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SYMBOLIC DYNAMICS AND THE CATEGORY OF GRAPHS

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ABSTRACT. Symbolic dynamics is partly the study of walks in a directed graph. By a walk, here we mean a morphism to the graph from the Cayley graph of the monoid of non-negative integers. Sets of these walks are also important in other areas, such as stochastic processes, automata, combinatorial group theory, C^*-algebras, etc. We put a Quillen model structure on the category of directed graphs, for which the weak equivalences are those graph morphisms which induce bijections on the set of walks. We determine the resulting homotopy category. We also introduce a "finite-level" homotopy category which respects the natural topology on the set of walks. To each graph we associate a basal graph, well defined up to isomorphism. We show that the basal graph is a homotopy invariant for our model structure, and that it is a finer invariant than the zeta series of a finite graph. We also show that, for finite walkable graphs, if B is basal and separated then the walk spaces for X and B are topologically conjugate if and only if X and B are homotopically equivalent for our model structure.

1. Introduction

Symbolic dynamics is partly the study of walks in a directed graph; see the discussion in Kitchens [1998] or Lind and Marcus [1995], for instance. Sets of these walks are also important in other areas, such as stochastic processes, automata, combinatorial group theory, C^*-algebras, etc., as can be seen from references such as Kemeny-Snell-Knapp [1976], Sakarovitch [2009], Epstein [1992], and Raeburn [2005].

Let **Gph** denote the category of directed graphs. In this paper we investigate **Gph** as a framework for analyzing symbolic dynamics of walks. By a walk in a directed graph X we mean a graph morphism from $\mathbf{N}$ to X, where $\mathbf{N}$ is the graph which, for each non-negative integer n, has a node n and an arc from n to $n+1$. So $\mathbf{N}$ is a Cayley graph, of the following simple type. Any monoid G, together with some subset $A \subset G$, determines a *Cayley graph* which, for each element $x \in G$, has a node x and an arc from x to xa for every $a \in A$. Our methods here probably have uses with more general categories of G-sets and Cayley graphs, but we leave that for further work.

In Section 2 we give our precise definitions and background.

In Section 3 we discuss the notion of Quillen model structure on a category, which expedites the description of an associated homotopy category. We define a model struc-

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ture on **Gph**, for which the weak equivalences are those graph morphisms which induce bijections on the set of walks.

In Section 4 we determine the resulting homotopy category.

In Section 5 we use the arc graph construction to introduce a “finite-level” homotopy category of graphs.

Then in Section 6 we describe the natural topology on the set of walks, and show that finite-level homotopies respect the topology.

In Section 7 we explore some applications of covering morphisms, inspired by a paper of Boldi and Vigna [2002]. We say that a graph is *basal* if the only epic covering morphisms with it as domain are the isomorphisms. To each graph we associate a basal graph, well defined up to isomorphism. We show that the basal graph is a homotopy invariant for our model structure, and that it is a finer invariant than the zeta series of a finite graph. We also show that, for finite walkable graphs, if B is basal and separated then the walk spaces for X and B are topologically conjugate if and only if X and B are homotopically equivalent for our model structure.

The Quillen model on graphs that we investigate here seems to be a particular example of the following general construction. Let $\mathcal{E}$ be a topos, and $\mathcal{I}$ a family of objects of $\mathcal{E}$. A closed model can be defined on $\mathcal{E}$ for which the class of weak equivalences are morphisms $f : X \to Y$ such that $\mathrm{Hom}_{\mathcal{E}}(I, X) \to \mathrm{Hom}_{\mathcal{E}}(I, Y)$ is a bijection for every $I \in \mathcal{I}$. In this paper, we study the particular example of this situation when $\mathcal{E}$ is the topos of directed graphs and $\mathcal{I}$ has the single object N. It seems likely that the general construction can be applied in other presheaf categories of combinatorial interest (see Lawvere [1989], for instance).

2. The set of walks and N-equivalence of graphs.

Let **Gph** denote the category of directed and possibly infinite graphs, with loops and multiple arcs allowed. This category is also studied in Bisson and Tsemo [2008], [2011].

Let us make precise the objects and morphisms in the category **Gph**. A *graph* is a data-structure $X = (X_0, X_1, s, t)$ with a set X_0 of *nodes*, a set X_1 of *arcs*, and a pair of functions $s, t : X_1 \to X_0$ which specify the *source* and *target* node of each arc. We may say that $a \in X_1$ is an arc *from node* $s(a)$ *to node* $t(a)$; a *loop* is just an arc a with $s(a) = t(a)$. A *graph morphism* $f : X \to Y$ is a pair of functions $f_1 : X_1 \to Y_1$ and $f_0 : X_0 \to Y_0$ such that $s \circ f_1 = f_0 \circ s$ and $t \circ f_1 = f_0 \circ t$. For X and Y in **Gph**, we may sometimes denote the set of graph morphisms from X to Y by $[X, Y]$.

The category **Gph** is a very nice category to work with. In particular, it is a presheaf topos (see Mac Lane and Moerdijk [1994], for instance, for a nice survey). As such, it has all limits and colimits, including the initial graph 0 (with no nodes and no arcs) and the terminal graph 1 (with one node and one loop). Here are some other standard graphs that we will be using. Let $\mathbf{N}$ denote the graph with nodes the natural numbers and arcs the pairs $(n, n+1)$ for $n \geq 0$, with $s(n, n+1) = n$ and $t(n, n+1) = n+1$. Let $\mathbf{Z}$ have nodes the integers and arcs $(n, n+1)$ for all integers, with source and target as above. Similarly, let $\mathbf{P}_n$ have set of nodes $\{0, \cdots, n\}$ and arcs $(k-1, k)$, for $1 \leq k \leq n$. We may call $\mathbf{P}_n$ the *path with n arcs*, and use the notations $\mathbf{D} = \mathbf{P}_0$ and $\mathbf{A} = \mathbf{P}_1$. For $n > 0$, let $\mathbf{C}_n$ have the nodes the integers mod n, and arcs $(k-1, k)$, for $1 \leq k \leq n$. We may call $\mathbf{C}_n$ the cyclic graph with n arcs. Note that $\mathbf{C}_1 = 1$.

For any graph X, a *path of length n* is just a graph morphism $\alpha : \mathbf{P}_n \to X$; its *source* $s(\alpha)$ is the image in X of node 0 in $\mathbf{P}_n$; its *target* $t(\alpha)$ is the image in X of node n in $\mathbf{P}_n$. Let $\alpha\beta$ denote the *concatenation* of paths, defined when $t(\alpha) = s(\beta)$. We may denote the set of paths $[\mathbf{P}_n, X]$ by $P_n(X)$.

A *walk* ω in a graph X is just a graph morphism $\omega : \mathbf{N} \to X$; its *source* $s_0(\omega)$ is the image in X of the node 0 in $\mathbf{N}$. Let $N(X) = [\mathbf{N}, X]$ denote the set of walks in X. A graph morphism $f : X \to Y$ gives a natural function $N(f) : N(X) \to N(Y)$ by $\omega \mapsto f \circ \omega$, giving a functor from **Gph** to **Set**.

But $N(X)$ also has a natural *shift operation*, as follows. Let $\sigma : \mathbf{N} \to \mathbf{N}$ denote the graph morphism given on nodes by $\sigma(n) = n + 1$. Let the shift operation $\tau : N(X) \to N(X)$ be given by $\omega \mapsto \omega \circ \sigma$ for $\omega \in N(X)$: the shift of a walk just deletes the first arc in the walk. For any graph morphism f the function $N(f)$ preserves τ, in that $N(f) \circ \tau = \tau \circ N(f)$. So $N(X)$ is naturally an *N-set*, and N is a functor from **Gph** to **NSet**, in the following sense.

2.1. Definition. An *N-set* is a pair (S, τ) with τ a function from S to S; and a *map of N-sets* from (S, τ) to (S', τ') is a function $f : S \to S'$ such that $\tau' \circ f = f \circ \tau$. Let **NSet** denote the category of N-sets, with functor $N : \mathbf{Gph} \to \mathbf{NSet}$. An *N-equivalence* is a graph morphism $f : X \to Y$ for which $N(f) : N(X) \to N(Y)$ is an isomorphism of N-sets.

There is a more general point of view about the category **NSet**. Let G be a monoid, with associative binary operation $G \times G \to G : (g, h) \mapsto g * h$ and with neutral element e; a G-set is a set S together with an *action*, that is a function $\mu : G \times S \to S$ such that $\mu(e, x) = x$ and $\mu(g, \mu(h, x)) = \mu(g * h, x)$. For any monoid G, the category of G-sets is a presheaf category, and thus a topos; see Mac Lane and Moerdijk [1994], for instance. Then **NSet** can be viewed as the category of N-sets, where N is the monoid of natural numbers, under addition; a set S together with an arbitrary function $\tau : S \to S$ corresponds exactly to an action of the monoid N, by $\mu(n, x) = \tau^n(x)$ for $n \in N$. Thus we can view **NSet** as a presheaf topos, with all products, and all coproducts (sums) formed "elementwise", etc.

2.2. Definition. The *arc graph* $A(X)$ of a graph X is the graph with the arcs of X as its nodes, and with length 2 paths in X as its arcs; and with source and target given by $s(a_1, a_2) = a_1$ and $t(a_1, a_2) = a_2$. Let $s_{1,0} : A(X) \to X$ denote the graph morphism given on nodes by $a \mapsto s(a)$, and on arcs by $(a', a) \mapsto a'$. This is a graph morphism since each arc (a', a), from node a' to node a in $A(X)$, maps to the arc a', from node $s(a')$ to node $s(a) = t(a')$ in X.

The arc graph is sometimes called "the line digraph" or "the line graph for directed graphs"; see for instance Kotani and Sunada [2000], where it is used in connection with zeta series. In Section 5 we show that $s_{1,0} : A(X) \to X$ is an N-equivalence, as part of a more general analysis. There we also generalize $s_{1,0}$ to a family $s_{n,m}$ of N-equivalences.

Here are some examples of arc graphs. We have $A(\mathbf{P}_n) = \mathbf{P}_{n-1}$; in particular, $A(\mathbf{D}) = 0$ and $A(\mathbf{A}) = \mathbf{D}$. Also, $A(\mathbf{N}) = \mathbf{N}$ and $A(\mathbf{Z}) = \mathbf{Z}$, and $A(\mathbf{C}_n) = \mathbf{C}_n$; in particular, $A(1) = 1$. For any set S, let $\mathbf{B}(S)$ denote the "bouquet of loops" with one node and with S as its set of arcs. Then $A(\mathbf{B}(S)) = \mathbf{K}(S)$ is the "very complete graph" with nodes S and arcs S^2, and with exactly one arc between any two nodes (including a unique loop from each node to itself). The equal signs above are really denoting natural isomorphisms, of course.

Not every graph arises as an arc graph; for instance, $A(X)$ is always a graph with no parallel arcs (where two arcs a and a' with $s(a) = s(a')$ and $t(a) = t(a')$ are said to be *parallel*).

3. A model structure for N-equivalence of graphs.

In two previous papers (Bisson, Tsemo [2008], [2011]) we developed a Quillen model structure on the category **Gph**, based on the set of cycles in a graph; we may refer to this as the C_*-equivalence model, since here we will develop a different (simpler) Quillen model structure for **Gph**, based on the set of walks in a graph.

We will use the following convenient terminology to explain Quillen model structures. Let $\ell : X \to Y$ and $r : A \to B$ be morphisms in a category $\mathcal{E}$. We say that ℓ is *weak orthogonal* to r (abbreviated by $\ell \dagger r$) when all squares with r on the right and ℓ on the left can be filled:

$$\text{if } \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\ell"'] & A \arrow[d, "r"] \\ Y \arrow[r, "g"'] & B \end{tikzcd} \quad \text{commutes, then} \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\ell"'] & A \arrow[d, "r"] \\ Y \arrow[r, "g"'] \arrow[ur, "h"] & B \end{tikzcd} \quad \text{commutes for some } h.$$

Given a class $\mathcal{F}$ of morphisms we define $\mathcal{F}^\dagger = \{r : f \dagger r,\ \forall f \in \mathcal{F}\}$ and ${}^\dagger\mathcal{F} = \{\ell : \ell \dagger f,\ \forall f \in \mathcal{F}\}$. A *weak factorization system* in $\mathcal{E}$ is given by two classes $\mathcal{L}$ and $\mathcal{R}$, such that $\mathcal{L}^\dagger = \mathcal{R}$ and $\mathcal{L} = {}^\dagger\mathcal{R}$ and such that, for any morphism c in $\mathcal{E}$, there exist $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$ with $c = r \circ \ell$.

We may express Quillen's notion [1967] of "model category structure" via the following axioms, which we learned from Section 7 of Joyal and Tierney [2007].

3.1. Definition. A *model structure* on a category $\mathcal{E}$ with finite limits and colimits is a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ of classes of morphisms in $\mathcal{E}$ which satisfy

- "three for two": if two of the three morphisms $a, b, a \circ b$ belong to $\mathcal{W}$ then so does the third,

- the pair $(\underline{\mathcal{C}}, \mathcal{F})$ is a weak factorization system (where $\underline{\mathcal{C}} = \mathcal{C} \cap \mathcal{W}$),
- the pair $(\mathcal{C}, \underline{\mathcal{F}})$ is a weak factorization system (where $\underline{\mathcal{F}} = \mathcal{W} \cap \mathcal{F}$).

For instance, the *trivial* model structure (for any suitable category $\mathcal{E}$) is given by the triple $(\mathrm{All}, \mathrm{Iso}, \mathrm{All})$.
The morphisms in $\mathcal{W}$ are called *weak equivalences*. The morphisms in $\mathcal{C}$ are called *cofibrations*, and the morphisms in $\underline{\mathcal{C}}$ are called *acyclic cofibrations*. The morphisms in $\mathcal{F}$ are called *fibrations*, and the morphisms in $\underline{\mathcal{F}}$ are called *acyclic fibrations*. An object X in $\mathcal{E}$ is called *cofibrant* when $0 \to X$ is in $\mathcal{C}$ (a cofibration), where 0 is an initial object. Dually, X is called *fibrant* when $X \to 1$ is in $\mathcal{F}$ (a fibration), where 1 is a terminal object.

We will show that the following three morphism classes give a model structure on the category **Gph**:

- the fibrations are $\mathcal{F}_N = \mathrm{All}$, the collection of all graph morphisms,
- the weak equivalences are $\mathcal{W}_N$, the collection of all N-equivalences, and
- the cofibrations are $\mathcal{C}_N = {}^{\dagger}\mathcal{W}_N$.

In an Appendix we give a direct proof, using a "small object" argument, that $(\mathcal{C}_N, \mathcal{W}_N, \mathcal{F}_N)$ is a model structure on **Gph**. We may call it the *N-equivalence model structure* on **Gph**; the subscripts here are optional, but serve to distinguish these classes from the C_*-equivalence model structure from Bisson and Tsemo [2008], [2011]).

In this section we will show that $(\mathcal{C}_N, \mathcal{W}_N, \mathcal{F}_N)$ is a model structure, by identifying it with a "transport" of the trivial model structure from the category **NSet**. This will also show that the N-equivalence model structure is *cofibrantly generated*. The transport will be along an *adjunction* (pair of adjoint functors) between **Gph** and **NSet**; Section 2.1 in Hovey [1999], for example, has a nice discussion of cofibrant generation, and other concepts which will be used in the following.

Let $\mathcal{E}$ be a category with all limits and colimits. Briefly, a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on $\mathcal{E}$ is *cofibrantly generated* when there are sets I and J of morphisms which generate $\mathcal{C}$ and $\underline{\mathcal{C}}$, in the sense that ${}^{\dagger}(I^{\dagger}) = \mathcal{C}$ and ${}^{\dagger}(J^{\dagger}) = \underline{\mathcal{C}}$; thus we also have $I^{\dagger} = \underline{\mathcal{F}}$ and $J^{\dagger} = \mathcal{F}$. For a set H of morphisms in $\mathcal{E}$, let $\mathrm{cell}(H)$ denote the class of all transfinite compositions of pushouts of morphisms in H; the morphims in $\mathrm{cell}(H)$ are called *relative H-cell complexes*. For background and references on the proof of the following general result, see Berger and Moerdijk [2003], for instance.

Transport Theorem: Let $\mathcal{E}$ be a model category which is cofibrantly generated, with cofibrations generated by I and acyclic cofibrations generated by J. Let $\mathcal{E}'$ be a category with all limits and colimits, and suppose that we have an adjunction

$$L : \mathcal{E} \rightleftharpoons \mathcal{E}' : R \quad \text{with} \quad R(\mathrm{cell}\ L(J)) \subseteq \mathcal{W}.$$

Also, assume that the sets $L(I)$ and $L(J)$ each permit the small object argument. Then there is a cofibrantly generated model structure on $\mathcal{E}'$ with generating cofibrations $L(I)$

and generating acyclic cofibrations $L(J)$. Moreover, the model structure $(\mathcal{C}', \mathcal{W}', \mathcal{F}')$ satisfies $f \in \mathcal{W}'$ iff $R(f) \in \mathcal{W}$, and $f \in \mathcal{F}'$ iff $R(f) \in \mathcal{F}$.

We apply the transport theorem with $\mathcal{E}$ as the category **NSet**, and with $\mathcal{E}'$ as the category **Gph**. We use an adjunction

$$D : \mathbf{NSet} \rightleftharpoons \mathbf{Gph} : N$$

which plays a central role throughout this paper. We have already defined the functor N. For any N-set (S, τ), let $X = D(S, \tau)$ denote the graph with nodes $X_0 = S$ and arcs $X_1 = S$, where the source and target functions $s, t : X_1 \to X_0$ are given by $s(x) = x$ and $t(x) = \tau(x)$ for each $x \in S$. Thus the elements in the N-set S give the nodes and the arcs in the graph X, and each arc x has target $\tau(x)$ and source x; we think of $\tau(x)$ as telling the unique “target” of each element x in the N-set S.

It is easy to check directly that (D, N) is an adjoint pair of functors; the adjunction is also proved in Bisson and Tsemo [2011], but there we used the functor from **NSet** to **Gph** which assigned to (S, τ) the graph directed opposite to $D(S, \tau)$. Here we are directing our arcs in the way that seems natural in graphical representation of dynamical systems (see Article III in Lawvere and Schanuel [1997], for instance).

3.2. Proposition. *The trivial model structure on* **NSet***, when transported along the adjunction* (D, N)*, gives the N-equivalence model structure* $(\mathcal{C}_N, \mathcal{W}_N, \mathcal{F}_N)$ *on* **Gph**. *This model structure is cofibrantly generated by* $\mathbf{I} = \{\mathbf{i}, \mathbf{j}\}$ *and by* $\mathbf{J} = \{\mathbf{0}\}$*, where* $\mathbf{i} : 0 \to \mathbf{N}$ *and* $\mathbf{j} : \mathbf{N} + \mathbf{N} \to \mathbf{N}$ *are the initial and co-diagonal graph morphisms, and* $\mathbf{0}$ *is the identity graph morphism* $\mathbf{0} : 0 \to 0$.

Proof. First we make precise our terminology for morphisms $\mathbf{i}$ and $\mathbf{j}$. Any object X in a category with coproducts has *initial* morphism $0 \to X$ (where 0 is the initial object), and *co-diagonal* morphism $X + X \to X$ (the morphism from the coproduct $X + X$ determined by the pair of identity morphisms). The category of N-sets has coproducts; the initial object 0 is the empty set. We (temporarily) let N denote the N-set of natural numbers with shift map $\tau(n) = n + 1$, and consider the sets $I = \{\mathrm{i}, \mathrm{j}\}$ and $J = \{0\}$ of N-set maps, with initial N-set maps $0 : 0 \to 0$ and $\mathrm{i} : 0 \to \mathrm{N}$, and co-diagonal N-set map $\mathrm{j} : \mathrm{N} + \mathrm{N} \to \mathrm{N}$. We have $J^\dagger = \mathrm{All}$, so that ${}^\dagger(J^\dagger) = \mathrm{Iso}$; and we have $I^\dagger = \mathrm{Iso}$, so that ${}^\dagger(I^\dagger) = \mathrm{All}$. This shows that the trivial model structure on **NSet** is cofibrantly generated. The smallness conditions in the Transport Theorem are automatically satisfied in our presheaf categories (see the proof at Example 2.1.5 in Hovey [1999], for instance). Now, let $\mathbf{I} = D(I)$ and $\mathbf{J} = D(J)$; then $\mathbf{I} = \{\mathbf{i}, \mathbf{j}\}$ and $\mathbf{J} = \{\mathbf{0}\}$. So, every morphism in $\mathrm{cell}(\mathbf{J})$ is a graph isomorphism, and the Transport Theorem applies, since we have $f \in \mathrm{cell}\, D(J)$ implies $N(f) \in \mathcal{W}$. We immediately have $\mathbf{J}^\dagger = \mathrm{All} = \mathcal{F}_N$ and ${}^\dagger(\mathbf{J}^\dagger) = \mathrm{Iso} = \underline{\mathcal{C}}_N$. Moreover, the definitions (in terms of filling conditions) show that $\mathbf{I}^\dagger = \mathcal{W}_N = \underline{\mathcal{F}}_N$, so that ${}^\dagger(\mathbf{I}^\dagger) = {}^\dagger\mathcal{W}_N = \mathcal{C}_N$. It follows that our morphism classes $(\mathcal{C}_N, \mathcal{W}_N, \mathcal{F}_N)$ are cofibrantly generated by $\mathbf{I}$ and $\mathbf{J}$. ∎

As part of the definition of the N-equivalence model structure, every graph morphism is a fibration. In particular, every graph is fibrant. We can use the transport definition of the model structure to get partial information about the class of cofibrations. The following seems convenient.

3.3. Definition. A graph X is a *dynamic graph* when every node in X has exactly one arc leaving it. Let **DGph** denote the full subcategory of dynamic graphs.

Thus the dynamic graphs are those which are isomorphic to $D(S,\tau)$ for some N-set (S,τ).

3.4. Proposition. *For the N-equivalence model structure on category* **Gph**, *every graph morphism between dynamic graphs is a cofibration. In particular, every dynamic graph is cofibrant.*

Proof. Let I denote the set $\{\mathrm{i},\mathrm{j}\}$ of N-set maps, as in the proof of the previous proposition. We showed there that the cofibrations in our N-equivalence model are generated by the set $D(I)$ of morphisms in **Gph**, so that $\mathrm{cell}(D(I)) \subseteq \mathcal{C}_N$. Since the functor D is a left adjoint, it preserves all colimits; so $D(\mathrm{cell}(I)) \subset \mathrm{cell}(D(I))$. But every map $f : S \to T$ of N-sets is in $\mathrm{cell}(I)$, as follows: let $S' = S + \sum_{x\in T} \mathrm{N}$; then $S \to S'$ is a pushout of a sum of copies of i; and $S' \to T$ is a pushout of copies of j (this is just like the argument that all functions between sets are in $\mathrm{cell}(\{1+1 \to 1, 0 \to 1\})$). It follows that $D(\mathrm{cell}(I))$ is the class of graph morphisms between dynamic graphs, and these are cofibrations. ∎

Let us make explicit some aspects of the adjunction (D,N). Each graph morphism $D(S,\tau) \to X$ corresponds to an adjoint N-set map $(S,\tau) \to N(X)$. For every N-set (S,τ) the unit $(S,\tau) \to N(D(S,\tau))$, which is adjoint to the identity morphism $D(S,\tau) \to D(S,\tau)$, is an isomorphism of N-sets; there is a unique walk starting at each node in a dynamic graph, and every N-set map which is a bijection is an N-set isomorphism.

For any graph X the counit $D(N(X)) \to X$ is the graph morphism adjoint to the identity N-set map $N(X) \to N(X)$. This comes up often in what follows, so we give it a name.

3.5. Definition. The *walk graph* $W(X)$ of a graph X is the dynamic graph which has the walks in X as both its nodes and its arcs, with $s(\omega) = \omega$ and $t(\omega) = \tau(\omega)$, for ω any walk in X. Let $s_0 : W(X) \to X$ denote the graph morphism which, on nodes assigns to each walk ω its first node; and on arcs assigns to ω its first arc. We may refer to s_0 as the *source truncation.*

We identify $D(N(X))$ with the walk graph $W(X)$, and identify s_0 with the counit of the (D,N) adjunction.

3.6. Proposition. *For any graph X, the graph $W(X)$ is cofibrant and the graph morphism $s_0 : W(X) \to X$ is an N-equivalence. Also, $W(f) : W(Y) \to W(X)$ is a graph isomorphism for any N-equivalence $f : Y \to X$.*

PROOF. Since $W(X)$ is a dynamic graph, it is cofibrant. Also $W(X) \to X$ is an N-equivalence, since $N(W(X)) = N(D(N(X)) = N(X)$, through the identification $N(D(S, \tau)) = (S, \tau)$ for every N-set (S, τ). The second statement follows from the fact that $D(N(f))$ is an isomorphism when $N(f)$ is an isomorphism. ∎

The above proposition shows that $W : \mathbf{Gph} \to \mathbf{Gph}$ is the *coreflection* of **Gph** into the full subcategory **DGph**. See Mac Lane [1971] for definitions of the general concepts. Results in Bisson and Tsemo [2011] show, essentially, that **DGph** is a full reflective and coreflective subcategory of **Gph**.

3.7. COROLLARY. *The dynamic graphs are the cofibrant objects for the N-equivalence model structure on graphs.*

PROOF. We have already shown that every dynamic graph is cofibrant. For the converse, suppose that graph X is a cofibrant graph. Since $s_0 : W(X) \to X$ is an N-equivalence, we have a filling f for the diagram

$$\begin{tikzcd} 0 \arrow[r, "f"] \arrow[d, "\ell"'] & W(X) \arrow[d, "s"] \\ X \arrow[r, "\mathrm{id}"'] \arrow[ur, "f"] & X \end{tikzcd}$$

This implies that s is an epic graph morphism and that f is a monic graph morphism. Suppose that X is not a dynamic graph; then the set $X(x, *)$ of arcs leaving some node x in X has cardinality other than one. But $X(x, *)$ can't be empty, since then there would be no walk in X leaving x, and x would not be in the image of $s_0 : W(X) \to X$, which contradicts s being epic. So $X(x, *)$ must have more than one element. But $W(X)$ is a dynamic graph, so f must map every arc in $X(x, *)$ to the unique arc leaving $f(x)$ in $W(X)$, which contradicts f being monic. ∎

4. The N-equivalence homotopy category.

The purpose of giving a model structure on a category $\mathcal{E}$ is to construct and study a new category $\mathrm{Ho}(\mathcal{E})$ which inverts the weak equivalences of the model category. Let us explain.

Suppose that $\mathcal{E}$ is a model category. A functor with domain $\mathcal{E}$ is said to be a *homotopy functor* when it takes every $f \in \mathcal{W}$ to an isomorphism. This involves just the class $\mathcal{W}$ of weak equivalences in the model structure. Quillen [1967] used the classes $\mathcal{C}$ and $\mathcal{F}$ to describe a particular category $\mathrm{Ho}(\mathcal{E})$, together with a functor $\gamma : \mathcal{E} \to \mathrm{Ho}(\mathcal{E})$ which is *initial* among homotopy functors on $\mathcal{E}$. This means that γ is a homotopy functor and that any homotopy functor $\Phi : \mathcal{E} \to \mathcal{D}$ factors uniquely through γ, in that $\Phi = \Phi' \circ \gamma$ for a unique functor $\Phi' : \mathrm{Ho}(\mathcal{E}) \to \mathcal{D}$.

In fact, Quillen constructs the category $\mathrm{Ho}(\mathcal{E})$ to have the same objects as $\mathcal{E}$, and describes the set $\mathrm{Ho}(X, Y)$ of "homotopy arrows" from X to Y in $\mathrm{Ho}(\mathcal{E})$, for any objects

X and Y in $\mathcal{E}$. His construction uses the following notions. A *cofibrant replacement* for an object X in $\mathcal{E}$ is a morphism $f : X' \to X$ where X' is cofibrant and f is a weak equivalence and a fibration ($f \in \underline{\mathcal{F}} = \mathcal{W} \cap \mathcal{F}$). Dually, a *fibrant replacement* for X is a morphism $g : X \to X''$ where X'' is fibrant and g is a weak equivalence and a cofibration ($g \in \underline{\mathcal{C}} = \mathcal{W} \cap \mathcal{C}$). It follows from the model category axioms that each object in $\mathcal{E}$ has a cofibrant replacement and a fibrant replacement.

The homotopy functor $\gamma : \mathcal{E} \to \mathrm{Ho}(\mathcal{E})$ carries morphisms in $\mathcal{E}$ to homotopy arrows in $\mathrm{Ho}(\mathcal{E})$, but there are usually homotopy arrows in $\mathrm{Ho}(\mathcal{E})$ which are not equal to $\gamma(f)$ for any morphism f in $\mathcal{E}$. So morphisms in $\mathcal{E}$ may become invertible in $\mathrm{Ho}(\mathcal{E})$, and objects which are not isomorphic in $\mathcal{E}$ may become isomorphic in $\mathrm{Ho}(\mathcal{E})$. We may say that two objects X and Y in $\mathcal{E}$ are *homotopy-equivalent* when X and Y become isomorphic in $\mathrm{Ho}(\mathcal{E})$; and that a morphism $f : X \to Y$ in $\mathcal{E}$ is a *homotopy equivalence* when $\gamma(f)$ becomes invertible in $\mathrm{Ho}(\mathcal{E})$. Also, we may say that morphisms $f, g : X \to Y$ in $\mathcal{E}$ are *homotopic* when they become equal in $\mathrm{Ho}(\mathcal{E})$, with $\gamma(f) = \gamma(g)$.

Let us see how these ideas work out for our N-equivalence model structure on **Gph**. Recall that every graph morphism is a fibration and that every graph is fibrant; every graph is its own fibrant replacement. Moreover, our results at the end of Section 3 show that the natural graph morphism $s_0 : W(X) \to X$ gives a cofibrant replacement for every graph X.

Now we are ready to describe precisely the various notions of homotopy for the N-equivalence model structure on **Gph**.

4.1. Proposition.

a) $N : \mathbf{Gph} \to \mathbf{NSet}$ *induces an equivalence of categories* $\mathrm{Ho}(\mathbf{Gph}) \to \mathbf{NSet}$.

b) *Graphs X and Y are homotopy-equivalent if and only if the N-sets $N(X)$ and $N(Y)$ are isomorphic.*

c) *A graph morphism f is a homotopy equivalence if and only if it is an N-equivalence.*

Proof. For the first statement, we note that the functor $N : \mathbf{Gph} \to \mathbf{NSet}$ factors through $\gamma : \mathbf{Gph} \to \mathrm{Ho}(\mathbf{Gph})$, and $N : \mathrm{Ho}(\mathbf{Gph}) \to \mathbf{NSet}$ gives the desired equivalence. Note that the unit $N(D(S, \tau)) \to (S, \tau)$ is already an isomorphism and it is only necessary to recall that the N-equivalence $W(X) \to X$ can be viewed as the counit $D(N(X)) \to X$. The second statement follows from the first: objects X and Y are isomorphic in $\mathrm{Ho}(\mathbf{Gph})$ if and only if $N(X)$ and $N(Y)$ are isomorphic in **NSet**. For the third statement, we use the following general result. From Quillen's description of the category $\mathrm{Ho}(\mathcal{E})$, for any model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$, it follows that $\gamma(f)$ is invertible in $\mathrm{Ho}(\mathcal{E})$ if and only if f is in $\mathcal{W}$ (see Hovey [1999], Theorem I.2.10, for instance). So, a graph morphism $f : X \to Y$ has $\gamma(f)$ invertible in $\mathrm{Ho}(\mathbf{Gph})$ if and only if $N(f)$ is an isomorphism of N-sets; and these N-equivalences are taken to form the class $\mathcal{W}_N$ of weak equivalences for our N-model structure on **Gph**. So f is an N-equivalence if and only if it is a homotopy equivalence. ∎

For any graph X, consider the subgraph of X which is the image of the natural graph morphism $s_0 : W(X) \to X$. We will call it the *walkable subgraph* of X. We use the following lemma about the graph morphism $s_0 : W(X) \to X$ to characterize when two graph morphisms are homotopic.

4.2. LEMMA. *The natural map $s_* : [W(X), W(Y)] \to [W(X), Y]$ given by $s_*(f) = s_0 \circ f$ is a bijection.*

PROOF. The adjoint pair (D, N) gives a natural bijection

$$\mathbf{NSet}[N(X), N(Y)] \cong [D(N(X)), Y].$$

We showed that the counit of the adjoint pair (D, N) gives a natural identification between $N \circ D$ and the identity functor; it follows that the functor D gives a natural bijection

$$\mathbf{NSet}[N(X), N(Y)] \cong [D(N(X)), D(N(Y))].$$

Recall that $W = D \circ N$. The resulting bijection $[D(N(X)), D(N(Y))] \cong [D(N(X)), Y]$ can be identified with $s_* : [W(X), W(Y)] \to [W(X), Y]$. ∎

4.3. PROPOSITION. *Graph morphisms $f, g : X \to Y$ are homotopic if and only if they agree on the walkable subgraph of X.*

PROOF. We have shown that f and g are homotopic if and only $N(f) = N(g)$. The lemma shows that $N(f) = N(g)$ if and only if the graph morphisms $s_0 \circ W(f), s_0 \circ W(g) : W(X) \to Y$ are equal. Let $s : W(X) \to w(X)$ denote the epic graph morphism onto the image $w(X)$ of the graph morphism $s_0 : W(X) \to X$. Then $s_0 \circ W(f) = f_| \circ s$, where $f_|$ denotes f restricted to $w(X)$. So, $s_0 \circ W(f) = s_0 \circ W(g)$ if and only if $f_| \circ s = g_| \circ s$, which is equivalent to $f_| = g_|$ since s is epic. ∎

By the above, any graph is homotopy equivalent to its walkable subgraph. So, if a graph X has no walks, then $N(X)$ is empty, and the walkable subgraph of X is empty; in this case, X is homotopy equivalent to 0, and any two graph morphisms from X to Y are homotopic (for any graph Y). In particular, the graphs 0 and 1 are homotopy equivalent. In fact, X and Y may be homotopy equivalent even when there is no graph morphism between them, in either direction. For example, let X have nodes x, x_1, x_2 with arcs a_i from x to x_i; let Y have nodes y, y_1, y_2 with arcs b_i from y_i to y.

Recall that a functor F defined on **Gph** will be a homotopy functor for the N-equivalence model structure if and only if $F(f) : F(X) \to F(Y)$ is an isomorphism whenever $f : X \to Y$ is an N-equivalence. For instance, the functor $\gamma : \mathbf{Gph} \to \mathrm{Ho}(\mathbf{Gph})$ is initial among homotopy functors; and it is equivalent to the functor $N : \mathbf{Gph} \to \mathbf{NSet}$. This also shows that the cofibrant replacement functor $W : \mathbf{Gph} \to \mathbf{NSet}$ is a homotopy functor, since W is $D \circ N$, and composing a homotopy functor with another functor gives a homotopy functor.

Many natural functors from **Gph** to **Set** are *not* homotopy functors. For instance $X \mapsto [\mathbf{D}, X] = X_0$ is not a homotopy functor, since 0 and 1 are homotopy equivalent graphs, but $[\mathbf{D}, 0] \neq [\mathbf{D}, 1]$. Similar reasoning applies to $X \mapsto \pi_0(X)$, the set of components of the graph X, formed as the coequalizer of the functions $s, t : X_1 \to X_0$. But dynamic graphs give representable homotopy functors, as follows.

4.4. PROPOSITION. *Let F be a dynamic graph:*

a) there is a natural graph morphism $\sigma : F \to F$ determined by $s(\sigma(a)) = t(a)$ on arcs;

b) the functor from **Gph** *to* **NSet** *given by $X \mapsto ([F, X], \sigma^*)$, with $\sigma^*(f) = f \circ \sigma$, is a homotopy functor.*

PROOF. We may identify $F = D(S, \tau)$ for some N-set (S, τ). The function τ is in fact an N-set map $\tau : (S, \tau) \to (S, \tau)$, and gives a graph morphism $D(\tau) : D(S, \tau) \to D(S, \tau)$. This gives $\sigma : F \to F$, and a functor F from **Gph** to **NSet**, with $F(X) = ([F, X], \sigma^*)$. We must show that if a graph morphism $f : X \to Y$ is an N-equivalence then $F(f)$ is an isomorphism of N-sets. But $F = D(S, \tau)$, and the adjunction (D, N) shows that $F(X)$ can be identified with the set of N-set maps from (S, τ) to $N(X)$, so that the functor $X \mapsto F(X)$ factors through $N : \mathbf{Gph} \to \mathbf{NSet}$. ■

For example, the functor $Z : \mathbf{Gph} \to \mathbf{NSet}$ given by $X \mapsto [\mathbf{Z}, X]$ is a homotopy functor, since $\mathbf{Z} = D(Z, +1)$ is the dynamic graph with nodes the integers. We may refer to elements of $[\mathbf{Z}, X]$ as two-way walks in X.

As another example, for any $n > 0$ the functor $\mathbf{Gph} \to \mathbf{NSet}$ given by $X \mapsto [\mathbf{C}_n, X]$ is a homotopy functor, since $\mathbf{C}_n = D(Z/n, +1)$ is the dynamic graph with nodes the integers mod n. It follows that the functors $C_n : \mathbf{Gph} \to \mathbf{Set}$, with $C_n(X) = [\mathbf{C}_n, X]$, are homotopy functors.

We refer to elements of $[\mathbf{C}_n, X]$ as cycles of length n in X; they can be identified with the set of $\omega \in N(X)$ such that $\tau^n(\omega) = \omega$. For a *finite graph* X (finitely many nodes and arcs), the *zeta series* of X is the formal power series

$$\mathrm{Zeta}(u) = \exp\left(\sum_{m=1}^{\infty} c_m \frac{u^m}{m}\right),$$

where $c_m = |C_m(X)|$ for $m > 0$. Clearly, if X and Y are N-equivalent finite graphs then they have the same zeta series.

In Bisson and Tsemo [2011], we studied a model structure whose weak equivalences were the graph morphisms $f : X \to Y$ for which $C_n(f) : C_n(X) \to C_n(Y)$ is a bijection for every $n > 0$. Here we may call this the *cyclic model structure* on **Gph**. Our main result in that paper said that finite graphs X and Y have the same zeta series if and only if they are homotopy equivalent in the cyclic model structure. Let us write $X \sim_C Y$ for this situation, and write $X \sim_N Y$ when X and Y are homotopy equivalent for the N-equivalence model structure.

4.5. PROPOSITION. *If X and Y are finite graphs, then $X \sim_N Y$ implies $X \sim_C Y$.*

PROOF. If $X \sim_N Y$ then there is an isomorphism of N-sets $\phi : N(X) \to N(Y)$. For each $n > 0$ this restricts to give a bijection $\phi : C_n(X) \to C_n(Y)$. These are finite sets if X and Y are finite graphs; and then we have $c_n(X) = c_n(Y)$ for all $n > 0$. Thus X and Y have the same zeta series, so that we have $X \sim_C Y$. ∎

In Section 7 we give an example of finite graphs X and Y which have the same zeta series but are not N-equivalent, so that we have $X \sim_C Y$ but not $X \sim_N Y$.

5. Arc graphs and finite-level homotopy.

In Section 2 we defined the arc graph $A(X)$ for any graph X. In Section 3 we defined the walk graph $W(X)$ and showed that it provides a cofibrant replacement for the N-equivalence model structure. Here we extend and relate these constructions, by the following general considerations.

Any pair of arrows $i_s, i_t : E_0 \to E_1$ in a category $\mathcal{E}$ gives a *representable functor* $E_* : \mathcal{E} \to \mathbf{Gph}$, as follows. Let $\mathcal{E}[E', E]$ denote the set of morphisms from object E' to object E in the category $\mathcal{E}$. For any object $E \in \mathcal{E}$, let $E_*(X)$ be the graph with node $E_0(X) = \mathcal{E}[E_0, E]$ and with arcs is $E_1(X) = \mathcal{E}[E_1, E]$; the source and target of arcs $\alpha : E_1 \to E$ in $E_1(X)$ are given by $s(\alpha) = \alpha \circ i_s$ and $t(\alpha) = \alpha \circ i_t$.

For instance, our cofibrant replacement functor $W : \mathbf{Gph} \to \mathbf{Gph}$ comes in this way from $i_s, i_t : \mathbf{N} \to \mathbf{N}$ in $\mathbf{Gph}$, where i_s is the identity graph morphism and i_t is the shift graph morphism.

For each $n \geq 0$ we define a functor $A^n : \mathbf{Gph} \to \mathbf{Gph}$ by the pair $i_s, i_t : \mathbf{P}_n \to \mathbf{P}_{n+1}$, where the graph morphisms are given on nodes by $i_s(k) = k$ and by $i_t(k) = k + 1$. For $n = 0, 1$ we have natural isomorphisms $A^0(X) = X$ and $A^1(X) = A(X)$, from $\mathbf{D} = \mathbf{P}_0$ and $\mathbf{A} = \mathbf{P}_1$.

Returning to general considerations, suppose that $i'_s, i'_t : E'_0 \to E'_1$ in $\mathcal{E}$ is giving another representable graph functor. A representable natural transformation from functor E_* to functor E'_* is given by any pair $f_0 : E_0 \to E'_0$ and $f_1 : E_1 \to E'_1$ of arrows in $\mathcal{E}$, such that $f_1 \circ i_s = i'_s \circ f_0$ and $f_1 \circ i_t = i'_t \circ f_0$.

For instance, the natural graph morphism $s_0 : W(X) \to X$ comes from $f_0 : \mathbf{P}_0 \to \mathbf{N}$ and $f_1 : \mathbf{P}_1 \to \mathbf{N}$. More generally, for each $n \geq 0$ we have natural transformations $s_n : W(X) \to A^n(X)$ given by $f_0 : \mathbf{P}_n \to \mathbf{N}$ and $f_1 : \mathbf{P}_{n+1} \to \mathbf{N}$; and for $n, m \geq 0$ we have natural transformations $s_{m,n} : A^{n+m}(X) \to A^n(X)$ given by $f_0 : \mathbf{P}_n \to \mathbf{P}_{n+m}$ and $f_1 : \mathbf{P}_{n+1} \to \mathbf{P}_{n+m+1}$. In all these cases, the graph morphisms f_i are determined by the condition that they take node 0 to node 0. We may call s_n and $s_{m,n}$ the length n "source truncations". In particular, we have $s_{m,0} : A^m(X) \to X$ and $s = s_0 : W(X) \to X$.

5.1. PROPOSITION. *For any graph X we may identify:*

1) $s_{m,n} \circ s_{n+m} = s_n : W(X) \to A^n(X)$,

2) $s_{m+k,n} = s_{m,n} \circ s_{k,n+m} : A^{n+m+k}(X) \to A^n(X)$,

3) $W(X) = \lim_n A^n(X)$,

4) $A^n(A^m(X)) = A^{n+m}(X)$, *and*

5) $A^n(W(X)) = W(X) = W(A^n(X))$

Proof. For parts 1 and 2, we check compatibility of the representing graph morphisms. For part 3, we verify the universal limit condition for the representing graph morphisms $\mathbf{P}_n \to \mathbf{N}$. For part 4, we use that every path of length $n+m$ is uniquely the concatenation of a path of length n and a path of length m. The following lemma shows that the natural graph morphisms $W(s_{n,0}) : W(A^n(X)) \to W(X)$ and $s_{n,0} : A^n(W(X)) \to W(X)$ are graph isomorphisms, proving part 5:

5.2. Lemma. *If Y is a dynamic graph then $s_n : W(Y) \to A^n(Y)$ and $s_{m,n} : A^{n+m}(Y) \to A^n(Y)$ are graph isomorphisms.*

Proof. We use the fact that a graph morphism between dynamic graphs is a graph isomorphism if and only if it is bijective on nodes. This is true since any graph morphism between dynamic graphs has the form $D(f)$ for some N-set map $f : S_1 \to S_2$; but an N-set map is an isomorphism if and only if it is a bijection on elements, and elements in S correspond to nodes in $D(S)$. Then we note that s_0 is a graph morphism between dynamic graphs; and it is clearly bijective on nodes. The other parts are similar. ■

■

By part 3 of the proposition, we may think of A^n as an iterated composition of the functor A with itself, and we may refer to $A^n(X)$ as the *n-fold, or length n, arc graph* on X. We also extend our examples of N-equivalences as follows.

5.3. Corollary. *The natural graph morphisms $s_n : W(X) \to A^n(X)$ and $s_{m,n} : A^{m+n}(X) \to A^n(X)$ are N-equivalences.*

Proof. We can see that $s_n : W(X) \to A^n(X)$ is an N-equivalence by identifying it with $W(A^n(X)) \to A^n(X)$ (using $W(A^n(X)) = W(X)$). Then $A^n(X) \to X$ is an N-equivalence by the 2/3 property for N-equivalences (this could also be shown by induction on n, of course). Finally, $A^{m+n}(X) \to A^n(X)$ is an N-equivalence, since $W(X) \to A^{m+n}(X)$ and $W(X) \to A^n(X)$ are N-equivalences. ■

Recall that $W(X) \to X$ gives a cofibrant replacement for our model structure, and every graph is its own fibrant replacement. It follows that the homotopy arrows from X to Y are represented by graph morphisms from $W(X)$ to Y. In the following, recall that $s_0 : W(W(X)) \to W(X)$ is a graph isomorphism, for any graph X, so that $\gamma(s_0) : W(X) \to X$ is an isomorphism in $\mathrm{Ho}(\mathbf{Gph})$.

5.4. Definition. For graph morphisms $f : W(X) \to Y$ and $g : W(Y) \to Z$, let $g \odot f$ denote the graph morphism $g \circ W(f) \circ s_0^{-1}$:

$$\begin{tikzcd} W(W(X)) \arrow[r, "W(f)"] \arrow[d, "s_0"'] & W(Y) \arrow[d, "g"] \\ W(X) \arrow[r, "g\odot f"] & Z \end{tikzcd}$$

For any graph morphism $f : W(X) \to Y$, let $\gamma'(f)$ denote $\gamma(f) \circ \gamma(s_0)^{-1}$ in Ho(**Gph**):

$$\begin{tikzcd} W(X) \arrow[r, "\gamma(f)"] \arrow[d, "\gamma(s_0)"'] & Y \\ X \arrow[ur, "\gamma'(f)"'] & \end{tikzcd}$$

5.5. Proposition. *The function γ' gives a bijection between the set of graph morphisms from $W(X)$ to Y, and the set of homotopy arrows from X to Y in Ho(**Gph**). For graph morphisms $f : W(X) \to Y$ and $g : W(Y) \to Z$, we have $\gamma'(g \odot f) = \gamma'(g) \circ \gamma'(f)$ in Ho(**Gph**).*

Proof. We use $[X, Y]$ as notation for the set of graph morphisms from X to Y, etc. In Section 4 we showed the equivalence of Ho(**Gph**) and **NSet**, giving natural bijections

$$[W(X), W(Y)] \cong \mathbf{NSet}[N(X), N(Y)].$$

Let **WGph** denote the category with the same objects as **Gph**, but with the new set of morphisms

$$\mathbf{WGph}[X, Y] = [W(X), W(Y)]$$

for objects X and Y. The functor $\mathbf{Gph} \to \mathbf{WGph}$ given by $f \mapsto W(f)$ is a homotopy functor; in fact, $W(f)$ is a graph isomorphism if and only if $N(f)$ is an isomorphism. It follows that Ho(**Gph**) and **WGph** are isomorphic as categories. The natural bijection

$$[W(X), W(Y)] \cong [W(X), Y]$$

allows us also to describe Ho(**Gph**) as the category whose objects are the graphs, but with morphism sets $\mathrm{Ho}(X, Y) = [W(X), Y]$. Then the homotopy functor $\gamma : \mathbf{Gph} \to \mathrm{Ho}(\mathbf{Gph})$ is described by the natural functions $s_0^* : [X, Y] \to [W(X), Y]$, where $s_0^*(f) = f \circ s_0$. The composition in the category Ho(**Gph**) corresponds to the associative "composition"

$$(f, g) \mapsto g \odot f \qquad [W(X), Y] \times [W(Y), Z] \to [W(X), Z].$$

■

Thus the category Ho(**Gph**) has been described directly in terms of graph morphisms defined on dynamic graphs, which are the cofibrant objects for our model structure. For this reason, we think of the above as giving a "cofibrant description of the homotopy category". As an application, we note that a graph morphism $f : X \to Y$ is a homotopy equivalence if and only if there exists a graph morphism $q : W(Y) \to X$ (thought of as a "homotopy arrow from Y to X"), with $f \odot q = \mathrm{id}$ and $q \odot f = \mathrm{id}$. This says that q makes the following diagram commute:

$$\begin{tikzcd} W(X) \arrow[r, "s"] \arrow[d, "W(f)"'] & X \arrow[d, "f"] \\ W(Y) \arrow[ur, "q"] \arrow[r, "s"] & Y \end{tikzcd}$$

The existence of such a q also shows that $W(f) : W(X) \to W(Y)$ is an isomorphism of graphs.

The above cofibrant description of Ho(**Gph**) suggests the following notion of "finite-level homotopy".

5.6. Definition. Define $\gamma_n : [A^n(X), Y] \to [W(X), Y] = \mathrm{Ho}(X, Y)$ by $\gamma_n(f) = f \circ s_n$, where $s_n : A^n(X) \to X$. A homotopy arrow from X to Y in Ho(**Gph**) is a *homotopy arrow of level* n when it has the form $\gamma_n(f)$ for some graph morphism $f : A^n(X) \to Y$. Letting n vary gives the *finite-level homotopy arrows*.

5.7. Proposition. *The finite-level homotopy arrows form a subcategory of* Ho(**Gph**).

Proof. The identity graph morphisms are homotopy arrows of level 0, by the identification $A^0(X) = X$. Consider the functions $[A^n(X), Y] \times [A^m(Y), Z] \to [A^{m+n}(X), Z]$, defined by $(f, g) \mapsto g \circ A^m(f)$ for $f : A^n(X) \to Y$ and $g : A^m(Y) \to Z$ and $A^m(f) : A^{m+n}(X) \to A^m(Y)$. These give a "composition" which is compatible with the composition in Ho(**Gph**), by the natural graph morphisms from walk graphs to arc graphs. This shows that the finite-level homotopy arrows are closed under composition, and form a subcategory of Ho(**Gph**). ∎

We may call this the *finite-level subcategory* of Ho(**Gph**). Let us say that a graph morphism $f : X \to Y$ is a *level n homotopy equivalence* when there exists a graph morphism $q : A^n(Y) \to X$ which fills the diagram:

$$\begin{tikzcd} A^n(X) \arrow[r, "s"] \arrow[d, "A^n(f)"'] & X \arrow[d, "f"] \\ A^n(Y) \arrow[ur, "q"] \arrow[r, "s"] & Y \end{tikzcd}$$

If f is a level n homotopy equivalence and g is a level m homotopy equivalence then $f \circ g$ is a level $n + m$ homotopy equivalence. Also, if f is a level n homotopy equivalence then f is a level $n + 1$ homotopy equivalence. For example, for every $n, m \geq 0$, the graph morphism $s_{n,m} : A^{n+m}(X) \to A^m(X)$ is a level n homotopy equivalence. In particular, $s_{1,0} : A(X) \to X$ is a level 1 homotopy equivalence.

Recall that any homotopy equivalence of graphs corresponds to an isomorphism of N-sets; and we have picked out a subcategory of finite-level homotopy arrows and finite-level homotopy equivalences. In the next section we show that a finite-level homotopy equivalence corresponds to a special kind of N-isomorphism, called a "topological conjugacy".

6. Symbolic dynamics and topological conjugacy of walk spaces.

In this section we want to relate our results to traditional questions and methods in symbolic dynamics. Symbolic dynamics originated as a tool for studying the sequence of state transitions (through discrete time) in the evolution (or trajectory) of a point in a dynamic system.

The study of dynamical systems often concentrates on a (compact) metrizable space S with a continuous transition map $\tau : S \to S$. This leads to the notion of "topological conjugacy" of such objects (S, τ), as we will discuss below. First we describe the well-known topological and metric structure on the set of walks in any graph.

For $\omega \in N(X)$, let $U_n(\omega)$ denote the set of all walks in $N(X)$ which agree with ω for the first n steps. This set depends only on the path given by the first n steps of ω; more precisely, $U_n(\omega) = U(\alpha)$, where $\alpha = s_n(\omega)$ and $U(\alpha)$ denotes the preimage of α under the source truncation $s_n : N(X) \to P_n(X)$. Note that $U(\alpha)$ is empty unless α is is the source truncation of some walk.

The sets $U(\alpha)$ are the "cylinder sets" used to study Markov chains and dynamical systems, as in Kemeny-Snell-Knapp [1976], Lind and Marcus [1995], Kitchens [1998], etc. Note that, for any $\omega \in U_{n'}(\omega') \cap U_{n''}(\omega'')$, we have $U_n(\omega) \subseteq U_{n'}(\omega') \cap U_{n''}(\omega'')$ where $n = \min(n', n'')$. This shows that the collection of all unions of sets of the form $U_n(\omega)$ is closed under arbitrary unions and finite intersections, and thus gives a topology on $N(X)$. We may refer to $N(X)$ with this topology as the *walk space* for graph X.

There is also a nice distance function on $N(X)$, given as follows: let $d(\omega, \omega) = 0$; if ω and ν are distinct walks in X, let $d(\omega, \nu) = 2^{-n}$, where n is the smallest natural number such that $s_n(\omega) \neq s_n(\nu)$. For example, we always have $d(\omega, \nu) \leq 1$; but $d(\omega, \nu) < 1$ if and only if $d(\omega, \nu) \leq 1/2$ if and only if $s_0(\omega) = s_0(\nu)$ (ω and ν have the same source node). To show that $N(X)$ is a metric space, we merely check the metric axioms: $0 = d(\omega, \nu)$ iff $\omega = \nu$, $d(\omega, \nu) = d(\nu, \omega)$, and $d(\omega, \nu) \leq d(\omega, \mu) + d(\mu, \nu)$, for all ω, ν, μ.

In fact, d satisfies the stronger *ultrametric condition*, $d(\omega, \nu) \leq \max(d(\omega, \mu), d(\mu, \nu))$ for all ω, ν, μ, as is easy to check. So the above distance function makes $N(X)$ an *ultrametric space*. Then $U_n(\omega)$ is $\{\nu \in N(X) : d(\omega, \nu) < 2^{-n}\}$, the open ball of radius 2^{-n} around ω, and the walk space topology has as its open sets the arbitrary unions of open balls for the ultrametric.

6.1. Proposition.

1) $N(X)$ is a totally disconnected topological space.

2) $N(X)$ is a complete metric space for the ultrametric structure.

3) If X is a finite graph, then $N(X)$ is compact and separable.

4) If X has finitely many arcs leaving each node, then $N(X)$ is locally compact.

PROOF. For part 1, one shows that any subset of $N(X)$ with more than one element is not connected; more precisely, if $\omega' \neq \omega$ then $\omega' \notin U_n(\omega)$, and $U_n(\omega)$ is open and closed. For part 2, one constructs the limit of any cauchy sequence of walks. For part 3, since $N(X)$ is metrizable, it suffices to show that every sequence has a convergent subsequence; this is easy to do. Also, $N(X)$ is separable since the periodic walks give a countable dense set in it. For part 4, one uses the fact that if $X(x, *)$ is finite for every node x, then the set of paths of given length leaving x is finite; it follows that $U(\alpha)$ is a compact subspace of $N(X)$ for every path α of positive length. ∎

For example, if X is a dynamic graph then $N(X)$ is a discrete topological space, since if ω and ν are distinct walks in the dynamic graph X, then $s_0(\omega) \neq s_0(\nu)$ and so $d(\omega, \nu) = 2^{-0} = 1$.

As a rather different example, for any set S let $X = \mathbf{B}(S)$, the bouquet with S as its set of loops; the topology on $N(X)$ is the product topology on $S^{\mathbf{N}}$, where S is given the discrete topology.

We have the following general results for the walk space topology. The shift map $\tau : N(X) \to N(X)$ is continuous, since $\tau : N(X) \to N(X)$ satisfies $d(\tau(\omega), \tau(\nu)) \leq 2 \cdot d(\omega, \nu)$ for all walks ω and ν in $N(X)$. Also, if $f : X \to Y$ is a graph morphism, then $N(f) : N(X) \to N(Y)$ is continuous, since $N(f)$, as given by $\omega \mapsto f \circ \omega$, is "distance decreasing": $d(f \circ \omega, f \circ \nu) \leq d(\omega, \nu)$ for all walks ω and ν in X.

6.2. DEFINITION. Graphs X and Y are *topologically N-equivalent* (denoted $X \sim_{tN} Y$) when there exists an isomorphism of N-spaces $\phi : N(X) \to N(Y)$ which is a homeomorphism. Then $N(X)$ and $N(Y)$ are said to be *topologically conjugate*, and ϕ is said to be a *topological conjugacy.* A graph morphism $f : X \to Y$ is a *topological N-equivalence* when $N(f)$ is a topological conjugacy.

This definition treats the walk space as a functor which associates to each graph a topological space with a continuous self-map. In this paper we have not tried to analyze this latter "dynamical systems" category (although we list some questions about it at the end of Section 7). We have preferred here to investigate aspects of the situation which can be captured within the category of graphs, worked with as a presheaf category where everything is strictly combinatorial.

6.3. PROPOSITION. *For any graph X, the graph morphism $s_{m,n} : A^{n+m}X \to A^n X$ is a topological N-equivalence for all $n, m \geq 0$. In particular, $s_{n,0} : A^n X \to X$ is a topological N-equivalence. But $s_n : WX \to A^n X$ is not in general a topological N-equivalence.*

PROOF. We have already shown that $N(s_{n,m}) : N(A^{n+m}X) \to N(A^m X)$ is an isomorphism of N-sets. Consider $s_{n,0} : A^n X \to X$. To show that $N(s_{n,0})$ is a homeomorphism, we observe that if walks ω and ν in X correspond to walks ω' and ν' in $A^n X$, then $d(\omega', \nu') = k \cdot d(\omega, \nu)$, where $k = 2^n$. The first statement follows when we replace X by $A^n X$. Taking $X = \mathbf{B}(\{a, b\})$, the bouquet on two loops, shows that $N(s_0)$ is not a homeomorphism, since the topological space $\{a, b\}^{\mathbf{N}}$ is not discrete, while $N(WX)$ has the discrete topology for any graph X. ∎

Any finite-level homotopy arrow from graph X to graph Y indirectly determines a continuous N-set map from $N(X)$ to $N(Y)$, as follows. A graph morphism $f : A^n X \to Y$ gives a continuous N-set map $N(f) : N(A^n X) \to N(Y)$, and $N(s_{n,0}) : N(A^n X) \to N(X)$ is a homeomorphism. Thus, the equivalence of categories from Ho(**Gph**) to **NSet** actually carries the finite-level homotopy subcategory into a topologized category of N-sets. In particular, we have the following.

6.4. COROLLARY. *If graphs X and Y are finite-level homotopy-equivalent then they are topologically N-equivalent.*

For finite graphs we have the following result, of the type attributed to Curtis, Lyndon, and Hedlund in Lind and Marcus [1995] (page 186); they use the terminology "finite-type shift space" for $N(X)$, and "sliding block code" for ϕ.

6.5. PROPOSITION. *Let X be a finite graph. If $\phi : N(X) \to N(Y)$ is a continuous N-map, then there exists a natural number n and a graph morphism $f : A^n X \to Y$ such that $\phi \circ N(s_{n,0}) = N(f)$.*

PROOF. Since X is finite, the space $N(X)$ is compact; so the continuous function $\phi : N(X) \to N(Y)$ is uniformly continuous. In particular, there exists a constant n so that, for every $\omega \in N(X)$,

$$\phi(U_n(\omega)) \subseteq U_0(\phi(\omega)) \quad \text{and} \quad \phi(U_{n+1}(\omega)) \subseteq U_1(\phi(\omega)).$$

We define $f : A^n X \to Y$ on nodes by $\alpha \mapsto s_0(\phi(\omega))$ where $\alpha = s_n(\omega)$; and on arcs by $\beta \mapsto s_1(\phi(\omega))$ where $\beta = s_{n+1}(\omega)$. The definition on nodes is independent of choice of ω since $s_n(\nu) = \alpha$ implies $\nu \in U_n(\omega)$, which implies that $\phi(\nu) \in U_0(\phi(\omega))$ and $s_0(\phi(\nu)) = s_0(\phi(\omega))$. The definition on arcs is similarly independent of choice. ∎

6.6. COROLLARY. *Finite graphs are finite-level homotopy equivalent if and only if they are topologically N-equivalent.*

PROOF. In one direction, this is Corollary 6.4. For the other direction, if X and Y are finite graphs which are topologically N-equivalent, then Proposition 6.5 gives graph morphisms $A^n X \to Y$ and $A^m Y \to X$ which show that X and Y are finite-level homotopy equivalent. ∎

We finish this section with two other applications of the above ideas to the study of N-equivalence of finite graphs.

6.7. Proposition. *If X is a finite graph, then any N-equivalence $f : X \to Y$ is a topological N-equivalence.*

Proof. Since X is finite, $N(X)$ is compact; and $N(Y)$ is metrizable and thus Hausdorf. So $N(f)$ is a continuous bijection which carries closed sets to closed sets. Thus $N(f)$ is a homeomorphism. ∎

6.8. Warning. An N-equivalence is a special type of graph morphism. So graphs may be N-equivalent even when there is no N-equivalence between them. For instance, any two graphs whose set of walks is empty are N-equivalent, but there may not exist any graph morphism between them. The situation for topological N-equivalence is especially subtle, because a topological conjugacy between the walk spaces of two graphs may not be induced by any graph morphism from one to the other.

6.9. Proposition. *For finite graphs X and Y:*

1) *X and Y are topologically N-equivalent if and only if there exists a finite graph E with N-equivalences $f : E \to X$ and $g : E \to Y$.*

2) *X and Y are topologically N-equivalent if there exist N-equivalences $X \to B$ and $Y \to B$.*

Proof. For part 1, if $f : E \to X$ and $g : E \to Y$ are N-equivalences with E finite, then $N(f)$ and $N(g)$ are topological conjugacies, so that $N(X)$ and $N(Y)$ are topologically conjugate. Conversely, if $\phi : N(X) \to N(Y)$ is a topologically conjugacy, then the continuous map of N-sets ϕ comes from some graph morphism $g : A^nX \to Y$ (since X is finite). Let $E = A^nX$; so $s : E \to X$ is an N-equivalence, and thus $g : A^nX \to Y$ must be an N-equivalence since $\phi : N(X) \to N(Y)$ is an isomorphism. For part 2, assume that $X \to B$ and $Y \to B$ are N-equivalences. Consider the fiber-product (pullback) $E = X \times_B Y$. Then E is finite, since X and Y are finite, and we have isomorphisms of N-sets

$$N(E) = N(X) \times_{N(B)} N(Y) = N(X) = N(Y).$$

So by part 1 we see that X and Y are topologically N-equivalent. ∎

Note that graph B need not be finite in part 2 of the preceding proposition.

7. Some necessary and sufficient conditions for topological N-equivalence.

In this section we want to give some further conditions for N-equivalence and topological N-equivalence. In particular, we will explore connections between symbolic dynamics and the following special type of graph morphism. For a node x in a graph X, let $X(*, x)$ denote the set of arcs in X with target the node x.

7.1. Definition. A graph morphism $f : X \to Y$ is a *covering* when $f : X(*, x) \to Y(*, f(x))$ is a bijection for every node x in X. We say it is an *epic covering* when f is also surjective on nodes (and thus on arcs).

According to the historical sketch given in Boldi and Vigna [2002], this basic concept has independently arisen many times in graph theory. Other names for covering include *divisor*, *fibration*, *equitable partition*, etc. Many of the natural graph morphisms in this paper are coverings.

7.2. Proposition. *For any graph X, the source truncations $s_0 : WX \to X$ and $s_{n,0} : A^nX \to X$ are coverings. Also, $s_n : WX \to A^nX$ and $s_{m,n} : A^{n+m}X \to A^nX$ are coverings, for all $m, n \geq 0$.*

Proof. A node in WX is a walk $\omega \in N(X)$. Let $x = s(\omega)$. Each arc in $WX(*, \omega)$ is a concatenated walk $a\omega$ with $a \in X(*, x)$; so $WX(*, \omega) \to X(*, x)$ is a bijection. A similar argument applies to A^nX, etc. The final statement follows by applying the first results to the graph A^mX. ∎

7.3. Proposition. *If X is walkable and $f : X \to Y$ is an N-equivalence then f is a covering.*

Proof. Since f is an N-equivalence, $W(f) : WX \to WY$ is a graph isomorphism. Since X is walkable, for any node x in X there is some walk ω with source x. Considering ω as a node in WX, we have bijections $s_X : WX(*, \omega) \to X(*, x)$ and $s_Y : WY(*, f\omega) \to Y(*, fx)$, and $W(f) : WX(*, \omega) \to WY(*, f\omega)$. Moreover, $f \circ s_X = s_Y \circ W(f)$. It follows that $f : X(*, x) \to Y(*, fx)$ must be a bijection. ∎

Recall that a graph morphism $f : X \to Y$ is a *level n homotopy equivalence* when there exists a graph morphism $q : A^nY \to X$ which fills the diagram

$$\begin{tikzcd} A^n(X) \arrow[r, "s"] \arrow[d, "A^n(f)"'] & X \arrow[d, "f"] \\ A^n(Y) \arrow[r, "s"] \arrow[ur, "q"] & Y \end{tikzcd}$$

7.4. Proposition. *If $f : X \to Y$ is a level n homotopy equivalence and every node is the source of some path α of length n in X, then f is a covering.*

Proof. By hypothesis, for any node x in X there is some path α of length n with source x. The graph morphism $q : A^nY \to X$ satisfies $q \circ A^n(f) = s_X$ and $f \circ q = s_Y$. Considering α as a node in A^nX and $f\alpha$ as a node in A^nY, we have bijections $s_X : A^nX(*, \alpha) \to X(*, x)$ and $s_Y : A^nY(*, f\alpha) \to Y(*, fx)$. Since $q \circ A^n(f) = s_X$, we know that $q(f\alpha) = x$. Consider the function $q : A^nY(*, f\alpha) \to X(*, x)$. Since $f \circ q = s_Y : A^nY(*, f(\alpha)) \to Y(*, f(x))$ is a bijection, we know that $f : X(*, x) \to Y(*, f(x))$ is surjective. Since $q \circ A^n(f) = s_X : A^nY(*, f(\alpha)) \to Y(*, f(x))$ is a bijection, we know that $f : X(*, x) \to Y(*, f(x))$ is injective. Thus $f : X(*, x) \to Y(*, fx)$ is a bijection. ∎

We will use the above to derive a necessary condition for N-equivalence.

7.5. Definition. Let x be a node in graph X; consider the graph $T(X, x)$ given as follows. The nodes in $T(X, x)$ are the finite paths in X with target x (where x is considered as a path of length 0 in X); the arcs in $T(X, x)$ are the triples $(a\alpha, a, \alpha)$ where $a\alpha$ is the concatenation of path α and arc a in X; and $s(a\alpha, a, \alpha) = a\alpha$ and $t(a\alpha, a, \alpha) = \alpha$. There is a natural graph morphism $s : T(X, x) \to X$ given by $\alpha \mapsto s(\alpha)$ and $(a\alpha, a, \alpha) \mapsto a$.

The arcs in $T(X, x)$ which have the node α as target are those of the form $(a\alpha, a, \alpha)$ for $a \in X(*, s(\alpha))$; it follows that the graph morphism $s : T(X, x) \to X$ is a covering. Moreover, the graph $T(X, x)$ is a rooted tree, which we may call the *tree at x*. Here by a *rooted tree*, we mean a graph T with node r such that there is a unique path in T from x to r, for each each node x in T. Notice that in this paper we are directing rooted trees *toward* their roots; we used the opposite convention in Bisson, Tsemo [2008] and [2011].

An induction argument shows that if $f : X \to Y$ is a covering then $T(X, x) \to T(Y, f(x))$ is a graph isomorphism for every node x in X. It follows that if f is a covering and nodes x and x' have $f(x) = f(x')$, then $T(X, x)$ and $T(X, x')$ are isomorphic graphs.

7.6. Definition. A graph B is *basal* when the only epic coverings $B \to B'$ are isomorphisms. A *basing for X* is an epic covering $p : X \to B$ where B is basal.

The next three propositions are modeled on the discussion in Boldi and Vigna [2002]. In their terminology, a basing is a "minimal fibration". We give the proofs here in our language (and with some added details). We will refer to the graphs $T(B, x)$, for nodes x in B, as the trees of B.

7.7. Proposition. *If no two trees in B are isomorphic then B is basal.*

Proof. If an epic covering is an injection on nodes then it must be an isomorphism. So if $p : B \to B'$ is an epic covering which is not an isomorphism, then there must be at least two distinct nodes x_1 and x_2 in B with $p(x_1) = p(x_2)$. But this would say that B has two trees which are isomorphic. ∎

7.8. Proposition. *Any graph X has a basing $p : X \to B$.*

Proof. We define an equivalence relation on the nodes of X by saying that nodes are equivalent when they have isomorphic trees. Then we choose $B_0 \subseteq X_0$ such that each equivalence class contains exactly one element of B_0. Let $p_0 : X_0 \to B_0$ assign to each node in X the element of B_0 in its equivalence class. Define $B_1 \subseteq X_1$ to be the disjoint union $B_1 = \sum_{b \in B_0} X(*, b)$. If we identify B_1 with the set of ordered pairs (b, a) having $b \in B_0$ and $a \in X(*, b)$, then we may define $s, t : B_1 \to B_0$ by $s(b, a) = p_0(s(a))$ and $t(b, a) = b$. The epic graph morphism $p : X \to B$ is given by function p_0 on nodes and by function $p_1(a) = (t(a), a)$ on arcs. To show that p is a covering, we use the bijection between $X(*, x)$ and $B(*, p(x))$ given by the isomorphism between $T(X, x)$ and $T(X, p(x))$. To show that B is basal, we use the fact that if nodes b, b' in B have isomorphic trees, then the corresponding trees $T(X, b)$ and $T(X, b')$ are isomorphic, so that $b = b'$. ∎

7.9. Corollary. *If B is basal then no two trees in B are isomorphic.*

Proof. If two trees in B were isomorphic, then the above construction would give an epic covering $p : B \to B'$ which identifies the two nodes. This would not be an isomorphism, contradicting the definition of basal graph. ∎

Similar reasoning shows that if $p : X \to B$ is a basing and X has isomorphic trees at nodes x and x', then $p(x) = p(x')$. We will use this in the next proof. We will also use the notation $f \approx_0 g$ to indicate that two graph morphisms f and g agree on nodes.

7.10. Proposition. *If $p : X \to B$ is a basing and $f : X \to Y$ is an epic covering then there exists an epic covering $h : Y \to B$ such that $(h \circ f) \approx_0 p$.*

Proof. Given an epic covering $f : X \to Y$ and a basing $p : X \to B$, we want to define a graph morphism $h : Y \to B$ such that, on the level of nodes, $p_0 = h_0 \circ f_0$. Choose any section $\phi : Y_0 \to X_0$ for the surjective function $f_0 : X_0 \to Y_0$, so that $f(\phi(y)) = y$ for each node $y \in Y_0$. Define h on nodes by $h_0(y) = p(\phi(y))$; note that we have $p(x) = h_0(y)$ for any node x with $f(x) = y$, since then $T(X, x)$ is isomorphic to $T(X, \phi(y))$, and p is a basing. But ϕ also determines a section $\phi_1 : Y_1 \to X_1$ of the surjective function $f_1 : X_1 \to Y_1$, by inverting each of the bijections $f : X(*, \phi(y)) \to Y(*, y)$. Define h on arcs by $h_1(a) = p_1(\phi_1(a))$. Let us check that this defines a graph morphism $h : X \to B$. Let $y = t(a)$ and $y' = s(a)$; then

$$t(h(a)) = t(p(\phi(a))) = p(t(\phi(a))) = p(\phi(y)) = h(y) = h(t(a))$$

$$s(h(a)) = s(p(\phi(a))) = p(s(\phi(a))) = p(\phi(y')) = h(y') = h(s(a))$$

note that $p(s(\phi(a)) = p(\phi(y'))$ since $f(s(\phi(a))) = y' = f(\phi(y'))$. In fact, h is an epic covering since h is a surjection on nodes, and $h : Y(*, y) \to B(*, h(y))$, for each $y \in Y_0$, is the composition of bijections $\phi_1 : Y(*, y) \to X(*, \phi(y))$ and $p_1 : X(*, \phi(y)) \to B(*, p(\phi(y)))$. ∎

7.11. Corollary. *If $p : X \to B$ and $p' : X \to B'$ are basings then B and B' are isomorphic graphs. More precisely, there exists an isomorphism of graphs $h : B' \to B$ with $(h \circ p') \approx_0 p$.*

Proof. The previous proposition, applied to the epic covering $p' : X \to B'$ and the basing $p : X \to B$, gives the existence of an epic covering $h : B' \to B$, which must be an isomorphism, since B' is basal. ∎

So, we may speak of "the basal graph of X", as this is well-defined up to isomorphism of graphs. But here is a cautionary example.

7.12. Example. Let $B = B'$ be the basal graph having one node x and arcs b, c (the bouquet with two loops). Let X have nodes x_0 and x_1 with arcs b', b'', c', c'' where $b' : x_0 \to x_1$, $b'' : x_1 \to x_0$, $c' : x_0 \to x_0$, and $c'' : x_1 \to x_1$. Consider the graph morphism $p : X \to B$ which takes b', b'' to b and c', c'' to c, and consider the graph morphism $p' : X \to B'$ which takes b', c' to b and b'', c'' to c. Note that $p : X \to B$ and $p' : X \to B'$ are epic coverings, and are thus basings; but there is no graph morphism $f : B' \to B$ making $p = p' \circ f$. So here are two basings $f : X \to B$ and $f' : X \to B'$ which are not "isomorphic" (as graph morphisms), even though their codomain basal graphs are isomorphic.

7.13. PROPOSITION. *If X and Y are walkable graphs which are N-equivalent, then the basal graphs of X and Y are isomorphic.*

PROOF. Let $p : X \to B$ and $p' : Y \to B'$ be basings for X and Y. Consider the source truncations $s : WX \to X$ and $s : WY \to Y$. These are coverings which are epic since X and Y are walkable. Since the graphs X and Y are N-equivalent, there exists a graph isomorphism $f : WX \to WY$. Thus we have epic coverings $p \circ s : WX \to B$ and $p' \circ s \circ f : WX \to B'$, which are basings since B and B' are basal. Thus B and B are isomorphic, by the previous corollary. ∎

7.14. EXAMPLE. Note that each cycle graph $\mathbf{C}_n$ has a basing to the terminal graph 1, but they are not N-equivalent unless $n = 1$, since their zeta series are different. This shows that the converse of the above proposition is not true.

So isomorphism of basal graphs is a necessary condition for two graphs to be N-equivalent. The following example shows that the basal graph is a finer invariant than the zeta series, in that it can distinguish between graphs which have the same zeta series.

7.15. EXAMPLE. We exhibit two finite graphs which have the same zeta series but non-isomorphic basal graphs. Let X be the graph with nodes $0, 1, 2, 3, 4$ and arcs $(0, i)$ and $(i, 0)$ for $i = 1, 2, 3, 4$. Let Y be the graph with nodes the integers mod 4, with arcs $(i, i + 1)$ and $(i, i - 1)$ for all i mod 4, and with source and target given by $s(i, j) = i$ and $t(i, j) = j$. The characteristic polynomial of Y is $x^4 - 4x^2$ and the characteristic polynomial of X is $x^5 - 4x^3$; so X and Y have the same zeta series (see the discussion at the end of Bisson and Tsemo [2011]). But X has a basing to the graph B with nodes x and x' and with four arcs from x to x' and one arc from x' to x; while Y has a basing to the graph B' with one node and two loops. Since B does not have the same number of nodes as B', it follows that X and Y are not N-equivalent (so that $N(X)$ and $N(Y)$ are not isomorphic as N-sets).

7.16. DEFINITION. Two arcs a and a' in graph Y are said to be *parallel* when $s(a) = s(a')$ and $t(a) = t(a')$. A graph Y is said to be *separated* when it has no parallel arcs.

This terminology comes from Vigna [1997], where he discusses some of the features of the full subcategory of separated graphs. Note also that if Y is a separated graph, then graph morphisms $f, g : X \to Y$ are equal if and only if $f \approx_0 g$.

Recall the warning, near the end of the previous section, that characterizing topological N-equivalence of graphs is subtle because a topological conjugacy between the walk spaces of two graphs may not be induced by any graph morphism from one to the other. So the following proposition, which gives necessary and sufficient conditions for topological N-equivalence of some graphs, seems of value.

7.17. PROPOSITION. *If X and B are finite and walkable, and B is separated and basal, then X and B are topologically N-equivalent if and only if they are N-equivalent.*

PROOF. Clearly topological N-equivalence implies N-equivalence. Assume that X and B are N-equivalent graphs which are finite and walkable; and assume also that B is separated and basal. So we have a graph isomorphism $f : WX \to WB$ and $s : WB \to B$ is a basing (since B is walkable, s is an epic covering). So $s \circ f : WX \to B$ is a basing. Let $p' : X \to B'$ be a basing. Then $s : WX \to X$ is an epic covering since X is walkable, and so $p' \circ s : WX \to B'$ is a basing. Since $s \circ f$ and $p' \circ s$ are both basings of WX, it follows that there exists an isomorphism of graphs $h : B' \to B$ such that $(h \circ p' \circ s) \approx_0 (s \circ f)$. But B is separated, so we must have $h \circ p' \circ s = s \circ f$. Since $s : WX \to X$ and $s \circ f : WX \to B$ are N-equivalences, the graph morphism $h \circ p' : X \to B$ must be an N-equivalence. Since X and B are finite graphs, $h \circ p'$ must be a topological N-equivalence. ∎

We end with a few questions:

1) Does topological N-equivalence come from a model structure on the category of graphs?

2) Is there a particular category of dynamical systems linked by adjoint functors to the category of graphs? There seem to be some choices here.

3) Is "finite-level homotopy" part of a model structure on the category of graphs? Is there a fruitful analogy with stable homotopy, where one works with maps defined on some finite suspension of a space?

8. Appendix: A direct verification of the N-model structure on **Gph**.

Let us show that the three classes $(\mathcal{C}_N, \mathcal{W}_N, \mathcal{F}_N)$ of graph morphisms, from Section 3, satisfy the axioms for a model structure.

Recall that $\mathcal{W}_N$ is the class of N-equivalences, which has the 2/3 property.

Since $\mathcal{C}_N = {}^{\dagger}\mathcal{W}_N$, it follows that $\underline{\mathcal{C}}_N = \mathcal{C}_N \cap \mathcal{W}_N$ is the class Iso of isomorphisms in **Gph**, and $(\underline{\mathcal{C}}_N, \mathcal{F}_N) = (\mathrm{Iso}, \mathrm{All})$ is a weak factorization system.

Since $\mathcal{F}_N$ is the class of all graph morphisms, we have $\underline{\mathcal{F}}_N = \mathcal{F}_N \cap \mathcal{W}_N = \mathcal{W}_N$. To verify that $(\mathcal{C}_N, \underline{\mathcal{F}}_N)$ is a weak factorization system, we must show that an arbitrary graph morphism $f : X \to Y$ factors as $f = g \circ h$ with $h \in \mathcal{C}_N$ and $g \in \mathcal{W}_N$. Consider the graph morphisms $\mathbf{i} : 0 \to \mathbf{N}$ and $\mathbf{j} : \mathbf{N} + \mathbf{N} \to \mathbf{N}$, which are easily seen to be in ${}^{\dagger}\mathcal{W}_N$. We will construct h as a transfinite composition of pushouts of copies of $\mathbf{i}$, and $\mathbf{j}$, from which $h \in \mathcal{C}_N$ follows, by general principles. We give a complete description of the construction, which is a Quillen "small object argument" (as described in Section 2.1 of Hovey [1999], for instance).

First we produce a graph X' and graph morphisms $f' : X \to X'$ and $g' : X' \to Y$, with $f = g' \circ f'$, and with $f' \in \mathcal{C}_N$ and $N(g')$ a surjection. Consider the inclusion

$f' : X \to X + \sum_I \mathbf{N}$ where $I = N(Y)$, with graph morphism $k : \sum_I \mathbf{N} \to Y$. So $f : X \to Y$ and k give $g' : X' \to Y$ with $N(g')$ surjective. Note that f' is essentially a pushout along f of copies of $\mathbf{i} : 0 \to \mathbf{N}$, and f' is certainly in $\mathcal{C}_N = {}^\dagger\mathcal{W}_N$.

Next we factor g' through a possibly transfinite sequence of compositions, indexed by a well-ordered set, an *ordinal*. We take each ordinal to be the set of all smaller ordinals (see Chapter II, Section 3 in Cohen [1966], for instance). Then each ordinal α has a *successor*, defined as $\alpha + 1 = \alpha \cup \{\alpha\}$.

Let Λ be an ordinal so large that there is no injective function $\Lambda \to X'_0 \times X'_0$. We also assume that Λ is not the successor of any ordinal, so that $\lambda \in \Lambda$ implies $\lambda + 1 \in \Lambda$. We use transfinite induction to define, for each $\lambda \in \Lambda$, a graph X^λ and graph morphisms $f^\lambda : X' \to X^\lambda$ and $g^\lambda : X^\lambda \to Y$ with $g' = g^\lambda \circ f^\lambda$, and with f^λ epic graph morphism in $\mathcal{C}_N$ and with $N(g^\lambda)$ surjective. We may refer to this as a Λ-*sequence* (for g'). The transfinite inductive definition goes as follows.

For the minimal element $0 \in \Lambda$, let $X^0 = X'$ and $f^0 = \mathrm{id}$ and $g^0 = g'$, so that $g' = g^0 \circ f^0$.

Assume that we have defined X^λ and f^λ and g^λ with $f^\lambda \circ g^\lambda = g'$, for every $\lambda < \nu$, for some $\nu \in \Lambda$.

For ν a limit ordinal (not the successor of any ordinal), we define $X^\nu = \mathrm{colim}_{\lambda<\nu} X^\lambda$ (viewing the ordinal ν as a category). The graph morphism $f^\nu : X' \to X^\nu$, the *transfinite composition* of epimorphisms in $\mathcal{C}_N$, is an epimorphism in $\mathcal{C}_N$. The colimit also determines a unique graph morphism $g^\nu : X^\nu \to Y$, with $g' = g^\nu \circ f^\nu$.

For $\nu = \lambda + 1$ and $N(g^\lambda)$ is a bijection, we define $X^{\lambda+1} = X^\lambda$ and $f^{\lambda+1} = f^\lambda$ and $g^{\lambda+1} = g^\lambda$.

For $\nu = \lambda + 1$ and $N(g^\lambda)$ is not a bijection, we define $X^\lambda \to X^{\lambda+1}$ by pushout with copies of $\mathbf{j}$, indexed by the set J of all (ω', ω'') such that $N(g^\lambda)$ carries ω' and ω'' to the same walk in $N(Y)$. Here we are gluing together along $\mathbf{j} : \mathbf{N} + \mathbf{N} \to \mathbf{N}$ in each summand of $\sum_J(\mathbf{N} + \mathbf{N}) \to X^\lambda$, to produce an epimorphism $f^{\lambda+1} : X^\lambda \to X^{\lambda+1}$, and a unique graph morphism $g^\nu : X^\nu \to Y$ with $g' = g^\nu \circ f^\nu$. Since it is a pushout of

$$\sum_J(\mathbf{N} + \mathbf{N}) \to X^\lambda \quad \text{and} \quad \sum_J(\mathbf{N} + \mathbf{N}) \to \mathbf{N},$$

we see that $f^{\lambda+1}$ is in $\mathcal{C}_N = {}^\dagger\mathcal{W}_N$.

Note that if g^λ is an N-equivalence, then we will have $X^\lambda = X^{\lambda'}$ for all $\lambda' > \lambda$, and we may say that the Λ-*sequence stabilizes at* λ. Let us verify that our Λ-sequence stabilizes at some $\lambda \in \Lambda$, so that $g' = g^\lambda \circ f^\lambda$; then $f = g \circ h$ with $h = f^\lambda \circ f'$ and $g = g^\lambda$ gives our desired factorization, with $h \in \mathcal{C}_N$ and $g \in \mathcal{W}_N$.

Each graph epimorphism $f^\lambda : X' \to X^\lambda$ determines an equivalence relation $E^\lambda \subseteq X'_0 \times X'_0$ on the nodes of X'. So long as g^λ is not an N-equivalence, we have $E^\lambda \subset E^{\lambda+1}$, a strict inclusion. This shows that the Λ-sequence constructed above eventually stabilizes, since otherwise we could choose a Λ-parametrized family of elements $p^\lambda \in X'_0 \times X'_0$ with $p^{\lambda+1} \in E^{\lambda+1} - E^\lambda$. This would give an injective function $\Lambda \to X'_0 \times X'_0$, which is impossible by our assumption about the size of Λ.

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