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TENSOR PRODUCTS OF SUP-LATTICES AND GENERALIZED SUP-ARROWS

T. KENNEY AND R.J. WOOD

Abstract. An alternative description of the tensor product of sup-lattices is given with yet another description provided for the tensor product in the special case of CCD sup-lattices. In the course of developing the latter, properties of sup-preserving functions and the totally below relation are generalized to not-necessarily-complete ordered sets.

1. Introduction

We write **sup** for the category of complete lattices and sup-preserving functions and speak of its objects as sup-lattices. When **sup** is considered as a category over **set**, the category of sets, by the obvious forgetful functor, bi-sup-preserving functions make sense. Given **sup**-lattices M, N, and L, a function $\phi : M \times N \to L$ is *bi-sup-preserving* if it preserves suprema in each variable separately. Every sup-preserving $\phi : M \times N \to L$ is bi-sup-preserving (unlike the corresponding situation for abelian groups) but the converse is not true. If $\phi : M \times N \to L$ is bi-sup-preserving and $l : L \to L'$ is sup-preserving, then the composite $l\phi : M \times N \to L'$ is bi-sup-preserving. The tensor product $M \otimes N$ for **sup**-lattices M and N is the codomain for a universal bi-sup-preserving function $\iota : M \times N \to M \otimes N$, composition with which provides a natural bijection between sup-preserving functions $f : M \otimes N \to L$ and bi-sup-preserving functions $\phi : M \times N \to L$, as in:

$$\begin{tikzcd} M \times N \arrow[r, "\iota"] \arrow[dr, "\phi"'] & M \otimes N \arrow[d, "f"] \\ & L \end{tikzcd}$$

It is now classical that $M \otimes N$ can be constructed as the quotient of the free sup-lattice on $M \times N$, obtained from the smallest congruence $\equiv$ with $(\bigvee_i m_i, n) \equiv \bigvee_i (m_i, n)$ and $(m, \bigvee_i n_i) \equiv \bigvee_i (m, n_i)$. The free functor $\mathscr{P} : \mathbf{set} \to \mathbf{sup}$ is given by the power set and

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direct images. For our purposes, the best references for this approach are [J&T] and [PIT], but the story is much older than even the first of those papers.

The quotient $\mathscr{P}(M \times N) \twoheadrightarrow M \otimes N$, being an arrow in **sup**, has a right adjoint in **ord**, the 2-category of ordered sets, so that in **ord**, $M \otimes N$ is a full reflective subobject of $\mathscr{P}(M \times N)$. It should be, and is, easier to give an explicit description of $M \otimes N$ as a full reflective subobject, and this is our purpose in Section 3. We find it convenient to regard **sup** as a 2-category over **ord**. The 2-functor **sup** $\twoheadrightarrow$ **ord** has a left 2-adjoint, and bi-sup-preserving functions are automatically in **ord** so that we *could* simply lift the classical approach to **ord**. However, the 2-dimensional structure of **ord** allows us to exploit the calculus of adjoints *within* **ord**, and this simplifies the description of the tensor product considerably. We first arrive at our description of the tensor product using some of its known properties, but we also show that our description allows a direct verification of the defining universal property.

In Section 4, we study sup-preserving functions in terms of upper and lower bounds, arriving quickly at a definition of sup-preserving function that makes sense in the absence of suprema (and infima). We study the sup-completion of an ordered set in the category of sup-preserving functions. We build on this work in Section 5, to describe and study the *totally below relation* for orders with no completeness properties. We isolate a property of ordered sets, which we call STB, that captures the essence of completely distributive (CCD) lattices, in the sense that a sup-lattice is CCD if and only if its underlying ordered set is STB. More remarkably, we show, in Theorem 5.9, that an ordered set is STB if and only if its sup-preserving sup-completion is CCD.

In Section 6 we apply our study of the totally below relation to give a very simple description of the tensor product of sup-lattices in the case that they are CCD sup-lattices. Section 2 sets some notation and recalls a few of the tools that we need.

2. Preliminaries

2.1. It is convenient to take an object $(X, \leq)$ of **ord** to be a set X together with a reflexive, transitive relation $\leq$. From our perspective, antisymmetry is an unnecessary and unnatural requirement. If we have $x \leq y$ and $y \leq x$ in X, then x and y are isomorphic elements and we could write $x \cong y$. But since $x \cong y$ looks both pedantic and irritating, we will usually write $x = y$ in this case and treat it as an abuse of notation.

The arrows of **ord** are order-preserving functions, which we freely call *functors*. Given our interests here, we note that a bi-sup-preserving function $\phi\colon M \times N \twoheadrightarrow L$ is necessarily a functor. For, if $(m, n) \leq (m', n')$ then

$$\phi(m,n) \leq \phi(m,n) \vee \phi(m,n') \vee \phi(m',n) \vee \phi(m',n') = \phi(m \vee m', n \vee n') = \phi(m',n')$$

The 2-cells of **ord** are (pointwise) inequalities of functors.

2.2. The free **sup**-lattice on $(X, \leq)$ is $D(X, \leq)$, the set of subsets S of X for which $x \leq y \in S$ implies $x \in S$, ordered by inclusion. We call the elements of DX, *downsets* of

$(X, \leq)$. For a functor $f : X \to A$ and a downset $S \in DX$, we have

$$Df(S) = \{a \in A \mid (\exists x)(a \leq fx \;\&\; x \in S)\}$$

For all f, Df has a right adjoint $\mathscr{D}f : DA \to DX$ given by inverse image. In turn, $\mathscr{D}f$ has a right adjoint $\mathbf{D}f : DX \to DA$ given by:

$$\mathbf{D}f(S) = \{a \in A \mid (\forall x)(fx \leq a \Longrightarrow x \in S)\}$$

Thus for any $f : X \to A$ we have $Df \dashv \mathscr{D}f \dashv \mathbf{D}f : DX \to DA$.

All functors in **ord** are faithful, so if f is full then $Df \subseteq \mathbf{D}f$ follows from general adjunction calculations. However, it is easy to argue directly with the quantifiers. Suppose that $a \leq fx_0$ and $x_0 \in S$. Then for any x, if $fx \leq a$ then $fx \leq fx_0$, which gives $x \leq x_0 \in S$ by fullness, and $x \in S$ because S is a downset.

The *inverter* of an inequality $f \leq g : X \to A$ in **ord** is just the full suborder of X determined by the set $\{x \in X \mid g(x) \leq f(x)\}$. In particular:

2.3. Proposition. *If $f : X \to A$ and $Df \subseteq \mathbf{D}f$ then the inverter is*

$$\{Y \in DX \mid \mathbf{D}f(Y) \subseteq Df(Y)\} = \{Y \in DX \mid (\forall B \in DA)(\mathscr{D}f(B) \subseteq Y \Longrightarrow B \subseteq Df(Y))\}$$

■

We remark that the implication in the equation of the proposition can be replaced by "if and only if" because the other implication holds automatically.

2.4. The unit for the 2-adjunction $D \dashv |-| : \mathbf{sup} \to \mathbf{ord}$ is the full (and faithful) downsegment functor $\downarrow_X : X \to |DX|$ in **ord**, where $\downarrow_X(x) = \{y \in X \mid y \leq x\}$. Writing D also for the resulting 2-monad on **ord**, we recall that it has the KZ-property which, as characterized in [MAR], means that its multiplication components $\bigcup_X : DDX \to DX$ satisfy $D\downarrow_X \dashv \bigcup_X \dashv \downarrow_{DX}$. We can verify this condition using subsection 2.2, for we have

$$(\mathbf{D}\underset{X}{\downarrow})(S) = \{T \in DX \mid (\forall x)(\underset{X}{\downarrow}(x) \subseteq T \Longrightarrow x \in S)\} = \{T \in DX \mid T \subseteq S\} = \underset{DX}{\downarrow}(S)$$

Since suprema for DX are given by union, we have $\bigcup_X \dashv \downarrow_{DX}$. Thus the equality $\mathbf{D}\downarrow_X = \downarrow_{DX}$ just established (a special case of a key result in [S&W]) shows that $\bigcup_X = \mathscr{D}\downarrow_X \dashv \mathbf{D}\downarrow_X$ and hence $D\downarrow_X \dashv \bigcup_X$. It is convenient to record here that

$$(D\underset{X}{\downarrow})(S) = \{T \in DX | (\exists x)(T \subseteq \underset{X}{\downarrow} x \;\&\; x \in S)\} = \{T \in DX | (\exists x)(T \subseteq \underset{X}{\downarrow} x \subseteq S)\}$$

2.5. Because **sup** is the 2-category of algebras $\mathbf{ord}^D$ for the KZ-monad, it follows that, for each sup-lattice M, we have a reflexive coinverter diagram in **sup**.

$$\begin{tikzcd} M & DM \arrow[l, "\bigvee"'] & DDM \arrow[l, bend right, "D\bigvee"'] \arrow[l, bend left, "\bigcup_M"] \arrow[l, phantom, "\Uparrow"] \end{tikzcd}$$

Reflexivity is provided by $D\downarrow_M : DM \to DDM$, and we note $D\bigvee \dashv D\downarrow_M \dashv \bigcup_M$. Moreover, the coinverter is $|-|$-contractible with data provided by the right adjoints in the adjunctions $\bigvee \dashv \downarrow_M : M \to DM$ and $\bigcup_M \dashv \downarrow_{DM} : DM \to DDM$.

3. Tensor Products of Sup-Lattices

3.1. We know that the tensor product of sup-lattices has a right adjoint in each variable separately, so that, for general reasons, a tensor product of reflexive coinverters is a reflexive coinverter. Thus for sup-lattices M and N, we have $M \otimes N$ the coinverter of

$$\begin{tikzcd} DM \otimes DN \arrow[r, phantom, "\Uparrow"] & DDM \otimes DDN \arrow[l, bend right, "D\bigvee \otimes D\bigvee"'] \arrow[l, bend left, "\bigcup_M \otimes \bigcup_N"] \end{tikzcd}$$

However, the tensor product of free lattices simplifies:

3.2. Lemma. *For X and Y in* **ord**, $DX \otimes DY \xrightarrow{\simeq} D(X \times Y)$ *in* **sup** *where the isomorphism corresponds to the bi-sup-preserving functor $\gamma : DX \times DY \to D(X \times Y)$, the* downset comparison functor *for the left exact functor $\Gamma = - \times - :$ **set** $\times$ **set** $\to$ **set**, as defined and studied in [RW1].*

Proof. (Sketch) In general, $\gamma_X : \Gamma DX \to D\Gamma X$ corresponds to the order ideal $\Gamma DX \dashrightarrow \Gamma X$ obtained by applying Γ to the order ideal $\downarrow_X^+ : DX \dashrightarrow X$, arising from $\downarrow : X \to DX$. In the case at hand, $\gamma(S,T) = S \times T$. We establish the isomorphism by a Gentzen-Lawvere proof tree. We follow Kelly's notation (see [KEL]) in using $\mathbf{sup}_0(-,-)$ for the **ord** hom for **sup** and $\mathbf{sup}(-,-)$ for the **sup**-enriched hom. So $\mathbf{sup}_0(L, L') = |\mathbf{sup}(L, L')|$. For L in **sup** and Y in **ord**, we write L^Y for the cotensor in the sense of enriched category theory and $|L|^Y$ for the exponential in **ord**. Thus $|L^Y| \cong |L|^Y$.

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D(X \times Y) \to L \text{ in } \mathbf{sup}}{X \times Y \to |L| \text{ in } \mathbf{ord}}}{X \to |L|^Y \text{ in } \mathbf{ord}}}{X \to |L^Y| \text{ in } \mathbf{ord}}}{DX \to L^Y \text{ in } \mathbf{sup}}}{Y \to \mathbf{sup}_0(DX, L) \text{ in } \mathbf{ord}}}{Y \to |\mathbf{sup}(DX, L)| \text{ in } \mathbf{ord}}}{DY \to (\mathbf{sup}(DX, L)) \text{ in } \mathbf{sup}}}{DX \times DY \to L \text{ bi-sup-preserving}}$$

Taking $L = D(X \times Y)$ and starting with the identity, we leave the reader the task of showing that the last line results in $\gamma : DX \times DY \to D(X \times Y) : (S,T) \mapsto S \times T$. ∎

We recall from [RW1] that $\gamma.\Gamma\downarrow_X = \downarrow_{\Gamma X}$, so $\downarrow x \times \downarrow y = \gamma(\downarrow x, \downarrow y) = \downarrow(x, y)$.

The next, well-known lemma recalls that coinverters in **sup** are calculated easily via inverters in **ord**.

3.3. Lemma. *For*

$$\begin{tikzcd} X \arrow[r, phantom, "\Uparrow"] & A \arrow[l, bend right, "f"'] \arrow[l, bend left, "g"] \end{tikzcd}$$

in **sup** *(where we use double-shafted arrows for instances of inequality) with* $f \dashv \phi$ *and* $g \dashv \gamma$ *in* **ord**, *and*

$$\begin{tikzcd} I \arrow[r, "\kappa"] & X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\gamma"'] \arrow[r, phantom, "\Downarrow"] & A \end{tikzcd}$$

an inverter in **ord**, *the full (and faithful)* κ *has a left adjoint* $k : X \to I$ *which provides a coinverter for* $g \to f$. *Moreover, if* $h : X \to J$ *in* **sup** *coinverts* $g \to f$, *then the unique* $l : I \to J$ *satisfying* $\ell k = h$ *is given by* $\ell = h\kappa$.

■

3.4. Note that the inverter of $D\downarrow_X \subseteq \mathbf{D}\downarrow_X$ is

$$\{Y \in DX | (\exists x)(Y = \underset{X}{\downarrow} x)\}$$

which, as shown in [RW2], is the Cauchy completion of X. (It is also the antisymmetrization of X.)

3.5. Theorem. *The tensor product of sup-lattices* M *and* N *can be calculated as the inverter*

$$\begin{tikzcd} M \otimes N \arrow[r, "\kappa"] & D(M \times N) \arrow[r, bend left, "D(\downarrow_M \times \downarrow_N)"] \arrow[r, bend right, "\mathbf{D}(\downarrow_M \times \downarrow_N)"'] \arrow[r, phantom, "\Downarrow"] & D(DM \times DN) \end{tikzcd}$$

Explicitly

$$M \otimes N = \{W \in D(M \times N) | (\forall (S,T) \in DM \times DN)(S \times T \subseteq W \Longrightarrow (\bigvee S, \bigvee T) \in W)\}$$

and this subset of $D(M \times N)$ *is reflective with the reflector providing the coinverter of the diagram in 3.1.*

Proof. The first and last parts of the statement follow from using Lemma 3.3 to calculate the coinverter of the the diagram in 3.1, after rewriting the domain of the 2-cell using Lemma 3.2. Since the domain of the relevant 2-cell in 3.1 is the right adjoint of the right adjoint of the codomain, the codomain of the inverter 2-cell must be the right adjoint of the right adjoint of $D(\downarrow_M \times \downarrow_N)$, which by subsection 2.2 is $\mathbf{D}(\downarrow_M \times \downarrow_N)$.

For the explicit description: using 2.2 we have

$$D(\underset{M}{\downarrow} \times \underset{N}{\downarrow})(U) = \{(S,T) \in DM \times DN \mid (\exists (m,n))((S \subseteq \downarrow m \;\&\; T \subseteq \downarrow n) \;\&\; (m,n) \in U)\}$$

and

$$\mathbf{D}(\underset{M}{\downarrow} \times \underset{N}{\downarrow})(U) = \{(S,T) \in DM \times DN \mid (\forall(m,n))((\downarrow m \subseteq S \;\&\; \downarrow n \subseteq T) \Longrightarrow (m,n) \in U)\}$$

$$= \{(S,T) \in DM \times DN \mid S \times T \subseteq U\}$$

From these descriptions, we see that $M \otimes N$, calculated in **ord**, is the subset of $D(M \times N)$ consisting of those W satisfying

$$(\forall(S,T) \in DM \times DN)(S \times T \subseteq W \Longrightarrow (\exists(m,n))(S \subseteq \downarrow m \;\&\; T \subseteq \downarrow n \;\&\; (m,n) \in W))$$

Because M and N are sup-lattices, we can replace $S \subseteq \downarrow m$ with $\bigvee S \leq m$ and $T \subseteq \downarrow n$ with $\bigvee T \leq n$ and the condition above simplifies to

$$(\forall(S,T) \in DM \times DN)(S \times T \subseteq W \Longrightarrow (\bigvee S, \bigvee T) \in W)$$

■

3.6. We will write $(-)^{\curlyvee\!\curlyvee}$ for the left adjoint to the inclusion $\kappa : M \otimes N \twoheadrightarrow D(M \times N)$. Of course, infima in $M \otimes N$ are given by intersection, as in $D(M \times N)$, while for any $\mathscr{S}$ in $D(M \otimes N)$, we have $\bigvee \mathscr{S} = (\bigcup \mathscr{S})^{\curlyvee\!\curlyvee}$.

For any $W \in M \otimes N$, the special rectangles $M \times \emptyset = \emptyset$ and $\emptyset \times N = \emptyset$ are contained in W so that we have $(\top_M, \bot_N) \in W$ and $(\bot_M, \top_N) \in W$. Since W is a downset, the axis wedge $M \times \{\bot_N\} \cup \{\bot_M\} \times N$ is contained in W. Moreover, at least using Boolean logic, it is clear that $M \times \{\bot_N\} \cup \{\bot_M\} \times N \in M \otimes N$, so that the bottom element of $M \otimes N$ is $\bot_{M \otimes N} = M \times \{\bot_N\} \cup \{\bot_M\} \times N$.

Any downset is the union of the principal downsegments determined by its elements. Thus for any W in $D(M \times N)$, we have $W = \bigcup\{\downarrow(m,n) | (m,n) \in W\}$. For any W in $M \otimes N$, we have

$$W = W^{\curlyvee\!\curlyvee} = (\bigcup\{\downarrow(m,n) | (m,n) \in W\})^{\curlyvee\!\curlyvee} = \bigvee\{(\downarrow(m,n))^{\curlyvee\!\curlyvee} | (m,n) \in W\} \tag{1}$$

Define $\iota : M \times N \twoheadrightarrow M \otimes N$ by $\iota(m,n) = (\downarrow(m,n))^{\curlyvee\!\curlyvee}$. At least using Boolean logic, it is easy to see that

$$(\downarrow(m,n))^{\curlyvee\!\curlyvee} = \bot_{M \otimes N} \cup \downarrow(m,n) = M \times \{\bot_N\} \cup \downarrow(m,n) \cup \{\bot_M\} \times N$$

It is suggestive to write $m \otimes n$ for $\iota(m,n) = (\downarrow(m,n))^{\curlyvee\!\curlyvee}$, so $W = \bigvee\{m \otimes n | (m,n) \in W\}$, for any $W \in M \otimes N$.

From Equation (1) it follows that, for any sup-preserving $f, g : M \otimes N \twoheadrightarrow L$, $f\iota = g\iota$ implies $f = g$. In fact, for any sup-preserving f such that $f\iota = \phi$, we must have $f(W) = \bigvee\{\phi(m,n) \mid (m,n) \in W\}$. Since $(m,n) \in W$ if and only if $\iota(m,n) \leq W$ in $M \otimes N$, f is the left Kan extension of ϕ along ι. In particular, the identity is the left Kan extension of ι along ι, so ι is dense.

Using known properties of tensor products of sup-lattices, we have deduced that tensor products can be described as in Theorem 3.5. However, we can sharpen our understanding of the concepts involved, by showing that the universal property of the tensor product follows directly from the description in the theorem.

3.7. Proposition. *For sup-lattices M, N, and L, a functor $\phi\colon M\times N \to L$ is bi-sup-preserving if and only if the following equation holds, where γ is the downset comparison functor mentioned in Lemma 3.2:*

$$\begin{tikzcd} DM\times DN \arrow[r, "\gamma"] \arrow[d, "\bigvee\times\bigvee"'] & D(M\times N) \arrow[r, "D\phi"] & DL \arrow[d, "\bigvee"] \\ M\times N \arrow[rr, "\phi"'] & & L \end{tikzcd}$$

Proof. Assume that ϕ is bi-sup-preserving. For any (S,T) in $DM\times DN$, we have (using $(-)^{\downarrow}$ to denote the down-closure of a subset):

$$\begin{aligned} \phi(\bigvee S, \bigvee T) &= \bigvee\{\phi(s,\bigvee T) \mid s\in S\} \\ &= \bigvee\{\bigvee\{\phi(s,t) \mid t\in T\} \mid s\in S\} \\ &= \bigvee\{\phi(s,t) \mid (s,t)\in S\times T\} \\ &= \bigvee\{\phi(s,t) \mid (s,t)\in S\times T\}^{\downarrow} \\ &= \bigvee D\phi(\gamma(S,T)) \end{aligned}$$

Conversely, assume that the equation given by the diagram holds, and consider an arbitrary subset A of M and an element b of N. Now

$$\begin{aligned} \phi(\bigvee A, b) &= \phi(\bigvee A^{\downarrow}, \bigvee{\downarrow} b) \\ &= \bigvee D\phi(\gamma(A^{\downarrow}, {\downarrow} b)) \\ &= \bigvee D\phi(\gamma(A^{\downarrow}, \{b\}^{\downarrow})) \\ &= \bigvee D\phi(A\times\{b\})^{\downarrow} \\ &= \bigvee\{\phi(a,b)|a\in A\}^{\downarrow} \\ &= \bigvee\{\phi(a,b)|a\in A\} \end{aligned}$$

where the second equation is the assumption. Similarly, for any $a\in M$ and $B\subseteq N$, $\phi(a,\bigvee B) = \bigvee\{\phi(a,b)|b\in B\}$. ■

In the introduction, we remarked that if $\phi : M\times N \to L$ is bi-sup-preserving and $l\colon L\to L'$ is sup-preserving, then $l\phi$ is bi-sup-preserving. This follows immediately from the characterization of bi-sup-preservation provided by Proposition 3.7, since "l preserves sups" is expressed by the following equation:

$$\begin{tikzcd} DL \arrow[r, "Dl"] \arrow[d, "\bigvee"'] & DL' \arrow[d, "\bigvee"] \\ L \arrow[r, "l"'] & L' \end{tikzcd}$$

which can be pasted to that of Proposition 3.7 along the edge $\bigvee : DL \to L$. We want to show that $\iota : M \times N \to M \otimes N$ is bi-sup-preserving. Our next lemma builds on our remarks in 3.6.

3.8. Lemma. *The following equation holds:*

$$\begin{tikzcd} DM \times DN \arrow[r, "\gamma"] \arrow[d, "\bigvee\times\bigvee"'] & D(M \times N) \arrow[d, "(-)^{\vee\vee}"] \\ M \times N \arrow[r, "\iota"'] & M \otimes N \end{tikzcd}$$

Proof. Let (S, T) be an element of $DM \times DN$. We must show $(S \times T)^{\vee\vee} = ({\downarrow}(\bigvee S, \bigvee T))^{\vee\vee}$. From $S \times T \subseteq (S \times T)^{\vee\vee}$, we have $(\bigvee S, \bigvee T) \in (S \times T)^{\vee\vee}$, which is the same as ${\downarrow}(\bigvee S, \bigvee T) \subseteq (S \times T)^{\vee\vee}$ and hence $({\downarrow}(\bigvee S, \bigvee T))^{\vee\vee} \subseteq (S \times T)^{\vee\vee}$. On the other hand,

$$S \times T \subseteq {\downarrow}\bigvee S \times {\downarrow}\bigvee T = {\downarrow}(\bigvee S, \bigvee T)$$

so $(S \times T)^{\vee\vee} \subseteq ({\downarrow}(\bigvee S, \bigvee T))^{\vee\vee}$ ■

3.9. Corollary. *The functor $\iota : M \times N \to M \otimes N$ is bi-sup-preserving.*

Proof. Consider the following diagram in the light of the diagrams in Proposition 3.7 and Lemma 3.8:

$$\begin{tikzcd} DM \times DN \arrow[r, "\gamma"] \arrow[d, "\bigvee\times\bigvee"'] & D(M \times N) \arrow[r, "D\iota"] \arrow[dr, "(-)^{\vee\vee}"'] & D(M \otimes N) \arrow[d, "\bigvee"] \\ M \times N \arrow[rr, "\iota"'] & & M \otimes N \end{tikzcd}$$

Commutativity of the outer square expresses the statement of the corollary. It remains to establish the equation given by the triangle. Since all arrows in the triangle preserve suprema, it suffices to show that the composites agree on principal downsets of $M \times N$. In other words we have only to show

$$({\downarrow}(m, n))^{\vee\vee} = \bigvee\{({\downarrow}(a, b))^{\vee\vee} \mid (a, b) \leq (m, n)\}$$

which is trivial. ■

3.10. Lemma 3.7 shows that if a functor ϕ preserves suprema then, unlike the situation for abelian groups, ϕ is bi-sup-preserving. To see this, just observe that the triangle in the following diagram commutes:

$$\begin{tikzcd}
DM \times DN \arrow[r, "\gamma"] \arrow[d, "\bigvee\times\bigvee"'] & D(M \times N) \arrow[r, "D\phi"] \arrow[dl, "\bigvee"'] & DL \arrow[d, "\bigvee"] \\
M \times N \arrow[rr, "\phi"'] & & L
\end{tikzcd}$$

Also, since $\gamma : DM \times DN \to D(M \times N)$ has a left adjoint, given in terms of the product projections p and r by $\langle Dp, Dr\rangle : D(M \times N) \to DM \times DN$, and $\gamma.(\downarrow_M \times \downarrow_N) = \downarrow_{M\times N}$, we have

$$\begin{tikzcd}
DM \times DN \arrow[d, "\bigvee\times\bigvee"'] & D(M \times N) \arrow[l, "\langle Dp,Dr\rangle"'] \arrow[dl, "\bigvee"] \\
M \times N &
\end{tikzcd}$$

Thus general suprema for $M \times N$ are suprema of rectangles. Of course, this does *not* say that bi-sup-preserving implies sup-preserving. (Given U in $D(M \times N)$, $Dp(U) \times Dr(U)$ is the smallest rectangle that contains U.)

Writing $\mathbf{bisup}(M\times N, L)$ for the ordered set of bi-sup-preserving functors from $M\times N$ to L, and recalling our remark in 3.6 that a sup-preserving f with $f\iota = \phi$ is necessarily the left Kan extension of ϕ along ι, we have

$$\mathbf{sup}(M \otimes N, L) \xrightarrow{-\cdot\iota} \mathbf{bisup}(M \times N, L)$$

one to one.

3.11. Theorem. *The sup-lattice $M \otimes N$, as given by Theorem 3.5, classifies functors that are bi-sup-preserving, in the sense that*

$$\mathbf{sup}(M \otimes N, L) \xrightarrow{-\cdot\iota} \mathbf{bisup}(M \times N, L)$$

is a bijection.

Proof. Any $\phi : M \times N \to L$ gives rise to a unique sup-functor $F : D(M \times N) \to L$ for which $F{\downarrow_{M\times N}} = \phi$, namely the left Kan extension of ϕ along $\downarrow_{M\times N}$, which is given by

$F = \bigvee \cdot D\phi$. Now consider

$$\begin{tikzcd}
M \otimes N & D(M \times N) \arrow[l, "(-)^{\vee\vee}"'] \arrow[dl, "D\phi"] & D(DM \times DN) \arrow[l, bend right, "D(\bigvee\times\bigvee)"'] \arrow[l, bend left, "\mathbf{D}(\bigvee\times\bigvee)"] \arrow[l, phantom, "\Uparrow"] \\
DL \arrow[d, "\bigvee"] & & \\
L & &
\end{tikzcd}$$

To show that $- \cdot \iota$ of the theorem statement is surjective, it suffices to show that if $\phi : M \times N \twoheadrightarrow L$ is bi-sup-preserving then $F = \bigvee D\phi$ coinverts the inequality. For, in that case, we have a sup-preserving $f : M \otimes N \twoheadrightarrow L$ with $f.(-)^{\vee\vee} = F$, and hence

$$f\iota = f.(-)^{\vee\vee}. \underset{M\times N}{\downarrow} = F \underset{M\times N}{\downarrow} = \phi$$

To show that $\bigvee D\phi$ coinverts the inequality is, by Theorem 3.5, to show that its right adjoint takes values in $M \otimes N$, which is to show, for all $l \in L$, that $\phi^{-1}(\downarrow l) \in M \otimes N$. So assume that, for $(S, T) \in DM \times DN$, we have $S \times T \subseteq \phi^{-1}(\downarrow l)$, which is equivalent to assuming that $D\phi(S \times T) \subseteq \downarrow l$. We must show that $(\bigvee S, \bigvee T) \in \phi^{-1}(\downarrow l)$. By Proposition 3.7, $\phi(\bigvee S, \bigvee T) = \bigvee D\phi(S \times T)$, but by applying $\bigvee$ to the assumption, $\bigvee D\phi(S \times T) \leq \bigvee \downarrow l = l$. So we have $\phi(\bigvee S, \bigvee T) \leq l$ and hence $(\bigvee S, \bigvee T) \in \phi^{-1} \downarrow(l)$. ∎

4. Sup-Arrows

4.1. We will soon turn to a description of $M \otimes N$, for CCD lattices M and N, in terms of the *totally below relation*. For a and b in L a complete lattice, we define

$$a \ll b \quad \text{iff} \quad (\forall S \in DL)(b \leq \bigvee S \Longrightarrow a \in S)$$

and read *"a is totally below b"* for $a \ll b$, as in [RW2]. (We caution that other authors use $a \ll b$ for the *way below* relation, which requires that the S in our definition be an *up-directed* downset. The two relations are not the same. Totally below trivially implies way below, but the converse is false. For example, in any power set lattice, to say that S is totally below T is to say that S is a sub-singleton subset of T while S is way below T if and only if S is a finite subset of T.) We will provide an interesting extension of the totally below relation to ordered sets that are not necessarily complete. Before doing so, we define a few other concepts, without completeness, that are familiar for complete lattices.

4.2. From [RW3], for any X in **ord**, we have $(-)^+ : DX \twoheadrightarrow UX = (D(X^{op}))^{op}$, where, for S in DX, we define $S^+ = \{u \in X \mid (\forall s \in S)(s \leq u)\}$ as the set of upper bounds for S. Similarly, we have $(-)^- : UX \twoheadrightarrow DX$, where T^- is the set of lower bounds for T. We always have $(-)^+ \dashv (-)^-$ and the two equations on the left below. The two equations on the right hold if X is complete (equivalently cocomplete).

$$
\begin{tikzcd}
 & DX \arrow[dd, shift right=2, "(-)^+"'] \arrow[dd, phantom, "\dashv"] \\
X \arrow[ur, "\downarrow"] \arrow[dr, "\uparrow"'] & \\
 & UX \arrow[uu, shift right=2, "(-)^-"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & DX \arrow[dl, "\bigvee"'] \arrow[dd, shift right=2, "(-)^+"'] \arrow[dd, phantom, "\dashv"] \\
X & \\
 & UX \arrow[ul, "\bigwedge"] \arrow[uu, shift right=2, "(-)^-"']
\end{tikzcd}
$$

For the monad $(-)^{+-}$ on DX, we will write $(DX)^{+-}$ for the $(-)^{+-}$-closed subsets of DX. In **ord**, $(DX)^{+-}$ is a full reflective subobject of DX, so it is also a complete lattice, and $\downarrow : X \twoheadrightarrow DX$ factors through $(DX)^{+-}$. We will write $d : X \twoheadrightarrow (DX)^{+-}$ for the first such factor.

4.3. Lemma. *For x in X and S in DX, if $\bigvee S$ exists then*

$$x \in S^{+-} \Longleftrightarrow x \leq \bigvee S$$

.

Proof. Observe that $\bigvee S$ exists if and only if $\bigwedge S^+$ exists, in which case they are equal.

$$
\begin{array}{c}
x \in S^{+-} \\ \hline
(\forall u \in S^+)(x \leq u) \\ \hline
x \leq \bigwedge S^+ \\ \hline
x \leq \bigvee S
\end{array}
$$

■

4.4. Corollary. *For S in DX, $\bigvee S$ exists if and only if $S^{+-} \cap S^+$ is non-empty.*

■

4.5. Corollary. *For S in DX, if $\bigvee S$ exists then $S^{+-} = \downarrow \bigvee S$.*

Proof. Trivially, $x \leq \bigvee S$ if and only if x is in $\downarrow \bigvee S$. ■

4.6. Corollary. *An ordered set X is complete if and only if $d : X \twoheadrightarrow (DX)^{+-}$ is an equivalence.*

■

For any $f : X \to A$ in **ord**, we have $Df : DX \to DA$. If S in DX has a supremum in X and $Df(S)$ has a supremum in A, then f *preserves* $\bigvee S$ if and only if $f(\bigvee S) \le \bigvee Df(S)$ (the opposite inequality holding automatically). Lemma 4.3 allows us to express sup preservation for an arrow $f : X \to A$ in **ord** without requiring existence of any suprema.

4.7. Definition. *An arrow $f : X \to A$ in* **ord** *will be called a* **sup**-arrow *if, for $x \in X$ and $S \in DX$,*

$$x \in S^{+-} \Longrightarrow f(x) \in (Df(S))^{+-}$$

We write $\mathbf{ord_{sup}}$ *for the locally full sub-***ord***-category of* **ord** *determined by the* **sup***-arrows.*

Note that a **sup**-arrow preserves any suprema that exist. For, if $\bigvee S$ exists, then from $\bigvee S \in S^{+-}$, we have $f(\bigvee S) \in (Df(S))^{+-}$, while it is trivial that $f(\bigvee S) \in (Df(S))^{+}$, so by Corollary 4.4, $f(\bigvee S) = \bigvee Df(S)$.

Hence, if A is complete, then f is a **sup**-arrow if and only if

$$x \in S^{+-} \Longrightarrow f(x) \le \bigvee Df(S)$$

Clearly, there is an inclusion functor, $I : \mathbf{sup} \to \mathbf{ord_{sup}}$. We should also note that (after extending the definition of Df to arbitrary functions between ordered sets) **sup**-arrows are automatically order preserving.

While $\downarrow : X \to DX$ preserves only trivial suprema, we have:

4.8. Theorem. *For X an ordered set, the arrow $d : X \to (DX)^{+-}$ is a* **sup***-arrow and provides the unit for an adjunction $(D-)^{+-} \dashv I : \mathbf{sup} \to \mathbf{ord_{sup}}$.*

Proof. For the first clause, let S be a downset of X, and take $x \in S^{+-}$. We must show $d(x) \in (Dd(S))^{+-}$. First observe that

$$(Dd(S))^{+} = \{T \in (DX)^{+-} \mid (\exists s \in S)(T \subseteq d(s))\}^{+} = \{U \in (DX)^{+-} \mid S \subseteq U\}$$

It follows that, to show $d(x) \in (Dd(S))^{+-}$ is to show $d(x) \subseteq U$ for all $U \in (DX)^{+-}$ which contain S. But this is to show $x \in U$ for all $U \in (DX)^{+-}$ which contain S. But this says precisely that $x \in S^{+-}$.

For the second clause, observe that any $S \in (DX)^{+-}$ satisfies $S \cong \bigvee\{d(s) \mid s \in S\}$. It follows that, for any $f : X \to A$ in $\mathbf{ord_{sup}}$ with A complete, there is at most one arrow $\widehat{f} : (DX)^{+-} \to A$ in **sup** (to within isomorphism) satisfying the equation

$$\begin{tikzcd} X \arrow[r, "d"] \arrow[dr, "f"'] & (DX)^{+-} \arrow[d, "\widehat{f}"] \\ & A \end{tikzcd}$$

and it is given by $\widehat{f}(S) = \bigvee\{f(s) \mid s \in S\}$. To see that $\widehat{f}\colon (DX)^{+-} \to A$ as defined is an arrow in **sup**, we must show that $\widehat{f}(\bigvee \mathscr{S}) \leq \bigvee D\widehat{f}(\mathscr{S})$, for any $\mathscr{S} \in D((DX)^{+-})$. But

$$\begin{aligned} \widehat{f}(\bigvee \mathscr{S}) &= \widehat{f}((\bigcup \mathscr{S})^{+-}) \\ &= \bigvee\{f(x) \mid x \in (\bigcup \mathscr{S})^{+-}\} \end{aligned}$$

and

$$\begin{aligned} \bigvee D\widehat{f}(\mathscr{S}) &= \bigvee\{\widehat{f}(S) \mid S \in \mathscr{S}\} \\ &= \bigvee\{\bigvee\{f(s) \mid s \in S\} \mid S \in \mathscr{S}\} \\ &= \bigvee\{f(s) \mid s \in \bigcup \mathscr{S}\} \end{aligned}$$

so it suffices to show that, for $x \in (\bigcup \mathscr{S})^{+-}$, $f(x) \leq \bigvee\{f(s) \mid s \in \bigcup \mathscr{S}\}$. We have this because f is a **sup**-arrow and $\bigcup \mathscr{S}$ is a downset. ∎

The theorem tells us that $d\colon X \to (DX)^{+-}$ is the completion of X that preserves any existing suprema in X. Indeed, it is the completion by one-sided Dedekind cuts.

5. The Totally Below Relation

5.1. Definition. *For y and x in X in* **ord**, *we define*

$$y \ll x \quad \textit{iff} \quad (\forall S \in DX)(x \in S^{+-} \Longrightarrow y \in S)$$

read $y \ll x$ as "y is totally below x", *and write* $\Downarrow x = \{y \mid y \ll x\}$.

By Lemma 4.3, the definition of the totally below relation for general orders agrees with the previous definition for complete X. The elementary properties of $\ll$ for complete orders persist: for any $(X, \leq)$, $\ll_X$ is an order ideal from X to X (so $b \leq y \ll x \Longrightarrow b \ll x$ and $y \ll x \leq a \Longrightarrow y \ll a$) and $y \ll x \Longrightarrow y \leq x$. It follows that $\ll$ is transitive.

5.2. We recall that a completely distributive, CD, lattice is a complete lattice L which satisfies

$$(\forall \mathscr{S} \subseteq \mathscr{P}L)(\bigwedge\{\bigvee S \mid S \in \mathscr{S}\} = \bigvee\{\bigwedge\{T(S) \mid S \in \mathscr{S}\} \mid T \in \Pi\mathscr{S}\}$$

where we have written $\Pi\mathscr{S}$ for the set of choice functions T on $\mathscr{S}$ so that, for each $S \in \mathscr{S}$, $T(S) \in S$. A complete lattice is *constructively completely distributive*, CCD, if it satisfies the above but with "$\forall \mathscr{S} \subseteq \mathscr{P}L$" replaced by "$\forall \mathscr{S} \subseteq DL$". Evidently, CD implies CCD, and the converse holds in the presence of the axiom of choice, AC. In fact, we have

$$(\text{AC}) \Longleftrightarrow ((\text{CD}) \Longleftrightarrow \text{CCD})$$

which surely motivates the terminology. Many familiar theorems for CD lattices have been proven constructively for CCD lattices, so that they become theorems about CCD lattices in a topos. In particular we have, constructively, the Raney-Büchi theorem that a complete lattice is CCD if and only if it is a complete-homomorphic image of a complete ring of sets.

For our present purposes, we recall from [RW2] that a complete lattice L is CCD if and only if, for all $x \in L$, $x \leq \bigvee \Downarrow x$. In other words, every element of L is the supremum of all the elements totally below it. Since we have generalized the totally below relation from complete lattices to general ordered sets, the characterization of CCD lattices in this paragraph suggests the following:

5.3. Definition. *An ordered set $(X, \leq)$ is said to be STB if, for all $x \in X$, we have $x \in (\Downarrow x)^{+-}$. We write* **stb** *for the full subcategory of* $\mathbf{ord_{sup}}$ *determined by the $(X, \leq)$ with the STB property.*

It follows from Lemma 4.3 that a complete lattice is CCD if and only if it is STB as an ordered set. Hence the inclusion functor $I : \mathbf{sup} \to \mathbf{ord_{sup}}$ restricts to an inclusion functor $I : \mathbf{ccd_{sup}} \to \mathbf{stb}$ where $\mathbf{ccd_{sup}}$ is the full subcategory of **sup** determined by the CCD lattices. Clearly, the following diagram is a pullback:

$$\begin{tikzcd}
\mathbf{ccd_{sup}} \arrow[r, tail] \arrow[d, tail, "I"'] & \mathbf{sup} \arrow[d, tail, "I"] \\
\mathbf{stb} \arrow[r, tail] & \mathbf{ord_{sup}}
\end{tikzcd}$$

The STB condition allows a simple, useful characterization of $(-)^{+-}$-closed downsets. In fact, this characterization of $(-)^{+-}$-closed downsets characterizes STB orders:

5.4. Lemma. *For $(X, \leq)$ in* **ord***, $(X, \leq)$ is STB if and only if, for all $S \in DX$, $S^{+-} = \{x \in X \mid \Downarrow x \subseteq S\}$.*

Proof. Assume $(X, \leq)$ is STB. If $\Downarrow x \subseteq S$ then $x \in (\Downarrow x)^{+-} \subseteq S^{+-}$ shows $x \in S^{+-}$, while if $x \in S^{+-}$ then for any $y \ll x$ we have $y \in S$, so that $\Downarrow x \subseteq S$.

Conversely, assume the condition and, for any $x \in X$, consider the downset $(\Downarrow x)^{+-} = \{y \in X \mid \Downarrow y \subseteq \Downarrow x\}$. Since $\Downarrow x \subseteq \Downarrow x$, $x \in (\Downarrow x)^{+-}$ and X is STB. ∎

5.5. Lemma. *(Interpolation) If $(X, \leq)$ is STB and $y \ll x$ then $(\exists z)(y \ll z \ll x)$.*

Proof. Let $S = \{u | (\exists z)(u \ll z \ll x)\}$ and assume $y \ll x$, so that $\Downarrow y \subseteq S$. By STB for X, $y \in S^{+-}$ and since y is arbitrary $\Downarrow x \subseteq S^{+-}$. By STB for X again, $x \in S^{+-+-} = S^{+-}$. But now $y \ll x \in S^{+-}$ implies $y \in S$. ∎

Thus if $(X, \leq)$ is STB, $\ll$ is *idempotent* as a relation from X to X. (To say that $\ll \circ \ll = \ll$ is to say that $\ll \circ \ll \subseteq \ll$, transitivity, and that $\ll \subseteq \ll \circ \ll$, interpolativity in the sense of Lemma 5.5.) Recall the **ord**-category **idm** studied in detail in [MRW]. The objects of **idm** are pairs $(X, <)$ where X is a set and $<$ is an idempotent relation on X. An arrow $f : (X, <) \to (A, <)$ is a function $f : X \to A$ for which $x < y$ in X implies $fx < fy$ in A. If $f, g : X \to A$ in **idm** then $f \leq g$ if and only if, for all $x < y$ in X, we have $fx < gy$ in A. Notice that **ord** is a 2-full sub-**ord**-category of **idm**. Clearly, if $(X, \leq)$ is STB then $(X, \ll)$ is an object of **idm** and the identity function provides an arrow $(X, \ll) \to (X, \leq)$ in **idm**.

There is an alternative description of the arrows in **stb**.

5.6. Proposition. *For STB objects X and A and $f : X \to A$ in* **ord***, f is in* $\mathbf{ord_{sup}}$ *(and hence in* **stb***) if and only if*

$$(\forall a \in A, x \in X)(a \ll fx \Longrightarrow (\exists y \in X)(a \leq fy \ \& \ y \ll x))$$

Proof. Assume $f \in \mathbf{ord_{sup}}$ and $a \ll fx$. Since X is STB, $x \in (\Downarrow x)^{+-}$, and then since f is a sup-arrow, $f(x) \in (Df(\Downarrow x))^{+-}$. By definition of $\ll$, we have $a \in (Df(\Downarrow x))$ so $(\exists y)(a \leq f(y)$ and $y \ll x)$.

Conversely, assume the condition and $x \in S^{+-}$. To show $f \in \mathbf{ord_{sup}}$ we must show $f(x) \in (Df(S))^{+-}$. By Lemma 5.4, it is sufficient to show that, for $a \ll f(x)$, we have $a \in Df(S)$. But by assumption we have y with $a \leq fy$ and $y \ll x$. Since $x \in S^{+-}$, the second conjunct gives us $y \in S$ and hence $a \in Df(S)$. ∎

We recall from [MRW] that the 2-functor D extends to **idm**. For any $(X, <)$ in **idm**, we say that a subset S of X is a *downset of the idempotent* if

$$x \in S \Longleftrightarrow (\exists y)(x < y \in S)$$

We write $D(X, <)$ for the set of downsets of $(X, <)$, ordered by inclusion. In fact, see [RW2], $D(X, <)$ is a CCD lattice and every CCD lattice arises in this way. The 2-natural transformation $\downarrow$ also extends to **idm**. For any $(X, <)$ in **idm** we define $\downarrow : X \to D(X, <)$ by $\downarrow x = \{y \mid y < x\}$.

5.7. Lemma. *For $x, y \in (X, \leq)$ an ordered set, $x \ll y$ in X if and only if $\downarrow x \ll \downarrow y$ in $(D(X, \leq))^{+-}$.*

Proof. Assume $x \ll y$ and that, for $\mathscr{S} \in D((D(X, \leq))^{+-})$, we have $\downarrow y \subseteq \bigvee \mathscr{S}$ in $(D(X, \leq))^{+-}$. Now $\bigcup \mathscr{S}$ is certainly a downset of $D(X, \leq)$ and since $\bigvee \mathscr{S} = (\bigcup \mathscr{S})^{+-}$, $\downarrow y \subseteq (\bigcup \mathscr{S})^{+-}$, so $y \in (\bigcup \mathscr{S})^{+-}$ and from from $x \ll y$, we get $x \in \bigcup \mathscr{S}$. Thus $x \in S \in \mathscr{S}$, for some $S \in \mathscr{S}$. Now we have $\downarrow x \subseteq S \in \mathscr{S}$ in $(D(X, \leq))^{+-}$ so that $\downarrow x \in \mathscr{S}$ since $\mathscr{S}$ is a downset of elements of $(D(X, \leq))^{+-}$. This shows that $\downarrow x \ll \downarrow y$.

For the converse, assume $\downarrow x \ll \downarrow y$ and $y \in S^{+-}$ for some $S \in D(X, \leq)$. We know that $S = \bigcup\{\downarrow s \mid s \in S\}$, so $y \in S^{+-}$ gives $y \in (\bigcup\{\downarrow s \mid s \in S\})^{+-}$, which means that $y \in \bigvee\{\downarrow s \mid s \in S\}$ in $(D(X, \leq))^{+-}$. So $\downarrow y \subseteq \bigvee\{\downarrow s \mid s \in S\} = \bigvee\{T \mid T \subseteq \downarrow s \ \& \ s \in S\}$ in $(D(X, \leq))^{+-}$ and $\{T \mid T \subseteq \downarrow s \ \& \ s \in S\}$, call it $\mathscr{S}$, is a downset of $(D(X, \leq))^{+-}$.

Since $\downarrow x \ll \downarrow y$, we have $\downarrow x \in \mathscr{S}$. But now $\downarrow x \subseteq \downarrow s$, for some $s \in S$. Hence $x \leq s \in S$, and thus $x \in S$, which shows that $x \ll y$. ∎

5.8. Remark. Before stating the next theorem, it is convenient to point out that an STB order $(X, \leq)$ allows us to construct $D(X, \ll)$, for $\ll$ an idempotent, in addition to the usual $D(X, \leq)$. Every $\ll$-downset is easily seen to be a $\leq$-downset and we leave it as an exercise for the reader to show that $S^\circ = \{x \in X \mid (\exists y)(x \ll y \in S)\}$ describes a right adjoint $(-)^\circ : D(X, \leq) \to D(X, \ll)$ to the inclusion $i : D(X, \ll) \to D(X, \leq)$, and $S^\circ = \Downarrow \bigvee S$.

5.9. Theorem. *For $(X, \leq)$ an ordered set, the following are equivalent:*

(i) $(X, \leq)$ is STB;

(ii) The composites

$$D(X, \ll) \underset{(-)^\circ}{\overset{i}{\rightleftarrows}} D(X, \leq) \underset{j}{\overset{(-)^{+-}}{\rightleftarrows}} (D(X, \leq))^{+-}$$

are inverse isomorphisms;

(iii) $D(X, \ll) \cong D(X, \leq))^{+-}$;

(iv) $(D(X, \leq))^{+-}$ is CCD.

Proof. *(i) $\implies$ (ii)* For $T \in D(X, \ll)$, we have $T \subseteq (T^{+-})^\circ$ from the composite of the adjunctions $i \dashv (-)^\circ$ and $(-)^{+-} \dashv j$. For $S \in (D(X, \leq))^{+-}$, we have $(S^\circ)^{+-} \subseteq S$, also from the composite adjunction. Using the characterization given in Lemma 5.4, we have $(T^{+-})^\circ = \{x \mid (\exists y)(x \ll y \;\&\; \Downarrow y \subseteq T)\} \subseteq T$. On the other hand, again by Lemma 5.4, $(S^\circ)^{+-} = \{y \mid \Downarrow y \subseteq S^\circ\}$. Take $y \in S$. Then for any $x \ll y$, we have $x \in S^\circ$. Thus $\Downarrow y \subseteq S^\circ$, and so $y \in (S^\circ)^{+-}$. So $S \subseteq (S^\circ)^{+-}$.

(ii) $\implies$ (iii) is trivial.

(iii) $\implies$ (iv) follows from the fact that all lattices of the form $D(A, <)$ for $<$ an idempotent on a set A are CCD.

(iv) $\implies$ (i) Assume $(D(X, \leq))^{+-}$ is CCD, and take $x \in X$. We have

$$\begin{aligned}
x \in \downarrow x &= \bigvee\{S \in (D(X, \leq))^{+-} \mid S \ll \downarrow x\} \\
&= \bigvee\{\bigvee\{\downarrow s \mid s \in S\} \mid S \ll \downarrow x\} \\
&= \bigvee\{\downarrow s \mid s \in S \ll \downarrow x\} \\
&= \bigvee\{\downarrow s \mid \downarrow s \ll \downarrow x\} \\
&= \bigvee\{\downarrow s \mid s \ll x\} \\
&= (\bigcup\{\downarrow s \mid s \ll x\})^{+-} \\
&= \{t \mid t \ll x\}^{+-} \\
&= (\Downarrow x)^{+-}
\end{aligned}$$

where the fifth equality uses Lemma 5.7. ∎

5.10. REMARK. In the special case where X of the theorem is taken to be a CCD lattice L, then a minor replacement in *(ii)*, using Corollary 4.5, gives us that the composites

$$D(L,\ll) \underset{(-)^\circ}{\overset{i}{\rightleftarrows}} D(L,\le) \underset{\downarrow}{\overset{\bigvee}{\rightleftarrows}} L$$

are inverse isomorphisms. This is Proposition 13 of [RW2].

Since $D(X,\ll)$ is a CCD lattice it follows that $(D-)^{+-} : \mathbf{ord}_{\mathbf{sup}} \to \mathbf{sup}$ restricts to give $(D-)^{+-} : \mathbf{stb} \to \mathbf{ccd}_{\mathbf{sup}}$ and the following corollary follows immediately from Theorems 4.8 and 5.9.

5.11. COROLLARY. *For X an STB order, the arrow $d: X \to (DX)^{+-}$ is a $\mathbf{sup}$-arrow and provides the unit for a 2-adjunction $(D-)^{+-} \dashv I : \mathbf{ccd}_{\mathbf{sup}} \to \mathbf{stb}$.* ∎

We deduce further:

5.12. COROLLARY. *The mate with respect to the adjunctions $(D-)^{+-} \dashv I$ in the pullback diagram preceding Lemma 5.4 is also an equality and the resulting diagram*

$$\begin{tikzcd}
\mathbf{ccd}_{\mathbf{sup}} \arrow[r, tail] & \mathbf{sup} \\
\mathbf{stb} \arrow[r, tail] \arrow[u, "(D-)^{+-}"] & \mathbf{ord}_{\mathbf{sup}} \arrow[u, "(D-)^{+-}"']
\end{tikzcd}$$

is also a pullback. ∎

Observe that the condition of Proposition 5.6 is implied by

$$(\forall a \in A, x \in X)(a \ll fx \Longrightarrow (\exists y \in X)(a \ll fy \;\&\; y \ll x))$$

simply because $a \ll fy$ implies $a \le fy$.

5.13. LEMMA. *If X and A are STB orders and $f: X \to A$ is a function that preserves merely $\ll$, then the condition above is equivalent to the condition of Proposition 5.6.*

PROOF. Assume the condition of Proposition 5.6, and let $a \ll fx$. We have z with $a \le fz$ and $z \ll x$. From the second conjunct, we have $z \ll y \ll x$, and since f preserves $\ll$, we have $a \le fz \ll fy$ and hence $a \ll fy$ (and $y \ll x$). ∎

Since the displayed condition above is the condition for an arrow $f:(X,\ll) \to (A,\ll)$ in **idm** *between STB orders* $(X,\leq)$ and $(Y,\leq)$ to be a **sup**-arrow and it makes no mention of order preservation, we generalize one step further and say:

5.14. Definition. *An arrow* $f:(X,<) \to (A,<)$ *in* **idm** *is a* **sup**-*arrow if*

$$(\forall a \in A,\ x \in X)(a < fx \Longrightarrow (\exists y \in X)(a < fy \ \&\ y < x))$$

We caution however that a **sup**-arrow in **idm** does not speak about preserving suprema with respect to the idempotents $<$, even if the idempotents should happen to be reflexive relations and hence orders. We have not defined suprema for general idempotents here (and have no need to do so) but it can be done using the 2-structure of **idm**.

5.15. We write $\mathbf{idm_{sup}}$ for the locally-full sub-2-category of **idm** determined by the **sup**-arrows. We recall from [RW2] that an arrow $f: L \to A$ in $\mathbf{ccd_{sup}}$ preserves the totally below relation if and only if the right adjoint of f has a right adjoint, so that f is a *map* in **sup**, meaning that f has a right adjoint *in the 2-category* **sup**. We will write $\mathbf{ccd_{mapsup}}$ for the locally full sub-2-category of $\mathbf{ccd_{sup}}$ determined by the maps. It follows from Lemma 5.13 that there is a forgetful functor $(-,\ll):\mathbf{ccd_{mapsup}} \to \mathbf{idm_{sup}}$, which sends a CCD lattice L to the idempotent $(L,\ll)$ given by its totally below relation, and regards a map $f:L \to A$ in **sup** as an arrow $f:(L,\ll) \to (A,\ll)$ in $\mathbf{idm_{sup}}$.

5.16. Theorem. *For* $(X,<)$ *an idempotent, the arrow* $\downarrow: X \to D(X,<)$ *in* **idm** *gives an arrow* $\downarrow:(X,<) \to (D(X,<),\ll)$ *in* $\mathbf{idm_{sup}}$*, and provides the unit for a 2-adjunction* $D \dashv (-,\ll):\mathbf{ccd_{mapsup}} \to \mathbf{idm_{sup}}$.

Proof. Since $D(X,<)$ is a CCD lattice, $(D(X,<),\ll)$ is also an idempotent. It was shown in [RW2] that if $x < y$ in X then $\downarrow x \ll \downarrow y$ in $D(X,<)$. Now assume $S \ll \downarrow x$. We have $S \subseteq \downarrow t$ and $t < x$. We interpolate to get $t < y < x$ and now $S \subseteq \downarrow t$, $t \in \downarrow y$, and $y < x$ provides $S \ll \downarrow y$ and $y < x$, which shows $\downarrow:(X,<) \to (D(X,<),\ll)$ in $\mathbf{idm_{sup}}$. Next assume that we are given an arbitrary $f:(X,<) \to (A,\ll)$ in $\mathbf{idm_{sup}}$, with A a CCD lattice. To finish the proof of the theorem, we must show that there is a unique (to within isomorphism) arrow $\widehat{f}:D(X,<) \to A$ in $\mathbf{ccd_{mapsup}}$ for which $(\widehat{f},\ll)$ satisfies the following equation in $\mathbf{idm_{sup}}$.

$$\begin{tikzcd}
(X,<) \arrow[r, "\downarrow"] \arrow[dr, "f"'] & (D(X,<),\ll) \arrow[d, "\widehat{f}"] \\
& (A,\ll)
\end{tikzcd}$$

For every element S in $D(X,<)$, we have $S = \bigvee\{\downarrow s \mid s \in S\}$. Thus any $\widehat{f}$ satisfying our requirements must have

$$\widehat{f}(S) = \widehat{f}(\bigvee\{\downarrow s \mid s \in S\}) = \bigvee\{\widehat{f}(\downarrow s) \mid s \in S\} = \bigvee\{fs \mid s \in S\}$$

Thus it remains to show that $\widehat{f}(S) = \bigvee\{fs \mid s \in S\}$ meets all of our requirements. To show that the equation $\widehat{f}.{\downarrow} = f$ holds, we have

$$\widehat{f}({\downarrow} s) = \bigvee\{fx \mid x < s\} = \bigvee\{a \mid a \leq fx \;\&\; x < s\}$$

$$= \bigvee\{a \mid a \ll fs\} = fs$$

The last equality holds because A is CCD and the penultimate equality uses the properties of f being in $\mathbf{idm_{sup}}$. Now $\widehat{f}$ preserves all suprema because, taking $\mathscr{S}$ in $D(D(X,<))$, we have

$$\widehat{f}(\bigvee\mathscr{S}) = \widehat{f}(\bigcup\mathscr{S}) = \bigvee\{fs \mid s \in \bigcup\mathscr{S}\} = \bigvee\{\bigvee\{fs \mid s \in S\} \mid S \in \mathscr{S}\}$$

$$= \bigvee\{\widehat{f}(S) \mid S \in \mathscr{S}\}$$

Because $\widehat{f}$ preserves suprema and $D(X,<)$ is complete, it follows that $\widehat{f}$ has a right adjoint in **ord**. This right adjoint has a right adjoint (making $\widehat{f}$ an arrow in $\mathbf{ccd_{mapsup}}$) if and only if $\widehat{f}$ preserves $\ll$. (See [RW2].) So assume $S \ll T$ in $D(X,<)$. We have $S \subseteq {\downarrow} t$ for some $t \in T$ and hence also $t < u$ for some $u \in T$. Applying $\widehat{f}$ and f we have:

$$\widehat{f}(S) \leq \widehat{f}({\downarrow} t) = ft \ll fu \leq \widehat{f}(T)$$

and hence $\widehat{f}(S) \ll \widehat{f}(T)$. ∎

6. Tensor Products of CCD Lattices

6.1. The paper [RW2] exhibits a biequivalence between $\mathbf{ccd_{sup}}$ and the idempotent splitting completion of the bicategory of relations, **kar**(**rel**), which has a tensor product that is given on objects by cartesian product. The paper then shows that the tensor product of CCD lattices as sup-lattices agrees with the tensor product of **kar**(**rel**). We conclude now with yet another description of the tensor product of CCD lattices that uses our results about the totally below relation.

6.2. Lemma. *For CCD lattices M and N, the reflector $(-)^{\vee\!\vee} : D(M \times N) \to M \otimes N$ is given by $U^{\vee\!\vee} = \{(m,n) \mid \Downarrow m \times \Downarrow n \subseteq U\}$*

Proof. Provisionally write $\overline{U} = \{(m,n) \mid \Downarrow m \times \Downarrow n \subseteq U\}$. To show that $\overline{U} \in M \otimes N$, assume we have $S \times T \subseteq \overline{U}$ for $(S,T) \in DM \times DN$. We need to show $(\bigvee S, \bigvee T) \in \overline{U}$. For this we require $\Downarrow \bigvee S \times \Downarrow \bigvee T \subseteq U$. From Remark 5.8 this means precisely that we require $S^\circ \times T^\circ \subseteq U$ and we recall that $S^\circ = \{x \mid (\exists s)(x \ll s \in S)\}$. So if we have $(x,y) \in S^\circ \times T^\circ$, we have $x \ll s \in S$ and $y \ll t \in T$ with $(s,t) \in S \times T \subseteq \overline{U}$. So $(x,y) \in \Downarrow s \times \Downarrow t \subseteq U$. Thus $\overline{U} \in M \otimes N$. It is clear that, for any $U \in D(M \times N)$, we have $U \subseteq \overline{U}$. Assume now that $W \in M \otimes N$ and $U \subseteq W$. It suffices to show that $\overline{U} \subseteq W$. But if (m,n) satisfies $\Downarrow m \times \Downarrow n \subseteq U$ then $U \subseteq W$ implies $\Downarrow m \times \Downarrow n \subseteq W$ and hence $(m,n) = (\bigvee \Downarrow m, \bigvee \Downarrow n) \in W$. Since $\overline{(-)}$ is left adjoint to the inclusion $\kappa : M \otimes N \to D(M \times N, \leq)$, $\overline{U} = U^{\vee\!\vee}$. ∎

Generalizing very slightly what we observed in Remark 5.8, we note that every downset of $M \times N$ with respect to the idempotent $\ll_M \times \ll_N$ is a downset of $M \times N$ with respect to $\leq_{M \times N} = \leq_M \times \leq_N$ so that we have an inclusion

$$i: D(M \times N, \ll_M \times \ll_N) \to D(M \times N, \leq)$$

It has a right adjoint

$$(-)^\circ : D(M \times N, \leq) \to D(M \times N, \ll_M \times \ll_N)$$

which, for $U \in D(M \times N, \leq)$, is given by $U^\circ = \{(a,b) \mid \exists((x,y) \in U)(a \ll x \,\&\, b \ll y)\}$ Again, we leave the details to the reader.

6.3. Theorem. *For CCD lattices M and N CCD, $M \otimes N \cong D(M \times N, \ll_M \times \ll_N)$. The composites*

$$D(M \times N, \ll_M \times \ll_N) \underset{(-)^\circ}{\overset{i}{\rightleftarrows}} D(M \times N, \leq) \underset{\kappa}{\overset{(-)^{\Downarrow}}{\rightleftarrows}} M \otimes N$$

are inverse isomorphisms;

Proof. Write $<$ for the idempotent $\ll_M \times \ll_N$ on $M \times N$. For any $V \in D(M \times N, <)$, we have $V \subseteq V^{\Downarrow\circ}$ and, for any $W \in M \otimes N$, we have $W^{\circ\Downarrow} \subseteq W$, by adjointness. Now take $(x,y) \in V^{\Downarrow\circ}$. This implies $(x,y) < (m,n) \in V^{\Downarrow}$, which implies

$$(x,y) \in \Downarrow m \times \Downarrow n \;\&\; (m,n) \in V^{\Downarrow}$$

which implies $(x,y) \in V$, so that $V^{\Downarrow\circ} \subseteq V$. Finally, assume $(x,y) \in W$. We want to show that $(x,y) \in W^{\circ\Downarrow}$, which is to show $\Downarrow x \times \Downarrow y \subseteq W^\circ$. For any $(a,b) \in \Downarrow x \times \Downarrow y$, its membership in W° is witnessed by (x,y). ∎

6.4. Remark. The reader may have noticed that the proof of Theorem 6.3 is similar to that which establishes the isomorphism *(ii)* in Theorem 5.9. Both can be seen to follow from (a dual of) Eilenberg and Moore's theorem, Proposition 3.3 in [E&M]. This is the theorem which asserts that for $t \dashv g : A \to A$ in the 2-category of categories, with t underlying a monad and g a comonad, the category of algebras A^t is isomorphic to the category of coalgebras A_g, via a functor that identifies the forgetful functors. Eilenberg and Moore's theorem is easily seen to hold in any 2-category in which the objects A^t and A_g exist. In particular, it holds in each of the duals of the 2-category of categories and in the duals of **ord**. Thus if $g \dashv t$ then the Kleisli categories, A^g and A_t are isomorphic via a functor that identifies the free functors. If a monad t is idempotent, then the Eilenberg-Moore object and the Kleisli object coincide and the Kleisli arrow is the left adjoint of the Eilenberg-Moore arrow. In **ord** all monads and comonads are idempotent.

6.5. Remark. For M and N CCD lattices, the (fully faithful) adjoint string

$$D(\downarrow\times\downarrow)\dashv\mathscr{D}(\downarrow\times\downarrow)\dashv\mathbf{D}(\downarrow\times\downarrow)$$

gives rise to the longer adjoint string

$$D(\Downarrow\times\Downarrow)\dashv D(\bigvee\times\bigvee)\dashv D(\downarrow\times\downarrow)\dashv\mathscr{D}(\downarrow\times\downarrow)\dashv\mathbf{D}(\downarrow\times\downarrow)$$

In the terminology of [RW4], these are *distributive* adjoint strings and the inverter, λ, of the inequality $D(\Downarrow\times\Downarrow)\leq D(\downarrow\times\downarrow)$ is necessarily the left adjoint of the left adjoint of the inverter, κ, of $D(\downarrow\times\downarrow)\leq\mathbf{D}(\downarrow\times\downarrow)$. It follows that, for M and N CCD lattices, $M\otimes N$ can equally well be calculated as the inverter of $D(\Downarrow\times\Downarrow)\leq D(\downarrow\times\downarrow)$. Of course the inclusions λ and κ are in general different but it is easy to calculate and see that the inverter of $D(\Downarrow\times\Downarrow)\leq D(\downarrow\times\downarrow)$ reveals directly that $M\otimes N$ is the set of downsets of $(M\times N,\leq)$ that are also downsets for the idempotent $\ll_M\times\ll_N$ as already shown in Theorem 6.3. Moreover, the fully faithful adjoint string $\lambda\dashv(-)^{\vee\!\vee}\dashv\kappa$ reveals $M\otimes N$ to be a complete quotient of $D(M\times N)$ and hence CCD by Proposition 11 of [F&W].

$$\begin{tikzcd}[column sep=huge]
M\otimes N \arrow[r, shift left=4, "\lambda"] \arrow[r, shift right=4, "\kappa"'] & \mathscr{D}(M\times N) \arrow[l, "(-)^{\vee\!\vee}" description] \arrow[r, shift left=8, "D(\Downarrow\times\Downarrow)"] \arrow[r, "D(\downarrow\times\downarrow)" description] \arrow[r, shift right=8, "\mathbf{D}(\downarrow\times\downarrow)"'] & \mathscr{D}(\mathscr{D}M\times\mathscr{D}N) \arrow[l, shift right=4, "D(\bigvee\times\bigvee)" description] \arrow[l, shift left=4, "\mathscr{D}(\downarrow\times\downarrow)" description]
\end{tikzcd}$$

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Department of Mathematics and Statistics
Dalhousie University
Halifax, NS, B3H 3J5 Canada
Email: tkenney@mathstat.dal.ca, rjwood@dal.ca

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