathbf{i}_0^{\mathcal{C}}) \otimes 1]\bar{b}_2 = -[(\mathbf{i}_0^{\mathcal{C}} \otimes \mathbf{i}_0^{\mathcal{C}})\bar{b}_1 \otimes 1]\bar{b}_2 = -(\mathbf{i}_0^{\mathcal{C}} \otimes 1)\bar{b}_2 = 1,$$

because $\mathbf{i}_0^{\mathcal{C}}$ is the strict unit of $\mathcal{A} = \mathsf{D}(\mathcal{C}|\mathcal{B})$. If $X \notin \operatorname{Ob}\mathcal{B}$, but $Y \in \operatorname{Ob}\mathcal{B}$ we choose the contracting homotopies

$$\chi = 1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}} : s\mathsf{D}(\mathcal{C}|\mathcal{B})(X,Y) \to s\mathsf{D}(\mathcal{C}|\mathcal{B})(X,Y).$$

Using Proposition 7.2 we extend $\bar{\jmath}^{\mathcal{C}}$ to a unique unital strict $A_\infty$-functor $f : \mathcal{D} \to \mathcal{A} = \mathsf{D}(\mathcal{C}|\mathcal{B})$ which satisfies the equation

$$\big(s\mathcal{D} \xrightarrow{H} s\mathcal{D} \xrightarrow{f_1} s\mathsf{D}(\mathcal{C}|\mathcal{B})\big) = \big(s\mathcal{D} \xrightarrow{f_1} s\mathsf{D}(\mathcal{C}|\mathcal{B}) \xrightarrow{\chi} s\mathsf{D}(\mathcal{C}|\mathcal{B})\big)$$

whenever the left hand side is defined. In particular, $ef = \bar{\jmath}^{\mathcal{C}}$ and $\operatorname{Ob} f = \mathrm{id}_{\operatorname{Ob}\mathcal{C}}$. The composition $fh : \mathcal{D} \to \mathcal{D}$ satisfies equation $efh = \bar{\jmath}^{\mathcal{C}} h = e = e\,\mathrm{id}_{\mathcal{D}} : \mathcal{C} \to \mathcal{D}$. According to Theorem 1.3 the strict $A_\infty$-functor given by composition with $e$

$$(e \boxtimes 1)M : A_\infty^u(\mathcal{D},\mathcal{D}) \to A_\infty^u(\mathcal{C},\mathcal{D})_{\operatorname{mod}\mathcal{B}}, \qquad f \mapsto ef,$$

is an $A_\infty$-equivalence. Therefore, $fh \simeq \mathrm{id}_{\mathcal{D}}$ due to Lemma 7.3. We conclude that $f_1h_1$ is homotopy invertible, and $f_1$ is homotopy invertible on the right.

We claim that $\big(s\mathcal{A}(X,Y) \xrightarrow{\psi} s\mathcal{A}(X,Y) \xrightarrow{f_1} s\mathcal{A}(X,Y)\big) = \mathrm{id}$, where $\psi$ is constructed in Lemma 7.1. Indeed, consider this equation on the summand $s\mathcal{C}(X,C_1) \otimes s\mathcal{C}(C_1,C_2) \otimes \dots \otimes s\mathcal{C}(C_{n-2},C_{n-1}) \otimes s\mathcal{C}(C_{n-1},Y)$ of $s\mathcal{A}(X,Y)$. For $n = 1$ the equation $e_1f_1 = 1$ is obvious. Let us prove it by induction on $n$. If $n > 1$, then

$$\begin{aligned}
\psi_n f_1 = -\sum_{k=1}^{n-1}(e_1^{\otimes k} \otimes \psi_{n-k}H)b_{k+1}f_1 &= -\sum_{k=1}^{n-1}(1^{\otimes k} \otimes \chi)\bar{b}_{k+1} = -(1 \otimes \chi)\bar{b}_2 \\
&= [1 \otimes (\mathbf{i}_0^{\mathcal{C}} \otimes 1^{\otimes n-1})]\bar{b}_2 = (1 \otimes \mathbf{i}_0^{\mathcal{C}})b_2 \otimes 1^{\otimes n-1} = 1.
\end{aligned}$$

Since $f_1$ is homotopy invertible on the right and on the left, it is homotopy invertible. Since $\operatorname{Ob} f = \mathrm{id}_{\operatorname{Ob}\mathcal{C}}$, the $A_\infty$-functor $f : \mathsf{Q}(\mathcal{C}|\mathcal{B}) \to \mathsf{D}(\mathcal{C}|\mathcal{B})$ is an equivalence. ∎

## 8. The example of complexes

Let $\mathcal{A}$ be a $\Bbbk$-linear Abelian category, let $\mathcal{C} = \underline{\mathsf{C}}(\mathcal{A})$ be the differential graded category of complexes in $\mathcal{A}$, and let $\mathcal{B}$ be its full subcategory of acyclic complexes. Let $\mathcal{D} = \mathsf{Q}(\mathcal{C}|\mathcal{B})$ be the quotient unital $A_\infty$-category. The embedding $e : \mathcal{C} \hookrightarrow \mathcal{D}$ induces a $\Bbbk$-linear functor of homotopy categories

$$H^0 e : \operatorname{Ho}\mathcal{C} = H^0(\mathcal{C}, m_1) \to H^0(\mathcal{D}, m_1) = \operatorname{Ho}\mathcal{D},$$

where $m_1 = sb_1s^{-1}$. Morphisms of $\operatorname{Ho}\mathcal{C}$ are homotopy equivalence classes $[q]$ of chain morphisms $q : X \to Y$.

8.1. Proposition. *For a quasi-isomorphism $q$ its image $[qe] \in \operatorname{Ho}\mathcal{D}(X, Y)$ is invertible.*

Proof. If $q : X \to Y$ is a quasi-isomorphism, then $C = \operatorname{Cone} q$ is acyclic. The complex $C$ is the graded object $Y \oplus X[1]$ with the differential given by the matrix

$$d^C = \begin{pmatrix} d^Y & 0 \\ s^{-1}q & d^{X[1]} \end{pmatrix} = \begin{pmatrix} d^Y & 0 \\ s^{-1}q & -s^{-1}d^X s \end{pmatrix}.$$

There is a semisplit exact sequence of chain morphisms $0 \to Y \xrightarrow{n} C \xrightarrow{k} X[1] \to 0$, where $n = \mathrm{in}^Y = (1, 0)$, $k = \mathrm{pr}^{X[1]} = \binom{0}{1}$. Thus $n \in \mathcal{C}(Y, C)^0$, $k \in \mathcal{C}(C, X[1])^0$ are cycles, $nm_1 = 0$, $km_1 = 0$. The morphisms $s \in \mathcal{C}(X, X[1])^{-1}$, $s^{-1} \in \mathcal{C}(X[1], X)^1$ are also cycles, because in our conventions $sm_1 = sd^{X[1]} + d^X s = 0$, similarly $s^{-1}m_1 = 0$. Hence, $ks^{-1} \in \mathcal{C}(C, X)^1$ is also a cycle, $(ks^{-1})m_1 = 0$.

Since $C \in \operatorname{Ob}\mathcal{B}$, we have a map

$$\underline{H} = sHs^{-1} : \mathcal{C}(C, C) \longrightarrow \mathcal{D}(\mathfrak{t}_1)(C, C) \hookrightarrow \mathcal{D}(C, C), \qquad f \mapsto fsHs^{-1} = f\underline{H}.$$

It satisfies the equation $\underline{H}m_1 + m_1\underline{H} = e : \mathcal{C}(C, C) \to \mathcal{D}(C, C)$. In particular, there is an element $1_C\underline{H} \in \mathcal{D}(C, C)$. Define an element

$$p = (n \otimes 1_C\underline{H} \otimes ks^{-1})(1 \otimes m_2)m_2 \in \mathcal{D}(Y, X),$$

where $m_2 = (s \otimes s)b_2s^{-1}$. More generally, $m_n = s^{\otimes n}b_ns^{-1}$. We have $\deg p = 0$ and

$$pm_1 = -(n \otimes 1_C \otimes ks^{-1})(1 \otimes m_2)m_2 = nks^{-1} = 0.$$

Let us show that $[p] \in H^0\mathcal{D}(Y, X)$ is inverse to $[q] \in H^0\mathcal{D}(X, Y)$. Define $h = \big(X \xrightarrow{s} X[1] \xrightarrow{\mathrm{in}^{X[1]}} C\big)$, then $h \in \mathcal{C}(X, C)^{-1}$, and

$$hm_1 = hd^C + d^X h = (0, s)\begin{pmatrix} d^Y & 0 \\ s^{-1}q & -s^{-1}d^X s \end{pmatrix} + d^X(0, s) = (q, 0) = qn.$$

One can check that

$$
\begin{aligned}
&(h\otimes 1_C\underline{H}\otimes ks^{-1})(1\otimes m_2)m_2m_1-(q\otimes n\otimes 1_C\underline{H}\otimes ks^{-1})(1\otimes 1\otimes m_2)m_3m_1\\
&\qquad =-hks^{-1}+(q\otimes n\otimes 1_C\underline{H}\otimes ks^{-1})(1\otimes 1\otimes m_2)(1\otimes m_2)m_2=-1_X+(q\otimes p)m_2,
\end{aligned}
$$

because $hk=s$.

Denote by $z$ the morphism $z=\mathrm{pr}^Y:C\to Y$, then $z\in\mathcal{C}(C,Y)^0$ and

$$
zm_1=zd^Y-d^Cz=\begin{pmatrix}1\\0\end{pmatrix}d^Y-\begin{pmatrix}d^Y & 0\\ s^{-1}q & d^{X[1]}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=-\begin{pmatrix}0\\ s^{-1}q\end{pmatrix}=-ks^{-1}q.
$$

One can check that

$$
\begin{aligned}
&(n\otimes 1_C\underline{H}\otimes z)(1\otimes m_2)m_2m_1-(n\otimes 1_C\underline{H}\otimes ks^{-1}\otimes q)(1\otimes m_3)m_2m_1\\
&\qquad\qquad -(n\otimes 1_C\underline{H}\otimes ks^{-1}\otimes q)(1\otimes m_2\otimes 1)m_3m_1\\
&\qquad =nz-(n\otimes 1_C\underline{H}\otimes ks^{-1}\otimes q)(1\otimes m_2\otimes 1)(m_2\otimes 1)m_2=1_Y-(p\otimes q)m_2,
\end{aligned}
$$

because $nz=1_Y$. Therefore, the cycles $p\in\mathcal{D}(Y,X)^0$ and $q\in\mathcal{D}(X,Y)^0$ are inverse to each other modulo boundaries. ∎

8.2. Corollary. *The functor $H^0e$ factors as $H^0\mathcal{C}\xrightarrow{Q_{\text{Verdier}}}H^0\mathcal{C}/H^0\mathcal{B}\xrightarrow{g}H^0(\mathsf{Q}(\mathcal{C}|\mathcal{B}))$, where the Verdier quotient $H^0\mathcal{C}/H^0\mathcal{B}=D(\mathcal{A})$ is the derived category of $\mathcal{A}$, and the functor $g:D(\mathcal{A})\to H^0\mathcal{D}$ is identity on objects.*

8.3. Consequences of further research. We shall use the results of the forthcoming book [BLM07] to draw more conclusions for the example of complexes. The above differential graded categories of complexes $\mathcal{C}$, $\mathcal{B}$ are pretriangulated, see [BK90]. Therefore, $\mathsf{Q}(\mathcal{C}|\mathcal{B})$ is a pretriangulated $A_\infty$-category by results of [BLM07, Chapters 15, 16]. The $A_\infty$-equivalent to it (see Section 7) differential graded category $\mathsf{D}(\mathcal{C}|\mathcal{B})$ is pretriangulated as well by [loc. cit.]. The differential graded category $\mathsf{D}(\mathcal{C}|\mathcal{B})$ is precisely the Drinfeld's quotient $\mathcal{C}/\mathcal{B}$ introduced in [Dri04, Section 3.1].

The isomorphism of $A_\infty$-functors from Section 7 yields an isomorphism of triangulated functors

$$
H^0\mathcal{J}\simeq\big[H^0\mathcal{C}\xrightarrow{H^0e}H^0(\mathsf{Q}(\mathcal{C}|\mathcal{B}))\xrightarrow{H^0f}H^0(\mathsf{D}(\mathcal{C}|\mathcal{B}))\big] \tag{57}
$$

by [BLM07, Chapter 18]. Notice that $H^0e$ and $H^0\mathcal{J}$ take objects of $H^0\mathcal{B}$ to zero objects. Hence, (57) can be presented as the pasting

$$
\begin{tikzcd}
 & H^0(\mathsf{Q}(\mathcal{C}|\mathcal{B})) \arrow[ddr, "H^0f", "\text{equiv}"'] & \\
 \arrow[r, phantom, "="] & H^0\mathcal{C}/H^0\mathcal{B} \arrow[u, "g"] \arrow[dr, "\Psi"] \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "="] & \\
H^0\mathcal{C} \arrow[uur, "H^0e"] \arrow[ur, "Q_{\text{Verdier}}"] \arrow[rr, "H^0\mathcal{J}"] & & H^0(\mathsf{D}(\mathcal{C}|\mathcal{B}))
\end{tikzcd} \tag{58}
$$

for a unique triangulated functor $\Psi$ by the properties of the Verdier quotient/localization $Q_{\text{Verdier}}$ [Ver96, Section 2.2].

Denote by $\mathcal{E}^{\mathsf{tr}}$ the pretriangulated envelope of an $A_\infty$-category $\mathcal{E}$ [BLM07, Chapter 18]. The $A_\infty$-functor $u_{\mathsf{tr}} : \mathcal{E} \to \mathcal{E}^{\mathsf{tr}}$ is the natural embedding. The commutative square

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\bar{\jmath}"] \arrow[d, "u_{\mathsf{tr}}"'] & \mathsf{D}(\mathcal{C}|\mathcal{B}) \arrow[d, "u_{\mathsf{tr}}"] \\ \mathcal{C}^{\mathsf{tr}} \arrow[r, "\bar{\jmath}^{\mathsf{tr}}"] & \mathsf{D}(\mathcal{C}|\mathcal{B})^{\mathsf{tr}} \end{tikzcd}$$

whose vertical arrows are $A_\infty$-equivalences implies the commutative diagram

$$\begin{tikzcd} H^0\mathcal{C} \arrow[r, "Q_{\text{Verdier}}"] \arrow[d, "H^0(u_{\mathsf{tr}})"'] \arrow[dr, phantom, "="] & H^0\mathcal{C}/H^0\mathcal{B} \arrow[r, "\Psi"] \arrow[d, "H^0(u_{\mathsf{tr}})"'] \arrow[dr, phantom, "="] & H^0(\mathsf{D}(\mathcal{C}|\mathcal{B})) \arrow[d, "H^0(u_{\mathsf{tr}})"] \\ H^0(\mathcal{C}^{\mathsf{tr}}) \arrow[r, "Q_{\text{Verdier}}"] & H^0(\mathcal{C}^{\mathsf{tr}})/H^0(\mathcal{B}^{\mathsf{tr}}) \arrow[r, "\Phi"] & H^0(\mathsf{D}(\mathcal{C}|\mathcal{B})^{\mathsf{tr}}) \end{tikzcd}$$

whose rows compose to $H^0\bar{\jmath}$ and $H^0(\bar{\jmath}^{\mathsf{tr}})$, and columns are equivalences.

When $\Bbbk$ is a field, the functor $\Phi$ is an equivalence by Theorem 3.4 of Drinfeld [Dri04]. In this case $\Psi$ is an equivalence, as well as $H^0 f$ from diagram (58). Hence, the triangulated functor $g : D(\mathcal{A}) = H^0\mathcal{C}/H^0\mathcal{B} \to H^0(\mathsf{Q}(\mathcal{C}|\mathcal{B}))$ from Corollary 8.2 is also an equivalence.

## A. The Yoneda Lemma for unital $A_\infty$-categories

A.1. Basic identities in symmetric closed monoidal category of complexes. We want to work out in detail a system of notations suitable for computations in symmetric closed monoidal categories. Actually we need only the category of $\mathbb{Z}$-graded $\Bbbk$-modules with a differential of degree 1. The corresponding system of notations was already used in [Lyu03, LO06].

There exists a $\mathscr{U}'$-small set $S$ of $\mathscr{U}$-small $\Bbbk$-modules such that any $\mathscr{U}$-small $\Bbbk$-module $M$ is isomorphic to some $\Bbbk$-module $N \in S$ (due to presentations $\Bbbk^{(P)} \to \Bbbk^{(Q)} \to M \to 0$). We turn $S$ into a category of $\Bbbk$-modules $\Bbbk$-mod with $\operatorname{Ob}\Bbbk\text{-mod} = S$. Thus $\Bbbk$-mod is an Abelian $\Bbbk$-linear symmetric closed monoidal $\mathscr{U}'$-small $\mathscr{U}$-category. The category $\mathsf{C}_\Bbbk = \mathsf{C}(\Bbbk\text{-mod})$ of complexes in $\Bbbk$-mod inherits all these properties from $\Bbbk$-mod, except that the symmetry becomes $c : X \otimes Y \to Y \otimes X$, $x \otimes y \mapsto (-)^{xy} y \otimes x = (-)^{\deg x \cdot \deg y} y \otimes x$. Therefore, we may consider the category of complexes enriched in $\mathsf{C}_\Bbbk$ (a differential graded category), and it is denoted by $\underline{\mathsf{C}}_\Bbbk$ in this case. The (inner) hom-object between complexes $X$ and $Y$ is the complex

$$\underline{\mathsf{C}}_\Bbbk(X,Y)^n = \prod_{i\in\mathbb{Z}} \operatorname{Hom}_\Bbbk(X^i, Y^{i+n}), \qquad (f^i)_{i\in\mathbb{Z}} d = (f^i d^{i+\deg f} - (-)^{\deg f} d^i f^{i+1})_{i\in\mathbb{Z}}.$$

The product $X = \prod_{\iota\in I} X_\iota$ in the category of complexes of $\Bbbk$-modules of the family of objects $(X_\iota)_{\iota\in I}$ coincides with the product in the category of $\mathbb{Z}$-graded $\Bbbk$-modules (and differs from the product in the category of $\Bbbk$-modules). It is given by $X^m = \prod_{\iota\in I} X_\iota^m$.

Given a complex $Z$ and an element $a \in \underline{\mathsf{C}}_\Bbbk(X,Y)$, we assign to it elements $1 \otimes a \in \underline{\mathsf{C}}_\Bbbk(Z \otimes X, Z \otimes Y)$, $(z \otimes x)(1 \otimes a) = z \otimes xa$, and $a \otimes 1 \in \underline{\mathsf{C}}_\Bbbk(X \otimes Z, Y \otimes Z)$, $(x \otimes z)(a \otimes 1) = (-)^{za} xa \otimes z$. Clearly, $(1 \otimes a)c = c(a \otimes 1) \in \underline{\mathsf{C}}_\Bbbk(Z \otimes X, Y \otimes Z)$ and $(a \otimes 1)c = c(1 \otimes a) \in \underline{\mathsf{C}}_\Bbbk(X \otimes Z, Z \otimes Y)$. If $g \in \underline{\mathsf{C}}_\Bbbk(Z,W)$, then we have $(1 \otimes a)(g \otimes 1) = (-)^{ag}(g \otimes 1)(1 \otimes a) \in \underline{\mathsf{C}}_\Bbbk(Z \otimes X, W \otimes Y)$ (Koszul's rule).

For any pair of complexes $X$, $Y \in \operatorname{Ob} \mathsf{C}_\Bbbk$ denote by

$$\mathrm{ev}_{X,Y} : X \otimes \underline{\mathsf{C}}_\Bbbk(X,Y) \to Y, \qquad \mathrm{coev}_{X,Y} : Y \to \underline{\mathsf{C}}_\Bbbk(X, X \otimes Y)$$

the canonical evaluation and coevaluation maps respectively. Then the adjunction isomorphisms are explicitly given as follows:

$$\begin{aligned} \mathsf{C}_\Bbbk(Y, \underline{\mathsf{C}}_\Bbbk(X,Z)) &\longleftrightarrow \mathsf{C}_\Bbbk(X \otimes Y, Z), \\ f &\longmapsto (1_X \otimes f)\,\mathrm{ev}_{X,Z}, \\ \mathrm{coev}_{X,Y}\, \underline{\mathsf{C}}_\Bbbk(X,g) &\longleftarrow\!\shortmid\; g. \end{aligned} \tag{59}$$

Given a complex $Z$ and an element $a \in \underline{\mathsf{C}}_\Bbbk(X,Y)$, we assign to it the element $\underline{\mathsf{C}}_\Bbbk(1,a) = \underline{\mathsf{C}}_\Bbbk(Z,a) = a\phi$ of $\underline{\mathsf{C}}_\Bbbk(\underline{\mathsf{C}}_\Bbbk(Z,X), \underline{\mathsf{C}}_\Bbbk(Z,Y))$ obtained from the equation

$$m_2^{\mathsf{C}_\Bbbk} = \bigl(\underline{\mathsf{C}}_\Bbbk(Z,X) \otimes \underline{\mathsf{C}}_\Bbbk(X,Y) \xrightarrow[\exists!]{1\otimes\phi} \underline{\mathsf{C}}_\Bbbk(Z,X) \otimes \underline{\mathsf{C}}_\Bbbk(\underline{\mathsf{C}}_\Bbbk(Z,X), \underline{\mathsf{C}}_\Bbbk(Z,Y)) \xrightarrow{\mathrm{ev}} \underline{\mathsf{C}}_\Bbbk(Z,Y)\bigr),$$

which holds for a unique chain map $\phi$. Despite that the map $a$ is not a chain map we write this element as $a : X \to Y$, and we write $a\phi$ as

$$\underline{\mathsf{C}}_\Bbbk(1,a) : \underline{\mathsf{C}}_\Bbbk(Z,X) \to \underline{\mathsf{C}}_\Bbbk(Z,Y), \qquad (f^i)_{i\in\mathbb{Z}} \mapsto (f^i a^{i+\deg f})_{i\in\mathbb{Z}}.$$

Similarly, given a complex $X$ and an element $g \in \underline{\mathsf{C}}_\Bbbk(W,Z)$, we assign to it the element $\underline{\mathsf{C}}_\Bbbk(g,1) = \underline{\mathsf{C}}_\Bbbk(g,X) = g\psi \in \underline{\mathsf{C}}_\Bbbk(\underline{\mathsf{C}}_\Bbbk(Z,X), \underline{\mathsf{C}}_\Bbbk(W,X))$ obtained from the diagram

$$\begin{tikzcd} \underline{\mathsf{C}}_\Bbbk(Z,X) \otimes \underline{\mathsf{C}}_\Bbbk(W,Z) \arrow[r, "c"] \arrow[d, "1\otimes\psi", "\exists!"'] & \underline{\mathsf{C}}_\Bbbk(W,Z) \otimes \underline{\mathsf{C}}_\Bbbk(Z,X) \arrow[d, "m_2^{\mathsf{C}_\Bbbk}"] \\ \underline{\mathsf{C}}_\Bbbk(Z,X) \otimes \underline{\mathsf{C}}_\Bbbk(\underline{\mathsf{C}}_\Bbbk(Z,X), \underline{\mathsf{C}}_\Bbbk(W,X)) \arrow[r, "\mathrm{ev}"] & \underline{\mathsf{C}}_\Bbbk(W,X) \end{tikzcd}$$

commutative for a unique chain map $\psi$. Although the map $\underline{\mathsf{C}}_\Bbbk(g,1)$ is not a chain map we write it as

$$\underline{\mathsf{C}}_\Bbbk(g,1) : \underline{\mathsf{C}}_\Bbbk(Z,X) \to \underline{\mathsf{C}}_\Bbbk(W,X), \qquad (f^i)_{i\in\mathbb{Z}} \mapsto \bigl((-)^{\deg f \cdot \deg g} g^i f^{i+\deg g}\bigr)_{i\in\mathbb{Z}}.$$

For each pair of homogeneous elements $a \in \underline{\mathsf{C}}_\Bbbk(X,Y)$, $g \in \underline{\mathsf{C}}_\Bbbk(W,Z)$ we have

$$\begin{aligned} &\bigl(\underline{\mathsf{C}}_\Bbbk(Z,X) \xrightarrow{\underline{\mathsf{C}}_\Bbbk(Z,a)} \underline{\mathsf{C}}_\Bbbk(Z,Y) \xrightarrow{\underline{\mathsf{C}}_\Bbbk(g,Y)} \underline{\mathsf{C}}_\Bbbk(W,Y)\bigr) \\ &\qquad = (-)^{ag}\bigl(\underline{\mathsf{C}}_\Bbbk(Z,X) \xrightarrow{\underline{\mathsf{C}}_\Bbbk(g,X)} \underline{\mathsf{C}}_\Bbbk(W,X) \xrightarrow{\underline{\mathsf{C}}_\Bbbk(W,a)} \underline{\mathsf{C}}_\Bbbk(W,Y)\bigr). \end{aligned}$$

This equation follows from one of the standard identities in symmetric closed monoidal categories [EK66], and can be verified directly. We also have $\underline{\mathsf{C}}_{\Bbbk}(1,a)\underline{\mathsf{C}}_{\Bbbk}(1,h) = \underline{\mathsf{C}}_{\Bbbk}(1,ah)$ and $\underline{\mathsf{C}}_{\Bbbk}(g,1)\underline{\mathsf{C}}_{\Bbbk}(e,1) = (-)^{ge}\underline{\mathsf{C}}_{\Bbbk}(eg,1)$, whenever these maps are defined.

One easily sees that $m_1^{\underline{\mathsf{C}}_{\Bbbk}} = d : \underline{\mathsf{C}}_{\Bbbk}(X,Y) \to \underline{\mathsf{C}}_{\Bbbk}(X,Y)$ coincides with $\underline{\mathsf{C}}_{\Bbbk}(1,d_Y) - \underline{\mathsf{C}}_{\Bbbk}(d_X,1)$.

Let $f : A \otimes X \to B$, $g : B \otimes Y \to C$ be two homogeneous $\Bbbk$-linear maps of arbitrary degrees. Then the following holds:

$$
\begin{aligned}
&\bigl(X \otimes Y \xrightarrow{\mathrm{coev}_{A,X}\otimes \mathrm{coev}_{B,Y}} \underline{\mathsf{C}}_{\Bbbk}(A, A\otimes X) \otimes \underline{\mathsf{C}}_{\Bbbk}(B, B\otimes Y) \\
&\qquad \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(A,f)\otimes\underline{\mathsf{C}}_{\Bbbk}(B,g)} \underline{\mathsf{C}}_{\Bbbk}(A,B)\otimes \underline{\mathsf{C}}_{\Bbbk}(B,C) \xrightarrow{m_2} \underline{\mathsf{C}}_{\Bbbk}(A,C)\bigr) \\
&= \bigl(X\otimes Y \xrightarrow{\mathrm{coev}_{A,X\otimes Y}} \underline{\mathsf{C}}_{\Bbbk}(A, A\otimes X\otimes Y) \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(A,f\otimes 1)} \underline{\mathsf{C}}_{\Bbbk}(A, B\otimes Y) \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(A,g)} \underline{\mathsf{C}}_{\Bbbk}(A,C)\bigr).
\end{aligned} \tag{60}
$$

Indeed, $(\mathrm{coev}_{A,X}\otimes \mathrm{coev}_{B,Y})(\underline{\mathsf{C}}_{\Bbbk}(A,f)\otimes\underline{\mathsf{C}}_{\Bbbk}(B,g)) = (\mathrm{coev}_{A,X}\,\underline{\mathsf{C}}_{\Bbbk}(A,f)\otimes \mathrm{coev}_{B,Y}\,\underline{\mathsf{C}}_{\Bbbk}(B,g))$, for coev has degree 0. Denote $\bar f = \mathrm{coev}_{A,X}\,\underline{\mathsf{C}}_{\Bbbk}(A,f)$, $\bar g = \mathrm{coev}_{B,Y}\,\underline{\mathsf{C}}_{\Bbbk}(B,g)$. The morphisms $\bar f$ and $\bar g$ correspond to $f$ and $g$ by adjunction. Further, the morphism $m_2$ comes by adjunction from the following map:

$$
A\otimes \underline{\mathsf{C}}_{\Bbbk}(A,B)\otimes \underline{\mathsf{C}}_{\Bbbk}(B,C) \xrightarrow{\mathrm{ev}_{A,B}\otimes 1} B\otimes \underline{\mathsf{C}}_{\Bbbk}(B,C) \xrightarrow{\mathrm{ev}_{B,C}} C,
$$

in particular the following diagram commutes:

$$
\begin{tikzcd}
A\otimes \underline{\mathsf{C}}_{\Bbbk}(A,B)\otimes \underline{\mathsf{C}}_{\Bbbk}(B,C) \arrow[r, "\mathrm{ev}_{A,B}\otimes 1"] \arrow[d, "1\otimes m_2"'] & B\otimes \underline{\mathsf{C}}_{\Bbbk}(B,C) \arrow[d, "\mathrm{ev}_{B,C}"] \\
A\otimes \underline{\mathsf{C}}_{\Bbbk}(A,C) \arrow[r, "\mathrm{ev}_{A,C}"] & C
\end{tikzcd}
$$

Thus we have a commutative diagram

$$
\begin{tikzcd}
A\otimes X\otimes Y \arrow[r, "1\otimes\bar f\otimes 1"] & A\otimes \underline{\mathsf{C}}_{\Bbbk}(A,B)\otimes Y \arrow[r, "\mathrm{ev}_{A,B}\otimes 1"] \arrow[d, "1\otimes 1\otimes\bar g"'] & B\otimes Y \arrow[d, "1\otimes\bar g"] \\
& A\otimes \underline{\mathsf{C}}_{\Bbbk}(A,B)\otimes \underline{\mathsf{C}}_{\Bbbk}(B,C) \arrow[r, "\mathrm{ev}_{A,B}\otimes 1"] \arrow[d, "1\otimes m_2"'] & B\otimes \underline{\mathsf{C}}_{\Bbbk}(B,C) \arrow[d, "\mathrm{ev}_{B,C}"] \\
& A\otimes \underline{\mathsf{C}}_{\Bbbk}(A,C) \arrow[r, "\mathrm{ev}_{A,C}"] & C
\end{tikzcd}
$$

The top row composite coincides with $f\otimes 1$ and the right-hand side vertical composite coincides with $g$ (by adjunction). Thus $(1\otimes \bar f\otimes 1)(1\otimes 1\otimes \bar g)(1\otimes m_2)\,\mathrm{ev}_{A,C}$ coincides with $(f\otimes 1)g$. But the mentioned morphism comes from $(\bar f\otimes\bar g)m_2$ by adjunction, so that $(\bar f\otimes \bar g)m_2 = \mathrm{coev}_{A,X\otimes Y}\,\underline{\mathsf{C}}_{\Bbbk}(A,(f\otimes 1)g)$ (the latter morphism is the image of $(f\otimes 1)g$ under the adjunction), and we are done.

One verifies similarly the following assertion [EK66]: given $f \in \underline{\mathsf{C}}_{\Bbbk}(A,B)$, then the diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{coev}_{A,X}"] \arrow[d, "\mathrm{coev}_{B,X}"'] & \underline{\mathsf{C}}_{\Bbbk}(A, A\otimes X) \arrow[d, "\underline{\mathsf{C}}_{\Bbbk}(A,f\otimes 1)"] \\ \underline{\mathsf{C}}_{\Bbbk}(B, B\otimes X) \arrow[r, "\underline{\mathsf{C}}_{\Bbbk}(f,B\otimes X)"] & \underline{\mathsf{C}}_{\Bbbk}(A, B\otimes X) \end{tikzcd} \tag{61}$$

commutes.

If $P$ is a $\mathbb{Z}$-graded $\Bbbk$-module, then $sP = P[1]$ denotes the same $\Bbbk$-module with the grading $(sP)^d = P^{d+1}$. The "identity" map $P \to sP$ of degree $-1$ is also denoted $s$. The map $s$ commutes with the components of the differential $b$ in an $A_\infty$-category ($A_\infty$-algebra) in the following sense: $s^{\otimes n} b_n = m_n s$. The main identity $b^2 = 0$ written in components takes the form

$$\sum_{r+n+t=k} (1^{\otimes r} \otimes b_n \otimes 1^{\otimes t}) b_{r+1+t} = 0 : T^k s\mathcal{A} \to s\mathcal{A}. \tag{62}$$

A.2. $A_\infty$-FUNCTOR $h^X$. Let $\mathcal{A}$ be an $A_\infty$-category. Following Fukaya [Fuk02, Definition 7.28] for any object $X$ of $\mathcal{A}$ we define a cocategory homomorphism $h^X : Ts\mathcal{A} \to Ts\underline{\mathsf{C}}_{\Bbbk}$ as follows. It maps an object $Z$ to the complex $h^X Z = (s\mathcal{A}(X,Z), -b_1)$. The minus sign is explained by the fact that $H^X Z \overset{\text{def}}{=} (h^X Z)[-1] = (\mathcal{A}(X,Z), m_1)$. Actually, $h^X Z$ is some fixed complex of $\Bbbk$-modules from $\operatorname{Ob}\mathsf{C}_{\Bbbk}$, with a fixed isomorphism $h^X Z \xrightarrow{\sim} (s\mathcal{A}(X,Z), -b_1)$. These isomorphisms enter implicitly into the structure maps of $h^X$, however, we shall pretend that they are identity morphisms. The closed monoidal structure of $\mathsf{C}_{\Bbbk}$ gives us the right to omit these isomorphisms in all the formulae.

We require $h^X$ to be pointed, that is, $(T^0 s\mathcal{A}) h^X \subset T^0 s\underline{\mathsf{C}}_{\Bbbk}$. Therefore, $h^X$ is completely specified by its components $h^X_k$ for $k \geqslant 1$:

$$\begin{aligned} h^X_k = \Big[ & s\mathcal{A}(Z_0,Z_1) \otimes \cdots \otimes s\mathcal{A}(Z_{k-1},Z_k) \xrightarrow{\mathrm{coev}} \underline{\mathsf{C}}_{\Bbbk}(h^X Z_0, h^X Z_0 \otimes h^{Z_0} Z_1 \otimes \cdots \otimes h^{Z_{k-1}} Z_k) \\ & \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,b_{1+k})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z_0), s\mathcal{A}(X,Z_k)) \xrightarrow{s} s\underline{\mathsf{C}}_{\Bbbk}(h^X Z_0, h^X Z_k) \Big]. \end{aligned} \tag{63}$$

The composition $H^X = h^X \cdot [-1]$ is described in [LM07, eq. (A.1)]. It is proven in this work that $H^X : \mathcal{A} \to \underline{\mathsf{C}}_{\Bbbk}$ is an $A_\infty$-functor. Therefore, $h^X = H^X \cdot [1] : \mathcal{A} \to \underline{\mathsf{C}}_{\Bbbk}$ is an $A_\infty$-functor as well. A similar statement is known from Fukaya's work [Fuk02, Proposition 7.18] under slightly more restrictive general assumptions.

Let $\mathcal{A}$ be a unital $A_\infty$-category. Then for each object $X$ of $\mathcal{A}$ the $A_\infty$-functor $H^X : \mathcal{A} \to \underline{\mathsf{C}}_{\Bbbk}$ is unital by [LM07, Remark 5.19]. Hence, the $A_\infty$-functor $h^X : \mathcal{A} \to \underline{\mathsf{C}}_{\Bbbk}$ is unital as well.

A.3. THE OPPOSITE $A_\infty$-CATEGORY. Let $\mathcal{A}$ be a graded $\Bbbk$-quiver. Then its *opposite quiver* $\mathcal{A}^{\mathrm{op}}$ is defined as the quiver with the same class of objects $\operatorname{Ob}\mathcal{A}^{\mathrm{op}} = \operatorname{Ob}\mathcal{A}$, and with graded $\Bbbk$-modules of morphisms $\mathcal{A}^{\mathrm{op}}(X,Y) = \mathcal{A}(Y,X)$.

Let $\gamma : Ts\mathcal{A}^{\mathrm{op}} \to Ts\mathcal{A}$ denote the following cocategory anti-isomorphism:

$$\gamma = (-1)^k \omega_c^0 : s\mathcal{A}^{\mathrm{op}}(X_0, X_1) \otimes \dots \otimes s\mathcal{A}^{\mathrm{op}}(X_{k-1}, X_k) \to s\mathcal{A}(X_k, X_{k-1}) \otimes \dots \otimes s\mathcal{A}(X_1, X_0), \tag{64}$$

where the permutation $\omega^0 = \left(\begin{smallmatrix} 1 & 2 & \dots & k-1 & k \\ k & k-1 & \dots & 2 & 1 \end{smallmatrix}\right)$ is the longest element of $\mathfrak{S}_k$, and $\omega_c^0$ is the corresponding signed permutation, the action of $\omega^0$ in tensor products via standard symmetry. Clearly, $\gamma\Delta = \Delta(\gamma \otimes \gamma)c = \Delta c(\gamma \otimes \gamma)$, which is the anti-isomorphism property. Notice also that $(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{A}$ and $\gamma^2 = \mathrm{id}$.

When $\mathcal{A}$ is an $A_\infty$-category with the codifferential $b : Ts\mathcal{A} \to Ts\mathcal{A}$, then $\gamma b \gamma : Ts\mathcal{A}^{\mathrm{op}} \to Ts\mathcal{A}^{\mathrm{op}}$ is also a codifferential. Indeed,

$$\begin{aligned}\gamma b \gamma \Delta = \gamma b \Delta c(\gamma \otimes \gamma) = \gamma \Delta(1 \otimes b + b \otimes 1)c(\gamma \otimes \gamma) = \Delta(\gamma \otimes \gamma)c(1 \otimes b + b \otimes 1)c(\gamma \otimes \gamma) \\ = \Delta(\gamma \otimes \gamma)(b \otimes 1 + 1 \otimes b)(\gamma \otimes \gamma) = \Delta(\gamma b \gamma \otimes 1 + 1 \otimes \gamma b \gamma).\end{aligned}$$

A.4. Definition. [cf. Fukaya [Fuk02] Definition 7.8] *The* opposite $A_\infty$-category $\mathcal{A}^{\mathrm{op}}$ *to an $A_\infty$-category $\mathcal{A}$ is the opposite quiver, equipped with the codifferential $b^{\mathrm{op}} = \gamma b \gamma : Ts\mathcal{A}^{\mathrm{op}} \to Ts\mathcal{A}^{\mathrm{op}}$.*

The components of $b^{\mathrm{op}}$ are computed as follows:

$$\begin{aligned} b_k^{\mathrm{op}} = (-)^{k+1} \big[ & s\mathcal{A}^{\mathrm{op}}(X_0, X_1) \otimes \dots \otimes s\mathcal{A}^{\mathrm{op}}(X_{k-1}, X_k) \\ & \xrightarrow{\omega_c^0} s\mathcal{A}(X_k, X_{k-1}) \otimes \dots \otimes s\mathcal{A}(X_1, X_0) \xrightarrow{b_k} s\mathcal{A}(X_k, X_0) = s\mathcal{A}^{\mathrm{op}}(X_0, X_k) \big]. \end{aligned}$$

The sign $(-1)^k$ in (64) ensures that the above definition agrees with the definition of the opposite usual category.

A.5. The Yoneda $A_\infty$-functor. Since the considered $A_\infty$-category $\mathcal{A}$ is $\mathscr{U}$-small, and $\mathsf{C}_\Bbbk$ is a $\mathscr{U}'$-small $\mathscr{U}$-category, $A_\infty(\mathcal{A}, \underline{\mathsf{C}}_\Bbbk)$ is a $\mathscr{U}'$-small differential graded $\mathscr{U}$-category. Indeed, every its set of morphisms $sA_\infty(\mathcal{A}, \underline{\mathsf{C}}_\Bbbk)(f, g)$ is isomorphic to the product of graded $\Bbbk$-modules

$$\prod_{k=0}^{\infty} \prod_{X,Y \in \mathrm{Ob}\,\mathcal{A}} \underline{\mathsf{C}}_\Bbbk\big(T^k s\mathcal{A}(X, Y), s\underline{\mathsf{C}}_\Bbbk(Xf, Yg)\big),$$

that is $\mathscr{U}$-small.

The Yoneda $A_\infty$-functor exists in two versions: $Y$ and $\mathscr{Y} : \mathcal{A}^{\mathrm{op}} \to A_\infty(\mathcal{A}, \underline{\mathsf{C}}_\Bbbk)$ which differ by a shift: $Y = \mathscr{Y} \cdot A_\infty(1, [1])$. The pointed cocategory homomorphism $Y : Ts\mathcal{A}^{\mathrm{op}} \to TsA_\infty(\mathcal{A}, \underline{\mathsf{C}}_\Bbbk)$ is given as follows: on objects $X \mapsto h^X$, the components

$$Y_n : s\mathcal{A}^{\mathrm{op}}(X_0, X_1) \otimes \dots \otimes s\mathcal{A}^{\mathrm{op}}(X_{n-1}, X_n) \to sA_\infty(\mathcal{A}, \underline{\mathsf{C}}_\Bbbk)(h^{X_0}, h^{X_n}) \tag{65}$$

are determined by the following formulas. The composition of $Y_n$ with

$$\mathrm{pr}_k : sA_\infty(\mathcal{A}, \underline{\mathsf{C}}_\Bbbk)(h^{X_0}, h^{X_n}) \to \underline{\mathsf{C}}_\Bbbk(h^{Z_0} Z_1 \otimes \dots \otimes h^{Z_{k-1}} Z_k, s\underline{\mathsf{C}}_\Bbbk(h^{X_0} Z_0, h^{X_n} Z_k))$$

(that is, the $k$-th component of the coderivation $(x_1\otimes\dots\otimes x_n)Y_n)$ is given by the formula

$$
\begin{aligned}
Y_{nk} = (-)^n\bigl[s\mathcal{A}(X_1,X_0)\otimes\dots\otimes s\mathcal{A}(X_n,X_{n-1}) \xrightarrow{\mathrm{coev}} \\
\underline{\mathsf{C}}_{\Bbbk}(h^{X_0}Z_0\otimes h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k, h^{X_0}Z_0\otimes h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k\otimes h^{X_1}X_0\otimes\dots\otimes h^{X_n}X_{n-1}) \\
\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\tau_c b_{k+n+1})} \underline{\mathsf{C}}_{\Bbbk}(h^{X_0}Z_0\otimes h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k, h^{X_n}Z_k) \\
\xrightarrow{\sim} \underline{\mathsf{C}}_{\Bbbk}(h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k, \underline{\mathsf{C}}_{\Bbbk}(h^{X_0}Z_0, h^{X_n}Z_k)) \\
\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,s)} \underline{\mathsf{C}}_{\Bbbk}(h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k, s\underline{\mathsf{C}}_{\Bbbk}(h^{X_0}Z_0, h^{X_n}Z_k))\bigr],
\end{aligned}
$$

where the permutation $\tau\in\mathfrak{S}_{k+n+1}$ is $\tau^{k,n} = \bigl(\begin{smallmatrix} 0 & 1 & \dots & k & k+1 & \dots & k+n \\ n & 1+n & \dots & k+n & n-1 & \dots & 0 \end{smallmatrix}\bigr)$.

In other words, the coderivation $(x_1\otimes\dots\otimes x_n)Y_n$ has components $s\mathcal{A}(Z_0,Z_1)\otimes\dots\otimes s\mathcal{A}(Z_{k-1},Z_k)\to s\underline{\mathsf{C}}_{\Bbbk}(h^{X_0}Z_0,h^{X_n}Z_k)$, $(z_1\otimes\dots\otimes z_k)\mapsto(z_1\otimes\dots\otimes z_k\otimes x_1\otimes\dots\otimes x_n)Y'_{nk}$, where

$$
\begin{aligned}
Y'_{nk} = (-)^n\bigl[s\mathcal{A}(Z_0,Z_1)\otimes\dots\otimes s\mathcal{A}(Z_{k-1},Z_k)\otimes s\mathcal{A}(X_1,X_0)\otimes\dots\otimes s\mathcal{A}(X_n,X_{n-1}) \\
\xrightarrow{\mathrm{coev}} \underline{\mathsf{C}}_{\Bbbk}(h^{X_0}Z_0, h^{X_0}Z_0\otimes h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k\otimes h^{X_1}X_0\otimes\dots\otimes h^{X_n}X_{n-1}) \\
\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\tau_c b_{k+n+1})} \underline{\mathsf{C}}_{\Bbbk}(h^{X_0}Z_0, h^{X_n}Z_k) \xrightarrow{s} s\underline{\mathsf{C}}_{\Bbbk}(h^{X_0}Z_0, h^{X_n}Z_k)\bigr]. \quad (66)
\end{aligned}
$$

The pointed cocategory homomorphism $Y:\mathcal{A}^{\mathrm{op}}\to A_\infty(\mathcal{A},\underline{\mathsf{C}}_{\Bbbk})$ is an $A_\infty$-functor. An equivalent statement is already proved by Fukaya [Fuk02, Lemma 9.8] and by the authors in [LM07, Section 5.5].

A.6. THE YONEDA EMBEDDING. We claim that the Yoneda $A_\infty$-functor $Y$ is an equivalence of $\mathcal{A}^{\mathrm{op}}$ with its image. This is already proven by Fukaya in the case of strictly unital $A_\infty$-category $\mathcal{A}$ [Fuk02, Theorem 9.1]. This result extends to arbitrary unital $A_\infty$-categories as follows.

Let us define a full subcategory $\mathrm{Rep}A^u_\infty(\mathcal{A},\underline{\mathsf{C}}_{\Bbbk})$ of the $\mathscr{U}'$-small differential graded $\mathscr{U}$-category $A^u_\infty(\mathcal{A},\underline{\mathsf{C}}_{\Bbbk})$ as follows. Its objects are all $A_\infty$-functors $h^X:\mathcal{A}\to\underline{\mathsf{C}}_{\Bbbk}$ for $X\in\mathrm{Ob}\,\mathcal{A}$. As we know, they are unital. The differential graded category $\mathrm{Rep}A^u_\infty(\mathcal{A},\underline{\mathsf{C}}_{\Bbbk})$ is $\mathscr{U}$-small. Thus, the Yoneda $A_\infty$-functor $Y:\mathcal{A}^{\mathrm{op}}\to A_\infty(\mathcal{A},\underline{\mathsf{C}}_{\Bbbk})$ takes values in the $\mathscr{U}$-small subcategory $\mathrm{Rep}A^u_\infty(\mathcal{A},\underline{\mathsf{C}}_{\Bbbk})$.

A.7. THEOREM. *Let $\mathcal{A}$ be a unital $A_\infty$-category. Then the restricted Yoneda $A_\infty$-functor $Y:\mathcal{A}^{\mathrm{op}}\to\mathrm{Rep}A^u_\infty(\mathcal{A},\underline{\mathsf{C}}_{\Bbbk})$ is an equivalence.*

The theorem follows immediately from Corollary A.9 of [LM07] which states that $\mathscr{Y}:\mathcal{A}^{\mathrm{op}}\to\mathrm{Rep}A^u_\infty(\mathcal{A},\underline{\mathsf{C}}_{\Bbbk})$ is an $A_\infty$-equivalence. This is a corollary to a much stronger result, the $A_\infty$-version of the Yoneda Lemma [LM07, Theorem A.1].

A.8. COROLLARY. *Each $\mathscr{U}$-small unital $A_\infty$-category $\mathcal{A}$ is $A_\infty$-equivalent to a $\mathscr{U}$-small differential graded category $\mathrm{Rep}A^u_\infty(\mathcal{A},\underline{\mathsf{C}}_{\Bbbk})^{\mathrm{op}}$.*

A.9. Remark. We may use the surjective map $\operatorname{Ob} Y : \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\operatorname{Rep}A^u_\infty(\mathcal{A},\underline{\mathsf{C}}_\Bbbk)$ to transfer the differential graded category structure of $\operatorname{Rep}A^u_\infty(\mathcal{A},\underline{\mathsf{C}}_\Bbbk)$ to $\operatorname{Ob}\mathcal{A}$. This new $\mathscr{U}$-small differential graded category is denoted $\widetilde{\operatorname{Rep}}A^u_\infty(\mathcal{A},\underline{\mathsf{C}}_\Bbbk)$. Thus, its set of objects is $\operatorname{Ob}\mathcal{A}$, the sets of morphisms are

$$\widetilde{\operatorname{Rep}}A^u_\infty(\mathcal{A},\underline{\mathsf{C}}_\Bbbk)(X,Y) = A_\infty(\mathcal{A},\underline{\mathsf{C}}_\Bbbk)(h^X,h^Y),$$

and the operations are those of $A_\infty(\mathcal{A},\underline{\mathsf{C}}_\Bbbk)$. It is equivalent to $\operatorname{Rep}A^u_\infty(\mathcal{A},\underline{\mathsf{C}}_\Bbbk)$ by surjectivity of $\operatorname{Ob}Y$. The Yoneda $A_\infty$-functor can be presented as an $A_\infty$-equivalence $\widetilde{Y} : \mathcal{A}^{\mathrm{op}} \to \widetilde{\operatorname{Rep}}A^u_\infty(\mathcal{A},\underline{\mathsf{C}}_\Bbbk)$, identity on objects, whose components $\widetilde{Y}_n = Y_n$ are given by (65). A quasi-inverse to $\widetilde{Y}$ equivalence $\widetilde{\operatorname{Rep}}A^u_\infty(\mathcal{A},\underline{\mathsf{C}}_\Bbbk) \to \mathcal{A}^{\mathrm{op}}$ can be chosen so that it induces the identity map on objects as well by Corollary 1.9.

## B. Strict $A^u_\infty$-2-functor

The goal of this section is to show that the problem of representing the $A^u_\infty$-2-functor $\mathcal{A} \mapsto A^u_\infty(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}}$ for a pair $(\mathcal{C},\mathcal{B})$ of a unital $A_\infty$-category $\mathcal{C}$ and its full subcategory $\mathcal{B}$ reduces to the case of differential graded $\mathcal{C}$.

B.1. An $A_\infty$-functor. For arbitrary $A_\infty$-categories $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ the left hand side of the equation

$$\begin{aligned}&\big[TsA_\infty(\mathcal{Y},\mathcal{Z})\boxtimes TsA_\infty(\mathcal{X},\mathcal{Y}) \xrightarrow{c} TsA_\infty(\mathcal{X},\mathcal{Y})\boxtimes TsA_\infty(\mathcal{Y},\mathcal{Z}) \xrightarrow{M} TsA_\infty(\mathcal{X},\mathcal{Z})\big]\\ &= \big[TsA_\infty(\mathcal{Y},\mathcal{Z})\boxtimes TsA_\infty(\mathcal{X},\mathcal{Y}) \xrightarrow{1\boxtimes A_\infty(\_,\mathcal{Z})} TsA_\infty(\mathcal{Y},\mathcal{Z})\boxtimes TsA_\infty(A_\infty(\mathcal{Y},\mathcal{Z}),A_\infty(\mathcal{X},\mathcal{Z}))\\ &\qquad\xrightarrow{\alpha} TsA_\infty(\mathcal{X},\mathcal{Z})\big]\end{aligned} \tag{67}$$

is an $A_\infty$-functor. Therefore, by Proposition 5.5 of [Lyu03] there exists a unique $A_\infty$-functor

$$A_\infty(\_,\mathcal{Z}) : A_\infty(\mathcal{X},\mathcal{Y}) \to A_\infty(A_\infty(\mathcal{Y},\mathcal{Z}),A_\infty(\mathcal{X},\mathcal{Z}))$$

in the right hand side, which makes equation (67) hold true. The proof of Proposition 3.4 of [Lyu03] contains a recipe for finding the components of $A_\infty(\_,\mathcal{Z})$. Namely, the equation

$$(p\boxtimes 1)M = [p.A_\infty(\_,\mathcal{Z})]\theta \tag{68}$$

has to hold for all $p \in TsA_\infty(\mathcal{X},\mathcal{Y})$. In particular,

$$\begin{aligned}f.A_\infty(\_,\mathcal{Z}) &= (f\boxtimes 1)M = (f\boxtimes 1_{\mathcal{Z}})M : A_\infty(\mathcal{Y},\mathcal{Z}) \to A_\infty(\mathcal{X},\mathcal{Z}) \quad \text{for } f \in \operatorname{Ob}A_\infty(\mathcal{X},\mathcal{Y}),\\ r.A_\infty(\_,\mathcal{Z})_1 &= (r\boxtimes 1)M = (r\boxtimes 1_{\mathcal{Z}})M : (f\boxtimes 1)M \to (g\boxtimes 1)M \text{ for } r \in sA_\infty(\mathcal{X},\mathcal{Y})(f,g).\end{aligned}$$

Other components of $A_\infty(\_,\mathcal{Z})$ are obtained from the recurrent relation, which is equation (68) written for $p = p^1\otimes\dots\otimes p^n$:

$$\begin{aligned}(p^1\otimes\dots\otimes p^n)A_\infty(\_,\mathcal{Z})_n &= (p^1\otimes\dots\otimes p^n\boxtimes 1)M\\ &- \sum_{i_1+\dots+i_l=n}^{l>1} [(p^1\otimes\dots\otimes p^n).(A_\infty(\_,\mathcal{Z})_{i_1}\otimes A_\infty(\_,\mathcal{Z})_{i_2}\otimes\dots\otimes A_\infty(\_,\mathcal{Z})_{i_l})]\theta.\end{aligned} \tag{69}$$

In particular, for $r \otimes t \in T^2 sA_\infty(\mathcal{X},\mathcal{Y})$ we get

$$(r \otimes t)A_\infty(\_,\mathcal{Z})_2 = (r \otimes t \boxtimes 1)M - [(r \boxtimes 1)M \otimes (t \boxtimes 1)M]\theta.$$

Given $g^0 \xrightarrow{p^1} g^1 \xrightarrow{p^2} \dots g^{n-1} \xrightarrow{p^n} g^n$ we find from (69) the components of the $A_\infty$-transformation

$$(p^1 \otimes \dots \otimes p^n)A_\infty(\_,\mathcal{Z})_n \in sA_\infty(A_\infty(\mathcal{Y},\mathcal{Z}), A_\infty(\mathcal{X},\mathcal{Z}))((g^0 \boxtimes 1)M, (g^n \boxtimes 1)M)$$

in the form

$$[(p^1 \otimes \dots \otimes p^n)A_\infty(\_,\mathcal{Z})_n]_m = (p^1 \otimes \dots \otimes p^n \boxtimes 1)M_{nm}.$$

So they vanish for $m > 1$.

If $\mathcal{Z}$ is unital, then the $A_\infty$-functor $A_\infty(\_,\mathcal{Z})$ takes values in the subcategory

$$A^u_\infty(A_\infty(\mathcal{Y},\mathcal{Z}), A_\infty(\mathcal{X},\mathcal{Z})),$$

because the $A_\infty$-functors $(f \boxtimes 1_\mathcal{Z})M$ commute with $(1 \boxtimes \mathbf{i}^\mathcal{Z})M$, so they are unital.

B.2. PROPOSITION. *For arbitrary $A_\infty$-categories $\mathcal{X}$, $\mathcal{Y}$ and unital $A_\infty$-categories $\mathcal{C}$, $\mathcal{D}$ we have*

$$\begin{aligned}
\big[TsA^u_\infty(\mathcal{C},\mathcal{D}) \boxtimes TsA_\infty(\mathcal{X},\mathcal{Y}) &\xrightarrow{c} TsA_\infty(\mathcal{X},\mathcal{Y}) \boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D}) \xrightarrow{A_\infty(_,\mathcal{C})\boxtimes A_\infty(\mathcal{X},_)} \\
&TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}), A_\infty(\mathcal{X},\mathcal{C})) \boxtimes TsA^u_\infty(A_\infty(\mathcal{X},\mathcal{C}), A_\infty(\mathcal{X},\mathcal{D})) \\
&\xrightarrow{M} TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}), A_\infty(\mathcal{X},\mathcal{D}))\big] \\
= \big[TsA^u_\infty(\mathcal{C},\mathcal{D}) \boxtimes TsA_\infty(\mathcal{X},\mathcal{Y}) &\xrightarrow{A_\infty(\mathcal{Y},_)\boxtimes A_\infty(_,\mathcal{D})} \\
&TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}), A_\infty(\mathcal{Y},\mathcal{D})) \boxtimes TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{D}), A_\infty(\mathcal{X},\mathcal{D})) \\
&\xrightarrow{M} TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}), A_\infty(\mathcal{X},\mathcal{D}))\big].
\end{aligned} \tag{70}$$

*The same statement holds true if one removes the unitality superscript $u$, and do not assume $\mathcal{C}$, $\mathcal{D}$ unital. The same equation holds true if all four $A_\infty$-categories $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{C}$, $\mathcal{D}$ are unital and all $A_\infty$-categories $A_\infty(,)$ are replaced with their subcategories $A^u_\infty(,)$.*

PROOF. Due to Proposition 3.4 of [Lyu03] equation (70) is equivalent to the following one:

$$\begin{aligned}
\big[TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D})\boxtimes TsA_\infty(\mathcal{X},\mathcal{Y}) &\xrightarrow{1\boxtimes c} TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA_\infty(\mathcal{X},\mathcal{Y})\boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D}) \\
\xrightarrow{1\boxtimes A_\infty(_,\mathcal{C})\boxtimes A_\infty(\mathcal{X},_)} TsA_\infty(\mathcal{Y},\mathcal{C})&\boxtimes TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}), A_\infty(\mathcal{X},\mathcal{C}))\boxtimes TsA^u_\infty(A_\infty(\mathcal{X},\mathcal{C}), A_\infty(\mathcal{X},\mathcal{D})) \\
&\xrightarrow{1\boxtimes M} TsA_\infty(\mathcal{Y},\mathcal{C}) \boxtimes TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}), A_\infty(\mathcal{X},\mathcal{D})) \xrightarrow{\alpha} TsA_\infty(\mathcal{X},\mathcal{D})\big] \\
= \big[TsA_\infty(\mathcal{Y},\mathcal{C}) \boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D}) \boxtimes TsA_\infty(\mathcal{X},\mathcal{Y}) &\xrightarrow{1\boxtimes A_\infty(\mathcal{Y},_)\boxtimes A_\infty(_,\mathcal{D})} \\
TsA_\infty(\mathcal{Y},\mathcal{C}) \boxtimes TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}), &A_\infty(\mathcal{Y},\mathcal{D})) \boxtimes TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{D}), A_\infty(\mathcal{X},\mathcal{D})) \\
&\xrightarrow{1\boxtimes M} TsA_\infty(\mathcal{Y},\mathcal{C}) \boxtimes TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}), A_\infty(\mathcal{X},\mathcal{D})) \xrightarrow{\alpha} TsA_\infty(\mathcal{X},\mathcal{D})\big].
\end{aligned}$$

Using the definition of $M$ [Lyu03, diagram (4.0.1)] we transform this equation into

$$\begin{aligned}
&\big[TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D})\boxtimes TsA_\infty(\mathcal{X},\mathcal{Y}) \xrightarrow{1\boxtimes c} TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA_\infty(\mathcal{X},\mathcal{Y})\boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D})\\
&\xrightarrow{1\boxtimes A_\infty(_,\mathcal{C})\boxtimes A_\infty(\mathcal{X},_)} TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}),A_\infty(\mathcal{X},\mathcal{C}))\boxtimes TsA^u_\infty(A_\infty(\mathcal{X},\mathcal{C}),A_\infty(\mathcal{X},\mathcal{D}))\\
&\xrightarrow{\alpha\boxtimes 1} TsA_\infty(\mathcal{X},\mathcal{C})\boxtimes TsA^u_\infty(A_\infty(\mathcal{X},\mathcal{C}),A_\infty(\mathcal{X},\mathcal{D})) \xrightarrow{\alpha} TsA_\infty(\mathcal{X},\mathcal{D})\big]\\
&= \big[TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D})\boxtimes TsA_\infty(\mathcal{X},\mathcal{Y}) \xrightarrow{1\boxtimes A_\infty(\mathcal{Y},_)\boxtimes A_\infty(_,\mathcal{D})}\\
&TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}),A_\infty(\mathcal{Y},\mathcal{D}))\boxtimes TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{D}),A_\infty(\mathcal{X},\mathcal{D}))\\
&\xrightarrow{\alpha\boxtimes 1} TsA_\infty(\mathcal{Y},\mathcal{D})\boxtimes TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{D}),A_\infty(\mathcal{X},\mathcal{D})) \xrightarrow{\alpha} TsA_\infty(\mathcal{X},\mathcal{D})\big].
\end{aligned}$$

Using the definitions of $A_\infty(\_,\mathcal{C})$ and $A_\infty(\mathcal{Y},\_)$ [Lyu03, (6.1.2)] we rewrite this equation as follows:

$$\begin{aligned}
&\big[TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D})\boxtimes TsA_\infty(\mathcal{X},\mathcal{Y}) \xrightarrow{c_{(123)}} TsA_\infty(\mathcal{X},\mathcal{Y})\boxtimes TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D})\\
&\xrightarrow{M\boxtimes A_\infty(\mathcal{X},_)} TsA_\infty(\mathcal{X},\mathcal{C})\boxtimes TsA^u_\infty(A_\infty(\mathcal{X},\mathcal{C}),A_\infty(\mathcal{X},\mathcal{D})) \xrightarrow{\alpha} TsA_\infty(\mathcal{X},\mathcal{D})\big]\\
&= \big[TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D})\boxtimes TsA_\infty(\mathcal{X},\mathcal{Y}) \xrightarrow{M\boxtimes A_\infty(_,\mathcal{D})}\\
&TsA_\infty(\mathcal{Y},\mathcal{D})\boxtimes TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{D}),A_\infty(\mathcal{X},\mathcal{D})) \xrightarrow{\alpha} TsA_\infty(\mathcal{X},\mathcal{D})\big].
\end{aligned}$$

Now we use definitions of $A_\infty(\mathcal{X},\_)$ and $A_\infty(\_,\mathcal{D})$ to get an equivalent form of the required equation:

$$\begin{aligned}
&\big[TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D})\boxtimes TsA_\infty(\mathcal{X},\mathcal{Y}) \xrightarrow{c_{(123)}} TsA_\infty(\mathcal{X},\mathcal{Y})\boxtimes TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D})\\
&\xrightarrow{M\boxtimes 1} TsA_\infty(\mathcal{X},\mathcal{C})\boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D}) \xrightarrow{M} TsA_\infty(\mathcal{X},\mathcal{D})\big]\\
&= \big[TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D})\boxtimes TsA_\infty(\mathcal{X},\mathcal{Y}) \xrightarrow{M\boxtimes 1} TsA_\infty(\mathcal{Y},\mathcal{D})\boxtimes TsA_\infty(\mathcal{X},\mathcal{Y})\\
&\xrightarrow{c} TsA_\infty(\mathcal{X},\mathcal{Y})\boxtimes TsA_\infty(\mathcal{Y},\mathcal{D}) \xrightarrow{M} TsA_\infty(\mathcal{X},\mathcal{D})\big].
\end{aligned}$$

This equation holds true due to associativity of $M$, since $M$ has degree 0.

Other statements are similar or follow from the already proven one. ∎

B.3. An $A^u_\infty$-2-functor. Let $A_\infty$-category $\mathcal{A}$ be unital. The $A_\infty$-category $\mathcal{E}$ is pseudounital with distinguished elements equal to the unit elements of $\mathcal{C}$.

Strict $A^u_\infty$-2-functors are defined in [LM06, Definition 3.1]. There is a strict $A^u_\infty$-2-functor $F$, given by the following data:

1. the mapping of objects $F : \operatorname{Ob} A^u_\infty \to \operatorname{Ob} A^u_\infty$, $\mathcal{A} \mapsto F\mathcal{A} = A_\infty(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}}$ ($F\mathcal{A}$ is a full subcategory of the unital $A_\infty$-category $A_\infty(\mathcal{C},\mathcal{A})$, hence it is unital as well);

2. the strict unital $A_\infty$-functor $F = F_{\mathcal{A}_1,\mathcal{A}_2} : A^u_\infty(\mathcal{A}_1,\mathcal{A}_2) \to A^u_\infty(F\mathcal{A}_1,F\mathcal{A}_2)$ for each pair of unital $A_\infty$-categories $\mathcal{A}_1$, $\mathcal{A}_2$ given as follows:

$$\operatorname{Ob} F : g \mapsto gF = (1\boxtimes g)M|_{F\mathcal{A}_1},$$

where $(1\boxtimes g)M : A_\infty(\mathcal{C},\mathcal{A}_1)\to A_\infty(\mathcal{C},\mathcal{A}_2)$. Indeed, if $\mathcal{B}\hookrightarrow\mathcal{C}\xrightarrow{f}\mathcal{A}_1$ is a contractible $A_\infty$-functor, then so is $\mathcal{B}\hookrightarrow\mathcal{C}\xrightarrow{fg}\mathcal{A}_2$. Actually, if ${}_X\mathbf{i}_0 f_1 = w_X b_1$ for some $w_X\in(s\mathcal{A}_1)^{-2}(Xf,Xf)$, $X\in\operatorname{Ob}\mathcal{B}$, then ${}_X\mathbf{i}_0 f_1 g_1 = w_X b_1 g_1 = (w_X g_1)b_1$, where $w_X g_1\in(s\mathcal{A}_2)^{-2}(Xfg,Xfg)$. Furthermore,

$$\begin{gathered}F_1 : A^u_\infty(\mathcal{A}_1,\mathcal{A}_2)(g,h)\to A^u_\infty(F\mathcal{A}_1,F\mathcal{A}_2)(gF,hF),\\ (r:g\to h:\mathcal{A}_1\to\mathcal{A}_2)\mapsto rF_1=(1\boxtimes r)M|_{F\mathcal{A}_1},\end{gathered}$$

or more precisely,

$$\begin{aligned}&\Big[sF\mathcal{A}_1(f_0,f_1)\otimes\cdots\otimes sF\mathcal{A}_1(f_{n-1},f_n)\xrightarrow{[rF_1]_n}sF\mathcal{A}_2(f_0g,f_nh)\Big]\\ &=\Big[sA_\infty(\mathcal{C},\mathcal{A}_1)(f_0,f_1)\otimes\cdots\otimes sA_\infty(\mathcal{C},\mathcal{A}_1)(f_{n-1},f_n)\xrightarrow{(1\boxtimes r)M_{n1}}sA_\infty(\mathcal{C},\mathcal{A}_2)(f_0g,f_nh)\Big].\end{aligned}$$

The necessary equations for $F_1$ are consequences of those for $A_\infty(\mathcal{C},\_)$ [Lyu03, Proposition 6.2]. Clearly, $F$ is a unital $A_\infty$-functor. Let us check that the following diagram commutes:

$$\begin{tikzcd} TsA^u_\infty(\mathcal{A}_0,\mathcal{A}_1)\boxtimes TsA^u_\infty(\mathcal{A}_1,\mathcal{A}_2) \arrow[r, "M"] \arrow[d, "F_{\mathcal{A}_0,\mathcal{A}_1}\otimes F_{\mathcal{A}_1,\mathcal{A}_2}"'] \arrow[dr, phantom, "="] & TsA^u_\infty(\mathcal{A}_0,\mathcal{A}_2) \arrow[d, "F_{\mathcal{A}_0,\mathcal{A}_2}"] \\ TsA^u_\infty(F\mathcal{A}_0,F\mathcal{A}_1)\boxtimes TsA^u_\infty(F\mathcal{A}_1,F\mathcal{A}_2) \arrow[r, "M"] & TsA^u_\infty(F\mathcal{A}_0,F\mathcal{A}_2). \end{tikzcd}$$

It follows from a similar diagram for $A_\infty(\mathcal{C},\_)$ in place of $F$ (equation (3.3.1) of [LM06]). The commutativity is clear on objects; since both sides of the required identity are cocategory homomorphisms it suffices to show that

$$\begin{aligned}&(F_{\mathcal{A}_0,\mathcal{A}_1}\boxtimes F_{\mathcal{A}_1,\mathcal{A}_2})M\operatorname{pr}_1 = MF_{\mathcal{A}_0,\mathcal{A}_1}\operatorname{pr}_1 :\\ &\qquad TsA^u_\infty(\mathcal{A}_0,\mathcal{A}_1)\boxtimes TsA^u_\infty(\mathcal{A}_1,\mathcal{A}_2)\to sA^u_\infty(F\mathcal{A}_0,F\mathcal{A}_2).\end{aligned}$$

Since $F_{-,-}$ are strict $A_\infty$-functors, we must show that for any non-negative integers $n$, $m$

$$\begin{aligned}&((F_{\mathcal{A}_0,\mathcal{A}_1})_1^{\otimes n}\boxtimes(F_{\mathcal{A}_1,\mathcal{A}_2})_1^{\otimes m})M_{nm} = M_{nm}(F_{\mathcal{A}_0,\mathcal{A}_2})_1 :\\ &\qquad T^n sA^u_\infty(\mathcal{A}_0,\mathcal{A}_1)\boxtimes T^m sA^u_\infty(\mathcal{A}_1,\mathcal{A}_2)\to sA^u_\infty(F\mathcal{A}_0,F\mathcal{A}_2).\end{aligned}$$

Since $M_{nm}$ vanishes whenever $m>1$, we restrict our attention to $m=0$ and $m=1$. These cases are similar and we will give verification in the case $m=1$. Given diagrams of $A_\infty$-functors and $A_\infty$-transformations

$$f^0\xrightarrow{r^1}f^1\longrightarrow\dots\xrightarrow{r^n}f^n:\mathcal{A}_0\to\mathcal{A}_1,\qquad g^0\xrightarrow{t^1}g^1:\mathcal{A}_1\to\mathcal{A}_2,$$

we must show that $(1\boxtimes(r^1\otimes\dots\otimes r^n\boxtimes t^1)M)M|_{F\mathcal{A}_0}=((1\boxtimes r^1)M|_{F\mathcal{A}_0}\otimes\dots\otimes(1\boxtimes r^n)M|_{F\mathcal{A}_0}\boxtimes(1\boxtimes t^1)M|_{F\mathcal{A}_1})M$. This is a particular case of equation (3.3.1) of [LM06] since it coincides with a similar equation for $A_\infty(\mathcal{C},\_)$ in place of $F$.

Let $\mathcal{D}$ be the $A_\infty$-category defined in Section 5.2. We claim that restr : $A_\infty(\mathcal{D},\mathcal{A})\to F\mathcal{A}$ is an $A^u_\infty$-2-transformation as defined in [LM06, Definition 3.2]. The strict $A_\infty$-functor restr is induced by the inclusion $\iota:\mathcal{C}\hookrightarrow\mathcal{D}$:

$$f\mapsto(\mathcal{C}\hookrightarrow\mathcal{D}\xrightarrow{f}\mathcal{A})=f|_{\mathcal{C}},$$
$$\mathrm{restr}_1:A_\infty(\mathcal{D},\mathcal{A})(f,g)\to F\mathcal{A}(f|_{\mathcal{C}},g|_{\mathcal{C}})=A_\infty(\mathcal{C},\mathcal{A})(f|_{\mathcal{C}},g|_{\mathcal{C}}).$$

The restriction $f|_{\mathcal{B}}$ is a contractible $A_\infty$-functor, for ${}_X\mathbf{i}_0f_1=\varepsilon_Xb_1f_1=(\varepsilon_Xf_1)b_1$ for $X\in\mathcal{B}$. Let us check that the following diagram of $A_\infty$-functors commutes:

$$\begin{tikzcd}
A^u_\infty(\mathcal{A}_1,\mathcal{A}_2) \arrow[r, "A_\infty(\mathcal{D},-)"] \arrow[d, "F"'] \arrow[dr, phantom, "="] & A^u_\infty(A_\infty(\mathcal{D},\mathcal{A}_1),A_\infty(\mathcal{D},\mathcal{A}_2)) \arrow[d, "(1\boxtimes\mathrm{restr}_{\mathcal{A}_2})M"] \\
A^u_\infty(F\mathcal{A}_1,F\mathcal{A}_2) \arrow[r, "(\mathrm{restr}_{\mathcal{A}_1}\boxtimes 1)M"] & A^u_\infty(A_\infty(\mathcal{D},\mathcal{A}_1),F\mathcal{A}_2)
\end{tikzcd}$$

All functors in the diagram above are strict (the proof is given in [LM06, Section 3.4]). We must verify the equation

$$A_\infty(\mathcal{D},-)(1\boxtimes\mathrm{restr}_{\mathcal{A}_2})M=F(\mathrm{restr}_{\mathcal{A}_1}\boxtimes 1)M.$$

On objects: given a unital $A_\infty$-functor $g:\mathcal{A}_1\to\mathcal{A}_2$, we are going to check that

$$[(1\boxtimes g)M]_n\cdot\mathrm{restr}_1=\mathrm{restr}_1^{\otimes n}\cdot[(1\boxtimes g)M]_n$$

for any $n\geqslant 1$. Indeed, for any $n$-tuple of composable $A_\infty$-transformations

$$f^0\xrightarrow{r^1}f^1\longrightarrow\dots\xrightarrow{r^n}f^n:\mathcal{D}\to\mathcal{A}_1$$

we have

$$\begin{aligned}
\{(r^1\otimes\dots\otimes r^n)[(1\boxtimes g)M]_n\}_k\big|_{\mathcal{C}}&=[(r^1\otimes\dots\otimes r^n\boxtimes g)M_{n0}]_k\big|_{\mathcal{C}}\\
&=\Big\{\sum_l(r^1\otimes\dots\otimes r^n)\theta_{kl}g_l\Big\}\big|_{\mathcal{C}}\\
&=\sum_l(r^1|_{\mathcal{C}}\otimes\dots\otimes r^n|_{\mathcal{C}})\theta_{kl}g_l|_{\mathcal{C}}\\
&=\{(r^1|_{\mathcal{C}}\otimes\dots\otimes r^n|_{\mathcal{C}})[(1\boxtimes g|_{\mathcal{C}})M]_n\}_k.
\end{aligned}$$

The coincidence of $A_\infty$-transformations $((1\boxtimes t)M)\cdot\mathrm{restr}_{\mathcal{D}}=\mathrm{restr}_{\mathcal{C}}\cdot((1\boxtimes t)M)$ follows similarly from the computation:

$$\begin{aligned}\{(r^1\otimes\cdots\otimes r^n)[(1\boxtimes t)M]_n\}_k\big|_{\mathcal C} &= [(r^1\otimes\cdots\otimes r^n\boxtimes t)M_{n1}]_k\big|_{\mathcal C}\\ &= \Big\{\sum_l (r^1\otimes\cdots\otimes r^n)\theta_{kl}t_l\Big\}\big|_{\mathcal C}\\ &= \sum_l (r^1|_{\mathcal C}\otimes\cdots\otimes r^n|_{\mathcal C})\theta_{kl}t_l|_{\mathcal C}\\ &= \{(r^1|_{\mathcal C}\otimes\cdots\otimes r^n|_{\mathcal C})[(1\boxtimes t|_{\mathcal C})M]_n\}_k.\end{aligned}$$

We are more interested in the following $A^u_\infty$-2-subfunctor of $F$, denoted $G:\operatorname{Ob}A^u_\infty\to\operatorname{Ob}A^u_\infty$, $\mathcal A\mapsto G\mathcal A=A^u_\infty(\mathcal C,\mathcal A)_{\operatorname{mod}\mathcal B}\subset F\mathcal A$. All the structure data of $G$ are restrictions of those of $F$. Hence, the conclusion that $\mathrm{restr}:A^{\psi u}_\infty(\mathcal D,\mathcal A)\to G\mathcal A$ is an $A^u_\infty$-2-natural transformation remains valid. We prove in Theorem 5.13 that this $A_\infty$-functor is an equivalence. Therefore, this restriction $A_\infty$-functor is a 2-natural $A_\infty$-equivalence. If $\mathcal C$ is strictly unital, then $\mathcal D=\mathsf Q(\mathcal C|\mathcal B)$ is unital by Theorem 6.5. Whenever $\mathcal D$ is unital, we say that $\mathcal D$ *unitally* represents $G$.

We are going to discuss how an $A^u_\infty$-2-functor represented by an $A_\infty$-category $\mathcal X$ depends on $\mathcal X$.

B.4. Proposition. *Let $f:\mathcal X\to\mathcal Y$ be an $A_\infty$-functor. Then*

$$\lambda_{\mathcal A}=(f\boxtimes 1)M:A_\infty(\mathcal Y,\mathcal A)\to A_\infty(\mathcal X,\mathcal A),\qquad g\mapsto fg$$

*is a strict $A^u_\infty$-2-transformation between two $A^u_\infty$-2-functors of $\mathcal A\in\operatorname{Ob}A^u_\infty$. If $f:\mathcal X\to\mathcal Y$ is a unital $A_\infty$-functor, then*

$$(f\boxtimes 1)M:A^u_\infty(\mathcal Y,\mathcal A)\to A^u_\infty(\mathcal X,\mathcal A)$$

*is also a strict $A^u_\infty$-2-transformation.*

Proof. The $A_\infty$-functor $(f\boxtimes 1)M$ strictly commutes with the unit transformations $(1\boxtimes\mathbf i^{\mathcal A})M$ in $A_\infty(\mathcal Y,\mathcal A)$ and $A_\infty(\mathcal X,\mathcal A)$:

$$(f\boxtimes 1_{\mathcal A})M\cdot(1_{\mathcal X}\boxtimes\mathbf i^{\mathcal A})M=(1_{\mathcal Y}\boxtimes\mathbf i^{\mathcal A})M\cdot(f\boxtimes 1_{\mathcal A})M$$

due to associativity of $M$. Therefore, $(f\boxtimes 1)M$ is unital.

We have to prove that the diagram of $A_\infty$-functors

$$\begin{tikzcd} A^u_\infty(\mathcal C,\mathcal D) \arrow[r, "A_\infty(\mathcal Y,\_)"] \arrow[d, "A_\infty(\mathcal X,\_)"'] & A^u_\infty(A_\infty(\mathcal Y,\mathcal C),A_\infty(\mathcal Y,\mathcal D)) \arrow[d, "(1\boxtimes\lambda_{\mathcal D})M"] \\ A^u_\infty(A_\infty(\mathcal X,\mathcal C),A_\infty(\mathcal X,\mathcal D)) \arrow[r, "(\lambda_{\mathcal C}\boxtimes 1)M"] & A^u_\infty(A_\infty(\mathcal Y,\mathcal C),A_\infty(\mathcal X,\mathcal D)) \end{tikzcd} \tag{71}$$

commutes. All four $A_\infty$-functors in this diagram are strict. So it suffices to check commutativity on objects and for the first components.

If $h : \mathcal{C} \to \mathcal{D}$ is a unital $A_\infty$-functor, then

$$h.A_\infty(\mathcal{Y}, \_)(1 \boxtimes \lambda_{\mathcal{D}})M = (1 \boxtimes h)M \cdot (f \boxtimes 1)M = (f \boxtimes 1 \boxtimes h)(1 \boxtimes M)M$$

equals to

$$h.A_\infty(\mathcal{X}, \_)(\lambda_{\mathcal{C}} \boxtimes 1)M = (f \boxtimes 1)M \cdot (1 \boxtimes h)M = (f \boxtimes 1 \boxtimes h)(M \boxtimes 1)M$$

due to associativity of $M$.

If $t \in sA^u_\infty(\mathcal{C}, \mathcal{D})(g, h)$, then it is mapped by the first components of $A_\infty$-functors in diagram (71) to

$$t.A_\infty(\mathcal{Y}, \_)_1(1 \boxtimes \lambda_{\mathcal{D}})M_{10} = (1 \boxtimes t)M \cdot (f \boxtimes 1)M = (f \boxtimes 1 \boxtimes t)(1 \boxtimes M)M$$

and

$$t.A_\infty(\mathcal{X}, \_)_1(\lambda_{\mathcal{C}} \boxtimes 1)M_{01} = (f \boxtimes 1)M \cdot (1 \boxtimes t)M = (f \boxtimes 1 \boxtimes t)(M \boxtimes 1)M.$$

These expressions are equal due to associativity of $M$.

The case of unital $f$ and $A^u_\infty$-2-subfunctors $A^u_\infty(\mathcal{Y}, \mathcal{A})$, $A^u_\infty(\mathcal{X}, \mathcal{A})$ follows from the general case. ∎

With two strict $A^u_\infty$-2-transformations $\lambda$, $\mu$ in $F \xrightarrow{\lambda} G \xrightarrow{\mu} H : A^u_\infty \to A^u_\infty$ is associated the third $A^u_\infty$-2-transformation $\lambda\mu : F \to H : A^u_\infty \to A^u_\infty$ – their composition, specified by the family of unital $A_\infty$-functors

$$(\lambda\mu)_{\mathcal{C}} = \lambda_{\mathcal{C}}\mu_{\mathcal{C}} : F\mathcal{C} \to H\mathcal{C}, \qquad \mathcal{C} \in \operatorname{Ob} A^u_\infty.$$

In order to verify that $\lambda\mu$ is indeed a strict $A^u_\infty$-2-transformation, we have to check equation (3.2.1) of [LM06]:

$$F \cdot (1 \boxtimes (\lambda\mu)_{\mathcal{D}})M = H \cdot ((\lambda\mu)_{\mathcal{C}} \boxtimes 1)M : A^u_\infty(\mathcal{C}, \mathcal{D}) \to A^u_\infty(F\mathcal{C}, H\mathcal{D}).$$

We do it as follows:

$$\begin{aligned}
&\big[A^u_\infty(\mathcal{C}, \mathcal{D}) \xrightarrow{F} A^u_\infty(F\mathcal{C}, F\mathcal{D}) \xrightarrow[(1\boxtimes\lambda_{\mathcal{D}})M\cdot(1\boxtimes\mu_{\mathcal{D}})M]{(1\boxtimes\lambda_{\mathcal{D}}\mu_{\mathcal{D}})M} A^u_\infty(F\mathcal{C}, H\mathcal{D})\big] \\
= &\big[A^u_\infty(\mathcal{C}, \mathcal{D}) \xrightarrow{G} A^u_\infty(G\mathcal{C}, G\mathcal{D}) \xrightarrow[(1\boxtimes\mu_{\mathcal{D}})M\cdot(\lambda_{\mathcal{C}}\boxtimes1)M]{(\lambda_{\mathcal{C}}\boxtimes1)M\cdot(1\boxtimes\mu_{\mathcal{D}})M} A^u_\infty(F\mathcal{C}, H\mathcal{D})\big] \\
= &\big[A^u_\infty(\mathcal{C}, \mathcal{D}) \xrightarrow{H} A^u_\infty(H\mathcal{C}, H\mathcal{D}) \xrightarrow[(\lambda_{\mathcal{C}}\mu_{\mathcal{C}}\boxtimes1)M]{(\mu_{\mathcal{C}}\boxtimes1)M\cdot(\lambda_{\mathcal{C}}\boxtimes1)M} A^u_\infty(F\mathcal{C}, H\mathcal{D})\big].
\end{aligned}$$

B.5. Definition. *A strict* modification $m : \lambda \to \mu : F \to G : A^u_\infty \to A^u_\infty$ *of strict* $A^u_\infty$*-2-transformations* $\lambda$, $\mu$ *is*

1. *a family of* $A_\infty$*-transformations* $m_{\mathcal{C}} : \lambda_{\mathcal{C}} \to \mu_{\mathcal{C}} : F\mathcal{C} \to G\mathcal{C}$ *for* $\mathcal{C} \in \operatorname{Ob} A^u_\infty$ *such that*

2. *for any pair of unital* $A_\infty$*-categories* $\mathcal{C}$, $\mathcal{D}$ *the* $A_\infty$*-transformations*

$$(F_{\mathcal{C},\mathcal{D}}\boxtimes\lambda_{\mathcal{D}})\cdot M = (\lambda_{\mathcal{C}}\boxtimes G_{\mathcal{C},\mathcal{D}})\cdot M \xrightarrow{(m_{\mathcal{C}}\boxtimes G_{\mathcal{C},\mathcal{D}})\cdot M} (\mu_{\mathcal{C}}\boxtimes G_{\mathcal{C},\mathcal{D}})\cdot M : A^u_\infty(\mathcal{C},\mathcal{D}) \to A^u_\infty(F\mathcal{C}, G\mathcal{D}),$$

$$(F_{\mathcal{C},\mathcal{D}}\boxtimes\lambda_{\mathcal{D}})\cdot M \xrightarrow{(F_{\mathcal{C},\mathcal{D}}\boxtimes m_{\mathcal{D}})\cdot M} (F_{\mathcal{C},\mathcal{D}}\boxtimes\mu_{\mathcal{D}})\cdot M = (\mu_{\mathcal{C}}\boxtimes G_{\mathcal{C},\mathcal{D}})\cdot M : A^u_\infty(\mathcal{C},\mathcal{D}) \to A^u_\infty(F\mathcal{C}, G\mathcal{D})$$

*are equal, in short,*

$$G_{\mathcal{C},\mathcal{D}}(m_{\mathcal{C}}\boxtimes 1)M = F_{\mathcal{C},\mathcal{D}}(1\boxtimes m_{\mathcal{D}})M : F_{\mathcal{C},\mathcal{D}}(1\boxtimes\lambda_{\mathcal{D}})M \to G_{\mathcal{C},\mathcal{D}}(\mu_{\mathcal{C}}\boxtimes 1)M.$$

*A modification $m$ is* natural *if all $A_\infty$-transformations $m_{\mathcal{C}}$ are natural.*

B.6. Example of strict modification. We claim that for an arbitrary $A_\infty$-transformation $r : f \to g : \mathcal{X} \to \mathcal{Y}$ the family of $A_\infty$-transformations

$$m_{\mathcal{C}} = (r\boxtimes 1)M : \lambda_{\mathcal{C}} = (f\boxtimes 1)M \to \mu_{\mathcal{C}} = (g\boxtimes 1)M : F\mathcal{C} = A_\infty(\mathcal{Y},\mathcal{C}) \to G\mathcal{C} = A_\infty(\mathcal{X},\mathcal{C})$$

is a strict modification. In order to prove it, we notice first of all that $(r\boxtimes 1)M$ is a $((f\boxtimes 1)M, (g\boxtimes 1)M)$-coderivation for an arbitrary $\mathcal{C}$, since $M$ is a cocategory homomorphism. When we want to indicate the category $\mathcal{C}$, we write this coderivation as $(r\boxtimes 1_{\mathcal{C}})M \in sA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}), A_\infty(\mathcal{X},\mathcal{C}))((f\boxtimes 1_{\mathcal{C}})M, (g\boxtimes 1_{\mathcal{C}})M)$. The equation to verify is

$$\begin{aligned}
&\big[TsA^u_\infty(\mathcal{C},\mathcal{D}) \xrightarrow{(r\boxtimes 1_{\mathcal{C}})M\boxtimes A_\infty(\mathcal{X},_)} T^1sA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}), A_\infty(\mathcal{X},\mathcal{C}))\boxtimes TsA^u_\infty(A_\infty(\mathcal{X},\mathcal{C}), A_\infty(\mathcal{X},\mathcal{D}))\\
&\qquad \xrightarrow{M} TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}), A_\infty(\mathcal{X},\mathcal{D}))\big]\\
&= \big[TsA^u_\infty(\mathcal{C},\mathcal{D}) \xrightarrow{A_\infty(\mathcal{Y},_)\boxtimes(r\boxtimes 1_{\mathcal{D}})M} TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}), A_\infty(\mathcal{Y},\mathcal{D}))\boxtimes T^1sA^u_\infty(A_\infty(\mathcal{Y},\mathcal{D}), A_\infty(\mathcal{X},\mathcal{D}))\\
&\qquad \xrightarrow{M} TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}), A_\infty(\mathcal{X},\mathcal{D}))\big].
\end{aligned}$$

It is an immediate consequence of equation (70) restricted to the element $r \in T^1sA_\infty(\mathcal{X},\mathcal{Y})$.

B.7. Lemma. *If $m : \lambda \to \mu : F \to G : A^u_\infty \to A^u_\infty$ is a strict modification, then so is $mB_1 : \lambda \to \mu : F \to G : A^u_\infty \to A^u_\infty$, where $(mB_1)_{\mathcal{C}} = m_{\mathcal{C}}B_1 : \lambda_{\mathcal{C}} \to \mu_{\mathcal{C}} : F\mathcal{C} \to G\mathcal{C}$ for all $\mathcal{C} \in \operatorname{Ob} A^u_\infty$. The $\Bbbk$-linear map*

$$sA_\infty(\mathcal{X},\mathcal{Y})(f,g) \ni r \mapsto (r\boxtimes 1)M \in sA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}), A_\infty(\mathcal{X},\mathcal{C}))((f\boxtimes 1)M, (g\boxtimes 1)M)$$

*is a chain map.*

Proof. Let us prove that the family $m_{\mathcal{C}}B_1$ constitutes a strict modification. The identity $(1\boxtimes B + B\boxtimes 1)M = MB$ implies that

$$\begin{aligned}
&(F_{\mathcal{C},\mathcal{D}}\boxtimes m_{\mathcal{D}}.B_1)M = (F_{\mathcal{C},\mathcal{D}}\boxtimes m_{\mathcal{D}})MB - (-)^m B(F_{\mathcal{C},\mathcal{D}}\boxtimes m_{\mathcal{D}})M = [(F_{\mathcal{C},\mathcal{D}}\boxtimes m_{\mathcal{D}})M].B_1\\
&= [(m_{\mathcal{C}}\boxtimes G_{\mathcal{C},\mathcal{D}})M].B_1 = (m_{\mathcal{C}}\boxtimes G_{\mathcal{C},\mathcal{D}})MB - (-)^m B(m_{\mathcal{C}}\boxtimes G_{\mathcal{C},\mathcal{D}})M = (m_{\mathcal{C}}.B_1\boxtimes G_{\mathcal{C},\mathcal{D}})M.
\end{aligned}$$

Here we use the fact that $m_{\mathcal{D}}.B = m_{\mathcal{D}}.B_1$ due to $m_{\mathcal{D}} \in T^1sA_\infty(F\mathcal{D}, G\mathcal{D})(\lambda_{\mathcal{D}}, \mu_{\mathcal{D}})$, and, similarly, $m_{\mathcal{C}}.B = m_{\mathcal{C}}.B_1$. The equation

$$[(r\boxtimes 1)M].B_1 = (r\boxtimes 1)MB - (-)^r B(r\boxtimes 1)M = (r.B\boxtimes 1)M = (r.B_1\boxtimes 1)M \tag{72}$$

proves that $r \mapsto (r\boxtimes 1)M$ is a chain map. ∎

B.8. LEMMA. *Let $m$, $n$ in $\lambda \xrightarrow{m} \mu \xrightarrow{n} \nu : F \to G : A^u_\infty \to A^u_\infty$ be strict modifications. Then*

$$\begin{aligned} &F_{\mathcal{C},\mathcal{D}}[1 \boxtimes (m_{\mathcal{D}} \otimes n_{\mathcal{D}})B_2]M = G_{\mathcal{C},\mathcal{D}}[(m_{\mathcal{C}} \otimes n_{\mathcal{C}})B_2 \boxtimes 1]M \\ &\quad + G_{\mathcal{C},\mathcal{D}}[(m_{\mathcal{C}} \otimes n_{\mathcal{C}})(1 \otimes B_1 + B_1 \otimes 1)A_\infty(\_, G\mathcal{D})_2] - G_{\mathcal{C},\mathcal{D}}[(m_{\mathcal{C}} \otimes n_{\mathcal{C}})A_\infty(\_, G\mathcal{D})_2 B_1] \end{aligned}$$

PROOF. Since $A_\infty(\_, G\mathcal{D})$ is an $A_\infty$-functor we have

$$\begin{aligned} &[(m_{\mathcal{C}} \boxtimes 1)M \otimes (n_{\mathcal{C}} \boxtimes 1)M]B_2 + (m_{\mathcal{C}} \otimes n_{\mathcal{C}}).A_\infty(\_, G\mathcal{D})_2 B_1 \\ &= [m_{\mathcal{C}}.A_\infty(\_, G\mathcal{D})_1 \otimes n_{\mathcal{C}}.A_\infty(\_, G\mathcal{D})_1]B_2 + (m_{\mathcal{C}} \otimes n_{\mathcal{C}}).A_\infty(\_, G\mathcal{D})_2 B_1 \\ &= (m_{\mathcal{C}} \otimes n_{\mathcal{C}})B_2.A_\infty(\_, G\mathcal{D})_1 + (m_{\mathcal{C}} \otimes n_{\mathcal{C}})(1 \otimes B_1 + B_1 \otimes 1)A_\infty(\_, G\mathcal{D})_2 \\ &= [(m_{\mathcal{C}} \otimes n_{\mathcal{C}})B_2 \boxtimes 1]M + (m_{\mathcal{C}} \otimes n_{\mathcal{C}})(1 \otimes B_1 + B_1 \otimes 1)A_\infty(\_, G\mathcal{D})_2. \end{aligned} \tag{73}$$

Using this identity we find

$$\begin{aligned} &F_{\mathcal{C},\mathcal{D}}[1 \boxtimes (m_{\mathcal{D}} \otimes n_{\mathcal{D}})B_2]M = F_{\mathcal{C},\mathcal{D}}[(1 \boxtimes m_{\mathcal{D}})M \otimes (1 \boxtimes n_{\mathcal{D}})M]B_2 \\ &= [F_{\mathcal{C},\mathcal{D}}(1 \boxtimes m_{\mathcal{D}})M \otimes F_{\mathcal{C},\mathcal{D}}(1 \boxtimes n_{\mathcal{D}})M]B_2 = [G_{\mathcal{C},\mathcal{D}}(m_{\mathcal{C}} \boxtimes 1)M \otimes G_{\mathcal{C},\mathcal{D}}(n_{\mathcal{C}} \boxtimes 1)M]B_2 \\ &= G_{\mathcal{C},\mathcal{D}}[(m_{\mathcal{C}} \boxtimes 1)M \otimes (n_{\mathcal{C}} \boxtimes 1)M]B_2 = G_{\mathcal{C},\mathcal{D}}[(m_{\mathcal{C}} \otimes n_{\mathcal{C}})B_2 \boxtimes 1]M \\ &\quad + G_{\mathcal{C},\mathcal{D}}[(m_{\mathcal{C}} \otimes n_{\mathcal{C}})(1 \otimes B_1 + B_1 \otimes 1)A_\infty(\_, G\mathcal{D})_2] - G_{\mathcal{C},\mathcal{D}}[(m_{\mathcal{C}} \otimes n_{\mathcal{C}})A_\infty(\_, G\mathcal{D})_2 B_1], \end{aligned}$$

so the lemma is proven. ■

B.9. PROPOSITION. *If unital $A_\infty$-categories $\mathcal{X}$, $\mathcal{Y}$ are equivalent, then $A^u_\infty$-2-functors $\mathcal{A} \mapsto A^u_\infty(\mathcal{X}, \mathcal{A})$ and $\mathcal{A} \mapsto A^u_\infty(\mathcal{Y}, \mathcal{A})$ are naturally $A^u_\infty$-2-equivalent.*

PROOF. Let $\phi : \mathcal{X} \to \mathcal{Y}$, $\psi : \mathcal{Y} \to \mathcal{X}$ be $A_\infty$-equivalences, quasi-inverse to each other. Then there are natural $A_\infty$-transformations

$$r : \phi\psi \to \mathrm{id}_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}, \qquad p : \mathrm{id}_{\mathcal{X}} \to \phi\psi : \mathcal{X} \to \mathcal{X},$$

inverse to each other, that is,

$$(r \otimes p)B_2 \equiv \phi\psi\mathbf{i}^{\mathcal{X}}, \qquad (p \otimes r)B_2 \equiv \mathbf{i}^{\mathcal{X}}.$$

The $A_\infty$-transformations

$$(r \boxtimes 1)M : (\psi \boxtimes 1)M(\phi \boxtimes 1)M \to \mathrm{id} : A^u_\infty(\mathcal{X}, \mathcal{A}) \to A^u_\infty(\mathcal{X}, \mathcal{A}), \tag{74}$$

$$(p \boxtimes 1)M : \mathrm{id} \to (\psi \boxtimes 1)M(\phi \boxtimes 1)M : A^u_\infty(\mathcal{X}, \mathcal{A}) \to A^u_\infty(\mathcal{X}, \mathcal{A}) \tag{75}$$

are also natural by (72). We are going to prove that these $A_\infty$-transformations are inverse to each other.

The natural $A_\infty$-transformation

$$(1 \boxtimes \mathbf{i}^{\mathcal{A}})M : \mathrm{id} \to \mathrm{id} : A^u_\infty(\mathcal{X}, \mathcal{A}) \to A^u_\infty(\mathcal{X}, \mathcal{A})$$

is a unit transformation of $A_\infty$-category $A^u_\infty(\mathcal{X},\mathcal{A})$ by Proposition 7.7 of [Lyu03]. Another unit transformation is given by

$$(\mathbf{i}^{\mathcal{X}}\boxtimes 1)M : \mathrm{id}\to\mathrm{id} : A^u_\infty(\mathcal{X},\mathcal{A})\to A^u_\infty(\mathcal{X},\mathcal{A}). \tag{76}$$

Indeed, values of 0-th components of the both natural transformations on an object $f$ of $A^u_\infty(\mathcal{X},\mathcal{A})$ differ by a boundary since

$$_f[(\mathbf{i}^{\mathcal{X}}\boxtimes 1)M]_0 = (\mathbf{i}^{\mathcal{X}}\boxtimes f)M_{10} = \mathbf{i}^{\mathcal{X}}f \equiv f\mathbf{i}^{\mathcal{A}} = (f\boxtimes \mathbf{i}^{\mathcal{A}})M_{01} = {}_f[(1\boxtimes \mathbf{i}^{\mathcal{A}})M]_0.$$

Therefore, $[1\otimes(\mathbf{i}^{\mathcal{X}}\boxtimes 1)M_{10}]B_2$ and $[(\mathbf{i}^{\mathcal{X}}\boxtimes 1)M_{10}\otimes 1]B_2$ are homotopy invertible.

Furthermore, by (73)

$$[(\mathbf{i}^{\mathcal{X}}\boxtimes 1)M\otimes(\mathbf{i}^{\mathcal{X}}\boxtimes 1)M]B_2 \equiv [(\mathbf{i}^{\mathcal{X}}\otimes\mathbf{i}^{\mathcal{X}})B_2\boxtimes 1]M \equiv (\mathbf{i}^{\mathcal{X}}\boxtimes 1)M$$

due to Lemma B.7. Therefore, homotopy idempotent (76) is a unit transformation of $A^u_\infty(\mathcal{X},\mathcal{A})$ by [Lyu03, Definition 7.6]. Since unit transformation is unique up to equivalence by [Lyu03, Corollary 7.10] we have

$$(\mathbf{i}^{\mathcal{X}}\boxtimes 1)M \equiv (1\boxtimes\mathbf{i}^{\mathcal{A}})M : \mathrm{id}\to\mathrm{id} : A^u_\infty(\mathcal{X},\mathcal{A})\to A^u_\infty(\mathcal{X},\mathcal{A}).$$

Composing natural $A_\infty$-transformations $(r\boxtimes 1)M$ and $(p\boxtimes 1)M$ given by (74) and (75) we get

$$\begin{aligned}[(r\boxtimes 1)M\otimes(p\boxtimes 1)M]B_2 &\equiv [(r\otimes p)B_2\boxtimes 1]M \equiv (\phi\psi\mathbf{i}^{\mathcal{X}}\boxtimes 1)M \\ &\equiv (\mathbf{i}^{\mathcal{X}}\phi\psi\boxtimes 1)M = (\psi\boxtimes 1)M(\phi\boxtimes 1)M(\mathbf{i}^{\mathcal{X}}\boxtimes 1)M,\end{aligned}$$

$$[(p\boxtimes 1)M\otimes(r\boxtimes 1)M]B_2 \equiv [(p\otimes r)B_2\boxtimes 1]M \equiv (\mathbf{i}^{\mathcal{X}}\boxtimes 1)M$$

by (73) and Lemma B.7. Since $(\mathbf{i}^{\mathcal{X}}\boxtimes 1)M$ is a unit transformation, the $A_\infty$-transformations $(r\boxtimes 1)M$ and $(p\boxtimes 1)M$ are inverse to each other.

The obtained statement together with one more statement in which $\phi$ and $\psi$ exchange their places implies that $A_\infty$-functors

$$\begin{aligned}A^u_\infty(\phi,\mathcal{A}) &= (\phi\boxtimes 1)M : A^u_\infty(\mathcal{Y},\mathcal{A})\to A^u_\infty(\mathcal{X},\mathcal{A}),\\ A^u_\infty(\psi,\mathcal{A}) &= (\psi\boxtimes 1)M : A^u_\infty(\mathcal{X},\mathcal{A})\to A^u_\infty(\mathcal{Y},\mathcal{A})\end{aligned}$$

are quasi-inverse to each other.

They form strict $A^u_\infty$-2-transformations by Proposition B.4. Therefore, $A^u_\infty(\phi,\mathcal{A})$ and $A^u_\infty(\psi,\mathcal{A})$ are natural $A^u_\infty$-2-equivalences. ∎

Given a pair $(\mathcal{C},\mathcal{B})$ consisting of a unital $A_\infty$-category $\mathcal{C}$ and its full subcategory $\mathcal{B}$, we shall construct another pair $(\widetilde{\mathcal{C}},\widetilde{\mathcal{B}})$ consisting of a differential graded category $\widetilde{\mathcal{C}}$ and its full subcategory $\widetilde{\mathcal{B}}$ as follows. Set $\widetilde{\mathcal{C}}$ to be the differential graded category $\widetilde{\mathrm{Rep}A^u_\infty}(\mathcal{C}^{\mathrm{op}},\underline{\mathsf{C}}_\Bbbk)$ of $A_\infty$-functors, represented by objects of $\mathcal{C}$, see Remark A.9. Thus, $\operatorname{Ob}\widetilde{\mathcal{C}} = \operatorname{Ob}\mathcal{C}$. The category $\widetilde{\mathcal{C}}$ is equivalent to $\mathcal{C}$, the Yoneda $A_\infty$-equivalence $\widetilde{Y}:\mathcal{C}\to\widetilde{\mathcal{C}}$ and its quasi-inverse $\Psi:\widetilde{\mathcal{C}}\to\mathcal{C}$ are identity on objects by Remark A.9. We take $\widetilde{\mathcal{B}}$ to be the full subcategory of $\widetilde{\mathcal{C}}$ with the set of objects $\operatorname{Ob}\widetilde{\mathcal{B}} = \operatorname{Ob}\mathcal{B}$. Therefore, the $A_\infty$-functors $\widetilde{Y}$ and $\Psi$ can be restricted to quasi-inverse to each other $A_\infty$-equivalences $Y' = \widetilde{Y}\big|_{\mathcal{B}}:\mathcal{B}\to\widetilde{\mathcal{B}}$ and $\Psi' = \Psi\big|_{\widetilde{\mathcal{B}}}:\widetilde{\mathcal{B}}\to\mathcal{B}$.

B.10. Corollary. *Let $\mathcal{A}$ be a unital $A_\infty$-category. The $A_\infty$-functors*

$$\begin{aligned} (\Psi\boxtimes 1)M &: A^u_\infty(\mathcal{C},\mathcal{A})\to A^u_\infty(\widetilde{\mathcal{C}},\mathcal{A}), &\qquad f&\mapsto \Psi f,\\ (\widetilde{Y}\boxtimes 1)M &: A^u_\infty(\widetilde{\mathcal{C}},\mathcal{A})\to A^u_\infty(\mathcal{C},\mathcal{A}), &\qquad g&\mapsto \widetilde{Y}g, \end{aligned}$$

*are quasi-inverse to each other $A_\infty$-equivalences. The first maps objects of $A^u_\infty(\mathcal{C},\mathcal{A})_{\mathrm{mod}\,\mathcal{B}}$ to objects of $A^u_\infty(\widetilde{\mathcal{C}},\mathcal{A})_{\mathrm{mod}\,\widetilde{\mathcal{B}}}$, the second does vice versa. Therefore, their restrictions determine quasi-inverse to each other $A_\infty$-equivalences*

$$\begin{aligned} (\Psi\boxtimes 1)M &: A^u_\infty(\mathcal{C},\mathcal{A})_{\mathrm{mod}\,\mathcal{B}}\to A^u_\infty(\widetilde{\mathcal{C}},\mathcal{A})_{\mathrm{mod}\,\widetilde{\mathcal{B}}},\\ (\widetilde{Y}\boxtimes 1)M &: A^u_\infty(\widetilde{\mathcal{C}},\mathcal{A})_{\mathrm{mod}\,\widetilde{\mathcal{B}}}\to A^u_\infty(\mathcal{C},\mathcal{A})_{\mathrm{mod}\,\mathcal{B}}. \end{aligned}$$

Proof. Let $f:\mathcal{C}\to\mathcal{A}$ be a unital $A_\infty$-functor such that $f\big|_{\mathcal{B}}$ is contractible. Then for each object $X$ of $\mathcal{B}$ the complex $(s\mathcal{A}(Xf,Xf),b_1)$ is contractible [LO06, Proposition 6.1]. Equivalently we may say that for each object $Z$ of $\widetilde{\mathcal{B}}$ the complex $(s\mathcal{A}(Z\Psi f,Z\Psi f),b_1)$ is contractible, as the following commutative diagram shows:

$$\begin{tikzcd} \widetilde{\mathcal{B}} \arrow[r, hook] \arrow[d, "\Psi'"'] & \widetilde{\mathcal{C}} \arrow[r, "\Psi f"] \arrow[d, "\Psi"'] & \mathcal{A} \arrow[d, equal] \\ \mathcal{B} \arrow[r, hook] & \mathcal{C} \arrow[r, "f"] & \mathcal{A} \end{tikzcd}$$

This implies contractibility of the $A_\infty$-functor $\Psi f\big|_{\widetilde{\mathcal{B}}}$. Similarly, if $g\in A^u_\infty(\widetilde{\mathcal{C}},\mathcal{A})_{\mathrm{mod}\,\widetilde{\mathcal{B}}}$, then $\widetilde{Y}g\in A^u_\infty(\mathcal{C},\mathcal{A})_{\mathrm{mod}\,\mathcal{B}}$. ■

B.11. Corollary. *The $A^u_\infty$-2-functors $\mathcal{A}\mapsto A^u_\infty(\mathcal{C},\mathcal{A})_{\mathrm{mod}\,\mathcal{B}}$ and $\mathcal{A}\mapsto A^u_\infty(\widetilde{\mathcal{C}},\mathcal{A})_{\mathrm{mod}\,\widetilde{\mathcal{B}}}$ are naturally $A^u_\infty$-2-equivalent. Therefore, if one of them is representable, then so is the other.*

## C. The Yoneda Lemma for 2-categories and bicategories

In this article we deal with bicategories of a particular kind – strict 2-categories. However, 2-functors and their transformations need to be weak for our purposes.

C.1. 2-categories. We recall the definitions of strict 2-categories and associated weak notions originating in [Bén67]. We use the form and the notation of [Lyu99].

C.2. Definition. *A (strict)* 2-category $\mathfrak{A}$ *consists of*

1. *a set of objects* $\mathrm{Ob}\,\mathfrak{A}$*;*
2. *for any pair of objects* $X, Y \in \mathrm{Ob}\,\mathfrak{A}$ *a category* $\mathfrak{A}(X, Y)$*;*
3. (a) *for any object* $X \in \mathrm{Ob}\,\mathfrak{A}$ *an object* $1_X$ *of* $\mathfrak{A}(X, X)$*;*
   (b) *for any triple of objects* $X, Y, Z \in \mathrm{Ob}\,\mathfrak{A}$ *a functor*

$$\bullet : \mathfrak{A}(X, Y) \times \mathfrak{A}(Y, Z) \to \mathfrak{A}(X, Z), \qquad (F, G) \mapsto FG = F \bullet G = G \circ F;$$

*such that the following functors are equal*

4. $F \bullet 1 = F = 1 \bullet F$, $F(GH) = (FG)H$.

The 2-category of ($\mathscr{U}$-small) categories is denoted $\mathcal{C}at$.

C.3. Definition. *A* weak 2-functor *(a homomorphism in [Bén67]) between 2-categories* $\mathfrak{A}$ *and* $\mathfrak{C}$ *consists of*

1. *a function* $F : \mathrm{Ob}\,\mathfrak{A} \to \mathrm{Ob}\,\mathfrak{C}$*;*
2. *a functor* $F = F_{X,Y} : \mathfrak{A}(X, Y) \to \mathfrak{C}(FX, FY)$ *for each pair of objects* $X, Y \in \mathrm{Ob}\,\mathfrak{A}$*;*
3. (a) *an isomorphism* $\phi_0 : 1_{FX} \to F1_X$*;*
   (b) *an invertible (natural) transformation*

$$\begin{tikzcd} \mathfrak{A}(X, Y) \times \mathfrak{A}(Y, Z) \arrow[r, "\bullet"] \arrow[d, "F_{X,Y} \times F_{Y,Z}"'] & \mathfrak{A}(X, Z) \arrow[d, "F_{X,Z}"] \\ \mathfrak{C}(FX, FY) \times \mathfrak{C}(FY, FZ) \arrow[r, "\bullet"'] \arrow[ur, Rightarrow, "\phi_2"] & \mathfrak{C}(FX, FZ) \end{tikzcd}$$

*for each triple* $X, Y, Z \in \mathrm{Ob}\,\mathfrak{A}$*;*

*such that*

4. (a) *for any object* $M \in \mathfrak{A}(X, Y)$ *the composites*

$$FM = FM \bullet 1_{FY} \xrightarrow{FM \bullet \phi_0} FM \bullet F1_Y \xrightarrow{\phi_2} F(M \bullet 1_Y) = FM \tag{77}$$
$$FM = 1_{FX} \bullet FM \xrightarrow{\phi_0 \bullet FM} F1_X \bullet FM \xrightarrow{\phi_2} F(1_X \bullet M) = FM \tag{78}$$

*are identity morphisms in* $\mathfrak{C}(FM, FM)$*;*

*(b) For any objects $W, X, Y, Z \in \operatorname{Ob}\mathfrak{A}$ and any object*

$$(K, L, M) \in \mathfrak{A}(W, X) \times \mathfrak{A}(X, Y) \times \mathfrak{A}(Y, Z)$$

*there is an equation*

$$\begin{aligned}\big(FK{\bullet}(FL{\bullet}FM) \xrightarrow{FK\bullet\phi_2} FK{\bullet}F(L{\bullet}M) \xrightarrow{\phi_2} F(K{\bullet}(L{\bullet}M))\big)\\ = \big((FK{\bullet}FL){\bullet}FM \xrightarrow{\phi_2\bullet FM} F(K{\bullet}L){\bullet}FM \xrightarrow{\phi_2} F((K{\bullet}L){\bullet}M)\big).\end{aligned}$$

*If $\phi_2$ and $\phi_0$ are identity isomorphisms, $F$ is called a* strict 2-functor.

C.4. Definition. *A* weak 2-transformation *(pseudo-natural transformation [Gra74]) $\lambda : (F, \phi_2, \phi_0) \to (G, \psi_2, \psi_0) : \mathfrak{A} \to \mathfrak{C}$ is*

1. *a family of 1-morphisms $\lambda_X : FX \to GX$, $X \in \operatorname{Ob}\mathfrak{A}$;*
2. *for any 1-morphism $f : X \to Y$ in $\mathfrak{A}$ a 2-isomorphism in $\mathfrak{C}$*

$$\lambda_f : Ff{\bullet}\lambda_Y \xrightarrow{\sim} \lambda_X{\bullet}Gf : FX \to GY,$$

*which is an isomorphism of functors*

$$\lambda_- : F - {\bullet}\lambda_Y \to \lambda_X{\bullet}G- : \mathfrak{A}(X, Y) \to \mathfrak{C}(FX, GY),$$

*that is, for any 2-morphism $\xi : f \to g : X \to Y$*

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\lambda_X"'] & FY \arrow[d, "\lambda_Y"] \arrow[dl, Rightarrow, "\lambda_f"] \\ GX \arrow[r, bend left, "Gf"] \arrow[r, bend right, "Gg"'] \arrow[r, phantom, "\Downarrow G\xi"] & GY \end{tikzcd} = \begin{tikzcd} FX \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow F\xi"] \arrow[d, "\lambda_X"'] & FY \arrow[d, "\lambda_Y"] \arrow[dl, Rightarrow, "\lambda_g"'] \\ GX \arrow[r, "Gg"'] & GY \end{tikzcd},$$

*such that*

3. *(a) for any object $X \in \operatorname{Ob}\mathfrak{A}$*

$$\begin{tikzcd} FX \arrow[r, bend left, "1_{FX}"] \arrow[r, bend right, "F1_X"'] \arrow[r, phantom, "\Downarrow \phi_0"] \arrow[d, "\lambda_X"'] & FX \arrow[d, "\lambda_X"] \arrow[dl, Rightarrow, "\lambda_{1_X}"'] \\ GX \arrow[r, "G1_X"'] & GX \end{tikzcd} = \begin{tikzcd} FX \arrow[r, "1_{FX}"] \arrow[d, "\lambda_X"'] \arrow[dr, phantom, "="] & FX \arrow[d, "\lambda_X"] \\ GX \arrow[r, bend left, "1_{GX}"] \arrow[r, bend right, "G1_X"'] \arrow[r, phantom, "\Downarrow \psi_0"] & GX \end{tikzcd};$$

*(b) for any pair of composable 1-morphisms $f, g \in \mathfrak{A}_1$*

$$
\begin{tikzcd}
& FY \arrow[dr, "Fg"] \arrow[d, phantom, "\Downarrow\phi_2"] & \\
FX \arrow[ur, "Ff"] \arrow[rr, "F(fg)"'] \arrow[d, "\lambda_X"'] & & FZ \arrow[d, "\lambda_Z"] \arrow[dll, Rightarrow, "\lambda_{fg}"] \\
GX \arrow[rr, "G(fg)"'] & & GZ
\end{tikzcd}
=
\begin{tikzcd}
& FY \arrow[dr, "Fg"] \arrow[d, "\lambda_Y"] \arrow[ddl, Rightarrow, "\lambda_f"'] & \\
FX \arrow[ur, "Ff"] \arrow[d, "\lambda_X"'] & GY \arrow[dr, "Gg"] \arrow[d, phantom, "\Downarrow\psi_2"] & FZ \arrow[l, Rightarrow, "\lambda_g"'] \arrow[d, "\lambda_Z"] \\
GX \arrow[ur, "Gf"] \arrow[rr, "G(fg)"'] & & GZ
\end{tikzcd}
.
$$

*If $\lambda_f$ are identity isomorphisms, $\lambda$ is called a* strict 2-transformation*. A weak 2-transformation $\lambda = (\lambda_X)$ for which $\lambda_X$ are equivalences is called a* 2-natural equivalence*.*

C.5. Definition. *A* modification *$m : \lambda \to \mu : F \to G : \mathfrak{A} \to \mathfrak{C}$ is*

1. *a family of 2-morphisms $m_X : \lambda_X \to \mu_X$, $X \in \operatorname{Ob} \mathfrak{A}$ such that*
2. *for any 1-morphism $f : X \to Y$ in $\mathfrak{A}$*

$$
\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, shift right=4, "\mu_X"'] \arrow[d, shift left=4, "\lambda_X"] \arrow[d, phantom, "\overset{m_X}{\Leftarrow}"] & FY \arrow[d, "\lambda_Y"] \arrow[dl, Rightarrow, "\lambda_f"'] \\
GX \arrow[r, "Gf"'] & GY
\end{tikzcd}
=
\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, "\mu_X"'] & FY \arrow[d, shift right=4, "\mu_Y"'] \arrow[d, shift left=4, "\lambda_Y"] \arrow[d, phantom, "\overset{m_Y}{\Leftarrow}"] \arrow[dl, Rightarrow, "\mu_f"'] \\
GX \arrow[r, "Gf"'] & GY
\end{tikzcd}
$$

C.6. Proposition. [Invertibility of 2-natural equivalences] *Let $\lambda : F \to G : \mathfrak{A} \to \mathfrak{C}$ be a 2-natural equivalence. Then there exist a weak 2-transformation $\mu : G \to F : \mathfrak{A} \to \mathfrak{C}$ and invertible modifications $\varepsilon : \lambda\mu \to 1_F : F \to F : \mathfrak{A} \to \mathfrak{C}$ and $\eta : 1_G \to \mu\lambda : G \to G : \mathfrak{A} \to \mathfrak{C}$. Thus, $\mu$ is quasi-inverse to $\lambda$.*

Proof. Since $\lambda_X : FX \to GX$ is an equivalence for every $X \in \operatorname{Ob} \mathfrak{A}$, we obtain: for every $X \in \operatorname{Ob} \mathfrak{A}$ there exist a 1-morphism $\mu_X : GX \to FX$ and invertible 2-morphisms $\varepsilon_X : \lambda_X \mu_X \to 1_{FX} : FX \to FX$, $\beta_X : 1_{GX} \to \mu_X \lambda_X : GX \to GX$.

C.7. Lemma. *There exist such invertible 2-morphisms $\eta_X : 1_{GX} \to \mu_X \lambda_X : GX \to GX$ that the following equations hold true:*

$$
\bigl(\mu_X \xrightarrow{\eta_X \bullet \mu_X} \mu_X \lambda_X \mu_X \xrightarrow{\mu_X \bullet \varepsilon_X} \mu_X\bigr) = 1_{\mu_X}, \tag{79}
$$

$$
\bigl(\lambda_X \xrightarrow{\lambda_X \bullet \eta_X} \lambda_X \mu_X \lambda_X \xrightarrow{\varepsilon_X \bullet \lambda_X} \lambda_X\bigr) = 1_{\lambda_X}. \tag{80}
$$

Proof. Consider the following functors:

$$
\begin{aligned}
\mathfrak{C}(GX,GX) &\longrightarrow \mathfrak{C}(GX,FX), & \mathfrak{C}(GX,GX) &\longrightarrow \mathfrak{C}(FX,GX), \qquad (81)\\
f &\longmapsto f\mu_X, & f &\longmapsto \lambda_X f,\\
\phi : f \to g &\longmapsto \phi{\scriptstyle\bullet}\mu_X : f\mu_X \to g\mu_X, & \phi : f \to g &\longmapsto \lambda_X{\scriptstyle\bullet}\phi : \lambda_X f \to \lambda_X g, \qquad (82)\\
\mathfrak{C}(GX,FX) &\longrightarrow \mathfrak{C}(GX,GX), & \mathfrak{C}(FX,GX) &\longrightarrow \mathfrak{C}(GX,GX),\\
h &\longmapsto h\lambda_X, & h &\longmapsto \mu_X h,\\
\chi : h \to k &\longmapsto \chi{\scriptstyle\bullet}\lambda_X : h\lambda_X \to k\lambda_X, & \chi : h \to k &\longmapsto \mu_X{\scriptstyle\bullet}\chi : \mu_X h \to \mu_X k.
\end{aligned}
$$

These functors are faithful. Let us prove it for the first one. Indeed, if $\phi{\scriptstyle\bullet}\mu_X = \psi{\scriptstyle\bullet}\mu_X : f\mu_X \to g\mu_X$, then $\phi{\scriptstyle\bullet}\mu_X{\scriptstyle\bullet}\lambda_X = \psi{\scriptstyle\bullet}\mu_X{\scriptstyle\bullet}\lambda_X : f\mu_X\lambda_X \to g\mu_X\lambda_X$ and $(f{\scriptstyle\bullet}\beta_X)(\phi{\scriptstyle\bullet}\mu_X{\scriptstyle\bullet}\lambda_X) = (f{\scriptstyle\bullet}\beta_X)(\psi{\scriptstyle\bullet}\mu_X{\scriptstyle\bullet}\lambda_X) : f \to g\mu_X\lambda_X$, i.e., $\phi{\scriptstyle\bullet}(g\beta_X) = \psi{\scriptstyle\bullet}(g\beta_X) : f \to g\mu_X\lambda_X$, hence $\phi = \psi$, because $\beta_X$ is invertible. Similarly or by symmetry the other 3 functors are also faithful.

Functors (81) are full. Let us prove it for the first one. Given $\psi : f\mu_X \to g\mu_X$, we set $\phi = (f{\scriptstyle\bullet}\beta)(\psi{\scriptstyle\bullet}\lambda_X)(g{\scriptstyle\bullet}\beta_X^{-1}) : f \to g$. Then $(f{\scriptstyle\bullet}\beta_X)(\phi{\scriptstyle\bullet}\mu_X{\scriptstyle\bullet}\lambda_X)(g{\scriptstyle\bullet}\beta_X^{-1}) = (f{\scriptstyle\bullet}\beta_X)(f{\scriptstyle\bullet}\beta_X^{-1})\phi = \phi = (f{\scriptstyle\bullet}\beta_X)(\psi{\scriptstyle\bullet}\lambda_X)(g{\scriptstyle\bullet}\beta_X^{-1})$. This yields $\phi{\scriptstyle\bullet}\mu_X{\scriptstyle\bullet}\lambda_X = \psi{\scriptstyle\bullet}\lambda_X$. Since multiplication with $\lambda_X$ is a faithful functor, we obtain $\phi{\scriptstyle\bullet}\mu_X = \psi$. Notice that if $\psi$ is a 2-isomorphism, then the composition of 2-isomorphisms $\phi$ is a 2-isomorphism as well. Similarly we prove that the second functor of (81) is full.

Now let us take $f = 1_{GX}$, $g = \mu_X\lambda_X$. By the first bijection of (82) we find a 2-isomorphism $\eta_X : 1_{GX} \to \mu_X\lambda_X$ corresponding to the 2-isomorphism $\psi = \mu_X{\scriptstyle\bullet}\varepsilon_X^{-1} : \mu_X \to \mu_X\lambda_X\mu_X$. Then $\eta_X{\scriptstyle\bullet}\mu_X = \mu_X{\scriptstyle\bullet}\varepsilon_X^{-1}$, that is, $(\eta_X{\scriptstyle\bullet}\mu_X)(\mu_X{\scriptstyle\bullet}\varepsilon_X) = 1_{\mu_X}$. By the second bijection of (82) we find a 2-isomorphism $\gamma_X : 1_{GX} \to \mu_X\lambda_X$ corresponding to the 2-isomorphism $\xi = \varepsilon_X^{-1}{\scriptstyle\bullet}\lambda_X : \lambda_X \to \lambda_X\mu_X\lambda_X$. Then $\lambda_X{\scriptstyle\bullet}\gamma_X = \varepsilon_X^{-1}{\scriptstyle\bullet}\lambda_X$, that is, $(\lambda_X{\scriptstyle\bullet}\gamma_X)(\varepsilon_X{\scriptstyle\bullet}\lambda_X) = 1_{\lambda_X}$. We have

$$
\begin{aligned}
\gamma = \bigl(1_{GX} \xrightarrow{\gamma} \mu\lambda \xrightarrow{1_\mu\lambda} \mu\lambda\bigr) &= \bigl(1_{GX} \xrightarrow{\gamma} \mu\lambda \xrightarrow{\eta\mu\lambda} \mu\lambda\mu\lambda \xrightarrow{\mu\varepsilon\lambda} \mu\lambda\bigr)\\
&= \bigl(1_{GX} \xrightarrow{\eta} \mu\lambda \xrightarrow{\mu\lambda\gamma} \mu\lambda\mu\lambda \xrightarrow{\mu\varepsilon\lambda} \mu\lambda\bigr) = \bigl(1_{GX} \xrightarrow{\eta} \mu\lambda \xrightarrow{\mu 1_\lambda} \mu\lambda\bigr) = \eta.
\end{aligned}
$$

Therefore, the 2-isomorphism $\eta$ fulfills both equations (79) and (80). ∎

For any 1-morphism $f : X \to Y$ in $\mathfrak{A}$ we define a 2-isomorphism $\mu_f$ as the pasting of

$$
\begin{tikzcd}
GX \arrow[r, "\mu_X"] \arrow[dr, "1_{GX}"'] & FX \arrow[r, "Ff"] \arrow[d, "\lambda_X"] & FY \arrow[d, "\lambda_Y"'] \arrow[dr, "1_{FY}"] & \\
& GX \arrow[r, "Gf"'] \arrow[u, phantom, "\eta_X \Uparrow" near start, shift right=4ex] \arrow[ur, phantom, "\Uparrow \lambda_f^{-1}"] & GY \arrow[r, "\mu_Y"'] \arrow[ur, phantom, "\varepsilon_Y \Uparrow" near end] & FY
\end{tikzcd}
.
$$

Since $\lambda_f$ determine an isomorphism of functors

$$
\lambda_{\_} : F - {\scriptstyle\bullet}\lambda_Y \to \lambda_X{\scriptstyle\bullet}G- : \mathfrak{A}(X,Y) \to \mathfrak{C}(FX,GY),
$$

for any 2-morphism $\xi : f \to g : X \to Y$ equation 2 of Definition C.4 holds. It implies

$$
\begin{tikzcd}
GX \arrow[r, "\mu_X"] \arrow[dr, "1_{GX}"'] \arrow[dr, phantom, "\eta_X", near end, shift left=2ex] & FX \arrow[r, bend left, "Fg"] \arrow[r, "Ff"'] \arrow[r, phantom, bend left=20, "\Uparrow F\xi"] \arrow[d, "\lambda_X"] & FY \arrow[d, "\lambda_Y"] \arrow[dr, "1_{FX}"] & \\
& GX \arrow[r, "Gf"'] \arrow[ur, Rightarrow, "\lambda_f^{-1}"'] & GY \arrow[r, "\mu_Y"'] \arrow[ur, phantom, "\varepsilon_Y"] & FY
\end{tikzcd}
=
\begin{tikzcd}
GX \arrow[r, "\mu_X"] \arrow[dr, "1_{GX}"'] \arrow[dr, phantom, "\eta_X", near end, shift left=2ex] & FX \arrow[r, "Fg"] \arrow[d, "\lambda_X"] & FY \arrow[d, "\lambda_Y"] \arrow[dr, "1_{FX}"] & \\
& GX \arrow[r, "Gg"] \arrow[r, bend right, "Gf"'] \arrow[r, phantom, bend right=20, "\Uparrow G\xi"] \arrow[ur, Rightarrow, "\lambda_g^{-1}"] & GY \arrow[r, "\mu_Y"'] \arrow[ur, phantom, "\varepsilon_Y"] & FY
\end{tikzcd}
.
$$

This shows that the collection of 2-isomorphisms $\mu_f$ determines an isomorphism of functors

$$
\mu_- : G - \bullet \mu_Y \to \mu_X \bullet F - : \mathfrak{A}(X, Y) \to \mathfrak{C}(GX, FY).
$$

Let us check conditions 3(a),(b) of Definition C.4 for $\mu$. For 3(a) we have

$$
\begin{tikzcd}
GX \arrow[r, "\mu_X"] \arrow[dr, "1_{GX}"'] \arrow[dr, phantom, "\eta_X", near end, shift left=2ex] & FX \arrow[r, "F1_X"] \arrow[d, "\lambda_X"] & FX \arrow[d, "\lambda_X"] \arrow[dr, "1_{FX}"] & \\
& GX \arrow[r, "G1_X"] \arrow[r, bend right, "1_{GX}"'] \arrow[r, phantom, bend right=20, "\Uparrow \psi_0"] \arrow[ur, Rightarrow, "\lambda_{1_X}^{-1}"] & GX \arrow[r, "\mu_X"'] \arrow[ur, phantom, "\varepsilon_X"] & FX
\end{tikzcd}
=
\begin{tikzcd}
GX \arrow[r, "\mu_X"] \arrow[dr, "1_{GX}"'] \arrow[dr, phantom, "\eta_X", near end, shift left=2ex] & FX \arrow[r, bend left, "F1_X"] \arrow[r, "1_{FX}"'] \arrow[r, phantom, bend left=20, "\Uparrow \phi_0"] \arrow[d, "\lambda_X"] \arrow[dr, phantom, "="] & FX \arrow[d, "\lambda_X"] \arrow[dr, "1_{FX}"] & \\
& GX \arrow[r, "1_{GX}"'] & GX \arrow[r, "\mu_X"'] \arrow[ur, phantom, "\varepsilon_X"] & FX
\end{tikzcd}
$$

and the required equation follows from (79).

Equation (80) and the corresponding property for $\lambda$ imply

$$
\begin{tikzcd}
& & GY \arrow[dr, "Gg"] \arrow[d, phantom, "\Downarrow \psi_2"] & & \\
& GX \arrow[ur, "Gf"] \arrow[rr, "G(fg)"'] \arrow[drr, Rightarrow, "\lambda_{fg}^{-1}"] & {} & GZ \arrow[r, "\mu_Z"] \arrow[dr, phantom, "\varepsilon_Z"] & FZ \\
GX \arrow[ur, "1_{GX}"] \arrow[r, "\mu_X"'] \arrow[ur, phantom, "\eta_X", near end, shift right=2ex] & FX \arrow[u, "\lambda_X"] \arrow[rr, "F(fg)"'] & & FZ \arrow[u, "\lambda_Z"] \arrow[ur, "1_{FZ}"'] &
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
& & GY \arrow[r, "1_{GY}"] \arrow[dr, "\mu_Y"] \arrow[d, phantom, "\varepsilon_Y", shift left=4ex] \arrow[dr, phantom, "\eta_Y", shift left=3ex] & GY \arrow[dr, "Gg"] \arrow[ddr, Rightarrow, "\lambda_g^{-1}"'] & & \\
& GX \arrow[ur, "Gf"] \arrow[r, Rightarrow, "\lambda_f^{-1}"] & FY \arrow[u, "\lambda_Y"] \arrow[r, "1_{FY}"'] \arrow[d, phantom, "\Downarrow \phi_2", shift left=6ex] & FY \arrow[u, "\lambda_Y"'] \arrow[dr, "Fg"'] & GZ \arrow[r, "\mu_Z"] \arrow[dr, phantom, "\varepsilon_Z"] & FZ \\
GX \arrow[ur, "1_{GX}"] \arrow[r, "\mu_X"'] \arrow[ur, phantom, "\eta_X", near end, shift right=2ex] & FX \arrow[u, "\lambda_X"] \arrow[ur, "Ff"'] \arrow[rrr, "F(fg)"'] & {} & & FZ \arrow[u, "\lambda_Z"] \arrow[ur, "1_{FZ}"'] &
\end{tikzcd}
$$

and the assertion 3(b) for $\mu$ follows.

Let us verify that collections $\varepsilon = (\varepsilon_X)$ and $\eta = (\eta_X)$ determine modifications. Equation (80) implies that

$$
\begin{tikzcd}
FX \arrow[rr, "Ff"] \arrow[dr, "\lambda_X"'] \arrow[drr, "\lambda_X"] \arrow[dd, "1_{FX}"'] & & FY \arrow[dr, "\lambda_Y"] \arrow[d, phantom, "\Downarrow \lambda_f"] & \\
\arrow[r, phantom, "\overset{\varepsilon_X}{\Longleftarrow}"] & GX \arrow[r, "1_{GX}"] \arrow[dl, "\mu_X"] \arrow[dl, phantom, "\Downarrow\eta_X"] & GX \arrow[r, "Gf"] \arrow[d, phantom, "\Downarrow \lambda_f^{-1}"] & GY \arrow[d, "\mu_Y"] \arrow[dl, phantom, "\Downarrow\varepsilon_Y"] \\
FX \arrow[urr, "\lambda_X"] \arrow[rr, "Ff"'] & & FY \arrow[ur, "\lambda_Y"] \arrow[r, "1_{FY}"'] & FY
\end{tikzcd}
=
\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[dd, "1_{FX}"'] & FY \arrow[dd, "= 1_{FY}"'] \arrow[dr, "\lambda_Y"] & \\
 & & GY \arrow[dl, "\mu_Y"] \arrow[l, phantom, "\overset{\varepsilon_Y}{\Longleftarrow}"] \\
FX \arrow[r, "Ff"'] & FY &
\end{tikzcd}
.
$$

This means that $\varepsilon : \lambda\mu \to 1_F : F \to F : \mathfrak{A} \to \mathfrak{C}$ is a modification.

Also equation (80) implies that

$$
\begin{tikzcd}
 & GX \arrow[r, "Gf"] \arrow[dl, "\mu_X"'] \arrow[dd, "1_{GX} ="] & GY \arrow[dd, "1_{GY}"] \\
FX \arrow[dr, "\lambda_X"'] & \arrow[l, phantom, "\overset{\eta_X}{\Longleftarrow}"] & \\
 & GX \arrow[r, "Gf"'] & GY
\end{tikzcd}
=
\begin{tikzcd}
GX \arrow[r, "1_{GX}"] \arrow[d, "\mu_X"'] \arrow[d, phantom, "\eta_X"] & GX \arrow[rr, "Gf"] \arrow[d, phantom, "\Downarrow \lambda_f^{-1}"] & & GY \arrow[dl, "\mu_Y"'] \arrow[dd, "1_{GX}"] \arrow[dl, phantom, "\varepsilon_Y"] \\
FX \arrow[ur, "\lambda_X"] \arrow[r, "Ff"'] \arrow[dr, "\lambda_X"'] & FY \arrow[urr, "\lambda_Y"] \arrow[r, "1_{FY}"] \arrow[drr, "\lambda_Y"'] \arrow[d, phantom, "\Downarrow \lambda_f"] & FY \arrow[dr, "\lambda_Y"] \arrow[r, phantom, "\overset{\eta_Y}{\Longleftarrow}"] \arrow[d, phantom, "="] & \\
 & GX \arrow[rr, "Gf"'] & & GY
\end{tikzcd}
$$

This means that $\eta : 1_G \to \mu\lambda : G \to G : \mathfrak{A} \to \mathfrak{C}$ is a modification.

Therefore, $\mu$ is a weak 2-transformation quasi-inverse to $\lambda$. ■

C.8. Remark. Clearly, if $\lambda : F \to G : \mathfrak{A} \to \mathfrak{C}$ and $\mu : G \to F : \mathfrak{A} \to \mathfrak{C}$ are weak 2-transformations, quasi-inverse to each other, then both are 2-natural equivalences.

We shall use the generalization of the classical Yoneda Lemma to 2-categories. If we were using strict 2-functors and strict 2-transformations, we would view 2-categories as *cat*-categories, where *cat* is the category of categories. This would allow to use one of the Yoneda structures on 2-categories defined by Street and Walters [SW78, Example 7(1)], as well as weak Yoneda Lemma for enriched categories by Eilenberg and Kelly [EK66, Theorem I.8.6], Kelly [Kel82, Section 1.9] and strong Yoneda Lemma for enriched categories by Kelly [Kel82, Section 2.4]. However, we need weak 2-transformations (and modifications), so we use the Yoneda Lemma for bicategories obtained by Street [Str80, (1.9)]. Let us recall the latter statement.

C.9. The 2-functor $\mathfrak{A}(A, \_)$. Let $\mathfrak{A}$ be a strict 2-category. An arbitrary object $A \in \operatorname{Ob}\mathfrak{A}$ gives rise to a strict 2-functor $\mathfrak{A}(A, \_) : \mathfrak{A} \to \mathcal{C}at$. It is specified by the following data:

1. the function $\operatorname{Ob}\mathfrak{A} \to \operatorname{Ob}\mathcal{C}at$, $X \mapsto \mathfrak{A}(A, X)$;

2. the functor $\mathfrak{A}(A,\_)_{XY} : \mathfrak{A}(X,Y) \to \mathcal{C}at(\mathfrak{A}(A,X),\mathfrak{A}(A,Y))$ for each pair of objects $X, Y \in \operatorname{Ob}\mathfrak{A}$.

The functor $\mathfrak{A}(A,\_)_{XY}$ is given as follows. For any 1-morphism $f : X \to Y$ the functor $\mathfrak{A}(A,f) : \mathfrak{A}(A,X) \to \mathfrak{A}(A,Y)$ is given by the following formulas:

$$(\phi : A \to X) \mapsto (\phi f : A \to Y),$$
$$(\pi : \phi \to \psi : A \to X) \mapsto (\pi{\centerdot}f : \phi f \to \psi f : A \to Y).$$

For any 2-morphism $\alpha : f \to g : X \to Y$ the natural transformation $\mathfrak{A}(A,\alpha) : \mathfrak{A}(A,f) \to \mathfrak{A}(A,g) : \mathfrak{A}(A,X) \to \mathfrak{A}(A,Y)$ is given explicitly by its components:

$$(\phi)\mathfrak{A}(A,\alpha) = \phi{\centerdot}\alpha : (\phi)\mathfrak{A}(A,f) = \phi f \to \phi g = (\phi)\mathfrak{A}(A,g), \qquad \phi \in \mathfrak{A}(A,X).$$

Let $[\mathfrak{A},\mathcal{C}at]$ denote the strict 2-category of weak 2-functors $\mathfrak{A} \to \mathcal{C}at$, their weak 2-transformations and their modifications, see e.g. [Lyu99, Appendix A.1.5].

C.10. Lemma. [Yoneda Lemma for bicategories, Street [Str80, (1.9)]] *For a homomorphism $G : \mathfrak{A} \to \mathcal{C}at$ of bicategories, evaluation at the identity for each object $A$ of $\mathfrak{A}$ provides the components $[\mathfrak{A},\mathcal{C}at](\mathfrak{A}(A,\_),G) \to GA$ of an equivalence in $[\mathfrak{A},\mathcal{C}at]$.*

We have not found a detailed published proof of the above result in the existing literature, since it has to be quite lengthy. Curiously, part of the required statements were formalized and verified by a computer proof–checker [Moh97]. On the other hand, in the case of strict 2-categories one can write down a complete proof in several pages. For convenience of the reader we decompose it into several detailed statements, written for a strict 2-category $\mathfrak{A}$, fixed till the end of this section.

A weak 2-transformation $\lambda : \mathfrak{A}(A,\_) \to G : \mathfrak{A} \to \mathcal{C}at$ involves, in particular, a functor $\lambda_A : \mathfrak{A}(A,A) \to GA$. Evaluating it on the object $1_A \in \operatorname{Ob}\mathfrak{A}(A,A)$ we get an object $(1_A)\lambda_A \in \operatorname{Ob} GA$. A modification $m : \lambda \to \mu : \mathfrak{A}(A,\_) \to G : \mathfrak{A} \to \mathcal{C}at$ involves, in particular, a natural transformation $m_A : \lambda_A \to \mu_A : \mathfrak{A}(A,A) \to GA$. Evaluating it on the object $1_A \in \operatorname{Ob}\mathfrak{A}(A,A)$ we get a morphism $(1_A)m_A : (1_A)\lambda_A \to (1_A)\mu_A$ of $GA$.

C.11. Proposition. *Let $G : \mathfrak{A} \to \mathcal{C}at$ be a weak 2-functor. Let $A$ be an object of $\mathfrak{A}$. Then the functor*

$$\begin{aligned} \mathrm{ev}_{1_A} : [\mathfrak{A},\mathcal{C}at](\mathfrak{A}(A,\_),G) &\longrightarrow GA, \\ \lambda &\longmapsto (1_A)\lambda_A, \\ m : \lambda \to \mu &\longmapsto (1_A)m_A : (1_A)\lambda_A \to (1_A)\mu_A, \end{aligned}$$

*is full and faithful.*

Proof. Clearly, the assignment $\mathrm{ev}_{1_A}$ gives a functor. Let us show that it is faithful. Let two modifications $m, n : \lambda \to \mu : \mathfrak{A}(A, \_) \to G : \mathfrak{A} \to \mathcal{C}at$ be given such that $(1_A)m_A = (1_A)n_A$. The modification $m$ is a family of natural transformations $(m_C)_{C \in \mathrm{Ob}\,\mathfrak{A}}$ satisfying the equation

$$\begin{tikzcd}
\mathfrak{A}(A,B) \arrow[r, "\mathfrak{A}(A,f)"] \arrow[d, shift right=3, "\mu_B"'] \arrow[d, shift left=3, "\lambda_B"] \arrow[d, phantom, "\overset{m_B}{\Leftarrow}"] & \mathfrak{A}(A,C) \arrow[d, "\lambda_C"] \arrow[dl, Rightarrow, "\lambda_f"'] \\
GB \arrow[r, "Gf"'] & GC
\end{tikzcd}
=
\begin{tikzcd}
\mathfrak{A}(A,B) \arrow[r, "\mathfrak{A}(A,f)"] \arrow[d, "\mu_B"'] & \mathfrak{A}(A,C) \arrow[d, shift right=3, "\mu_C"'] \arrow[d, shift left=3, "\lambda_C"] \arrow[d, phantom, "\overset{m_C}{\Leftarrow}"] \arrow[dl, Rightarrow, "\mu_f"'] \\
GB \arrow[r, "Gf"'] & GC
\end{tikzcd}
\tag{83}$$

for an arbitrary 1-morphism $f : B \to C$ of $\mathfrak{A}$. In particular, it holds for $B = A$. Restrict this equation to the object $1_A$ of $\mathfrak{A}(A, A)$. Then it gives for an arbitrary 1-morphism $f : A \to C$ the equation

$$\begin{aligned}
\big[(f)\lambda_C \xrightarrow{(1_A)\lambda_f} ((1_A)\lambda_A)(Gf) \xrightarrow{((1_A)m_A)(Gf)} ((1_A)\mu_A)(Gf)\big]& \\
= \big[(f)\lambda_C \xrightarrow{(f)m_C} (f)\mu_C \xrightarrow{(1_A)\mu_f} ((1_A)\mu_A)(Gf)\big]&.
\end{aligned}$$

Therefore, the value of $m_C$ on an arbitrary object $f$ of $\mathfrak{A}(A, C)$ is completely determined by the morphism $(1_A)m_A$:

$$(f)m_C = \big[(f)\lambda_C \xrightarrow{(1_A)\lambda_f} ((1_A)\lambda_A)(Gf) \xrightarrow{((1_A)m_A)(Gf)} ((1_A)\mu_A)(Gf) \xrightarrow{(1_A)\mu_f^{-1}} (f)\mu_C\big].$$

Thus, $m = n$ and $\mathrm{ev}_{1_A}$ is faithful.

Let us prove that $\mathrm{ev}_{1_A}$ is full. Let $\lambda, \mu : \mathfrak{A}(A, \_) \to G : \mathfrak{A} \to \mathcal{C}at$ be weak 2-transformations. Let $\phi : (1_A)\lambda_A \to (1_A)\mu_A$ be a morphism of $GA$. We claim that there is a modification $m : \lambda \to \mu$ such that $(1_A)m_A = \phi$. The value of $m_C$ on an arbitrary 1-morphism $f : A \to C$ can be only

$$(f)m_C = \big[(f)\lambda_C \xrightarrow{(1_A)\lambda_f} ((1_A)\lambda_A)(Gf) \xrightarrow{(\phi)(Gf)} ((1_A)\mu_A)(Gf) \xrightarrow{(1_A)\mu_f^{-1}} (f)\mu_C\big], \tag{84}$$

as we have seen. Let us verify that, indeed, this formula determines a modification.

First of all, each $m_C$ is a natural transformation. Indeed, for each 2-morphism $\xi : f \to g : A \to C$ of $\mathfrak{A}$ the following diagram commutes:

$$\begin{tikzcd}
(f)\lambda_C \arrow[r, "(1_A)\lambda_f"] \arrow[d, "(\xi)\lambda_C"'] & ((1_A)\lambda_A)(Gf) \arrow[r, "(\phi)(Gf)"] \arrow[d, "((1_A)\lambda_A)(G\xi)"'] & ((1_A)\mu_A)(Gf) \arrow[r, "(1_A)\mu_f^{-1}"] \arrow[d, "((1_A)\mu_A)(G\xi)"] & (f)\mu_C \arrow[d, "(\xi)\mu_C"] \\
(g)\lambda_C \arrow[r, "(1_A)\lambda_g"] & ((1_A)\lambda_A)(Gg) \arrow[r, "(\phi)(Gg)"] & ((1_A)\mu_A)(Gg) \arrow[r, "(1_A)\mu_g^{-1}"] & (g)\mu_C
\end{tikzcd}
\tag{85}$$

The central square commutes because $G\xi : Gf \to Gg$ is a natural transformation. Condition 2 for $\lambda$ from Definition C.4 implies, in particular, equation

$$
\begin{tikzcd}
\mathfrak{A}(A,A) \arrow[r, "\mathfrak{A}(A,f)"] \arrow[d, "\lambda_A"'] & \mathfrak{A}(A,C) \arrow[d, "\lambda_C"] \arrow[dl, Rightarrow, "\lambda_f"] \\
GA \arrow[r, bend left, "Gf"] \arrow[r, bend right, "Gg"'] \arrow[r, phantom, "\Downarrow G\xi"] & GC
\end{tikzcd}
=
\begin{tikzcd}
\mathfrak{A}(A,A) \arrow[r, bend left, "\mathfrak{A}(A,f)"] \arrow[r, bend right, "\mathfrak{A}(A,g)"'] \arrow[r, phantom, "\Downarrow \mathfrak{A}(A,\xi)"] \arrow[d, "\lambda_A"'] & \mathfrak{A}(A,C) \arrow[d, "\lambda_C"] \arrow[dl, Rightarrow, "\lambda_g"'] \\
GA \arrow[r, "Gg"'] & GC
\end{tikzcd}
.
$$

Restricting this equation to the object $1_A$ of $\mathfrak{A}(A,A)$ we get an equation, which expresses precisely commutativity of the left square of diagram (85). The right square of (85) commutes by the same reasoning applied to $\mu$ instead of $\lambda$. Thus, $m_C$ is a natural transformation.

Secondly, we have to prove equation (83) for the family $(m_C)$ and for an arbitrary 1-morphism $f : B \to C$ of $\mathfrak{A}$. On an arbitrary object $g : A \to B$ of $\mathfrak{A}(A,B)$ this equation reads:

$$
\begin{aligned}
\bigl[(gf)\lambda_C \xrightarrow{(g)\lambda_f} ((g)\lambda_B)(Gf) \xrightarrow{((g)m_B)(Gf)} ((g)\mu_B)(Gf)\bigr] \\
= \bigl[(gf)\lambda_C \xrightarrow{(gf)m_C} (gf)\mu_C \xrightarrow{(g)\mu_f} ((g)\mu_B)(Gf)\bigr].
\end{aligned}
$$

Substituting definition (84) of $m_C$ we get an equation, which expresses commutativity of the exterior of the following diagram:

$$
\begin{tikzcd}
((g)\lambda_B)(Gf) \arrow[r, "((1_A)\lambda_g)(Gf)"] & ((1_A)\lambda_A)(Gg)(Gf) \arrow[r, "(\phi)(Gg)(Gf)"] \arrow[d, "((1_A)\lambda_A)(g,f)\psi_2"'] & ((1_A)\mu_A)(Gg)(Gf) \arrow[r, "((1_A)\mu_g^{-1})(Gf)"] \arrow[d, "((1_A)\mu_A)(g,f)\psi_2"] & ((g)\mu_B)(Gf) \\
(gf)\lambda_C \arrow[u, "(g)\lambda_f"] \arrow[r, "(1_A)\lambda_{gf}"] & ((1_A)\lambda_A)(G(gf)) \arrow[r, "(\phi)(G(gf))"] & ((1_A)\mu_A)(G(gf)) \arrow[r, "(1_A)\mu_{gf}^{-1}"] & (gf)\mu_C \arrow[u, "(g)\mu_f"']
\end{tikzcd}
\tag{86}
$$

The middle square commutes, because $(g,f)\psi_2 : (Gg)(Gf) \to G(gf)$ is a morphism of functors. Property 3(b) of Definition C.4 for $\lambda$ implies, in particular, the equation

$$
\begin{tikzcd}
 & \mathfrak{A}(A,B) \arrow[dr, "\mathfrak{A}(A,f)"] \arrow[d, phantom, "\Downarrow="] & \\
\mathfrak{A}(A,A) \arrow[ur, "\mathfrak{A}(A,g)"] \arrow[rr, "\mathfrak{A}(A,gf)"'] \arrow[d, "\lambda_A"'] & {} & \mathfrak{A}(A,C) \arrow[d, "\lambda_C"] \arrow[dll, Rightarrow, "\lambda_{gf}"] \\
GA \arrow[rr, "G(gf)"'] & & GC
\end{tikzcd}
=
\begin{tikzcd}
 & \mathfrak{A}(A,B) \arrow[dr, "\mathfrak{A}(A,f)"] \arrow[d, "\lambda_B"] \arrow[dl, Rightarrow, "\lambda_g"'] & \\
\mathfrak{A}(A,A) \arrow[ur, "\mathfrak{A}(A,g)"] \arrow[d, "\lambda_A"'] & GB \arrow[dr, "Gf"] \arrow[d, phantom, "\Downarrow\psi_2"] & \mathfrak{A}(A,C) \arrow[l, Rightarrow, "\lambda_f"'] \arrow[d, "\lambda_C"] \\
GA \arrow[ur, "Gg"] \arrow[rr, "G(gf)"'] & {} & GC
\end{tikzcd}
.
$$

Restricting this equation to the object $1_A$ of $\mathfrak{A}(A,A)$, we will get precisely the left square of diagram (86), therefore, it commutes. The right square of (86) commutes by the same reasoning applied to $\mu$ instead of $\lambda$. Therefore, $m$ is a modification, and $\mathrm{ev}_{1_A}$ is full. ■

C.12. Proposition. *For each object $x$ of $GA$ there is a weak 2-transformation*

$$\lambda^x = \lambda^{A,x} = {}^G\lambda^{A,x} : \mathfrak{A}(A,\_) \to (G,\psi_2,\psi_0) : \mathfrak{A} \to \mathcal{C}at,$$

*specified by the family of functors $\lambda^x_C$, $C \in \operatorname{Ob}\mathfrak{A}$:*

$$\begin{aligned} \lambda^x_C : \mathfrak{A}(A,C) &\longrightarrow GC, \\ f : A \to C &\longmapsto (x)(Gf), \\ \xi : f \to g : A \to C &\longmapsto (x)(G\xi) : (x)(Gf) \to (x)(Gg), \end{aligned}$$

*and by the family of invertible natural transformations*

$$\lambda^x_f : \mathfrak{A}(A,f)\centerdot\lambda^x_C \to \lambda^x_B\centerdot Gf : \mathfrak{A}(A,B) \to GC,$$

*$f \in \operatorname{Ob}\mathfrak{A}(B,C)$, which map an object $g \in \operatorname{Ob}\mathfrak{A}(A,B)$ to the isomorphism of $GC$:*

$$(g)\lambda^x_f \overset{def}{=} (x)(g,f)\psi_2^{-1} : (gf)\lambda^x_C = (x)(G(gf)) \to (x)(Gg)(Gf) = (g)\lambda^x_B(Gf). \tag{87}$$

*For each morphism $u : x \to y$ of $GA$ there is a modification*

$$\lambda^u = \lambda^{A,u} = {}^G\lambda^{A,u} : \lambda^x \to \lambda^y : \mathfrak{A}(A,\_) \to (G,\psi_2,\psi_0) : \mathfrak{A} \to \mathcal{C}at,$$

*specified by the family of natural transformations $\lambda^u_C$, $C \in \operatorname{Ob}\mathfrak{A}$:*

$$\lambda^u_C : \lambda^x_C \to \lambda^y_C : \mathfrak{A}(A,C) \to GC,$$
$$\lambda^u_C : (f : A \to C) \longmapsto \big((f)\lambda^u_C = (u)(Gf) : (f)\lambda^x_C = (x)(Gf) \to (y)(Gf) = (f)\lambda^y_C\big). \tag{88}$$

*The correspondence*

$$\begin{aligned} \Lambda : GA &\longrightarrow [\mathfrak{A},\mathcal{C}at](\mathfrak{A}(A,\_),G), \\ x &\longmapsto \lambda^x, \\ u : x \to y &\longmapsto \lambda^u : \lambda^x \to \lambda^y, \end{aligned}$$

*is a functor.*

Proof. As $G : \mathfrak{A}(A,C) \to \mathcal{C}at(GA,GC)$ is a functor, $G1_f = 1_{Gf}$ for the unit 2-morphism $1_f : f \to f : A \to C$ of $\mathfrak{A}$, and for each pair of composable 2-morphisms $f \xrightarrow{\xi} g \xrightarrow{\chi} h : A \to C$ of $\mathfrak{A}$ we have

$$G(\xi\chi) = \big(Gf \xrightarrow{G\xi} Gg \xrightarrow{G\chi} Gh\big).$$

Evaluating these equations on $x$ we get $(1_f)\lambda^x_C = (x)1_{Gf} = 1_{(x)(Gf)}$ and $(\xi\chi)\lambda^x_C = (\xi\lambda^x_C)(\chi\lambda^x_C)$, thus, $\lambda^x_C$ is a functor.

We claim that $\lambda^x_f$ given by (87) is a natural transformation. Indeed, naturality of $\psi_2$, expressed by

$$\begin{tikzcd} G(gf) \arrow[d, "G(\xi\centerdot f)"'] & (Gg)(Gf) \arrow[l, "(g,f)\psi_2"'] \arrow[d, "G\xi\centerdot Gf"] \arrow[dl, phantom, "="] \\ G(hf) & (Gh)(Gf) \arrow[l, "(h,f)\psi_2"] \end{tikzcd}$$

implies commutativity of

$$\begin{tikzcd}
(x)(G(gf)) \arrow[r, "(x)(g,f)\psi_2^{-1}"] \arrow[d, "(x)(G(\xi\bullet f))"'] \arrow[dr, phantom, "="] & (x)(Gg)(Gf) \arrow[d, "(x)(G\xi)(Gf)"] \\
(x)(G(hf)) \arrow[r, "(x)(h,f)\psi_2^{-1}"] & (x)(Gh)(Gf)
\end{tikzcd}$$

which is nothing else, but naturality of $\lambda^x_f$:

$$\begin{tikzcd}
(g)\mathfrak{A}(A,f)\lambda^x_C \arrow[r, "(g)\lambda^x_f"] \arrow[d, "(\xi)\mathfrak{A}(A,f)\lambda^x_C"'] \arrow[dr, phantom, "="] & (g)\lambda^x_B(Gf) \arrow[d, "(\xi)\lambda^x_B(Gf)"] \\
(h)\mathfrak{A}(A,f)\lambda^x_C \arrow[r, "(h)\lambda^x_f"] & (h)\lambda^x_B(Gf)
\end{tikzcd}$$

We claim that

$$\lambda^x_{-} : \mathfrak{A}(A,\_)\lambda^x_C \to \lambda^x_B \bullet G- : \mathfrak{A}(A,B) \to GC$$

is a morphism of functors. That is, for each 2-morphism $\xi : f \to g : B \to C$ of $\mathfrak{A}$ the following equation holds:

$$\begin{tikzcd}
\mathfrak{A}(A,B) \arrow[r, "\mathfrak{A}(A,f)"] \arrow[d, "\lambda^x_B"'] & \mathfrak{A}(A,C) \arrow[d, "\lambda^x_C"] \arrow[dl, Rightarrow, "\lambda^x_f"'] \\
GB \arrow[r, bend left, "Gf"] \arrow[r, bend right, "Gg"'] \arrow[r, phantom, "\Downarrow G\xi"] & GC
\end{tikzcd}
=
\begin{tikzcd}
\mathfrak{A}(A,B) \arrow[r, bend left, "\mathfrak{A}(A,f)"] \arrow[r, bend right, "\mathfrak{A}(A,g)"'] \arrow[r, phantom, "\Downarrow\mathfrak{A}(A,\xi)"] \arrow[d, "\lambda^x_B"'] & \mathfrak{A}(A,C) \arrow[d, "\lambda^x_C"] \arrow[dl, Rightarrow, "\lambda^x_g"'] \\
GB \arrow[r, "Gg"'] & GC
\end{tikzcd}
. \tag{89}$$

Indeed, for each 1-morphism $h : A \to B$ of $\mathfrak{A}$ we have

$$\begin{tikzcd}
(x)(G(hf)) \arrow[r, "(x)(h,f)\psi_2^{-1}"] \arrow[d, "(x)(G(h\bullet\xi))"'] \arrow[dr, phantom, "="] & (x)(Gh)(Gf) \arrow[d, "(x)(Gh)(G\xi)"] \\
(x)(G(hg)) \arrow[r, "(x)(h,g)\psi_2^{-1}"] & (x)(Gh)(Gg)
\end{tikzcd}$$

by naturality of $\psi_2$. Rewriting this equation in the form

$$\begin{tikzcd}
(hf)\lambda^x_C \arrow[r, "(h)\lambda^x_f"] \arrow[d, "(h\bullet\xi)\lambda^x_C"'] \arrow[dr, phantom, "="] & (h)\lambda^x_B(Gf) \arrow[d, "((h)\lambda^x_B)(G\xi)"] \\
(hg)\lambda^x_C \arrow[r, "(h)\lambda^x_g"] & (h)\lambda^x_B(Gg)
\end{tikzcd}$$

we deduce that (89) holds on $h$. Therefore, condition 2 of Definition C.4 is satisfied.

Let us verify condition 3(a) of Definition C.4, that is, equation

$$
\begin{tikzcd}
\mathfrak{A}(A,B) \arrow[r, "1_{\mathfrak{A}(A,B)}", "\mathfrak{A}(A,1_B)"'] \arrow[d, "\lambda^x_B"'] & \mathfrak{A}(A,B) \arrow[d, "\lambda^x_B"] \arrow[dl, Rightarrow, "\lambda^x_{1_B}"] \\
GB \arrow[r, "G1_B"'] & GB
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathfrak{A}(A,B) \arrow[r, "1_{\mathfrak{A}(A,B)}"] \arrow[d, "\lambda^x_B"'] \arrow[dr, phantom, "="] & \mathfrak{A}(A,B) \arrow[d, "\lambda^x_B"] \\
GB \arrow[r, bend left, "1_{GB}"] \arrow[r, bend right, "G1_B"'] \arrow[r, phantom, "\Downarrow\psi_0"] & GB
\end{tikzcd}
.
$$

On an object $f : A \to B$ of $\mathfrak{A}(A,B)$ it reads:

$$(f)\lambda^x_{1_B} = ((f)\lambda^x_B)\psi_0 : (f)\lambda^x_B \to (f)\lambda^x_B(G1_B). \tag{90}$$

It follows from condition (77) for $G$,

$$\big[Gf \xrightarrow{Gf\bullet\psi_0} (Gf)(G1_B) \xrightarrow{(f,1_B)\psi_2} Gf\big] = 1_{Gf} : Gf \to Gf : GA \to GB,$$

which, evaluated on $x \in \operatorname{Ob} GA$, can be written as

$$(x)(f,1_B)\psi_2^{-1} = ((x)(Gf))\psi_0 : (x)(Gf) \to (x)(Gf)(G1_B).$$

This is precisely (90).

Let us verify condition 3(b) of Definition C.4, that is, equation

$$
\begin{tikzcd}
 & \mathfrak{A}(A,C) \arrow[dr, "\mathfrak{A}(A,g)"] \arrow[d, phantom, "\Downarrow ="] & \\
\mathfrak{A}(A,B) \arrow[ur, "\mathfrak{A}(A,f)"] \arrow[rr, "\mathfrak{A}(A,fg)"'] \arrow[d, "\lambda^x_B"'] & {} & \mathfrak{A}(A,D) \arrow[d, "\lambda_D"] \arrow[dll, Rightarrow, "\lambda^x_{fg}"] \\
GB \arrow[rr, "G(fg)"'] & & GD
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & \mathfrak{A}(A,C) \arrow[dr, "\mathfrak{A}(A,g)"] \arrow[d, "\lambda^x_C"] \arrow[ddl, Rightarrow, "\lambda^x_f"'] & \\
\mathfrak{A}(A,B) \arrow[ur, "\mathfrak{A}(A,f)"] \arrow[d, "\lambda^x_B"'] & GC \arrow[dr, "Gg"] \arrow[d, phantom, "\Downarrow\psi_2"] & \mathfrak{A}(A,D) \arrow[l, Rightarrow, "\lambda^x_g"'] \arrow[d, "\lambda^x_D"] \\
GB \arrow[ur, "Gf"] \arrow[rr, "G(fg)"'] & {} & GD
\end{tikzcd}
\qquad (91)
$$

for arbitrary pair of composable 1-morphisms $B \xrightarrow{f} C \xrightarrow{g} D$ of $\mathfrak{A}$. We have to check this equation on an arbitrary 1-morphism $h : A \to B$. Condition 4(b) of Definition C.3 for $G$ is the equation

$$
\begin{tikzcd}
(Gh)(Gf)(Gg) \arrow[r, "Gh\bullet(f,g)\psi_2"] \arrow[d, "(h,f)\psi_2\bullet Gg"'] \arrow[dr, phantom, "="] & (Gh)\bullet G(fg) \arrow[d, "(h,fg)\psi_2"] \\
G(hf).Gg \arrow[r, "(hf,g)\psi_2"'] & G(hfg)
\end{tikzcd}
$$

Evaluating it on $x$ we get the equation

$$
\begin{aligned}
(x)(h,fg)\psi_2^{-1} = \big[(x)(G(hfg)) &\xrightarrow{(x)(hf,g)\psi_2^{-1}} (x)(G(hf))(Gg) \\
&\xrightarrow{((x)(h,f)\psi_2^{-1})(Gg)} (x)(Gh)(Gf)(Gg) \xrightarrow{((x)(Gh))(f,g)\psi_2} (x)(Gh)(G(fg))\big],
\end{aligned}
$$

which can be rewritten as

$$(h)\lambda^x_{fg} = \bigl[(hfg)\lambda^x_D \xrightarrow{(hf)\lambda^x_g} (hf)\lambda^x_C(Gg) \\ \xrightarrow{((h)\lambda^x_f)(Gg)} (h)\lambda^x_B(Gf)(Gg) \xrightarrow{((h)\lambda^x_B)(f,g)\psi_2} (h)\lambda^x_B(G(fg))\bigr].$$

And this is precisely (91), evaluated on $h : A \to B$.

Therefore, all conditions of Definition C.4 are satisfied, and $\lambda^x$ is a weak 2-transformation.

Let us show that correspondence (88) defines a natural transformation. Indeed, for each 2-morphism $\xi : f \to g : A \to C$ of $\mathfrak{A}$ the diagram

$$\begin{tikzcd}
(f)\lambda^x_C = (x)(Gf) \arrow[r, "(f)\lambda^u_C", "(u)(Gf)"'] \arrow[d, "(\xi)\lambda^x_C = "', "(x)(G\xi)"] & (y)(Gf) = (f)\lambda^y_C \arrow[d, "(y)(G\xi)"', "=(\xi)\lambda^y_C"] \\
(g)\lambda^x_C = (x)(Gg) \arrow[r, "(u)(Gg)", "(g)\lambda^u_C"'] & (y)(Gg) = (g)\lambda^y_C
\end{tikzcd}$$

commutes due to $G\xi : Gf \to Gg : GA \to GC$ being a natural transformation.

We claim that property 2 of Definition C.5 holds for $\lambda^u$. For an arbitrary 1-morphism $f : B \to C$ of $\mathfrak{A}$ we have to prove the equation

$$\begin{tikzcd}
\mathfrak{A}(A,B) \arrow[r, "\mathfrak{A}(A,f)"] \arrow[d, bend right, "\lambda^y_B"'] \arrow[d, bend left, "\lambda^x_B"] \arrow[d, phantom, "\overset{\lambda^u_B}{\Leftarrow}"] & \mathfrak{A}(A,C) \arrow[d, "\lambda^x_C"] \arrow[dl, Rightarrow, "\lambda^x_f"'] \\
GB \arrow[r, "Gf"] & GC
\end{tikzcd}
=
\begin{tikzcd}
\mathfrak{A}(A,B) \arrow[r, "\mathfrak{A}(A,f)"] \arrow[d, "\lambda^y_B"'] & \mathfrak{A}(A,C) \arrow[dl, Rightarrow, "\lambda^y_f"'] \arrow[d, bend right, "\lambda^y_C"'] \arrow[d, bend left, "\lambda^x_C"] \arrow[d, phantom, "\overset{\lambda^u_C}{\Leftarrow}"] \\
GB \arrow[r, "Gf"] & GC
\end{tikzcd} .$$

On the object $g : A \to B$ of $\mathfrak{A}(A,B)$ this equation reads

$$\begin{tikzcd}
(gf)\lambda^x_C = (x)(G(gf)) \arrow[r, "(g)\lambda^x_f", "(x)(g,f)\psi_2^{-1}"'] \arrow[d, "(gf)\lambda^u_C = "', "(u)(G(gf))"] \arrow[dr, phantom, "="] & (x)(Gg)(Gf) = (g)\lambda^x_B(Gf) \arrow[d, "(u)(Gg)(Gf)"', "=(g)\lambda^u_B(Gf)"] \\
(gf)\lambda^y_C = (y)(G(gf)) \arrow[r, "(y)(g,f)\psi_2^{-1}", "(g)\lambda^y_f"'] & (y)(Gg)(Gf) = (g)\lambda^y_B(Gf)
\end{tikzcd}$$

It holds due to $(g,f)\psi_2^{-1} : G(gf) \to (Gg)(Gf) : GA \to GC$ being a natural transformation. Therefore, $\lambda^u$ is a modification.

The unit morphism $1_x : x \to x$ of $GA$ goes to the identity transformation

$$\lambda^{1_x}_C : (f : A \to C) \longmapsto \bigl((1_x)(Gf) = 1_{(x)(Gf)} : (x)(Gf) \to (x)(Gf)\bigr),$$

because $Gf$ is a functor. For a pair of composable morphisms $x \xrightarrow{u} y \xrightarrow{v} z$ of $GA$ we have $\lambda^{uv}_C = \bigl(\lambda^x_C \xrightarrow{\lambda^u_C} \lambda^y_C \xrightarrow{\lambda^v_C} \lambda^z_C\bigr)$, since $(uv)(Gf) = (u)(Gf) \cdot (v)(Gf)$ due to $Gf$ being a functor. Therefore, $\Lambda$ is a functor. ■

The result of Yoneda Lemma C.10 for a strict 2-category $\mathfrak{A}$ can be made more precise as follows.

C.13. PROPOSITION. *Functors*

$$\mathrm{ev}_{1_A} : [\mathfrak{A}, \mathcal{C}at](\mathfrak{A}(A, \_), G) \to GA \quad \textit{and} \quad \Lambda : GA \to [\mathfrak{A}, \mathcal{C}at](\mathfrak{A}(A, \_), G)$$

*are equivalences, quasi-inverse to each other.*

PROOF. We have

$$\big[GA \xrightarrow{\Lambda} [\mathfrak{A}, \mathcal{C}at](\mathfrak{A}(A, \_), G) \xrightarrow{\mathrm{ev}_{1_A}} GA\big] = G1_A.$$

Indeed, for any object $x$ of $GA$

$$(x)\Lambda \,\mathrm{ev}_{1_A} = (\lambda^x)\,\mathrm{ev}_{1_A} = (1_A)\lambda^x_A = (x)(G1_A),$$

for any morphism $u : x \to y$ of $GA$

$$(u)\Lambda \,\mathrm{ev}_{1_A} = (\lambda^u)\,\mathrm{ev}_{1_A} = (1_A)\lambda^u_A = (u)(G1_A).$$

An isomorphism of functors $\psi_0 : 1_{GA} \to G1_A$ implies that an arbitrary object $x$ of $GA$ is isomorphic to $(x)(G1_A) = ((x)\Lambda)\,\mathrm{ev}_{1_A}$. Thus, $\mathrm{ev}_{1_A}$ is essentially surjective on objects. By Proposition C.11 $\mathrm{ev}_{1_A}$ is an equivalence. Therefore, $\Lambda$ is isomorphic to a functor quasi-inverse to $\mathrm{ev}_{1_A}$. Hence, $\Lambda$ itself is an equivalence quasi-inverse to $\mathrm{ev}_{1_A}$. ∎

C.14. EXAMPLE OF STRICT 2-FUNCTOR $G = \mathfrak{A}(B, \_)$. Applying Proposition C.12 to the strict 2-functor $G = \mathfrak{A}(B, \_) : \mathfrak{A} \to \mathcal{C}at$, we get the following. An arbitrary 1-morphism $f : B \to A$ gives rise to the strict 2-transformation $f^* = {}^{\mathfrak{A}(B,-)}\lambda^{A,f} : \mathfrak{A}(A, \_) \to \mathfrak{A}(B, \_)$. It is specified by the family of functors $f^*_C$, $C \in \mathrm{Ob}\,\mathfrak{A}$:

$$\begin{aligned} f^*_C = \mathfrak{A}(f, C) : \mathfrak{A}(A, C) &\longrightarrow \mathfrak{A}(B, C) \\ (\phi : A \to C) &\longmapsto (f)\mathfrak{A}(B, \phi) = f\phi : B \to C, \\ (\pi : \phi \to \psi : A \to C) &\longmapsto (f)\mathfrak{A}(B, \pi) = f{\bullet}\pi : f\phi \to f\psi : B \to C. \end{aligned}$$

An arbitrary 2-morphism $\alpha : f \to g : B \to A$ gives rise to the modification $\alpha^* = {}^{\mathfrak{A}(B,-)}\lambda^{A,\alpha} : f^* \to g^* : \mathfrak{A}(A, \_) \to \mathfrak{A}(B, \_)$ given by the family of natural transformations $\alpha^*_C : f^*_C \to g^*_C : \mathfrak{A}(A, C) \to \mathfrak{A}(B, C)$, $C \in \mathrm{Ob}\,\mathfrak{A}$. The transformation $\alpha^*_C$ is specified by its components:

$$\alpha^*_C = \mathfrak{A}(\alpha, C) : (\phi : A \to C) \longmapsto (\alpha)\mathfrak{A}(B, \phi) = \alpha{\bullet}\phi : (\phi)f^*_C = f\phi \to g\phi = (\phi)g^*_C.$$

By Proposition C.12 the correspondence $f \mapsto f^*$, $\alpha \mapsto \alpha^*$ determines a functor $Y_{AB} : \mathfrak{A}^{\mathrm{op}}(A, B) = \mathfrak{A}(B, A) \to [\mathfrak{A}, \mathcal{C}at](\mathfrak{A}(A, \_), \mathfrak{A}(B, \_))$. One easily verifies that in fact we have a strict 2-functor $Y : \mathfrak{A}^{\mathrm{op}} \to [\mathfrak{A}, \mathcal{C}at]$.

C.15. Corollary. *$Y$ is a local equivalence, i.e., for each pair of objects $A, B \in \operatorname{Ob} \mathfrak{A}^{\mathrm{op}}$ the functor $Y_{AB}$ is an equivalence.*

Let us recall also the notion of a birepresentable homomorphism $G : \mathfrak{A} \to \mathcal{C}at$ following Street [Str80, (1.11)]. He formulates the following statement for an arbitrary bicategory $\mathfrak{A}$, but we assume that $\mathfrak{A}$ is a strict 2-category as usual.

C.16. Proposition. *Let $G : \mathfrak{A} \to \mathcal{C}at$ be a weak 2-functor. Then the following conditions are equivalent:*

1. *there exists an object $A$ of $\mathfrak{A}$ and a 2-natural equivalence $\lambda : \mathfrak{A}(A, \_) \to G$;*
2. *there exists an object $A$ of $\mathfrak{A}$ and an object $x$ of $GA$ such that the weak 2-transformation $\lambda^x : \mathfrak{A}(A, \_) \to G$ is a 2-natural equivalence.*

Proof. Clearly, the second property implies the first one. Assume that condition 1) holds. By Proposition C.13 the weak 2-transformation $\lambda$ is isomorphic to $\lambda^x$ for some $x \in \operatorname{Ob} GA$. By Proposition C.6 $\lambda$ is a quasi-invertible 1-morphism of $[\mathfrak{A}, \mathcal{C}at]$, hence, so is $\lambda^x$. By Remark C.8 condition 2) holds. ∎

C.17. Definition. *A weak 2-functor $G : \mathfrak{A} \to \mathcal{C}at$ is* representable *(*birepresentable *in terminology of Street [Str80, (1.11)]) if it satisfies equivalent conditions of Proposition C.16. A pair $(A, x)$ consisting of an object $A$ of $\mathfrak{A}$ and an object $x$ of $GA$ is said to* represent *(*birepresent*) $G$, if $\lambda^x = \lambda^{A,x} = {}^G\lambda^{A,x} : \mathfrak{A}(A, \_) \to G$ is a 2-natural equivalence.*

C.18. Uniqueness of the representing pair. It is shown by Street that a representing pair is unique up to an equivalence in a certain bicategory [Str80, (1.10)-(1.11)]. Let us provide the details in our setting.

Let two pairs $(A, x)$ and $(B, y)$ represent $G$. Then there is a quasi-inverse to $\lambda^{B,y} : \mathfrak{A}(B, \_) \to G$ weak 2-transformation $\lambda^{B,y-} : G \to \mathfrak{A}(B, \_)$. Define a 2-natural equivalence $\mu = \lambda^{A,x} \bullet \lambda^{B,y-} : \mathfrak{A}(A, \_) \to \mathfrak{A}(B, \_)$. It is isomorphic to the 2-transformation ${}^{\mathfrak{A}(B,-)}\lambda^{A,f}$ for some $f \in \operatorname{Ob} \mathfrak{A}(B, A)$. There is an invertible modification $m$:

$$\begin{tikzcd}
\mathfrak{A}(A,_) \arrow[rr, "{}^G\lambda^{A,x}"] \arrow[dr, "{}^{\mathfrak{A}(B,-)}\lambda^{A,f}"'] & \arrow[d, phantom, "\Downarrow m"] & G \\
& \mathfrak{A}(B,_) \arrow[ur, "{}^G\lambda^{B,y}"'] &
\end{tikzcd}$$

Then $(1_A)m_A : (x)(G1_A) \to (y)(Gf)$ is an isomorphism of $GA$. Therefore,

$$x = (x)(1_{GA}) \xrightarrow{(x)\psi_0} (x)(G1_A) \xrightarrow{(1_A)m_A} (y)(Gf)$$

is an isomorphism of $GA$. By symmetry we get a 1-morphism $g : A \to B$ of $\mathfrak{A}$ and an isomorphism $y \xrightarrow{\sim} (x)(Gg)$ of $GB$. By construction the strict 2-transformations

$$ {}^{\mathfrak{A}(B,-)}\lambda^{A,f} : \mathfrak{A}(A, \_) \to \mathfrak{A}(B, \_), $$
$$ {}^{\mathfrak{A}(A,-)}\lambda^{B,g} : \mathfrak{A}(B, \_) \to \mathfrak{A}(A, \_) $$

are quasi-inverse to each other. In particular,

$$1_A \simeq ((1_A)^{\mathfrak{A}(B,-)}\lambda_A^{A,f})^{\mathfrak{A}(A,-)}\lambda_A^{B,g} = ((f)\mathfrak{A}(B,1_A))^{\mathfrak{A}(A,-)}\lambda_A^{B,g} = (f)^{\mathfrak{A}(A,-)}\lambda_A^{B,g} = (g)\mathfrak{A}(A,f) = gf,$$

and by symmetry $1_B \simeq fg$. Therefore, 1-morphisms $f$ and $g$ are quasi-inverse to each other.

Summing up, a pair $(A \in \operatorname{Ob}\mathfrak{A}, x \in \operatorname{Ob} GA)$ representing a weak 2-functor $G : \mathfrak{A} \to \mathcal{C}at$ is unique up to equivalence $f$ of the first objects, such that $Gf$ preserves the second object up to an isomorphism.

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