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ISOTROPY AND CROSSED TOPOSES

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In memory of Hugh Millington

Abstract. Motivated by constructions in the theory of inverse semigroups and étale groupoids, we define and investigate the concept of isotropy from a topos-theoretic perspective. Our main conceptual tool is a monad on the category of grouped toposes. Its algebras correspond to a generalized notion of crossed module, which we call a crossed topos. As an application, we present a topos-theoretic characterization and generalization of the 'Clifford, fundamental' sequence associated with an inverse semigroup.

Contents

1. Introduction

The purpose of this paper is to introduce the notion of isotropy in toposes, to explain how the isotropy group of an inverse semigroup, or more generally of an étale groupoid, is an instance of topos isotropy, and to give a general topos-theoretic account of the the so-called Clifford, fundamental sequence associated with an inverse semigroup. As it happens, these concepts are inextricably linked to a wide generalization of crossed modules, herein called crossed toposes. Thus, this paper presents a topos-theoretic investigation of isotropy and the Clifford sequence that reveals a central connection with, and generalization of, the

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concept of crossed module. On the one hand, all of this is part of a wider program of study clarifying the connections between topos theory and the theory of inverse semigroups, and more generally étale groupoids [Lawson-Steinberg '04, Funk-Steinberg '10]. On the other hand, this material can be seen as a contribution to topos theory in its own right as it introduces some ideas which exceed the immediate scope of their direct application to semigroups and groupoids.

1.1. Motivating examples Let us begin by explaining through three typical examples how a topos may have a canonical "isotropy" group in it.

1.2. Example. If C is an object of a groupoid $\mathbb{H} = (\mathbb{H}_0, \mathbb{H}_1)$, then the group $\mathrm{Aut}(C) = \mathbb{H}(C, C)$ is called the *isotropy group* of $\mathbb{H}$ at C. These groups form a presheaf $C \mapsto \mathrm{Aut}(C)$ on $\mathbb{H}$: the action $\mathrm{Aut}(C) \longrightarrow \mathrm{Aut}(D)$ by a morphism $h : D \longrightarrow C$ in $\mathbb{H}$ is given by conjugation with h. This operation is a group homomorphism, so that $\mathrm{Aut}(-)$ is a presheaf of groups: equivalently, it is a group object in the presheaf topos $\mathscr{B}(\mathbb{H}) = \mathsf{Set}^{\mathbb{H}^{\mathrm{op}}}$. When $\mathbb{H}$ is not a groupoid but just a category $\mathbb{C}$ there need not be a canonical way of making the assignment $C \mapsto \mathrm{Aut}(C) \subseteq \mathbb{C}(C, C)$ functorial because conjugation is not available. However, we shall see that a general presheaf topos always has an isotropy group that generalizes the groupoid case (Eg. 4.12).

1.3. Example. The traditional isotropy subgroups $G_x = \{g \mid xg = x\}$ associated with a right G-set X, where G is a group, may also be interpreted in topos-theoretic terms.[1] Consider the groupoid $\mathbb{H} = (X, X \times G)$. The isotropy group of this groupoid as an object of the slice topos $\mathscr{B}(G)/X$ of right G-sets over X, which is equivalent to $\mathscr{B}(\mathbb{H})$, is given by the coproduct $\coprod_X G_x \longrightarrow X$. The coproduct $\coprod_X G_x$ is a G-set with the action $(x, g)h = (xh, h^{-1}gh)$.

1.4. Example. Inverse semigroups describe partial symmetries of mathematical objects much in the same way as groups describe global symmetries [Lawson '98]. An *inverse semigroup* is a semigroup S (which is a set S with an associative binary operation) with the property that for each $s \in S$ there exists a unique s^* such that $ss^*s = s$ and $s^*ss^* = s^*$. We think of s^* as a partial inverse of s, the idempotent s^*s as the domain of s, and ss^* as the codomain. In fact, this forms an ordered groupoid $G(S)$ whose object set is the meet-semilattice of idempotents $E = \{s \in S \mid s^2 = s\}$, and whose morphism set is S.

An inverse semigroup has two canonical inverse subsemigroups: the meet-semilattice E, and the *centralizer of* E in S, defined by

$$Z(E) = \{s \in S \mid se = es, \text{ all } e \in E\} .$$

An easy calculation shows that if $s \in Z(E)$, then $s^*s = ss^*$, i.e., s has the same domain and codomain. If $S = Z(E)$, then S is called a *Clifford* semigroup; thus, $Z(E)$ is by definition the maximal Clifford semigroup in S. At the other extreme, when $E = Z(E)$, S is said to be *fundamental*.

[1] We thank Rick Blute for asking this question.

Before we explain how $Z(E)$ forms a group in the topos associated with S we note that there is a short exact sequence (in the appropriate semigroup-theoretic sense) associated with it, usually referred to as the Clifford, fundamental decomposition of S:

$$Z(E) \longrightarrow S \longrightarrow S/\mu \ . \tag{1}$$

Here μ is the so-called maximal idempotent-separating congruence on S, i.e., the largest congruence which does not identify distinct idempotents, so that the sequence is the identity when restricted to E. Part of the original motivation for the present work was to obtain a topos-theoretic account of this exact sequence and to see whether it admits generalizations to larger classes of structures.

Let $\mathscr{B}(S) = \mathscr{B}(G(S))$ denote the topos of ordered $G(S)$-sets, called *the classifying topos of* S. A somewhat simpler and more convenient description of $\mathscr{B}(S)$ is available in which a typical object consists of a set X equipped with an associative action by S, and a map $p : X \longrightarrow E$ satisfying $p(x \cdot t) = t^* p(x) t$ and $x \cdot p(x) = x$. Morphisms are S-equivariant maps in a commutative triangle over E. For instance, if we let $s \cdot t = t^* st$, then the map $Z(E) \xrightarrow{p} E$ such that $p(s) = s^* s = ss^*$ satisfies

$$p(s \cdot t) = p(t^* st) = (t^* st)^* t^* st = t^* s^* t t^* st = t^* s^* s t t^* t = t^* s^* st = t^* p(s) t \ ,$$

and $s \cdot p(s) = s$. The multiplication in S may be restricted to $Z(E)$

$$Z(E) \times_E Z(E) \longrightarrow Z(E) \ ; \quad (s, r) \mapsto sr$$

making $Z(E) \xrightarrow{p} E$ a group internal to $\mathscr{B}(S)$, called *the isotropy group of* S. Yet another equivalent way of presenting $\mathscr{B}(S)$ and its isotropy group is found in § 7.

1.5. Example. Our last example generalizes the previous two. An *étale groupoid* is a groupoid in the category of topological spaces for which the structure maps (domain, codomain, composition and inverse) are étale maps (i.e., local homeomorphisms). Standard examples of étale groupoids include orbit spaces (arising from the action of a discrete group on a topological space), étale equivalence relations (for example those arising from tilings), and groupoids of germs of partial homeomorphisms. Resende [Resende '06] provides many details and further references.

Consider the equalizer of the domain and codomain maps of an étale groupoid $H = (H_0, H_1)$:

$$A \longrightarrow H_1 \underset{c}{\overset{d}{\rightrightarrows}} H_0 \ .$$

The elements of A are those morphisms of H that have the same domain and codomain, so that A could be thought of as the isotropy of H. However, the composite $A \longrightarrow H_0$ may not be étale, so that we ought to consider its associated sheaf space $Z \longrightarrow H_0$. Because H is étale it follows that Z is found simply as the interior of A. Z carries an action by H given by conjugation, making it an object of the classifying topos $\mathscr{B}(H)$ of continuous

H-sets. Moreover, it follows that $Z \longrightarrow H_0$ is a group internal to that topos, which we again call *the isotropy group of* H .

In each of these cases we obtain a canonical group object in the topos in question. We are naturally interested in what distinguishes this group from all other groups in the topos, whether the construction of these groups are instances of a more general phenomenon, and whether there are more examples.

1.6. Contributions The central achievements of the paper are the introduction and development of crossed toposes and isotropy theory for toposes, and the application of these ideas to give a unified account and explanation of the motivating examples. In more detail the main contributions of the paper may be summarized as follows.

1. A monad on the category of grouped toposes is introduced whose algebras we call *crossed toposes*. We investigate the nature of these structures and explain how the motivating examples from the theory of inverse semigroups and étale groupoids give rise to crossed toposes.

2. We show that every topos has a canonical group object called the *isotropy group* of the topos, and that the isotropy groups arising in the motivating examples are instances of this general notion. We prove that the isotropy group of a topos $\mathscr{E}$ gives a 'standard' crossed topos structure on $\mathscr{E}$, which is in fact the terminal crossed topos structure on $\mathscr{E}$, thereby characterizing the role of these groups.

3. Crossed toposes on a presheaf topos are characterized in terms of a generalization of crossed modules. In other words, we define the notion of a crossed $\mathbb{C}$-module, where $\mathbb{C}$ is a small category, and prove that such a structure is the same as a crossed topos on $\mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$. We also show that this equivalence can be extended to crossed toposes on a sheaf topos when a subcanonical site is chosen.

4. The well known correspondence between crossed modules and categorical groups is extended to crossed $\mathbb{C}$-modules. This result has a parallel at the topos level because a crossed topos may be interpreted as a category object in the category of Grothendieck toposes. In addition, we investigate the category of discrete fibrations over this category object.

5. We prove an 'external/internal' theorem for crossed toposes (5.10), showing in effect that the externally defined category of crossed toposes on a fixed topos may be interpreted as a category of 'ordinary' crossed modules internal to that topos.

6. The general framework we develop is applied to make precise the sense in which the isotropy groups arising in the examples of interest are canonical, to interpret the external/internal theorem for inverse semigroups and étale groupoids (7.3, 7.13), and to give a topos-theoretic account of the Clifford construction (1).

1.7. Organization We assume that the reader is familiar with basic category theory; in particular, we assume that the reader is familiar with groupoids, adjunctions, monads, algebras for monads, sites and (pre)sheaves, and the beginnings of the theory of Grothendieck toposes and geometric morphisms. Further information on these concepts which we do not explain is readily found in the literature [MacLane '98, MacLane-Moerdijk '92, Johnstone '02]. On occasion we need concepts from 2-dimensional category theory. Indeed, the main ideas and developments of the paper are best understood with the help of 2-categories, but a reader who is not familiar with 2-categories should not be terribly hindered. Almost all relevant concepts may be found in [Lack '09], the minor exception being the full definition of a pseudo-algebra, for which we refer to [Cheng et al. '04]. In § 5.13 we shall also encounter (strict) double categories; no knowledge beyond the basic definitions is required here, which can be found in [Grandis-Paré '99]. We also use some elementary notions from the theory of fibered categories. A standard reference for this material is [Jacobs '99].

In § 2 we introduce our universe of discourse, which is the category of grouped toposes, and the isotropy monad that is defined on it. We also briefly discuss a purely algebraic explanation of the existence of this monad.

§ 3 begins an investigation of the algebras for the isotropy monad. First we show that the motivating examples can indeed be recast as algebras, and then we prove a technical result about pseudo-algebras, stating roughly that the associativity is a formal consequence of the other data and requirements for a pseudo-algebra. We refer to the category of algebras for the isotropy monad for a fixed topos as the isotropy category of the topos, and to an algebra as a crossed topos. We show that a localic topos is anisotropic in the sense that it does not admit non-trivial algebra structures, or equivalently, that its isotropy category is equivalent to the category of Abelian groups internal to the topos.

Our study of crossed toposes continues in § 4 where we show that every topos has an isotropy group which carries a 'standard' crossed topos structure, and that the standard crossed topos is the terminal one. We also show that a crossed topos gives rise to a "fundamental" quotient in the form of a connected, atomic geometric morphism out of the topos. This is part of an adjointness between crossed toposes and locally connected quotients.

In § 5 we introduce the notion of a crossed $\mathbb{C}$-module, and prove that such objects are the same as algebras on the presheaf topos $\mathsf{Set}^{\mathbb{C}^{op}}$ for the isotropy monad; this is easily extended to (subcanonical) sites. Next we generalize the correspondence between crossed modules and categorical groups to the level of crossed $\mathbb{C}$-modules. We prove an 'external/internal' theorem which recasts the isotropy category of a topos as a category of ordinary crossed modules internal to the topos.

§ 6 introduces and studies a category object internal to the category of Grothendieck toposes associated with a crossed topos. In turn, we consider discrete fibrations over such internal categories.

Finally, § 7 returns to the original motivating questions. The machinery at hand yields characterizations of the isotropy groups in the examples ultimately providing a topos-theoretic account of the Clifford construction, and consequently extending it to the context of étale groupoids and even to toposes.

We conclude this section with some further notation and terminology. We write $\mathsf{Sh}(\mathbb{C}, \mathcal{J})$ for the Grothendieck topos of sheaves on a site $(\mathbb{C}, \mathcal{J})$. A *geometric morphism* $\phi^* \dashv \phi_* : \mathscr{E} \longrightarrow \mathscr{F}$ is usually denoted simply ϕ. For instance, a Grothendieck topos $\mathscr{E}$ has an essentially unique structure geometric morphism denoted $\gamma : \mathscr{E} \longrightarrow \mathsf{Set}$, where γ_* is the global sections functor and γ^* is the constant sheaf functor. A geometric transformation

$$\begin{tikzcd} \mathscr{E} \arrow[r, bend left, "\psi"{name=U}] \arrow[r, bend right, "\phi"'{name=D}] & \mathscr{F} \arrow[from=U, to=D, phantom, "\Downarrow t"] \end{tikzcd}$$

between two geometric morphisms is a natural transformation $t : \psi^* \longrightarrow \phi^*$. There is just a set of geometric transformations between any two geometric morphisms. In fact, if $\mathscr{F} \simeq \mathsf{Sh}(\mathbb{C}, \mathcal{J})$, then the collection of geometric transformations $\psi \Rightarrow \phi$ is in bijection with a subset of the set $\prod_{C_0} \mathscr{E}(\psi^* C, \phi^* C)$. Grothendieck toposes, geometric morphisms and geometric transformations form a locally small 2-category $\mathfrak{BTop}$. At times we find it convenient to rely on the internal logic of a topos mainly in order to carry out equational reasoning or to define objects and morphisms [Johnstone '02]. Finally, we caution the reader that by a topos pullback we always mean a bipullback in the bicategorical sense.

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2. The isotropy monad

In this section we introduce the main conceptual idea of this paper, namely the isotropy monad on the category of grouped toposes. We begin by defining the ambient setting, which is the 2-category of grouped toposes, we establish some elementary facts, and then define the monad. We first provide a direct description of the monad, and then a purely group-theoretic one.

2.1. GROUPED TOPOSES We may consider group objects, and group homomorphisms between them, in any category with finite products and a terminal object. For our purposes we are only concerned with group objects in a Grothendieck topos, although many things in this paper also make sense for elementary toposes. Let $\mathsf{Grp}(\mathscr{E})$ denote the category of groups internal to a Grothendieck topos.

If $\phi : \mathscr{F} \longrightarrow \mathscr{E}$ denotes a geometric morphism and G a group in $\mathscr{E}$, then $\phi^* G$ is a group in $\mathscr{F}$ by virtue of the fact that inverse image functors preserve finite limits. This gives a functor $\mathsf{Grp}(\phi) : \mathsf{Grp}(\mathscr{E}) \longrightarrow \mathsf{Grp}(\mathscr{F})$. Hence we may regard Grp itself as a contravariant functor:

$$\mathsf{Grp}(-) : \mathfrak{BTop}^{\mathrm{op}} \longrightarrow \mathfrak{Cat} \,,$$

which corresponds to a fibration

$$\mathcal{U} = \int_{\mathfrak{BTop}} \mathsf{Grp} \longrightarrow \mathfrak{BTop} .$$

The total category $\mathcal{U}$ has the following explicit description:

Objects pairs $(\mathscr{E}, G)$, where G is a group in the topos $\mathscr{E}$. We call such a pair a *grouped topos*.

Morphisms a morphism of grouped toposes $(\mathscr{F}, H) \longrightarrow (\mathscr{E}, G)$ is a pair (ϕ, m), where $\phi : \mathscr{F} \longrightarrow \mathscr{E}$ is a geometric morphism and $m : H \longrightarrow \phi^* G$ is a group homomorphism. If m is an isomorphism, then (ϕ, m) is said to be *Cartesian*, and when $\phi = 1$ it is called *vertical*.

Composition $(\phi, m) : (\mathscr{F}, H) \longrightarrow (\mathscr{E}, G)$ and $(\psi, n) : (\mathscr{G}, K) \longrightarrow (\mathscr{F}, H)$ compose to the pair $(\phi\psi, \psi^*(m)n)$.

2-cells $\mathcal{U}$ becomes a 2-category by declaring a 2-cell $(\phi, m) \Rightarrow (\psi, n) : (\mathscr{F}, H) \longrightarrow (\mathscr{E}, G)$ to be a geometric transformation $\alpha : \phi \Rightarrow \psi$ such that $\alpha_G m = n$.

$$\begin{tikzcd} H \arrow[r, "m"] \arrow[dr, "n"'] & \phi^* G \arrow[d, "\alpha_G"] \\ & \psi^* G \end{tikzcd}$$

2.2. The endofunctor $\mathcal{I}$ We define the endofunctor part of the isotropy monad on the 2-category of grouped toposes $\mathcal{U}$. The starting point is that given a grouped topos $(\mathscr{E}, G)$ we may consider the topos of right G-objects in $\mathscr{E}$, denoted $\mathscr{B}(\mathscr{E}; G)$. It is also appropriate to denote this topos by $\mathscr{E}^{G^{\mathrm{op}}}$, which we sometimes do. (We do not use left actions.) Explicitly, an object of $\mathscr{B}(\mathscr{E}; G)$ is a pair (X, τ), where X is an object of $\mathscr{E}$ and $\tau : X \times G \longrightarrow X$ is a right (associative and unital) G-action on X. A morphism in $\mathscr{B}(\mathscr{E}; G)$ is a morphism in $\mathscr{E}$ that is G-equivariant in the sense that it respects the action by G.

So far we have only a functor from $\mathcal{U}$ to $\mathfrak{BTop}$. In order to get an endofunctor on $\mathcal{U}$ we must exhibit a group in the topos $\mathscr{B}(\mathscr{E}; G)$. This group is G itself regarded as an object of $\mathscr{B}(\mathscr{E}; G)$ by equipping it with the conjugation action. We shall denote this group in $\mathscr{B}(\mathscr{E}; G)$ by $\overline{G}$. (The trivial action is another possibility, but we shall soon see why that choice is not adequate.) This defines the object part of the endofunctor

$$\mathcal{I} : \mathcal{U} \longrightarrow \mathcal{U} ; \qquad (\mathscr{E}, G) \mapsto (\mathscr{B}(\mathscr{E}; G), \overline{G}) .$$

Given a morphism $(\phi, m) : (\mathscr{F}, H) \longrightarrow (\mathscr{E}, G)$ of grouped toposes, we note first that the following diagram is a pullback square of toposes.

$$\begin{tikzcd} \mathscr{B}(\mathscr{F}; \phi^* G) \arrow[r, "\overline{\phi}"] \arrow[d] & \mathscr{B}(\mathscr{E}; G) \arrow[d] \\ \mathscr{F} \arrow[r, "\phi"] & \mathscr{E} \end{tikzcd}$$

The vertical maps are the structure morphisms, whose inverse image functors equip an object with the trivial action. The inverse image functor $\overline{\phi}^*$ sends an object (X,τ) to $(\phi^*X,\phi^*\tau)$. The group homomorphism $m : H \longrightarrow \phi^*G$ induces an $\mathscr{F}$-essential geometric morphism ('change of scalars') also denoted m in the following diagram.

$$\mathscr{B}(\mathscr{F};H) \xrightarrow{m} \mathscr{B}(\mathscr{F};\phi^*G) \xrightarrow{\overline{\phi}} \mathscr{B}(\mathscr{E};G)$$

We define the underlying geometric morphism of $\mathcal{I}(\phi,m)$ to be the composite $\overline{\phi}m$. Explicitly, the inverse image functor of $\overline{\phi}m$ sends an object (X,τ) to the object ϕ^*X equipped with the following action by H:

$$\phi^*X \times H \xrightarrow{1\times m} \phi^*X \times \phi^*G \xrightarrow{\cong} \phi^*(X\times G) \xrightarrow{\phi^*\tau} \phi^*X\ .$$

In particular, it sends the group object $\overline{G}$ to the group $m^*\overline{\phi}^*(\overline{G})$. The underlying group is simply ϕ^*G, while the action by H is given by $(g,h)\mapsto m(h)^{-1}gm(h)$. The homomorphism $m : H \longrightarrow \phi^*G$ lifts to an H-equivariant map

$$\overline{m} : \overline{H} \longrightarrow m^*\overline{\phi}^*(\overline{G})$$

because a group homomorphism preserves conjugation. We therefore define

$$\mathcal{I}(\phi,m) = (\overline{\phi}m,\overline{m})\ .$$

The verification that this is functorial is straightforward and left to the reader.

On the level of 2-cells the action of $\mathcal{I}$ is also straightforward: given a 2-cell

$$\alpha : (\phi,m) \Rightarrow (\psi,n)$$

between two morphisms of grouped toposes, i.e., a natural transformation $\alpha : \phi^* \longrightarrow \psi^*$ with $\alpha_G m = n$, consider an object (X,τ) of $\mathscr{B}(\mathscr{E};G)$. Then the commutativity of the following diagram shows that the component α_X is H-equivariant:

$$\begin{tikzcd}
\phi^*X\times H \arrow[r, "1\times m"] \arrow[d, "\alpha_X\times H"'] & \phi^*X\times\phi^*G \arrow[r, "\cong"] \arrow[d, dashed, "\alpha_X\times\alpha_G"'] & \phi^*(X\times G) \arrow[r, "\phi^*\tau"] \arrow[d, dashed, "\alpha_{X\times G}"'] & \phi^*X \arrow[d, "\alpha_X"] \\
\psi^*X\times H \arrow[r, "1\times n"] & \psi^*X\times\psi^*G \arrow[r, "\cong"] & \psi^*(X\times G) \arrow[r, "\psi^*\tau"] & \psi^*X
\end{tikzcd}$$

The first square commutes by the fact that $\alpha_H m = n$, and the second and third by naturality. We thus get a natural transformation $\overline{\alpha} : (\overline{\phi}m)^* \longrightarrow (\overline{\psi}n)^*$. The remaining details are again straightforward.

For future reference we note the following result, the proof of which is almost immediate from the definitions given so far.

2.3. Lemma. *Let* $(\phi, m) : (\mathscr{F}, H) \longrightarrow (\mathscr{E}, G)$ *be a morphism of grouped toposes. Then*

$$\begin{tikzcd} \mathscr{B}(\mathscr{F}; H) \arrow[r, "\overline{\phi}m"] \arrow[d, "\gamma"'] & \mathscr{B}(\mathscr{E}; G) \arrow[d, "\gamma"] \\ \mathscr{F} \arrow[r, "\phi"] & \mathscr{E} \end{tikzcd}$$

is a topos pullback if and only if (ϕ, m) *is Cartesian, in which case* $\mathcal{I}(\phi, m)$ *is also Cartesian.*

2.4. Remark. The endofunctor $\mathcal{I}$ is not a fibered functor with regard to the fibration of grouped toposes over toposes: it does not preserve vertical maps.

2.5. The monad structure The underlying geometric morphism $\eta : \mathscr{E} \longrightarrow \mathscr{B}(\mathscr{E}; G)$ of the unit of $\mathcal{I}$ is the 'generic point' of $\mathscr{B}(\mathscr{E}; G)$; this is the étale geometric morphism induced by the identity $e : 1 \longrightarrow G$ of the group G. The inverse image of this geometric morphism forgets the action by sending an object (X, τ) to X. Therefore, at the object $\overline{G}$ we have that $\eta^*(\overline{G}) = G$, and we set

$$\eta_{(\mathscr{E},G)} = (\eta, 1_G) : (\mathscr{E}, G) \longrightarrow (\mathscr{B}(\mathscr{E}; G), \overline{G}) .$$

Thus in particular, the components of the unit of $\mathcal{I}$ are Cartesian morphisms of grouped toposes.

The multiplication $\mu : \mathcal{I}^2(\mathscr{E}, G) \longrightarrow (\mathscr{E}, G)$ for $\mathcal{I}$ is defined as follows. Note that we have

$$\mathcal{I}^2(\mathscr{E}, G) = (\mathscr{B}(\mathscr{B}(\mathscr{E}; G); \overline{G}), \overline{\overline{G}}) .$$

A typical object of the topos $\mathscr{B}(\mathscr{B}(\mathscr{E}; G), \overline{G})$ has the form (X, τ, σ), where τ is a G-action on X, and where σ is a $\overline{G}$-action on (X, τ), i.e., an action $X \times \overline{G} \longrightarrow X$ which is equivariant with respect to the action τ on X and conjugation on G. The group $\overline{\overline{G}}$ has the same underlying group as $\overline{G}$, but is equipped once more with the conjugation action. The inverse image μ^* of the (underlying geometric morphism of the) multiplication sends an object (X, τ) to (X, τ, μ_X), where the action by μ_X is simply that of τ. Abbreviating both τ and $\mu_X = \tau$ by juxtaposition, we have

$$\mu_X((x, g)h) = \mu_X(xh, h^{-1}gh) = xhh^{-1}gh = xgh = \mu_X(x, g)h .$$

This shows that μ_X is G-equivariant with respect to the action τ on X and the conjugation action on $\overline{G}$. It follows right away that $\mu^*(\overline{G}) = \overline{\overline{G}}$. Thus, the geometric morphism μ may be regarded as a Cartesian morphism of grouped toposes. This completes the definition of the monad $\mathcal{I}$.

2.6. Proposition. $\mathcal{I} = (\mathcal{I}, \eta, \mu)$ *is a strict 2-monad on the 2-category of grouped toposes.*

Proof. We have already shown that $\mathcal{I}$ is a 2-functor. Moreover, η and μ are strict 2-natural transformations. Verifying the monad identities is mostly a matter of spelling out the definitions; due to the fact that η and μ are Cartesian the required identities can be verified on the level of geometric morphisms, for which in turn it suffices to establish that the relevant diagrams of inverse image functors commute. ∎

An even easier proof that the monad identities hold can be extracted from the observations presented in § 2.7.

2.7. A cosimplicial group The following group-theoretic explanation of the isotropy monad is somewhat easier. Moreover, it makes more clear where the monad structure comes from.

Fix a group G, and consider the group $G \rtimes \overline{G}$; the elements of this group are simply those of $G \times G$, but multiplication is defined by

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, g_2^{-1} h_1 g_2 h_2) \ .$$

The semidirect product group $G \rtimes \overline{G}$ is a site of definition for the underlying topos of $\mathcal{I}^2(\mathscr{E}, G)$, which is

$$\mathscr{B}(\mathscr{B}(\mathscr{E}; G); \overline{G}) \simeq \mathscr{B}(\mathscr{E}; G \rtimes \overline{G}) \ .$$

(The text [Johnstone '02] explains the general construction of a site for a topos of internal sheaves. We discuss a related case in § 5.13.) $G \rtimes \overline{G}$ is equal to the ordinary Cartesian product $G \times G$ if and only if G is Abelian.

Multiplication $m : G \rtimes \overline{G} \longrightarrow G$ is a group homomorphism. Moreover, the unit element e of G induces two homomorphisms $d_0, d_1 : G \longrightarrow G \rtimes \overline{G}$, given by $d_0(g) = (e, g)$ and $d_1(g) = (g, e)$. These homomorphisms are part of a cosimplicial group

$$\begin{tikzcd} 1 \arrow[r, "e"] & G \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & G \rtimes \overline{G} \arrow[l, "m" description] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & G \rtimes \overline{G} \rtimes \overline{\overline{G}} \arrow[l, shift left=2] \arrow[l, shift right=2] & \cdots \end{tikzcd}$$

The two degeneracy maps $G \rtimes \overline{G} \rtimes \overline{\overline{G}} \longrightarrow G \rtimes \overline{G}$ are simply

$$(g, a, u) \mapsto (ga, u) \qquad \text{and} \qquad (g, a, u) \mapsto (g, au) \ .$$

The other face maps insert unit elements. The cosimplicial identities are readily seen to hold, and it is also easy to see that each of the face and degeneracy maps are in fact group homomorphisms.

2.8. Remark. The above cosimplicial group may be seen to arise from the nerve of a category object in the category of groups. This category object is in fact the one corresponding to the identity crossed module on a group G. Of course, a nerve is a simplicial object; the cosimplicial group used here arises by stripping away the outer face maps of this simplicial object.

The monad identities for $\mathcal{I}$ are an immediate consequence of the cosimplicial identities for this cosimplicial group. However, it is worth pointing out that there is no monad on the category of groups whose application to a group induces this cosimplicial group.

3. Algebras

We turn to algebras for the isotropy monad. Of course, because $\mathcal{I}$ is a 2-monad we can consider strict algebras, pseudo-algebras and lax/oplax algebras. We confine our attention to the strict and pseudo-algebras, starting with the motivating examples. We explain how algebras can be strictified, and spell out what pseudo-maps of algebras look like. Finally, we organize algebras on a fixed topos into a what we call the isotropy category of the topos, and determine the nature of the underlying geometric morphism of an algebra.

3.1. Examples of algebras Let us first give some intuition behind what an algebra structure on a grouped topos $(\mathscr{E}, G)$ amounts to. Suppose that

$$(\theta, m) : \mathcal{I}(\mathscr{E}, G) \longrightarrow (\mathscr{E}, G)$$

is a strict $\mathcal{I}$-algebra structure. This means first and foremost that $\theta : \mathscr{B}(\mathscr{E}; G) \longrightarrow \mathscr{E}$ is a geometric morphism; its inverse image functor $\theta^* : \mathscr{E} \longrightarrow \mathscr{B}(\mathscr{E}; G)$ thus sends an object X of $\mathscr{E}$ to an object $\theta^* X$ equipped with a G-action. But by the unit law we must have $\eta^*\theta^* X = X$. This means that $\theta^* X$ has the form (X, θ_X), where θ_X is a G-action on X. In other words, an algebra structure equips each object of $\mathscr{E}$ with a G-action in such a way that each morphism of $\mathscr{E}$ is equivariant for this action. However, this is not all because the algebra structure is not just a geometric morphism, but also a group homomorphism $m : \overline{G} \longrightarrow \theta^*(G)$. Again by the unit law we must have $\eta^*(m) = 1_G$. This forces right away that $m = 1$, so that in particular (θ, m) is Cartesian, and hence that θ must equip G with its conjugation action. We shall prove below that this suffices: a strict algebra is completely determined by specifying, for each object X, an action θ_X on X, where θ_G is conjugation, such that every morphism of $\mathscr{E}$ is equivariant. A suitably modified statement holds for pseudo-algebras.

Let us explain how the motivating examples of isotropy groups in toposes (§ 1.1) can indeed be regarded as algebras for the monad $\mathcal{I}$. These examples are all instances of a 'standard' one that every topos carries, which we introduce and explain in § 4.6.

3.2. Example. Continuing with Eg. 1.2, let $\mathscr{E} = \mathsf{Set}^{\mathbb{H}^{\mathrm{op}}} = \mathscr{B}(\mathbb{H})$, where $\mathbb{H}$ is a groupoid. Write $Z = \mathrm{Aut}$ for the isotropy group in $\mathscr{E}$: Z is the presheaf $C \mapsto \mathbb{H}(C, C)$ whose action on morphisms is given by conjugation. Then there is an $\mathcal{I}$-algebra structure θ on $(\mathscr{E}, Z)$, where the Z-action $\theta_X = \theta : X \times Z \longrightarrow X$ at an object C of $\mathbb{H}$ is given by

$$\theta_C : X(C) \times Z(C) \longrightarrow X(C) \; ; \qquad (x, f) \mapsto x_{|f} \; ,$$

where $x_{|f}$ stands for the restriction of the element $x \in X(C)$ along $f \in \mathbb{H}(C, C)$. By definition of how morphisms of $\mathbb{H}$ act on Z we find that θ_Z is indeed conjugation as required.

3.3. Example. Let S be an inverse semigroup, and let Z be the isotropy group of S, which is a group object in $\mathscr{B}(S)$, also denoted $Z(E) \longrightarrow E$ (Example 1.4). We equip an arbitrary object $X \longrightarrow E$ of $\mathscr{B}(S)$ with the action θ_X defined by

$$\theta_X : X \times_E Z(E) \longrightarrow X; \qquad (x, s) \mapsto xs \, ,$$

where xs denotes the action of S on X , but restricted to $Z(E)$.

3.4. Example. Continuing with Eg. 1.5, let H is an étale groupoid with isotropy group $Z \longrightarrow H_0$. *The standard algebra structure*, as we shall call it, is given by

$$\theta_X : X \times_{H_0} Z \longrightarrow X; \qquad (x,h) \mapsto xh = \sigma(x,h) \ ,$$

where (X,σ) is a typical object of $\mathscr{B}(H)$. Basically, θ_X is σ restricted to Z .

3.5. Example. We discuss trivial algebras. For any topos $\mathscr{E}$ and group G internal to it, the structure morphism $\mathscr{B}(\mathscr{E};G) \longrightarrow \mathscr{E}$ is an algebra if and only if G is Abelian. We say that such an algebra is *trivial*. There is a class of toposes (which we call *anisotropic*, Def. 4.3) for which conversely an Abelian group admits only the trivial algebra structure. This class includes all localic toposes. However, we do point out that it is possible for an Abelian group in a topos to admit a non-trivial algebra structure. Indeed, when G is any (non-trivial) Abelian group, then the standard algebra structure on the grouped topos $(\mathscr{B}(G),\overline{G})$ is not trivial even though the Abelian group $\overline{G}$ is constant.

3.6. Pseudo-algebras We unpack what pseudo-algebras for the isotropy monad amount to. Cheng et al. gives the full definition of a pseudo-algebra for a pseudo-monad [Cheng et al. '04]; however, because in our case the monad is strict, matters simplify slightly. Thus, a pseudo-algebra for $\mathcal{I}$ consists of the following data:

- a geometric morphism $\theta : \mathscr{B}(\mathscr{E};G) \longrightarrow \mathscr{E}$;
- a group homomorphism $m : \overline{G} \longrightarrow \theta^* G$;
- a unit isomorphism

$$\begin{tikzcd}
(\mathscr{E},G) \arrow[r, "\eta"] \arrow[dr, "\mathrm{id}"'] & \mathcal{I}(\mathscr{E},G) \arrow[d, "{(\theta,m)}"] \arrow[dl, phantom, "\Downarrow t"] \\
 & (\mathscr{E},G)\ ;
\end{tikzcd}$$

- an associativity isomorphism

$$\begin{tikzcd}
\mathcal{I}^2(\mathscr{E},G) \arrow[r, "{\mathcal{I}(\theta,m)=(\overline{\theta}m,\overline{m})}"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\Downarrow r"] & \mathcal{I}(\mathscr{E},G) \arrow[d, "{(\theta,m)}"] \\
\mathcal{I}(\mathscr{E},G) \arrow[r, "{(\theta,m)}"] & (\mathscr{E},G)\ .
\end{tikzcd}$$

This data is subject to certain coherence conditions [Cheng et al. '04], which for sake of expository simplicity we omit. A pseudo-algebra is called *strict* when the 2-cells t, r are identities.

Regarding the unit isomorphism, we observe that because $\eta^*\overline{G} = G$ we get that $t_G = \eta^* m : G \longrightarrow \eta^*\theta^* G$. In particular, since η^* reflects isomorphisms it follows that the group homomorphism m is in fact an isomorphism because t is one. Regarding the associativity isomorphism, we note that m and r_G must make the following diagram commutative:

$$\begin{tikzcd} \overline{\overline{G}} \arrow[r, "\overline{m}"] \arrow[dr, "1"'] & (\overline{\theta}m)^*\overline{G} \arrow[r, "(\overline{\theta}m)^*m"] & (\overline{\theta}m)^*\theta^*G \arrow[d, "r_G"] \\ & \mu^*\overline{G} \arrow[r, "\mu^*m"] & \mu^*\theta^*G \end{tikzcd} \tag{2}$$

We introduce the following notation: we write $\theta^*(X) = (X_\theta, \theta_X)$ for the action of the inverse image functor of θ. In this notation, we have $\eta^*\theta^*X = X_\theta$, and hence the coherence isomorphism t has components $t_X : X_\theta \longrightarrow X$.

The proof of Proposition 3.8 depends on the following lemma.

3.7. Lemma. *Suppose $\phi, \psi : \mathscr{B}(\mathscr{E}; G) \longrightarrow \mathscr{F}$ are geometric morphisms. A natural transformation $\alpha : \phi \Rightarrow \psi$ is fully determined by its composite with the unit η.*

Proof. The component of α_X at X is a morphism of the form $\alpha_X : \phi^*X \longrightarrow \psi^*X$. But this is simply a map in $\mathscr{E}$ of the form $\eta^*\phi^*X \longrightarrow \eta^*\psi^*X$ that is equivariant with respect to the G-actions; because η^* is faithful this means that natural transformations $\phi \Rightarrow \psi$ are simply 'pointwise equivariant' natural transformations $\phi\eta \Rightarrow \psi\eta$. ∎

The following expresses that the monad M is *algebraically free*, in the sense that its category of (pseudo-)algebras is equivalent to the category of (pseudo-)algebras for M qua pointed endofunctor.

3.8. Proposition. *If $(\mathscr{E}, G, \theta, m, t)$ is a pseudo-algebra for $\mathcal{I}$ qua pointed endofunctor, then there exists a unique r making $(\mathscr{E}, G, \theta, m, t, r)$ into a pseudo-algebra for $\mathcal{I}$ qua monad.*

Proof. We shall use Lemma 3.7 (twice) to show how the associativity isomorphism r can be defined in terms of the other data. We define

$$r : \theta \cdot \overline{\theta}m \Rightarrow \theta \cdot \mu$$

by defining its composite with the units

$$\begin{tikzcd} \mathscr{E} \arrow[r, "\eta"] & \mathscr{B}(\mathscr{E}; G) \arrow[r, "\eta"] & \mathscr{B}(\mathscr{E}; G)^{\overline{G}} \arrow[r, "\overline{\theta}m"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\Downarrow r"] & \mathscr{B}(\mathscr{E}; G) \arrow[d, "\theta"] \\ & & \mathscr{B}(\mathscr{E}; G) \arrow[r, "\theta"] & \mathscr{E}\ . \end{tikzcd}$$

Consider an object X of $\mathscr{E}$. Then using our notation $\theta^*(X) = (X_\theta, \theta_X)$ we have

$$\mu^*\theta^*(X) = \mu^*(X_\theta, \theta_X) = (X_\theta, \theta_X, \theta_X)\ ,$$

and hence

$$\eta^*\eta^*\mu^*\theta^*(X) = X_\theta\ .$$

On the other hand, the underlying object of $(\overline{\theta}m)^*\theta^*(X)$ is $(X_\theta)_\theta$, so that we have

$$\eta^*\eta^*(\overline{\theta}m)^*\theta^*(X) = (X_\theta)_\theta \ .$$

Thus, the component of r at X should be a morphism $X_\theta \longrightarrow (X_\theta)_\theta$. We may take this to be

$$r_X = t^{-1}_{X_\theta} \, .$$

It is straightforward to show that the r_X are natural in X , that these are in fact equivariant with respect to the actions of G and $\overline{G}$ (this follows from the fact that the components of t are equivariant), and that the component of r at the group G satisfies the requisite condition (2). Finally, the coherence conditions for a pseudo-algebra are also easily established. ∎

We obtain the following particularly easy description of strict algebras.

3.9. Corollary. *A strict $\mathcal{I}$-algebra determines, and is itself determined by, a section of the functor $\eta^* : \mathscr{B}(\mathscr{E}; G) \longrightarrow \mathscr{E}$ that sends G to $\overline{G}$.*

In the remainder of the paper we shall mainly work with strict algebras. However, in order for the following definition to be stable under isomorphism, it should be formulated in terms of pseudo-algebras, and not just the strict ones.

3.10. Definition. *We shall say that a group G in a topos $\mathscr{E}$ is* an isotropy group *if $(\mathscr{E}, G)$ admits a pseudo-algebra structure for the isotropy monad $\mathcal{I}$.*

3.11. Remark. An Abelian group G in any topos $\mathscr{E}$ is an isotropy group because in this case the trivial action is an algebra structure (Example 3.5). An isotropy group in a localic topos must be Abelian (Cor. 3.18). An isotropy group G has the 'Abelian-like' property that any subobject $S \hookrightarrow G$ is closed under conjugates because the morphism $S \hookrightarrow G$ must be equivariant. A group in Set with this property must be Abelian.

3.12. Algebra maps Usually one does not lose much by considering just the strict algebras for a 2-monad or pseudo-monad because of the coherence result that states that every algebra is equivalent to a strict one. In our case, it is particularly easy to see how to find a strict algebra isomorphic to a given pseudo-algebra. We must consider pseudo-maps of algebras because the isomorphism between an algebra and its strictification is necessarily a pseudo-map. The material in this section is included only for completeness and will not be used in the remainder of the paper.

We first spell out what a pseudo-map between algebras amounts to in our setting. (Cheng et al. explains the general notions [Cheng et al. '04].) A *pseudo-map* between two pseudo-algebras $(\mathscr{E}, G, \theta, m, t)$ and $(\mathscr{F}, H, \delta, n, s)$ consists of the following data:

- a geometric morphism $\phi : \mathscr{E} \longrightarrow \mathscr{F}$;
- a group homomorphism $k : G \longrightarrow \phi^* H$;

- a natural isomorphism

$$\begin{tikzcd} \mathcal{I}(\mathscr{E},G) \arrow[r, "\mathcal{I}(\phi,k)"] \arrow[d, "(\theta,m)"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathcal{I}(\mathscr{F},H) \arrow[d, "(\delta,n)"] \\ (\mathscr{E},G) \arrow[r, "(\phi,k)"] & (\mathscr{F},H) \end{tikzcd}$$

subject to a coherence condition with respect to the unit isomorphisms of the algebras (the condition regarding the associativity isomorphisms is then automatic, as the latter are defined in terms of the unit isomorphisms). Using again the notation $\theta^*(X) = (X_\theta, \theta_X)$, and $\delta^*(Y) = (Y_\delta, \delta_Y)$, we may write the unit isomorphisms as $s_Y : Y_\delta \longrightarrow Y$ and $t_X : X_\theta \longrightarrow X$, and the component α_Y as

$$\alpha_Y : \phi^*(Y_\delta) \longrightarrow (\phi^* Y)_\theta \ .$$

The coherence requirement is then the condition that the following diagram commute.

$$\begin{tikzcd} \phi^*(Y_\delta) \arrow[r, "\alpha_Y"] \arrow[dr, "\phi^*(s_Y)"'] & (\phi^*Y)_\theta \arrow[d, "t_{\phi^*Y}"] \\ & \phi^*Y \end{tikzcd}$$

We show how to strictify a pseudo-algebra. Fix an algebra $(\mathscr{E}, G, \theta, m, t)$. In particular, this gives a group isomorphism $m = t_G : G_\theta \longrightarrow G$ in $\mathscr{E}$, together with isomorphisms $t_X : X_\theta \longrightarrow X$. We define a strict algebra on $(\mathscr{E}, G_\theta)$ by setting (the underlying object of) $\delta^*(X)$ as X , with the action

$$X \times G_\theta \xrightarrow{t_X^{-1} \times t_G} X_\theta \times G \xrightarrow{\theta_X} X_\theta \xrightarrow{t_X} X \ .$$

It is easy to show that this definition is well-defined, that each morphism $f : X \longrightarrow Y$ in $\mathscr{E}$ is equivariant, and that $\delta^*(G_\theta) = \overline{G_\theta}$. Thus, δ is a strict algebra structure. Moreover, the group homomorphism $m : G_\theta \longrightarrow G$ together with the natural family $t_X : X_\theta \longrightarrow X$ induces a pseudo-map

$$\begin{tikzcd} \mathcal{I}(\mathscr{E},G_\theta) \arrow[r, "\mathcal{I}(\mathrm{id},m)"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\Downarrow t"] & \mathcal{I}(\mathscr{E},G) \arrow[d, "(\theta,m)"] \\ (\mathscr{E},G_\theta) \arrow[r, "(\mathrm{id},m)"] & (\mathscr{E},G) \ . \end{tikzcd}$$

This is a pseudo-map because m is an isomorphism.

3.13. The isotropy category of a topos We call an algebra for the isotropy monad *a crossed topos*. We shall sometimes write an $\mathcal{I}$-algebra δ on $(\mathscr{E}, G)$ simply as $\delta : G \longrightarrow \mathscr{E}$ by analogy with the crossed module notation of § 5. In this section we establish some topological properties of crossed toposes. In particular, we show that a localic topos does not admit non-trivial algebra structures.

We begin by organizing (strict) crossed toposes, their morphisms and 2-cells into a 2-category.

3.14. Definition. *The 2-category* XTop *has:*

Objects (strict) crossed toposes $\delta : G \longrightarrow \mathscr{E}$ *,*

Morphisms strict morphisms of crossed toposes,

2-cells an algebra 2-cell between strict morphisms

$$\xi : (\phi, k) \Rightarrow (\psi, l) : (\, G \xrightarrow{\theta} \mathscr{E}\,) \longrightarrow (\, H \xrightarrow{\delta} \mathscr{F}\,)$$

is simply a 2-cell of grouped topos morphisms $\xi : (\phi, k) \Rightarrow (\psi, l)$ *(by Remark 3.15).*

When a topos $\mathscr{E}$ *is fixed, then the locally discrete 2-category* $\mathsf{XTop}(\mathscr{E})$ *is the sub-2-category of the 2-category of strict* $\mathcal{I}$*-algebras on grouped toposes of the form* $(\mathscr{E}, G)$*, and all vertical algebra maps and 2-cells, i.e., all group homomorphisms* $k : G \longrightarrow H$ *in* $\mathscr{E}$ *such that*

$$\begin{array}{ccc} \mathcal{I}(\mathscr{E}, G) & \xrightarrow{\mathcal{I}(\mathrm{id}, k)} & \mathcal{I}(\mathscr{E}, H) \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\delta} \\ (\mathscr{E}, G) & \xrightarrow{(\mathrm{id}, k)} & (\mathscr{E}, H) \end{array}$$

commutes. We refer to $\mathsf{XTop}(\mathscr{E})$ *as* the isotropy category of $\mathscr{E}$. *Finally, when both* $\mathscr{E}$ *and* G *are fixed, we obtain a category of crossed toposes* $\mathsf{XTop}(\mathscr{E}, G)$.

3.15. Remark. The defining condition $\xi \cdot \theta = \delta \cdot \mathcal{I}(\xi)$ for an algebra 2-cell ξ is automatically satisfied, as may be seen by whiskering both sides with the unit of the monad: by Lemma 3.7, it suffices to prove that the two whiskered 2-cells coincide [Cheng et al. '04].

Let $\theta : \mathscr{B}(\mathscr{E}; G) \longrightarrow \mathscr{E}$ be a strict crossed topos, so that we may write $\theta^* X = (X, \theta_X)$. The left adjoint $\theta_!$ of θ^* may be defined as the coequalizer

$$X \times G \underset{\tau}{\overset{\theta_X}{\rightrightarrows}} X \longrightarrow\!\!\!\!\!\rightarrow \theta_!(X, \tau)$$

for any object (X, τ) of $\mathscr{B}(\mathscr{E}; G)$. Dually, the right adjoint θ_* may be described by the equalizer

$$\theta_*(X, \tau) \rightarrowtail X \underset{\widetilde{\tau}}{\overset{\widetilde{\theta_X}}{\rightrightarrows}} X^G \quad ,$$

where the two parallel maps are exponential transposes.

3.16. Proposition. *The underlying geometric morphism of a crossed topos is connected and atomic* (whence hyperconnected)*.*

Proof. The unit $\eta : \mathscr{E} \longrightarrow \mathscr{B}(\mathscr{E}; G)$ is an étale surjection because it is equivalent to the canonical projection

$$\mathscr{B}(\mathscr{E}; G)/G \longrightarrow \mathscr{B}(\mathscr{E}; G) \ ,$$

where G denotes the representable G-action (which is G itself with right multiplication). In fact, the equivalence $\mathscr{E} \simeq \mathscr{B}(\mathscr{E}; G)/G$ associates with an object $v : X \longrightarrow G$ the object $v^{-1}(1)$. We appeal to the fact about geometric morphisms that if a composite

$$\mathscr{E}/X \longrightarrow\!\!\!\!\!\rightarrow \mathscr{E} \xrightarrow{\psi} \mathscr{F}$$

has any of the properties connected, locally connected or atomic, then so does ψ [Johnstone '02]. This applies to the composite

$$\mathscr{B}(\mathscr{E}; G)/G \longrightarrow\!\!\!\!\!\rightarrow \mathscr{B}(\mathscr{E}; G) \xrightarrow{\delta} \mathscr{E} \ ,$$

where δ is a crossed topos. Indeed, this composite is an equivalence by the unit law for δ, so certainly it is atomic and connected. Hence, so is δ. ∎

3.17. Remark. We emphasize that if $\delta : G \longrightarrow \mathscr{E}$ is a (strict) crossed topos, then $\delta_\Omega : \Omega \times G \longrightarrow \Omega$ is the trivial action, where Ω denotes the subobject classifier of $\mathscr{E}$, because the inverse image functor of an atomic geometric morphism preserves the subobject classifier.

3.18. Corollary. *A group in a localic topos is an isotropy group* (Def. 3.10) *if and only if it is Abelian and the algebra structure is trivial.*

Proof. Compose a given crossed topos $\delta : G \longrightarrow \mathscr{E}$ with the structure morphism for $\mathscr{E}$ to get the structure morphism for $\mathscr{B}(\mathscr{E}; G)$ over Set:

$$\mathscr{B}(\mathscr{E}; G) \xrightarrow{\delta} \mathscr{E} \longrightarrow \mathsf{Set} \ .$$

By Proposition 3.16, δ is hyperconnected, and $\mathscr{E} \longrightarrow \mathsf{Set}$ is localic by assumption. It follows from the fact that the hyperconnected-localic factorization is essentially unique that the first leg must be isomorphic to $\gamma : \mathscr{B}(\mathscr{E}; G) \longrightarrow \mathscr{E}$ because this (structure) geometric morphism is also hyperconnected and has the same composite with $\mathscr{E} \longrightarrow \mathsf{Set}$. ∎

4. Crossed toposes

We continue our study of the algebras of the isotropy monad. In § 4.1 we define the isotropy group of a topos, and show this group carries a canonical crossed topos structure which we call the standard one. This crossed topos is the terminal object of the isotropy category of the topos. We then proceed to the isotropy quotient of a crossed topos, and show how it is part of an adjointness between crossed toposes and locally connected quotients of a topos.

4.1. The isotropy group of a topos In each of our motivating examples (§ 1.1) the classifying topos contains a canonical group object. We now show that these groups are instances of a single concept that we call the isotropy group of a topos.

Let $\mathscr{E}$ be a topos. Define a functor

$$\mathcal{Z} : \mathscr{E}^{\mathrm{op}} \longrightarrow \mathsf{Grp}$$

such that $\mathcal{Z}(X)$ equals the group of automorphisms

$$\begin{tikzcd} \mathscr{E}/X \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow t"] & \mathscr{E} \end{tikzcd}$$

of the étale geometric morphism $\mathscr{E}/X \longrightarrow \mathscr{E}$, whose inverse image functor sends an object E of $\mathscr{E}$ to the projection $E \times X \longrightarrow X$. More explicitly, such an automorphism t consists of a component automorphism

$$t_E : E \times X \longrightarrow E \times X \ , \ (b, x) \mapsto (t_E(b, x), x)$$

over X for every object E of $\mathscr{E}$, natural in E . For instance, the component map t_1 must be the identity on X . We usually work with t_E as a map $E \times X \longrightarrow E$ in $\mathscr{E}$. One might think of such a map as a representation of X in E or an X-indexed family of automorphisms of E (even though X need not carry any group structure itself).

Transition in $\mathcal{Z}$ along a map $m : Y \longrightarrow X$ of $\mathscr{E}$ is given by 'whiskering' with the geometric morphism $\mathscr{E}/Y \longrightarrow \mathscr{E}/X$: we denote the induced homomorphism

$$\widehat{m} : \mathcal{Z}(X) \longrightarrow \mathcal{Z}(Y) \ .$$

For $t \in \mathcal{Z}(X)$, we have

$$\widehat{m}(t)_E : E \times Y \longrightarrow E \ , \ (b, y) \mapsto t_E(b, my) \ .$$

4.2. Lemma. *For any topos $\mathscr{E}$, $\mathcal{Z}$ preserves small limits.*

Proof. We claim that $\mathcal{Z}$ carries a colimit in $\mathscr{E}$ to the corresponding limit of groups. Note that a limit of groups is created in Set . $\mathcal{Z}(0)$ equals the trivial group because $\mathscr{E}/0$ has one object and one morphism. It is relatively straightforward to show that $\mathcal{Z}$ preserves small products:

$$\mathcal{Z}(\coprod X_\alpha) \cong \prod \mathcal{Z}(X_\alpha) \ .$$

Indeed, if (t^α) is a 'vector' consisting of an automorphism t^α of $\mathscr{E}/X_\alpha \longrightarrow \mathscr{E}$ for every α , then we may amalgamate (t^α) into a single automorphism $\langle t^\alpha \rangle$ of $\mathscr{E}/\coprod X_\alpha \longrightarrow \mathscr{E}$ in the obvious way:

$$\langle t^\alpha \rangle_E = \langle t^\alpha_E \rangle : E \times \coprod X_\alpha \cong \coprod E \times X_\alpha \longrightarrow E \ .$$

Conversely, an automorphism t of $\mathscr{E}/\coprod X_\alpha \longrightarrow \mathscr{E}$ may be decomposed into a vector (t_α) by whiskering with each inclusion $X_\alpha \longrightarrow \coprod X_\alpha$.

Next we show that $\mathcal{Z}$ preserves equalizers: if

$$Y \underset{n}{\overset{m}{\rightrightarrows}} X \xrightarrow{k} W$$

is a coequalizer in $\mathscr{E}$, then

$$\mathcal{Z}(W) \xrightarrow{\widehat{k}} \mathcal{Z}(X) \underset{\widehat{n}}{\overset{\widehat{m}}{\rightrightarrows}} \mathcal{Z}(Y)$$

is an equalizer of groups. The homomorphism $\widehat{k}$ is injective basically because k is an epimorphism: if s, t are automorphisms of $\mathscr{E}/W \longrightarrow \mathscr{E}$ such that $\widehat{k}(s) = \widehat{k}(t)$, then the following diagram shows that $s = t$. E denotes an arbitrary object of $\mathscr{E}$.

$$\begin{tikzcd}
E \times X \arrow[r, shift left, "\widehat{k}(s)_E"] \arrow[r, shift right, "\widehat{k}(t)_E"'] \arrow[d, two heads, "E\times k"'] & E \times X \arrow[d, two heads, "E\times k"] \\
E \times W \arrow[r, shift left, "s_E"] \arrow[r, shift right, "t_E"'] & E \times W
\end{tikzcd}$$

In element-style notation we have $\widehat{k}(s)_E(b, x) = (s_E(b, kx), x)$. Finally, suppose that s is an automorphism of $\mathscr{E}/X \longrightarrow \mathscr{E}$ such that $\widehat{m}(s) = \widehat{n}(s)$. Then there is an automorphism t of $\mathscr{E}/W \longrightarrow \mathscr{E}$ such that $\widehat{k}(t) = s$ because the endofunctor $E \times -$ preserves coequalizers: concretely, the component t_E is induced by

$$\begin{tikzcd}
E \times Y \arrow[r, shift left, "E\times m"] \arrow[r, shift right, "E\times n"'] & E \times X \arrow[r, two heads, "E\times k"] \arrow[dr, "s_E"'] & E \times W \arrow[d, dashed, "t_E"] \\
& & E.
\end{tikzcd}$$

∎

The fact (from topos theory 'folklore') that a limit-preserving functor $\mathscr{E}^{\mathrm{op}} \longrightarrow \mathsf{Set}$ is necessarily representable is easy to deduce: if $\mathbb{C}$ is a site for $\mathscr{E}$, then the restriction $\mathbb{C}^{\mathrm{op}} \longrightarrow \mathscr{E}^{\mathrm{op}} \longrightarrow \mathsf{Set}$ is a sheaf that represents the given functor. The same is true for a limit-preserving functor $\mathscr{E}^{\mathrm{op}} \longrightarrow \mathsf{Grp}$. Our functor $\mathcal{Z}$ is thus represented by a group internal to $\mathscr{E}$, and this group is unique in the sense that any two such representing groups are isomorphic by a unique isomorphism commuting with the isomorphisms with $\mathcal{Z}$.

4.3. Definition. *We shall call the group internal to a topos $\mathscr{E}$ that represents the functor $\mathcal{Z} : \mathscr{E}^{op} \longrightarrow \mathsf{Grp}$* the isotropy group of $\mathscr{E}$, *denoted Z. When Z is the trivial group* 1 *we say that $\mathscr{E}$ is* anisotropic.

4.4. The isotropy group of $\mathscr{B}(H)$ The isotropy group of a topos (Def. 4.3) does indeed capture the isotropy groups in the motivating examples, but we do not need to verify each example individually (although this is a straightforward exercise) because we can treat them all as instances of an étale localic groupoid [Johnstone '02]. After defining the isotropy group of an étale localic groupoid directly (in such a way that clearly generalizes the motivating examples) we show that it coincides with the isotropy group of its classifying topos. The material in this section assumes background knowledge on (étale) localic groupoids and their sheaf toposes, and occasionally invokes some folklore results from this area.

Throughout, we use the following standard notation: $O(X)$ denotes the frame of 'opens' of a locale X; we also identify $U \in O(X)$ with the open inclusion $U \hookrightarrow X$.

Let $H = (H_0, H_1)$ be an étale localic groupoid, with domain and codomain maps d and c. The multiplication, identities and inverse maps are denoted by

$$m : H_1 \times_{H_0} H_1 \longrightarrow H_1; \qquad i : H_0 \longrightarrow H_1; \qquad \iota : H_1 \longrightarrow H_1 .$$

The homeomorphism ι satisfies $\iota^2 = \mathrm{id}$, $c\iota = d$, $m(\mathrm{id}, \iota) = id$, and $m(\iota, \mathrm{id}) = ic$, where id is the identity map on H_1. An object $\langle F, \sigma \rangle$ of the topos $\mathscr{B}(H)$ of étale actions consists of a sheaf F on H_0 and a natural transformation $\sigma : c^*F \longrightarrow d^*F$ satisfying unit and associative conditions.

We define a sheaf Z on the locale H_0: for any $U \in O(H_0)$ let

$$Z(U) = \{\text{locale morphisms } s : U \longrightarrow H_1 \mid \forall W \in O(H_0)\ s^*d^*W = s^*c^*W = U \wedge W\ \} .$$

The commutativity of the following diagram characterizes in localic terms the conditions that an $s \in Z(U)$ satisfies.

$$\begin{tikzcd}
U \arrow[dr, "s"] \arrow[drr, hook, bend left] \arrow[ddr, hook, bend right] & & \\
& H_1 \arrow[r, "c"] \arrow[d, "d"] & H_0 \\
& H_0 &
\end{tikzcd}$$

The isotropy sheaf Z carries an action

$$\sigma : c^*Z \longrightarrow d^*Z \ ,$$

which is given by conjugation. Intuitively, we have $\sigma(z, g) = g^{-1}zg$, and indeed for spaces we define σ this way (Eg. 1.5). However, in general we must formulate conjugation using diagrams of locale morphisms. The pullback c^*Z of Z along c is given by

$$c^*Z(V) = \{s : V \longrightarrow H_1 \mid \forall W \in O(H_0)\ s^*d^*W = s^*c^*W = V \wedge c^*W\ \} \ , \ V \in O(H_1) .$$

d^*Z is similarly given. In terms of locale morphisms, the conditions for $s \in c^*Z(V)$ are given by the commutativity of the following diagram.

$$\begin{tikzcd}
V \arrow[r, hook] \arrow[d, hook] \arrow[dr, "s"] & H_1 \arrow[dr, "c"] & \\
H_1 \arrow[dr, "c"'] & H_1 \arrow[r, "c"] \arrow[d, "d"] & H_0 \\
& H_0 &
\end{tikzcd}$$

Therefore, by the commutativity of the parallelogram above (left) we may form the locale morphism

$$V \longrightarrow H_1 \times_{H_0} H_1 \xrightarrow{m} H_1 \ ,$$

which we denote simply ms. The codomain of a product in H equals the codomain of the second factor, which is s in this case. Therefore, the diagram below (left) commutes, which gives the square below (right).

$$\begin{tikzcd}
V \arrow[r, "s"] \arrow[d, "ms"'] & H_1 \arrow[d, "c"] \\
H_1 \arrow[r, "c"] & H_0
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, hook] \arrow[d, "ms"'] & H_1 \arrow[r, "\iota"] \arrow[d, "c"] & H_1 \arrow[dl, "d"] \\
H_1 \arrow[r, "c"] & H_0 &
\end{tikzcd}$$

Then $\sigma(s)$ is defined as the product in H of $V \hookrightarrow H_1 \xrightarrow{\iota} H_1$ with ms, which is indeed defined because $c = d\iota$. It follows that $\sigma(s) \in d^*Z(V)$. Moreover, σ defines a natural transformation, and $\langle Z, \sigma \rangle$ is an object of $\mathscr{B}(H)$. In fact, $\langle Z, \sigma \rangle$ is a group internal to $\mathscr{B}(H)$, which we denote simply Z.

The following proposition states that the forgoing definition of Z agrees with Def. 4.3. The proof makes use of the *domain object* D of the topos $\mathscr{B}(H)$; this is the étale map $d : H_1 \longrightarrow H_0$, equipped with the composition action.

4.5. PROPOSITION. *Let $H = (H_0, H_1)$ be an étale localic groupoid, with isotropy group* Z (as defined above) *and classifying topos $\mathscr{B}(H)$. Then for every object $X = \langle F, \sigma \rangle$ of $\mathscr{B}(H)$ there is an isomorphism*

$$\mathscr{B}(H)(X, Z) \cong \mathcal{Z}(X) \ ,$$

which is natural in X. In other words, the functors $\mathcal{Z}$ and $\mathscr{B}(H)(-, Z)$ are isomorphic, so that $\mathcal{Z}$ is represented by Z.

PROOF. $\mathscr{B}(H)(-, Z)$ and $\mathcal{Z}$ are two limit-preserving functors on $\mathscr{E}^{\mathrm{op}}$ (the latter by Lemma 4.2), so we have only to show that they agree on a generating full subcategory of $\mathscr{E}$. We make use of the following structural results about sheaves on étale groupoids. First, we have adjoint functors

$$\begin{tikzcd}
\mathsf{Sh}(H_0) \arrow[r, bend left, "D_!"] \arrow[r, phantom, "\perp"] & \mathscr{B}(H) \arrow[l, bend left, "D^*"]
\end{tikzcd} \ ,$$

where D^* forgets the action: $D^*\langle F, \sigma\rangle = F$, and $D_! F$ equips a sheaf F on H_0 with the free action by H. Next, for any $U \in O(H_0)$, the sheaf topos $\mathsf{Sh}(U)$ and the slice topos $\mathscr{B}(H)/D_! U$ are canonically equivalent. For instance, $\mathsf{Sh}(H_0)$ and $\mathscr{B}(H)/D$ are equivalent because $D \cong D_! H_0$. There is also a canonical equivalence $\mathsf{Sh}(H_1) \simeq \mathscr{B}(H)/D \times D$. For any $U \in O(H_0)$, we have, by the adjointness $D_! \dashv D^*$ and the Yoneda lemma,

$$\mathscr{B}(H)(D_! U, X) \cong \mathsf{Sh}(H_0)(U, D^* X) \cong X(U)\ .$$

(We abuse notation and write $X(U)$ for $D^* X(U)$.) We also note that the topos $\mathscr{B}(H)$ is generated by the (small) full subcategory on the objects $D_! U$, where $U \in O(H_0)$.

Returning to the main argument, an element of $Z(U)$ is by definition a locale morphism $s : U \longrightarrow H_1$ such that the diagram below (left) commutes. In terms of $\mathscr{B}(H)$ and the domain object D, by forming sheaf toposes this diagram corresponds to the diagram of geometric morphisms below (right).

$$\begin{tikzcd}
U \arrow[dr, "s"] \arrow[drr, hook, bend left] \arrow[dd, hook, bend right] & & \\
& H_1 \arrow[r, "c"] \arrow[d, "d"'] & H_0 \\
& H_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{B}(H)/D_! U \arrow[dr, "s"] \arrow[drr, hook, bend left] \arrow[ddr, hook, bend right] & & \\
& \mathscr{B}(H)/D \times D \arrow[r, "c"] \arrow[d, "d"'] & \mathscr{B}(H)/D \arrow[d] \\
& \mathscr{B}(H)/D \arrow[r] & \mathscr{B}(H)
\end{tikzcd} \tag{3}$$

Because the inside square in (3) (right) is a topos bipullback in the bicategorical sense such morphisms s correspond to automorphisms of the outside geometric morphism

$$\mathscr{B}(H)/D_! U \longrightarrow \mathscr{B}(H)/D \longrightarrow \mathscr{B}(H)\ .$$

Of course, this geometric morphism equals the étale one $\mathscr{B}(H)/D_! U \longrightarrow \mathscr{B}(H)$. We have thus proved that there is a canonical isomorphism

$$\mathscr{B}(H)(D_! U, Z) \cong \mathcal{Z}(D_! U)\ .$$

Moreover, these isomorphisms commute with transition along a morphism $D_! V \longrightarrow D_! U$ of $\mathscr{B}(H)$, concluding the proof. ■

4.6. The standard crossed topos Having defined the isotropy group of a topos, we now show that this group admits a canonical crossed topos structure. We prove that this 'standard' crossed topos is the terminal object of the isotropy category of the topos.

Let $\mathscr{E}$ denote a topos with isotropy group Z, and θ the automorphism of $\mathscr{E}/Z \longrightarrow \mathscr{E}$ corresponding to the identity 1_Z. For any object E of $\mathscr{E}$, we use the notation

$$\theta_E : E \times Z \longrightarrow E\ ,\ \ \theta_E(b, z) = b \cdot z\ .$$

By the naturality of the isomorphisms in Def. 4.3, the automorphism of $\mathscr{E}/X \longrightarrow \mathscr{E}$ corresponding to an 'element' $z : X \longrightarrow Z$ must be given by

$$\widehat{z}(\theta)_E : E \times X \longrightarrow E\ ,\ (b, x) \mapsto b \cdot z(x)\ . \tag{4}$$

4.7. Lemma. *If two maps* $m, n : X \longrightarrow Z$ *satisfy*

$$\forall b \in E\,,\ b \cdot m(x) = b \cdot n(x)$$

for every object E *of* $\mathscr{E}$, *then* $m = n$.

Proof. By (4), the hypothesis merely asserts that the automorphisms of $\mathscr{E}/X \longrightarrow \mathscr{E}$ corresponding to m and n coincide. Therefore, the two maps m and n must coincide. ∎

For any two elements $w, z : X \longrightarrow Z$ we write $wz : X \longrightarrow Z$ for multiplication in Z. Because the isomorphism $\mathscr{E}(X, Z) \cong \mathcal{Z}(X)$ takes multiplication in Z to composition of automorphisms, by (4) we have

$$b \cdot wz(x) = (b \cdot w(x)) \cdot z(x)\ .$$

We may suppress the parameter X and write this as

$$b \cdot (wz) = (b \cdot w) \cdot z\ . \tag{5}$$

In the same manner we also have

$$b \cdot e = b\ ,$$

where $e : 1 \longrightarrow Z$ denotes the unit element.

4.8. Lemma. *Let* $\theta_Z(w, z) = w * z$. *For any object* E *of* $\mathscr{E}$, *the action* θ_E *satisfies*

$$\forall w, z \in Z\, \forall b \in E\ ,\ b \cdot z(w * z) = b \cdot (wz)\ .$$

Proof. By the naturality of θ for the projection maps $E \times Z \longrightarrow Z$ and $E \times Z \longrightarrow E$, it follows that $\theta_{E\times Z}(b, w, z) = (b \cdot z, w * z)$. By the naturality of θ for the map θ_E

$$\begin{tikzcd}
(E \times Z) \times Z \arrow[r, "\theta_{E\times Z}"] \arrow[d, "\theta_E \times Z"'] & E \times Z \arrow[d, "\theta_E"] \\
E \times Z \arrow[r, "\theta_E"] & E
\end{tikzcd}$$

and using (5) we have $b \cdot z(w * z) = (b \cdot z) \cdot (w * z) = (b \cdot w) \cdot z = b \cdot (wz)$. The middle equality is naturality of θ for θ_E . ∎

4.9. Proposition. *For any object* E *of* $\mathscr{E}$, θ_E *is a group action internal to* $\mathscr{E}$, *and* θ_Z *is conjugation. Moreover, every map of* $\mathscr{E}$ *is equivariant with respect to these actions.*

Proof. We have only to show that θ_Z is the conjugation action. Consider the two morphisms $Z \times Z \longrightarrow Z$ in $\mathscr{E}$: $(w, z) \mapsto z(w * z)$ and multiplication $(w, z) \mapsto wz$. By Lemmas 4.8 and 4.7, the two maps are equal: $z(w * z) = wz$. Hence, $w * z = z^{-1}wz$. ∎

4.10. Remark. We may define the action $\theta_E : E \times Z \longrightarrow E$ another way. Given $(f, t) : X \longrightarrow E \times Z$, let t also denote the automorphism of $\mathscr{E}/X \longrightarrow \mathscr{E}$ corresponding to the map $t : X \longrightarrow Z$. Define

$$\theta_E(f, t) = f \cdot t : X \longrightarrow E \ , \quad f \cdot t(x) = t_E(fx, x) \ ,$$

where $t_E : E \times X \longrightarrow E$.

By Proposition 4.9, the automorphism θ of $\mathscr{E}/Z \longrightarrow \mathscr{E}$ corresponding to the identity $Z \longrightarrow Z$ defines a crossed topos

$$\theta : Z \longrightarrow \mathscr{E} \ ,$$

which we call *the standard crossed topos on $\mathscr{E}$.*

4.11. Proposition. *The standard crossed topos is the terminal object of* $\mathsf{XTop}(\mathscr{E})$.

Proof. Suppose we are given an arbitrary crossed topos $\delta : G \longrightarrow \mathscr{E}$. Define a morphism $\delta \longrightarrow \theta$ of crossed toposes as follows. Observe that δ itself provides an automorphism of $\mathscr{E}/G \longrightarrow \mathscr{E}$ because each component map $E \times G \longrightarrow E \times G$, $(b, g) \mapsto (bg, g)$, is an isomorphism, where $bg = \delta_E(b, g)$. Under the group isomorphism $\mathscr{E}(G, Z) \cong \mathcal{Z}(G)$ this automorphism corresponds to a morphism $m : G \longrightarrow Z$ in $\mathscr{E}$, such that

$$\forall b \in E \ , \quad bg = b \cdot m(g) \tag{6}$$

for every object E. This says that m is a morphism of crossed toposes, but of course we must show that m is a group homomorphism. To show that m preserves the unit element, let e_G denotes the unit of G. By (6) we have

$$\forall b \in E \ , \quad b = be_G = b \cdot m(e_G) \ .$$

This says that $m(e_G) : 1 \longrightarrow Z$ corresponds to the identity automorphism of the identity geometric morphism on $\mathscr{E}$. The latter corresponds to $e : 1 \longrightarrow Z$, so $m(e_G) = e$. As for preservation of multiplication, we have

$$b \cdot m(gh) = b(gh) = (bg)h = (b \cdot m(g)) \cdot m(h) = b \cdot (m(g)m(h)) \ .$$

By Lemma 4.7, we have $m(gh) = m(g)m(h)$. Finally, we claim that m is the only morphism of crossed toposes $\delta \longrightarrow \theta$. Let $n : G \longrightarrow Z$ be a morphism of crossed toposes $\delta \longrightarrow \theta$, so that $bg = b \cdot n(g)$. Therefore, $b \cdot m(g) = bg = b \cdot n(g)$, so that $m = n$ by Lemma 4.7. ∎

4.12. Example. We compute the standard crossed topos on a presheaf topos $\mathscr{E} = \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$. It is convenient to denote the representable presheaf associated with an object C of $\mathbb{C}$ with the same symbol C. The isotropy group is given by the presheaf

$$Z(C) = \{\text{automorphisms of } \mathscr{E}/C \longrightarrow \mathscr{E}\} \ .$$

The standard action $P \times Z \xrightarrow{\theta_P} P$ on a presheaf P is the following natural transformation: at stage C, $\theta_{P,C}(x,t)$ is the element

$$C \xrightarrow{(x,1_C)} P \times C \xrightarrow{t_P} P \ ,$$

where $x \in P(C)$ and $t \in Z(C)$. Briefly, $\theta_{P,C}(x,t) = t_P(x, 1_C)$. Because $\mathscr{E}/C$ is equivalent to presheaves on $\mathbb{C}/C$, automorphisms of $\mathscr{E}/C \longrightarrow \mathscr{E}$ correspond to automorphisms of $\mathbb{C}/C \longrightarrow \mathscr{E}/C \longrightarrow \mathscr{E}$; but this composite factors through the full and faithful Yoneda embedding $\mathbb{C} \longrightarrow \mathscr{E}$, so these correspond to automorphisms of $\mathbb{C}/C \longrightarrow \mathbb{C}$. Therefore, Z is equivalently given as the presheaf

$$Z(C) = \{\text{automorphisms of } \mathbb{C}/C \longrightarrow \mathbb{C}\} \ .$$

Moreover, the standard crossed topos is given by $\theta_{P,C}(x,t) = P(t_C)(x)$, where t_C is the component of t at 1_C. An element of $Z(C)$ can be regarded as an automorphism t of C together with, for each $f : D \longrightarrow C$, a 'restriction' automorphism $t_{|f} : D \longrightarrow D$ satisfying $t_{|f}g = gt_{|fg}$ for any $g : E \longrightarrow D$. In particular, $tf = ft_{|f}$. In the groupoid case (Egs. 1.2, 3.2) we have $t_{|f} = f^{-1}tf$. Moreover, the action by f (making Z a presheaf) sends $t \mapsto t_{|f}$, so that this is conjugation in the groupoid case.

4.13. Remark. Further to the previous example, we mention that the group of global sections of Z (i.e. the group $\mathsf{Set}^{\mathbb{C}^{op}}(1, Z)$) may be identified with the group of automorphisms of the identity functor on $\mathbb{C}$, which is sometimes referred to as the *centre*[2] of $\mathbb{C}$. We stress however that the latter may contain less information than the isotropy group: even in the case where $\mathbb{C}$ is a group G the centre may be trivial while the isotropy group $\overline{G}$ is not.

By Prop. 4.11, a topos $\mathscr{E}$ is anisotropic (Def. 4.3) if and only if the trivial crossed topos $1 \longrightarrow \mathscr{E}$ (which is the identity geometric morphism on $\mathscr{E}$) is the terminal object of its isotropy category $\mathsf{XTop}(\mathscr{E})$.

The functor

$$\mathsf{Ab}(\mathscr{E}) \longrightarrow \mathsf{XTop}(\mathscr{E}) \ , \tag{7}$$

which sends an Abelian group G to the trivial crossed topos $G \longrightarrow \mathscr{E}$, is full and faithful. We have the following.

4.14. Proposition. *A topos $\mathscr{E}$ is anisotropic if and only if the inclusion* (7) *is an equivalence.*

4.15. Remark. By Cor. 3.18, a localic topos is anisotropic, but not conversely. For instance, the classifying topos $\mathscr{B}(S/\mu)$ of the fundamental quotient of an inverse semigroup S is anisotropic but it may not be localic.

[2]We thank André Joyal for drawing our attention to this aspect.

4.16. Isotropy quotients We turn to another aspect of crossed toposes, namely their quotient toposes. We begin with a result which explains the geometric nature of such a quotient. Herein we shall refer to an isomorphism coinserter in the 2-category of Grothendieck toposes simply as a topos coequalizer: it is formed as the isomorphism inserter of inverse image functors. (However, in the case of interest below it actually agrees with the strict notion.)

4.17. Proposition. *Let $\delta : G \longrightarrow \mathscr{E}$ be a crossed topos. Let $\mathscr{E}_\delta$ be the full subcategory of $\mathscr{E}$ on all objects X such that δ_X is trivial. Then $\mathscr{E}_\delta$ is a topos and its inclusion into $\mathscr{E}$ is the inverse image of a connected, atomic geometric morphism $\psi : \mathscr{E} \longrightarrow \mathscr{E}_\delta$ making*

$$\mathscr{B}(\mathscr{E};G) \underset{\gamma}{\overset{\delta}{\rightrightarrows}} \mathscr{E} \xrightarrow{\psi} \mathscr{E}_\delta\ .$$

a topos coequalizer (in the above sense)*. Moreover, $\psi_* G$ is an Abelian group, and $\psi^*\psi_* G$ coincides with the centralizer of G in $\mathscr{E}$.*

Proof. First observe that if an object of $\mathscr{B}(\mathscr{E};G)$ is isomorphic to a trivial action, then it is trivial. In other words, the subcategory $\mathscr{E}_\delta$ is a topos coequalizer in the sense we mentioned above. The inclusion ψ^* of $\mathscr{E}_\delta$ in $\mathscr{E}$ has both adjoints: the right adjoint ψ_* associates with an object X the equalizer $\psi_* X$ of the transposes $X \longrightarrow X^G$ of δ_X and the projection: $\psi_* X$ consists of all x stabilized by δ_X. The left adjoint $\psi_! X = X/G$ is the coequalizer of δ_X and the projection $X \times G \longrightarrow X$, also called the orbit space of δ_X. Therefore, $\mathscr{E}_\delta$ is a topos by the well-known result that a coreflexive subcategory (not necessarily full) of a topos that is closed under finite limits is a topos, as the inclusion functor is therefore comonadic for a finite limit preserving comonad on the given topos [Johnstone '02].

We verify that ψ is locally connected: we must show that if the left square below is a pullback, then so is the right one.

$$\begin{array}{ccc} P & \longrightarrow & \psi^* X \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \psi^* f} \\ W & \longrightarrow & \psi^* Y \end{array} \qquad \begin{array}{ccc} \psi_! P & \longrightarrow & X \\ \downarrow{\scriptstyle \psi_! p} & & \downarrow{\scriptstyle f} \\ \psi_! W & \longrightarrow & Y \end{array}$$

Equivalently, we assert that in the diagram below (in $\mathscr{E}$) the right square is a pullback when the outside square is one.

$$\begin{array}{ccccc} P & \longrightarrow & P/G & \longrightarrow & X \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f/G} & & \downarrow{\scriptstyle f} \\ W & \longrightarrow & W/G & \longrightarrow & Y \end{array}$$

An intuitive explanation that the left square is a pullback is as follows: δ_X and δ_Y are trivial, and δ_P is given by $(w,x)g = (wg,xg) = (wg,x)$. Thus, an orbit of δ_P (i.e., an element of P/G) is given by an orbit of δ_W and an element of X that agree when mapped to Y. In other words, we have $P/G \cong W/G \times_Y X$. If the reader does not find this explanation entirely convincing, then (s)he may appreciate the following argument kindly offered by the referee. It suffices to show that the left square above is a pullback because its rows are stable coequalizers, and since the outer square in the same diagram is a pullback. Let m denote the intervening morphism, which is a morphism over P/G, from P to the pullback of $W \longrightarrow W/G$ and f/G. A diagram chase, using the fact that the kernel pair of $W \longrightarrow W/G$ is δ_W and the projection $W \times G \longrightarrow W$ (and likewise for P), shows that the pullback of m along $P \longrightarrow P/G$ is an isomorphism. Therefore, m is an isomorphism because pullback along an epimorphism reflects isomorphisms.

The subobject classifier $\Omega_{\mathscr{E}}$ is an object of $\mathscr{E}_\delta$ by Prop. 3.16. It follows that $\Omega_{\mathscr{E}}$ is the subobject classifier of $\mathscr{E}_\delta$, preserved by ψ^*. Hence, ψ is atomic. Finally, $\psi_* G$ is an Abelian group in $\mathscr{E}_\delta$, and $\psi^*\psi_* G$ is the centralizer of G because δ_G is conjugation. ∎

4.18. Definition. *We call* $\mathscr{E} \overset{\psi}{\twoheadrightarrow} \mathscr{E}_\delta$ the isotropy quotient of a crossed topos δ.

4.19. Remark. In general, the isotropy quotient of a crossed topos may not be anisotropic. For example, if G is an Abelian group in $\mathscr{E}$, so that the structure geometric morphism $\gamma : \mathscr{B}(\mathscr{E}; G) \longrightarrow \mathscr{E}$ is a (trivial) crossed topos, then the isotropy quotient $\mathscr{E}_\gamma$ is $\mathscr{E}$ itself. However, the isotropy quotient of the standard crossed topos on an étale groupoid is anisotropic (Prop. 7.11).

The remainder of this section briefly addresses the the connection between crossed toposes on a topos on the one hand and geometric morphisms out of that topos on the other. A full treatment of this topic would lead us too far astray from the main questions of the paper. We therefore confine ourselves here to a sketch of the basic adjointness between locally connected geometric morphisms and crossed toposes, leaving the general theory for a follow-up paper. This material is not used elsewhere in the paper.

The following proposition explains, at least for the locally connected case, how a geometric morphism between toposes gives rise to a morphism of standard crossed toposes.

4.20. Proposition. *Let $\phi : \mathscr{E} \longrightarrow \mathscr{F}$ be locally connected. Let $Z_{\mathscr{E}}$ and $Z_{\mathscr{F}}$ be the isotropy groups of $\mathscr{E}$ and $\mathscr{F}$, respectively. Then there is a group homomorphism*

$$m : Z_{\mathscr{E}} \longrightarrow \phi^* Z_{\mathscr{F}}$$

such that $(\phi, m) : \theta^{\mathscr{E}} \longrightarrow \theta^{\mathscr{F}}$ is a morphism of standard crossed toposes.

Proof. As usual, we write $\phi_! \dashv \phi^*$. Let $t : X \longrightarrow Z_{\mathscr{E}}$ be an element of $Z_{\mathscr{E}}$. In other words, let t be an automorphism of $\mathscr{E}/X \longrightarrow \mathscr{E}$. In the diagram

$$\begin{tikzcd} \mathscr{E}/X \arrow[r, "\overline{\phi}"] \arrow[d] & \mathscr{F}/\phi_! X \arrow[d] \\ \mathscr{E} \arrow[r, "\phi"] & \mathscr{F} \end{tikzcd}$$

the geometric morphism $\overline{\phi}$ is connected in the sense that $\overline{\phi}^*$ is full and faithful. Therefore, the whiskering ϕt, which is an automorphism of $\mathscr{E}/X \longrightarrow \mathscr{F}$, factors through $\overline{\phi}$ by an automorphism we shall denote $m(t)$. It follows that this defines a group homomorphism $m : Z_{\mathscr{E}} \longrightarrow \phi^* Z_{\mathscr{F}}$. For any object F of $\mathscr{F}$, the component

$$m(t)_F : F \times \phi_! X \longrightarrow F$$

of the automorphism $m(t)$ is equal to the transpose under $\phi_! \dashv \phi^*$ of

$$t_{\phi^* F} : \phi^* F \times X \longrightarrow \phi^* F \ .$$

Moreover, (ϕ, m) is a morphism of standard crossed toposes: for every object F of $\mathscr{F}$,

$$\begin{tikzcd}
\phi^* F \times Z_{\mathscr{E}} \arrow[d, "1\times m"'] \arrow[dr, "\theta_{\phi^* F}"] & \\
\phi^* F \times \phi^* Z_{\mathscr{F}} \arrow[r, "\phi^*\theta_F"'] & \phi^* F
\end{tikzcd}$$

commutes. ∎

For any topos $\mathscr{E}$, let $\mathsf{LC}(\mathscr{E})$ denote the category of locally connected geometric morphisms with domain $\mathscr{E}$. An object of this category is a locally connected geometric morphism $\mathscr{E} \longrightarrow \mathscr{F}$, and a morphism is a commutative triangle of geometric morphisms under $\mathscr{E}$. We describe an adjointness $\Psi \dashv \Psi_*$:

$$\begin{tikzcd}
\mathsf{XTop}(\mathscr{E}) \arrow[r, bend left, "\Psi"] \arrow[r, phantom, "\perp"] & \mathsf{LC}(\mathscr{E}) \arrow[l, bend left, "\Psi_*"]
\end{tikzcd} ,$$

where $\Psi(\delta)$ is the quotient $\psi : \mathscr{E} \longrightarrow \mathscr{E}_\delta$, for a crossed topos $\delta : G \longrightarrow \mathscr{E}$. On the other hand, if $\phi : \mathscr{E} \longrightarrow \mathscr{F}$ is locally connected, then define a crossed topos $\Psi_*\phi$ on $\mathscr{E}$ as the restriction of the standard crossed topos on $\mathscr{E}$ to the kernel of the canonical homomorphism associated with ϕ, as illustrated by the following diagram.

$$\begin{tikzcd}
K \arrow[dr, hook] \arrow[drr, bend left, "n"] \arrow[ddr, bend right, "\Psi_*\phi"'] & & \\
& Z_{\mathscr{E}} \arrow[r, "m"] \arrow[d, "\theta"] & \phi^* Z_{\mathscr{F}} \\
& \mathscr{E} &
\end{tikzcd}$$

Moreover, $(\phi, n) : \Psi_*\phi \longrightarrow \theta^{\mathscr{F}}$ is a morphism of crossed toposes.

In particular, a locally connected geometric morphism $\phi : \mathscr{E} \longrightarrow \mathscr{F}$ factors through the isotropy quotient of $\mathscr{E}$ associated with the crossed topos $\Psi_*\phi$. This observation suggests that isotropy quotients are part of a factorization system for geometric morphisms, but we leave a thorough investigation of the question for elsewhere.

5. Algebras as crossed objects

Having defined algebras for the isotropy monad on the level of grouped toposes, we now wish to give a more concrete characterization on the level of sites. It is soon apparent that the characterization that we give is a broad generalization of the notion of a crossed module (Porter explains this notion in detail and gives many examples and applications [Porter '10]).

We begin by defining crossed modules for a small category and prove that such structures are equivalent to crossed toposes on the associated presheaf topos. We then generalize to arbitrary subcanonical sites and make the equivalence functorial. After that, we prove one of the main results of the paper, namely the "External/Internal" Theorem, which relates crossed modules internal to a topos to crossed toposes on that topos. Finally, we consider crossed modules from a double category perspective and characterize the double categories arising from our generalized crossed modules.

5.1. Crossed $\mathbb{C}$-modules For the purposes of this section $\mathbb{C}$ denotes a small category. We shall consider the topos $\mathscr{E} = \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$ of presheaves on $\mathbb{C}$, as well as a group G in $\mathscr{E}$, i.e., a presheaf of groups on $\mathbb{C}$. When $x \in G(C)$ is an element of the group $G(C)$ and $f : D \longrightarrow C$ is a morphism in $\mathbb{C}$, we write $x_{|f} \in G(D)$ for the restriction of x along f.

Our aim is to give a full description of the isotropy algebras on $(\mathscr{E}, G)$ in elementary terms, i.e., in terms of the given category $\mathbb{C}$ and group G. We start by considering a strict $\mathcal{I}$-algebra θ on $(\mathscr{E}, G)$; by Corollary 3.9, θ is determined by the functor θ^*, which satisfies $\theta^* X = (X, \theta_X)$ for some action θ_X on X, while θ_G is conjugation. In turn, the functor θ^* is determined completely by its values on the representable presheaves C. Writing $\theta_C : C \times G \longrightarrow C$ for the action with which θ endows C we note that the component of θ_C at an object D of $\mathbb{C}$ is a function

$$\theta_{C,D} : \mathbb{C}(D, C) \times G(D) \longrightarrow \mathbb{C}(D, C) \; .$$

Because θ^* is functorial, a morphism $f : C \longrightarrow C'$ in $\mathbb{C}$ is equivariant with respect to the actions θ_C and $\theta_{C'}$. In other words, the diagram

$$\begin{array}{ccc} C \times G & \xrightarrow{\theta_C} & C \\ {\scriptstyle f \times G}\downarrow & & \downarrow{\scriptstyle f} \\ C' \times G & \xrightarrow{\theta_{C'}} & C' \end{array}$$

commutes; and when we evaluate at C we obtain

$$\begin{array}{ccc} \mathbb{C}(C, C) \times G(C) & \xrightarrow{\theta_{C,C}} & \mathbb{C}(C, C) \\ {\scriptstyle \mathbb{C}(C,f) \times G}\downarrow & & \downarrow{\scriptstyle \mathbb{C}(C,f)} \\ \mathbb{C}(C, C') \times G(C) & \xrightarrow{\theta_{C',C}} & \mathbb{C}(C, C') \; . \end{array}$$

It is clear from this that θ_C is determined by the value $1_C \in \mathbb{C}(C,C)$: we therefore define

$$\delta_C : G(C) \longrightarrow \mathbb{C}(C,C)\ ; \qquad \delta_C(x) = \theta_{C,C}(1_C, x)\ .$$

Because θ_C is an action, we find that δ_C is in fact a homomorphism from the group $G(C)$ to the monoid $\mathbb{C}(C,C)$. Of course, we may recover $\theta_{C,D}$ via

$$\theta_{C,D}(f,x) = f\delta_D(x)\ .$$

Next we use naturality of θ_C to obtain, for each $f : D \longrightarrow C$, a commutative diagram

$$\begin{tikzcd}
\mathbb{C}(C,C) \times G(C) \arrow[r, "\theta_{C,C}"] \arrow[d, "\mathbb{C}(f,C)\times G(f)"'] & \mathbb{C}(C,C) \arrow[d, "\mathbb{C}(f,C)"] \\
\mathbb{C}(D,C) \times G(D) \arrow[r, "\theta_{C,D}"] & \mathbb{C}(D,C)\ .
\end{tikzcd}$$

Evaluating at the identity and using the definition $\delta_C(x) = \theta_C(1_C, x)$ gives

$$\delta_C(x)f = f\delta_D(x_{|f})\ .$$

This means that δ is *extranatural* in the sense that for any $f : D \longrightarrow C$, the diagram

$$\begin{tikzcd}
G(C) \arrow[r, "\delta_C"] \arrow[dd, "G(f)"'] & \mathbb{C}(C,C) \arrow[dr, "\mathbb{C}(f,C)"] & \\
& & \mathbb{C}(D,C) \\
G(D) \arrow[r, "\delta_D"'] & \mathbb{C}(D,D) \arrow[ur, "\mathbb{C}(D,f)"'] &
\end{tikzcd}$$

commutes. Finally, consider a general object X of $\mathscr{E}$. The action θ_X on X evaluated at an object C may be re-expressed as

$$X(C) \times G(C) \xrightarrow{X(c)\times\delta_C} X(C) \times \mathbb{C}(C,C) \longrightarrow X(C)\ , \tag{8}$$

where the right-hand map is simply the action of the presheaf X. This follows once again from a Yoneda-like argument, using the fact that all morphisms $C \longrightarrow X$ are equivariant. Letting X be the group G itself, this action should be conjugation. This gives the condition:

$$g_{|\delta_C(h)} = h^{-1}gh\ .$$

Thus, we make the following definition.

5.2. Definition. *Let G be a presheaf of groups on a category $\mathbb{C}$. A* crossed $\mathbb{C}$-module *is a family of homomorphisms $\delta_C : G(C) \longrightarrow \mathbb{C}(C,C)$ indexed by objects of $\mathbb{C}$, such that the following two conditions hold:*

$$\begin{array}{lrcll} (1) & \delta_C(x)f & = & f\delta_D(x_{|f}) \quad \textit{for all } f : D \longrightarrow C, x \in G(C) & \textit{(extranaturality)} \\ (2) & g_{|\delta_C(h)} & = & h^{-1}gh \quad \textit{for all } g,h \in G(C) & \textit{(Peiffer identity)}\,. \end{array}$$

To say that δ_C is a homomorphism is to say that it takes the group operation in $G(C)$ to composition of morphisms in $\mathbb{C}(C,C)$. δ_C factors through $\mathrm{Aut}(C)$ because $G(C)$ is a group.

We denote such a crossed $\mathbb{C}$-module by $G \xrightarrow{\delta} \mathbb{C}$. This notation and nomenclature is justified by the fact that if $\mathbb{C}$ is a group, then we recover the classical notion of crossed module [Porter '10]. The preceding discussion shows most of the following fact.

5.3. Theorem. *Strict crossed toposes $G \longrightarrow \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$ are in 1-1 correspondence with crossed $\mathbb{C}$-modules $G \longrightarrow \mathbb{C}$.*

Proof. Given a crossed $\mathbb{C}$-module $\delta : G \longrightarrow \mathbb{C}$, we define the inverse image functor of a crossed topos θ to be $\theta^* X = (X, \theta_X)$ where the component of θ_X at C is given by (8). θ_X is functorial in $\mathbb{C}$ because δ is extranatural, and θ_X is an action because the δ_C are homomorphisms. It follows from the Peiffer identity that θ_G is conjugation. ■

Next we indicate how to generalize this to a subcanonical site $(\mathbb{C}, \mathcal{J})$, i.e., where the representable presheaves are sheaves. In this case the same reasoning as above gives the following result.

5.4. Theorem. *Let $\mathcal{J}$ be a subcanonical topology on $\mathbb{C}$, and let G be a $\mathcal{J}$-sheaf of groups. Then strict crossed toposes $G \longrightarrow \mathsf{Sh}(\mathbb{C}, \mathcal{J})$ are in 1-1 correspondence with crossed $\mathbb{C}$-modules $\delta : G \longrightarrow \mathbb{C}$.*

Proof. If $\delta : G \longrightarrow \mathbb{C}$ is a crossed $\mathbb{C}$-module, where G is now a $\mathcal{J}$-sheaf, then

$$\begin{tikzcd} \mathcal{I}(\mathsf{Sh}(\mathbb{C},\mathcal{J}), G) \arrow[r, "\mathcal{I}(i)"] \arrow[d, "\theta'"'] & \mathcal{I}(\mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}, i_* G) \arrow[d, "\theta"] \\ (\mathsf{Sh}(\mathbb{C},\mathcal{J}), G) \arrow[r, "i"] & (\mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}, i_* G) \end{tikzcd}$$

is a commutative square of grouped toposes. Here $i = (i^* \dashv i_*)$ is the subtopos inclusion of sheaves into presheaves, the crossed topos θ corresponds to δ, and θ' is the restriction of θ to the sheaf subtopos. Note that the diagram illustrates at the same time that the inclusion i is a morphism of crossed toposes. On the other hand, when $\mathcal{J}$ is subcanonical, a crossed topos $G \longrightarrow \mathsf{Sh}(\mathbb{C}, \mathcal{J})$ gives a crossed $\mathbb{C}$-module in just the same way as in the discussion preceding Def. 5.2. ■

Theorem 5.4 gives another explanation of the fact that a localic topos is anisotropic (Remark 4.15). For a crossed topos $G \longrightarrow \mathsf{Sh}(X)$ corresponds to a crossed $O(X)$-module $\delta_U : G(U) \longrightarrow O(X)(U, U)$, where $O(X)$ is the frame of a locale X. Because $O(X)(U, U)$ is trivial, we can only have trivial crossed $O(X)$-modules.

5.5. Morphisms of crossed $\mathbb{C}$-modules Our next aim is to make the passage from crossed $\mathbb{C}$-modules $G \longrightarrow \mathbb{C}$ to crossed toposes functorial. We shall restrict our attention to the untopologized version; this can be extended to crossed $\mathbb{C}$-modules when $\mathbb{C}$ carries a Grothendieck topology and G is a sheaf of groups, but we prefer to simplify the exposition.

The following definition of morphisms of crossed modules is a straightforward extension of the usual one.

5.6. Definition. *Let $G \xrightarrow{\delta} \mathbb{C}$ be a crossed $\mathbb{C}$-module, and $H \xrightarrow{\zeta} \mathbb{D}$ a crossed $\mathbb{D}$-module. A* morphism of crossed modules *$\delta \longrightarrow \zeta$ is a pair (F, m), where $F : \mathbb{C} \longrightarrow \mathbb{D}$ is a functor and $m_C : G(C) \longrightarrow HF(C)$ a family of group homomorphisms indexed by the objects of $\mathbb{D}$. These are subject to the conditions:*

1. *the family m_C is natural in C, i.e., for any $k : C' \longrightarrow C$ in $\mathbb{C}$ and any $x \in G(C')$, we have $m_{C'}(G(k)(x)) = HF(k)(m_C(x))$;*
2. *for every object C of $\mathbb{C}$, the diagram*

$$\begin{tikzcd} G(C) \arrow[r, "\delta_C"] \arrow[d, "m_C"'] & \mathbb{C}(C, C) \arrow[d, "F_{C,C}"] \\ HF(C) \arrow[r, "\zeta_{F(C)}"] & \mathbb{D}(F(C), F(C)) \end{tikzcd}$$

   commutes.

Given two such morphisms $(F, m), (F', m') : \delta \longrightarrow \zeta$, a 2-cell $\alpha : (F, m) \Rightarrow (F', m')$ is a natural transformation $\alpha : F \Rightarrow G$ for which $m' = H\alpha.m$.

We write XMod for the 2-category of crossed modules in this generalized sense. When we fix $\mathbb{C}$ there is a locally discrete sub-2-category $\mathsf{XMod}/\mathbb{C}$ of crossed $\mathbb{C}$-modules and morphisms for which the functor part is the identity.

5.7. Theorem. *The assignment sending a crossed module $\delta : G \longrightarrow \mathbb{C}$ to its crossed topos $G \longrightarrow \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$* (as in Theorem 5.3) *is the object part of a 2-functor $\mathsf{XMod} \longrightarrow \mathsf{XTop}$. This functor is faithful on 1-cells, and fully faithful on 2-cells.*

Proof. Consider a morphism $(F, m) : \delta \longrightarrow \zeta$ of crossed modules. The inverse image functor F^* of the geometric morphism $F : \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}} \longrightarrow \mathsf{Set}^{\mathbb{D}^{\mathrm{op}}}$ associated with F sends a presheaf P to $F^*P = PF$. The first condition on (F, m) implies that $m : G \longrightarrow F^*H$ is a morphism of presheaves of groups; hence that (F, m) is a morphism of grouped toposes $(\mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}, G) \longrightarrow (\mathsf{Set}^{\mathbb{D}^{\mathrm{op}}}, H)$.

The second condition on (F, m) states precisely that this morphism of grouped toposes is a morphism of algebras: indeed, consider a presheaf X and an object C of $\mathbb{C}$; in the commutative diagram

$$\begin{tikzcd}
XF(C) \times G(C) \arrow[r, "1\times\delta_C"] \arrow[d, "1\times m_C"'] & XF(C) \times \mathbb{C}(C,C) \arrow[d, "1\times F_{C,C}"] \arrow[dr, "\mathrm{act}"] & \\
XF(C) \times HF(C) \arrow[r, "1\times\zeta_{F(C)}"'] & XF(C) \times \mathbb{D}(F(C), F(C)) \arrow[r, "\mathrm{act}"'] & XF(C)
\end{tikzcd}$$

the two composites are the components at C of the G-actions of $\delta^* F^* X$ and of $(\overline{F}m)^*\zeta^*(X)$, respectively. The triangle commutes because F is a functor. It is immediately clear that this is faithful.

Finally, on the level of 2-cells we only have to note that the condition on 2-cells of crossed module morphisms says exactly that the induced natural transformation is actually a 2-cell between morphisms of grouped toposes, and hence also an algebra 2-cell (Def. 3.14). ∎

Of course, the construction is not full on 1-cells because the only morphisms in its image are those induced by morphisms of sites. However, for fixed $\mathbb{C}$, it gives an equivalence of locally discrete 2-categories

$$\mathsf{XMod}/\mathbb{C} \simeq \mathsf{XTop}(\mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}) \ ,$$

where the category on the right is defined in Def. 3.14. This equivalence may be extended to one

$$\mathsf{XMod}/(\mathbb{C}, \mathcal{J}) \simeq \mathsf{XTop}(\mathsf{Sh}(\mathbb{C}, \mathcal{J}))$$

when $\mathcal{J}$ is subcanonical, and where a crossed $\mathbb{C}$-module $G \longrightarrow \mathbb{C}$ is a crossed $(\mathbb{C}, \mathcal{J})$-module just when G is a sheaf.

5.8. The external/internal theorem This section presents a result which explains the relationship between crossed modules internal to a topos and crossed toposes on that topos. We begin with a result on base change along a group homomorphism.

5.9. Proposition. *Consider a crossed topos $\theta : G \longrightarrow \mathscr{E}$ and a group homomorphism $m : H \longrightarrow G$ in $\mathscr{E}$. Then the following are equivalent:*

1. *m (and θ_H) is a crossed module internal to $\mathscr{E}$;*

2. *the diagram*

$$\begin{tikzcd}
H \times H \arrow[d, "1\times m"'] \arrow[dr, "\mathrm{conj}"] & \\
H \times G \arrow[r, "\theta_H"'] & H
\end{tikzcd}$$

is commutative, i.e., the action θ_H restricts along m to the conjugation action;

3. *there is a unique crossed topos* $\delta = \theta m : H \longrightarrow \mathscr{E}$ *making the following diagram of grouped toposes commute:*

$$\begin{array}{ccc} (\mathscr{B}(\mathscr{E};H), \overline{H}) & \xrightarrow{\mathcal{I}(\mathrm{id},m)} & (\mathscr{B}(\mathscr{E};G), \overline{G}) \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\theta} \\ (\mathscr{E}, H) & \xrightarrow{(\mathrm{id},m)} & (\mathscr{E}, G) \end{array}$$

and hence making (id, m) *a morphism of crossed toposes.*

Proof. The equivalence between (1) and (2) is immediate: given $m : H \longrightarrow G$ in $\mathscr{E}$, the algebra θ makes H into a G-module and m into an equivariant map; the diagram in condition (2) expresses the Peiffer identity. Given condition (2), we define δ in the only possible manner, namely by setting $\delta^*(X) = (X, \delta_X)$ where δ_X is the action

$$X \times H \xrightarrow{1\times m} X \times G \xrightarrow{\theta_X} X \ .$$

This is clearly a retraction of the unit, and condition (2) says exactly that it is a morphism of grouped toposes. The converse is equally straightforward. ∎

Note that δ may be constructed as the following composite geometric morphism:

$$\mathscr{B}(\mathscr{E};H) \xrightarrow{\eta^{H^{\mathrm{op}}}} \mathscr{B}(\mathscr{E};G)^{H^{\mathrm{op}}} \longrightarrow \mathscr{B}(\mathscr{E};G) \xrightarrow{\theta} \mathscr{E}$$

where the morphism in the middle is the crossed topos associated with the crossed module m.

5.10. Corollary. ('External/Internal' theorem) *Let* $\delta : G \longrightarrow \mathscr{E}$ *be a crossed topos. Then* $\mathsf{XTop}(\mathscr{E})/\delta$ *is equivalent to the category whose objects are crossed modules* $H \longrightarrow G$ (with δ_H) *internal to* $\mathscr{E}$ *and whose morphisms are group homomorphisms* $H \longrightarrow K$ *over* G.

$\mathsf{XTop}(\mathscr{E})$ *is equivalent to the category whose objects are crossed modules* $H \longrightarrow Z$ (with θ_H) *internal to* $\mathscr{E}$ *and whose morphisms are group homomorphisms* $H \longrightarrow K$ *over* Z, *where* $\theta : Z \longrightarrow \mathscr{E}$ *is the standard crossed topos. The equivalence is mediated by the group isomorphism* $\mathscr{E}(H, Z) \cong \mathcal{Z}(H)$*: the maps* $H \longrightarrow Z$ *that are crossed modules internal to* $\mathscr{E}$ (Prop. 5.9, (2)) *correspond to those automorphisms of* $\mathscr{E}/H \longrightarrow \mathscr{E}$ *that give crossed toposes.*

It is well known in crossed module theory that even if a homomorphism $m : H \longrightarrow G$ (where H is a G-module via an action δ) does not satisfy the Peiffer identity, we can always force it by dividing by the so-called Peiffer commutator subgroup for δ, i.e., the subgroup $P = P_{\delta,m}$ of H generated by elements of the form

$$\delta_H(h, m(x))^{-1}(x^{-1}hx) \ .$$

This factors m as

$$H \longrightarrow H/P \xrightarrow{\widetilde{m}} G$$

where $\widetilde{m}$ is a crossed module. Putting this together we have the following result.

5.11. Corollary. *There is an action*

$$\mathsf{XTop}(\mathscr{E}, G) \times \mathsf{Grp}(\mathscr{E})/G \longrightarrow \mathsf{XTop}(\mathscr{E}) \;,$$

which sends $\delta : G \longrightarrow \mathscr{E}$ *and* $m : H \longrightarrow G$ *to the crossed topos* $\delta\widetilde{m} : H/P \longrightarrow \mathscr{E}$ *.*

Proof. The only aspect that needs addressing is functoriality. However, this follows readily from the fact that dividing by the Peiffer subgroup is functorial in the second variable, and that the construction of $\delta\widetilde{m}$, as explained above, is given simply by composition. ∎

5.12. Remark. A special case of Cor. 5.11 may be worth mentioning. An endomorphism $m : G \longrightarrow G$ is a crossed module for conjugation if and only if it is 'over the center,' meaning that

$$\begin{tikzcd} G \arrow[rr, "m"] \arrow[dr] & & G \arrow[dl] \\ & G/C & \end{tikzcd}$$

commutes, where C denotes the center of G. The action in Cor. 5.11 may be restricted to the monoid of endomorphisms of G over the center, thereby obtaining a functorial action of the monoid on $\mathsf{XTop}(\mathscr{E}, G)$.

5.13. Internal categories Ordinary crossed modules $\delta : G \longrightarrow H$, where H is a group, may be viewed as a one-dimensional incarnation of a 2-dimensional structure, namely a categorical group. More precisely, there is an isomorphism of categories

$$\mathsf{XMod} \simeq \mathfrak{Cat}(\mathsf{Grp}) \;,$$

where on the left we have the category of crossed modules, and on the right the category of categories internal to the category of groups. The equivalence makes use of the Grothendieck construction: it associates with a crossed module $\delta : G \longrightarrow H$ the category whose group of objects is H, and whose group of morphisms is $H \rtimes G$. The codomain map $H \rtimes G \longrightarrow H$ is the projection, while the domain map sends (h, g) to $h\delta(g)$. Conversely, it associates with a category $\mathbb{D}$ in Grp the crossed module

$$\mathrm{Ker}(c) \longrightarrow \mathbb{D}_1 \xrightarrow{d} \mathbb{D}_0 \;\; .$$

This readily generalizes: given a small category $\mathbb{C}$ and a presheaf of groups G on $\mathbb{C}$, we define a new category $\mathbb{C} \rtimes G$ as follows.

- The objects of the category $\mathbb{C} \rtimes G$ are the same as those of $\mathbb{C}$.
- A morphism $C \longrightarrow D$ in $\mathbb{C} \rtimes G$ is a pair (f, k), where $f : C \longrightarrow D$ is a morphism in $\mathbb{C}$, and where k is an element of the group $G(C)$.
- The composite of $(f, k) : C \longrightarrow D$ and $(g, l) : D \longrightarrow E$ is defined to be $(gf, l_{|f}k)$, where $l_{|f}$ is the restriction of l along f.

- In case $\mathbb{C}$ is equipped with a topology $\mathcal{J}$: a sieve R on an object C is defined to be covering when there is a $\mathcal{J}$-covering sieve S on C, such that R has the form

$$R = \{(f,k) \mid k \in G(C'), f : C' \longrightarrow C \in S\} .$$

5.14. Remark. It is well known [Johnstone '02] that

$$\mathscr{B}(\mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}; G) \simeq \mathsf{Set}^{(\mathbb{C} \rtimes G)^{\mathrm{op}}} .$$

The structure geometric morphism $\gamma : \mathsf{Set}^{(\mathbb{C} \rtimes G)^{\mathrm{op}}} \longrightarrow \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$ is the one associated with the projection functor $c : \mathbb{C} \rtimes G \longrightarrow \mathbb{C}$.

Generalized crossed modules may be described in terms of $\mathbb{C} \rtimes G$. There is a functor $\eta : \mathbb{C} \longrightarrow \mathbb{C} \rtimes G$, which is the identity on objects and sends a morphism $f : C \longrightarrow D$ to $(f, 1_C)$, where 1_C denotes the unit element of the group $G(C)$.

5.15. Lemma. *Crossed modules $\delta : G \longrightarrow \mathbb{C}$ correspond to functors $d : \mathbb{C} \rtimes G \longrightarrow \mathbb{C}$ such that the following diagrams commute.*

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\eta"] \arrow[dr, "1"'] & \mathbb{C} \rtimes G \arrow[d, "d"] \\ & \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} (\mathbb{C} \rtimes G)^{\mathrm{op}} \arrow[d, "d^{\mathrm{op}}"'] \arrow[dr, "\overline{G}"] & \\ \mathbb{C}^{\mathrm{op}} \arrow[r, "G"] & \mathsf{Set} \end{tikzcd}$$

(Here $\overline{G}$ is regarded as a presheaf on $\mathbb{C} \rtimes G$, which is the same as a presheaf on $\mathbb{C}$ together with a right G-action.)

Proof. Given a crossed module $\delta : G \longrightarrow \mathbb{C}$, define a functor

$$d : \mathbb{C} \rtimes G \longrightarrow \mathbb{C} \,; \qquad d(C) = C\,, \quad d(k,g) = k\delta_C(g)\,,$$

where $k : C \longrightarrow D, g \in G(C)$. The extranaturality of δ implies that d is functorial; the Peiffer identity forces $\overline{G} = Gd$.

Conversely, given a functor d satisfying the two conditions, set

$$\delta_C : G(C) \longrightarrow \mathbb{C}(C,C) \,; \qquad \delta_C(g) = d(c,g)\,.$$

The details are straightforward. In the topologized case, d is automatically cover-preserving. ∎

Just as in the group case, the functors $d, c : \mathbb{C} \rtimes G \longrightarrow \mathbb{C}$ are the domain and codomain maps of a category internal to $\mathfrak{Cat}$, i.e., a double category. Let $\mathfrak{DblCat}$ denote the 2-category of (strict) double categories, strict double functors and strict transformations.

5.16. Proposition. *There is a 2-functor from* XMod *to* $\mathfrak{DblCat}$*, which is full and faithful on 1-cells and on 2-cells.*

PROOF. We have explained the action of this functor on objects; on a morphism

$$(F, m) : (\mathbb{C}, G, \delta) \longrightarrow (\mathbb{D}, H, \delta')$$

it produces an internal functor whose object map is $F : \mathbb{C} \longrightarrow \mathbb{D}$, and whose morphism map is $F \rtimes m : \mathbb{C} \rtimes G \longrightarrow \mathbb{D} \rtimes H$. To show that this is full, consider an internal functor

$$\begin{tikzcd} \mathbb{C} \rtimes G \arrow[r, "F_1"] \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] & \mathbb{D} \rtimes H \arrow[d, shift right, "d'"'] \arrow[d, shift left, "c'"] \\ \mathbb{C} \arrow[r, "F"] & \mathbb{D} \end{tikzcd}$$

where the action on arrows of d, and of d' is given by $d(k, g) = k\delta(g)$ and $d'(l, h) = l\delta'(h)$, respectively. Since the vertical functors are the identity on objects, it must be that $F_1(C) = F(C)$. Because $c'F_1 = Fc$, we have $cF_1(k, g) = F(k)$, for $(k, g) : C \longrightarrow C'$. We therefore may write $F_1(k, g) = (Fk, m_{k,g})$. But the identity $d'F_1 = Fd$ gives

$$Fk \circ F(\delta_C(g)) = F(k\delta_C(g)) = Fd(k, g) = d'F_1(k, g) = d'(Fk, m_{k,g}) = Fk \circ \delta'_{FC}(m_{k,g}) .$$

In particular, taking $k = 1_C$, this gives, for each $g \in G(C)$, the equality

$$F(\delta_C(g)) = \delta'_{FC}(m_{1,g}) .$$

This suggests setting

$$m_C : G(C) \longrightarrow HF(C); \qquad m_C(g) = m_{1,g} .$$

It is straightforward to show that (F, m) is a morphism of crossed modules. Finally, it is straightforward to show that 2-cells $\alpha : (F, m) \Rightarrow (F', m')$ between crossed module morphisms correspond bijectively to internal natural transformations between the associated internal functors $(F, F \rtimes m)$ and $(F', F' \rtimes m')$. ■

We would like to characterize the essential image of this functor. The first property which double categories in the image have is that all the structure morphisms (i.e., the domain, codomain, identity and composition functors) are the identity on objects. Such double categories may be regarded as 2-categories. So we may reformulate the construction as the 2-functor

$$\mathsf{XMod} \longrightarrow 2\text{-}\mathfrak{Cat}$$

that associates with a crossed $\mathbb{C}$-module $\delta : G \longrightarrow \mathbb{C}$ the 2-category whose underlying category is simply $\mathbb{C}$, and whose 2-cells are exactly those of the form $(k, g) : k\delta(g) \Rightarrow k$. The vertical composition of such 2-cells is then determined by the horizontal composition (which in turn is the multiplication in G). The 2-categories so obtained are *locally groupoidal* in the sense that their 2-cells are invertible (with respect to the vertical composition of 2-cells): given a 2-cell $(1, g) : \delta(g) \Rightarrow 1$ (it suffices to consider this case) note that $g \in G(C)$ has an inverse h qua element of the group $G(C)$. Then the whiskering $\delta(g)(1, h)$ is the inverse of $(1, g)$ qua 2-cell $\delta(g) \longrightarrow 1$. This leads us to the following definition.

5.17. Definition. *A 2-category* $\mathbb{C}$ *is* left-generated by contractible loops *when every 2-cell* $\alpha : l \Rightarrow k$ *can be written in a unique way as* $\alpha = k\alpha'$*, where* α' *is a 2-cell with an identity morphism as codomain. Pictorially:*

$$\begin{tikzcd} C \arrow[r, bend left, "l"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \end{tikzcd} \qquad = \qquad \begin{tikzcd} C \arrow[r, bend left, "\delta(\alpha')"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\alpha'"] & C \arrow[r, "k"] & D \end{tikzcd}$$

Our main characterization is the following.

5.18. Theorem. *There is a 2-functor from the category* XMod *to the category of 2-categories, which is full and faithful on 1-cells and on 2-cells, and whose essential image consists of the locally groupoidal 2-categories that are left-generated by contractible loops.*

Proof. It is easily seen that every 2-category in the image of the functor has the properties mentioned. In the other direction, consider a 2-category $\mathcal{C}$ with the given properties. We define a presheaf of groups on the underlying category $\mathbb{C}$ of $\mathcal{C}$:

$$G : \mathbb{C}^{\mathrm{op}} \longrightarrow \mathsf{Set}\ ; \qquad G(C) = \{\alpha : \delta(\alpha) \Rightarrow 1_C\}\ ,$$

where we write $\delta(\alpha)$ for the domain of α. (This definition is the expected one from the group case.) We also put

$$\delta_C : G(C) \longrightarrow \mathbb{C}(C, C); \qquad \alpha \mapsto \delta(\alpha).$$

We must show first that G as defined is a presheaf. Given a morphism $f : C' \longrightarrow C$ and $\alpha \in G(C)$, consider

$$\begin{tikzcd} C' \arrow[r, "f"] & C \arrow[r, bend left, "\delta(\alpha)"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\alpha"] & C \end{tikzcd} \qquad = \qquad \begin{tikzcd} C' \arrow[r, bend left, "\delta(\alpha_{|f})"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\alpha_{|f}"] & C' \arrow[r, "f"] & C \end{tikzcd}\ .$$

Thus, we take $\alpha_{|f} \in G(C')$ to be the unique 2-cell factoring αf as $\alpha_{|f} f$ where the codomain of $\alpha_{|f}$ is 1. This makes G into a presheaf, and shows at the same time that δ_C is extranatural in C. Each $G(C)$ admits a composition law, via horizontal composition of 2-cells:

$$\begin{tikzcd} C \arrow[r, bend left, "\delta(\alpha)"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\alpha"] & C \arrow[r, bend left, "\delta(\beta)"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\beta"] & C \end{tikzcd} \qquad = \qquad \begin{tikzcd} C' \arrow[rr, bend left, "\delta(\beta*\alpha)=\delta(\beta)\delta(\alpha)"] \arrow[rr, bend right, "1"'] \arrow[rr, phantom, "\Downarrow\beta*\alpha"] & & C \end{tikzcd}\ .$$

This is well-defined by uniqueness of the factorization of 2-cells. The diagram also shows that δ becomes a homomorphism this way. Moreover, each element of $G(C)$ is invertible (meaning that $G(C)$ is a group): given $\beta : \delta(\beta) \longrightarrow 1)$ first observe that it has an inverse $\beta^{-1} : 1 \longrightarrow \delta(\beta)$ w.r.t. the vertical composition. Then β^{-1} can be rewritten as a whiskering $\delta(\beta)\alpha$ for a 2-cell $\alpha : \delta(\alpha) \longrightarrow 1$. In particular, this shows that $\delta(\alpha)\delta(\beta) = 1$; similarly, one finds that $\delta(\alpha)$ has a section, and hence is an isomorphism (and that $\delta(\beta)$ is its inverse). Using this, it is readily verified that β is the inverse of α w.r.t. the horizontal composition of 2-cells.

Finally, from the factorization

$$
\begin{tikzcd} C \arrow[r, "\delta(\beta)"] & C \arrow[r, bend left, "\delta(\alpha)"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\alpha"] & C \end{tikzcd}
\qquad = \qquad
\begin{tikzcd} C \arrow[r, bend left, "\delta(\alpha_{|\delta(\beta)})"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\alpha_{|\delta(\beta)}"] & C \arrow[r, "\delta(\alpha)"] & C \end{tikzcd}
$$

it is clear that the Peiffer identity holds, i.e., that G acts on itself via conjugation. ∎

6. The topos category of a crossed topos

We return to the general case of a crossed topos. It may come as no surprise, given the fact that on the level of sites a crossed topos is a generalized crossed module and hence an internal category, that a crossed topos is part of a higher-dimensional structure as well. Given a crossed topos $G \longrightarrow \mathscr{E}$ and a (subcanonical) site $\mathbb{C}$ for $\mathscr{E}$, then by our previous results we have a category internal to $\mathfrak{Cat}$ of the form $\mathbb{C} \rtimes G \rightrightarrows \mathbb{C}$, which we may lift to the (pre)sheaf level. However, what we shall call the *topos category* associated with a crossed topos may be defined directly without appealing to a site presentation of $\mathscr{E}$.

We begin in § 6.1 with some helpful technical observations concerning étendues and atomic geometric morphisms, then we define the topos category of a crossed topos and its discrete fibrations, and conclude by using these concepts to define the 'Clifford' sequence associated with a crossed topos.

6.1. Topos preliminaries We mention and prove some facts about torsion-free generators, and atomic geometric morphisms. An object X of a topos $\gamma_{\mathscr{E}} : \mathscr{E} \longrightarrow \mathsf{Set}$ is said to be *torsion-free* if $\mathscr{E}/X$ is a localic topos [Johnstone '02]. Equivalently, X is torsion-free if the terminal object $X \longrightarrow X$ is a generator of $\mathscr{E}/X$.

6.2. Lemma. *A torsion-free object with global support is a generator.*

Proof. Let X be a torsion-free object with global support in a topos $\mathscr{E}$. If Y is any object of $\mathscr{E}$, then $Y \times X$ is a subquotient of $\gamma_{\mathscr{E}}^* A \times X$ over X for some set A because the terminal object 1_X generates $\mathscr{E}/X$. Now compose with the projection $Y \times X \longrightarrow Y$ which is an epimorphism. ∎

The connected locally connected factor of the connected locally connected, discrete factorization of an atomic geometric morphism is atomic. If ψ is atomic, then $\psi_!$ preserves monomorphisms.

6.3. Proposition. *If $\psi : \mathscr{E} \longrightarrow \mathscr{F}$ is an atomic surjection, and X generates $\mathscr{E}$, then $\psi_! X$ generates $\mathscr{F}$.*

Proof. Let Y be an object of $\mathscr{F}$. Then $\psi^* Y$ is a subquotient in $\mathscr{E}$, depicted below (left), which we transpose to $\mathscr{F}$ under $\psi_! \dashv \psi^*$ noting that $\psi_!$ preserves monomorphisms.

$$
\begin{tikzcd} S \arrow[r, two heads] \arrow[d, tail] & \psi^* Y \\ \gamma_{\mathscr{E}}^* A \times X & \end{tikzcd}
\qquad
\begin{tikzcd} \psi_! S \arrow[r, two heads] \arrow[d, tail] & Y \\ \gamma_{\mathscr{F}}^* A \times \psi_! X & \end{tikzcd}
$$

The transpose $\psi_! S \longrightarrow Y$ is an epimorphism because ψ^* is faithful (by assumption). ∎

6.4. PROPOSITION. *If $\psi : \mathscr{E} \longrightarrow \mathscr{F}$ is atomic, and X is a torsion-free object of $\mathscr{E}$, then $\psi_! X$ is torsion-free. Moreover, the top horizontal morphism in the diagram below* (left)

$$\begin{tikzcd} \mathscr{E}/X \arrow[r, "\simeq"] \arrow[d] & \mathscr{F}/\psi_! X \arrow[d] \\ \mathscr{E} \arrow[r, "\psi"] & \mathscr{F} \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r] \arrow[d] & \psi^*\psi_! Y \arrow[d] \\ X \arrow[r] & \psi^*\psi_! X \end{tikzcd}$$

is an equivalence in this case. In particular, every adjunction square over X is a pullback, depicted above (right)*.*

PROOF. Let $f : Y \longrightarrow \psi_! X$ be given. Because X is torsion-free, we have a subquotient diagram in $\mathscr{E}/X$, where $p : P \longrightarrow X$ is a pullback.

$$\begin{tikzcd} S \arrow[r, two heads] \arrow[d, tail] & P \arrow[r] \arrow[d, "p"] & \psi^* Y \arrow[d, "\psi^* f"] \\ \gamma^*_{\mathscr{E}} A \times X \arrow[r] & X \arrow[r] & \psi^*\psi_! X \end{tikzcd}$$

Now transpose under $\psi_! \dashv \psi^*$ keeping in mind that $\psi_!$ preserves monomorphisms.

$$\begin{tikzcd} \psi_! S \arrow[r, two heads] \arrow[d, tail] & \psi_! P \arrow[r, "\cong"] \arrow[d, "\psi_! p"] & Y \arrow[d, "f"] \\ \gamma^*_{\mathscr{F}} A \times \psi_! X \arrow[r] & \psi_! X \arrow[r, "1"] & \psi_! X \end{tikzcd}$$

The right hand square in the above diagram is a pullback. This exhibits f as a subquotient of a constant object over $\psi_! X$, showing that $\psi_! X$ is torsion-free.

As for the second statement, the top horizontal in the diagram of geometric morphisms is the connected locally connected factor of the connected locally connected, discrete factorization of the composite $\mathscr{E}/X \longrightarrow \mathscr{E} \xrightarrow{\psi} \mathscr{F}$. In this case, because ψ is atomic, the connected factor is also atomic, whence hyperconnected. The top horizontal must be an equivalence because $\mathscr{E}/X$ and $\mathscr{F}/\psi_! X$ are both localic. ∎

6.5. COROLLARY. *If $\psi : \mathscr{E} \longrightarrow \mathscr{F}$ is an atomic surjection, and $\mathscr{E}$ is an étendue with torsion-free generator X, then $\mathscr{F}$ is an étendue with torsion-free generator $\psi_! X$. Moreover, the top horizontal morphism in the diagram*

$$\begin{tikzcd} \mathscr{E}/X \arrow[r, "\simeq"] \arrow[d] & \mathscr{F}/\psi_! X \arrow[d] \\ \mathscr{E} \arrow[r, "\psi"] & \mathscr{F} \end{tikzcd}$$

is an equivalence.

6.6. The definition of the topos category The underlying geometric morphism of a crossed topos $\theta : G \longrightarrow \mathscr{E}$ is the domain morphism of category object internal to $\mathfrak{BTop}$:

$$\begin{tikzcd} \mathscr{B}(\mathscr{E};G) \arrow[r, shift left=2, "\theta"] \arrow[r, shift right=2, "\gamma"'] & \mathscr{E} \arrow[l, "\eta" description] \end{tikzcd} .$$

The codomain morphism is the structure morphism γ and the identities morphism is the unit η of the monad $\mathcal{I}$, corresponding to the unit of the group $1 \longrightarrow G$. $\mathscr{B}(\mathscr{B}(\mathscr{E};G),\overline{G}) \simeq \mathscr{B}(\mathscr{E};G \rtimes \overline{G})$ is the object of composable pairs of this internal category because the following is a pullback of toposes: it is a pullback because $\theta^*(G) = \overline{G}$ (Lemma 2.3).

$$\begin{tikzcd} \mathscr{B}(\mathscr{B}(\mathscr{E};G),\overline{G}) \arrow[r, "\mathcal{I}(\theta)=\overline{\theta}"] \arrow[d, "\overline{\gamma}"'] & \mathscr{B}(\mathscr{E};G) \arrow[d, "\gamma"'] \\ \mathscr{B}(\mathscr{E};G) \arrow[r, "\theta"] & \mathscr{E} \end{tikzcd}$$

The composition morphism of the topos category is the multiplication of the isotropy monad $\mathcal{I}$:

$$\mu : \mathscr{B}(\mathscr{E};G \rtimes \overline{G}) \longrightarrow \mathscr{B}(\mathscr{E};G) .$$

The composition μ and second projection $\overline{\gamma}$ are engendered by group homomorphisms $(g,a) \mapsto ga$ and respectively $(g,a) \mapsto g$, but the first projection $\mathcal{I}(\theta)$ is not. In fact, if (X,σ) is a G-action, then $\mathcal{I}(\theta)^*(X,\sigma)$ is X with the $G \rtimes \overline{G}$-action $x(g,a) = \sigma(\theta_X(x,g),a)$.

6.7. Discrete fibrations on the topos category In order to explain what is a discrete fibration on the topos category of a crossed topos we digress briefly to consider a generic construction. If $\begin{tikzcd} \mathbb{D}_1 \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & \mathbb{D}_0 \end{tikzcd}$ is an internal category, then a discrete fibration over $\mathbb{D}$ is an internal functor $F : \mathbb{X} \longrightarrow \mathbb{D}$ such that the commuting square

$$\begin{tikzcd} \mathbb{X}_1 \arrow[r, "F_1"] \arrow[d, "c"] & \mathbb{D}_1 \arrow[d, "c"] \\ \mathbb{X}_0 \arrow[r, "F_0"] & \mathbb{D}_0 \end{tikzcd}$$

is a pullback. Morphisms of discrete fibrations are functors over $\mathbb{D}$. In particular, the free (or representable) discrete fibration $\widehat{f}$ on a morphism $f : X \longrightarrow \mathbb{D}_0$ is given by the top composite in the diagram below.

$$\begin{tikzcd} X \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r] \arrow[d] \arrow[rr, bend left, "\widehat{f}"] & \mathbb{D}_1 \arrow[r, "d"'] \arrow[d, "c"] & \mathbb{D}_0 \\ X \arrow[r, "f"] & \mathbb{D}_0 & \end{tikzcd}$$

The map $\widehat{f}$ is the object function of the free discrete fibration.

The generic construction just described applies to the topos category associated with a crossed topos. Starting with a crossed topos $\theta : G \longrightarrow \mathscr{E}$, we have in mind for f , at least for our immediate purposes, the étale geometric morphism associated with an object X of $\mathscr{E}$, as usual denoted $X : \mathscr{E}/X \longrightarrow \mathscr{E}$.

6.8. Definition. *We refer to the geometric morphism* $\widehat{X} = \theta \cdot \mathcal{I}(X)$ *as* the (representable) discrete fibration over θ *associated with the object* X *, depicted in the following topos pullback diagram.*

$$\begin{tikzcd}
\mathscr{B}(\mathscr{E}/X; X^*G) \arrow[r, "\mathcal{I}(X)=\overline{X}"'] \arrow[d, "\overline{\gamma}"'] \arrow[rr, bend left, "\widehat{X}"] & \mathscr{B}(\mathscr{E}; G) \arrow[r, "\theta"'] \arrow[d, "\gamma"] & \mathscr{E} \\
\mathscr{E}/X \arrow[r, "X"] & \mathscr{E} &
\end{tikzcd}$$

In Def. 6.8, we are regarding $\widehat{X}$ as the object map of a functor internal to toposes. But we may also regard $\widehat{X}$ as a Cartesian morphism of grouped toposes. To see how we do this, we first regard $\mathscr{E}/X \longrightarrow \mathscr{E}$ as a Cartesian morphism of grouped toposes by pairing $\mathscr{E}/X$ with the group X^*G , where X^*G is the projection $X \times G \longrightarrow X$. In other words, $(X, 1) : (\mathscr{E}/X, X^*G) \longrightarrow (\mathscr{E}, G)$ is a Cartesian morphism of grouped toposes. We shall write $(\mathscr{E}, G)_X$ for the slice grouped topos $(\mathscr{E}/X, X^*G)$. Applying $\mathcal{I}$ to the morphism $(X, 1) = X$ gives

$$\mathcal{I}(X) : \mathcal{I}((\mathscr{E}, G)_X) \longrightarrow \mathcal{I}(\mathscr{E}, G) ,$$

whose underlying étale geometric morphism is

$$\mathscr{B}(\mathscr{E}/X; X^*G) \simeq \mathscr{B}(\mathscr{E}; G)/\gamma^*X \longrightarrow \mathscr{B}(\mathscr{E}; G) .$$

Under the above equivalence, $\overline{X^*G}$ corresponds to the projection $\overline{G} \times \gamma^*X \longrightarrow \gamma^*X$, so that there is an equivalence of grouped toposes

$$\mathcal{I}((\mathscr{E}, G)_X) = (\mathscr{B}(\mathscr{E}/X; X^*G), \overline{X^*G}) \simeq (\mathscr{B}(\mathscr{E}; G)/\gamma^*X\ ,\ \overline{G} \times \gamma^*X \longrightarrow \gamma^*X) .$$

6.9. Proposition. *For any crossed topos* $\theta : G \longrightarrow \mathscr{E}$ *and any object* X *of* $\mathscr{E}$*, the discrete fibration* $\widehat{X}$ *is an atomic geometric morphism. As a morphism of grouped toposes,* $\widehat{X}$ *is Cartesian. Moreover, if* $\mathscr{E}$ *is an étendue with torsion-free generator* X*, then the topos* $\mathscr{B}(\mathscr{E}/X; X^*G)$ *is an étendue with torsion-free generator* X^*G *.*

Proof. The morphism $\mathcal{I}(X)$ is Cartesian (Lemma 2.3), and it is atomic (in fact, étale) because it is the pullback of an étale morphism. The algebra θ is also Cartesian and atomic, so $\widehat{X}$ is Cartesian (as a morphism of grouped toposes), and atomic. If X is torsion-free, then so is X^*G because we have a canonical equivalence $\mathscr{B}(\mathscr{E}/X; X^*G)/X^*G \simeq \mathscr{E}/X$. By Lemma 6.2, X^*G is a generator of $\mathscr{B}(\mathscr{E}/X; X^*G)$ over Set . ∎

We are now prepared to introduce what we mean by a discrete fibration on a crossed topos. We confine out attention to discrete fibrations whose object map is an atomic geometric morphism.

6.10. Definition. An atomic discrete fibration on a crossed topos *is a discrete fibration on the topos category* (internal to toposes) *of the crossed topos, such that the object geometric morphism is atomic. We denote the category of atomic discrete fibrations on a crossed topos θ by* $\mathsf{DFib}(\theta)$ *.*

By definition, a morphism of crossed toposes is in the first place a morphism of grouped toposes. On the other hand, a discrete fibration on a crossed topos is a functor internal to toposes only, not grouped toposes. The following proposition shows how the two are related. Its proof is mostly a matter of unraveling the definitions.

6.11. Proposition. *For any crossed topos $\theta : G \longrightarrow \mathscr{E}$, $\mathsf{DFib}(\theta)$ is equivalent to the full subcategory of the slice category XTop/θ on objects $(\phi, m) : \delta \longrightarrow \theta$ such that (ϕ, m) is Cartesian* (i.e., m is an isomorphism)*, and ϕ is atomic.*

Proof. If $(\phi, m) : \delta \longrightarrow \theta$ is a Cartesian morphism of crossed toposes (with ϕ is atomic), then the following two squares commute: the domain map square is the one with δ and θ and the other is the codomain map square.

$$\begin{tikzcd} \mathscr{B}(\mathscr{F}; H) \arrow[r, "\overline{\phi}m"] \arrow[d, "\delta"'] & \mathscr{B}(\mathscr{E}; G) \arrow[d, "\theta"] \\ \mathscr{F} \arrow[r, "\phi"] & \mathscr{E} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{B}(\mathscr{F}; H) \arrow[r, "\overline{\phi}m"] \arrow[d, "\gamma"] & \mathscr{B}(\mathscr{E}; G) \arrow[d, "\gamma"] \\ \mathscr{F} \arrow[r, "\phi"] & \mathscr{E} \end{tikzcd}$$

Moreover, the codomain map square, i.e., the square with the γ's, is a pullback (Lemma 2.3). This is precisely what we mean by an atomic discrete fibration on the topos category of θ: the domain topos category of the discrete fibration is the topos category of δ. The converse is just as easy: we leave the remaining details as an exercise. ∎

Definition 6.8 is a functorial construction, so that we have a functor

$$\mathscr{E} \longrightarrow \mathsf{DFib}(\theta) \; ; \qquad X \mapsto \widehat{X} \; .$$

6.12. The Clifford sequence of a crossed topos We shall see in § 7 how crossed toposes and their discrete fibrations are related to the Clifford construction in semigroup theory. For that purpose we note that with a topos $\mathscr{E}$, an object X of $\mathscr{E}$, and a crossed topos $\delta : G \longrightarrow \mathscr{E}$ there is associated a 'Clifford' sequence:

$$\mathscr{B}(\mathscr{E}/X; X^*G) \xrightarrow{\widehat{X}} \mathscr{E} \xrightarrow{\psi} \mathscr{E}_\delta \; , \tag{9}$$

where $\mathscr{E}_\delta$ denotes the isotropy quotient of δ. We may think of the crossed topos itself as the 'kernel' of ψ, but notice that a presentation of it appears in (9). In fact, the kernel has a presentation for every object X of $\mathscr{E}$. By contrast, the isotropy quotient is defined independently of a particular X.

If $\mathscr{E}$ is an étendue with torsion-free generator X , then the other two toposes in (9) are also étendue. Indeed, by Cor. 6.5 the orbit space $\psi_! X = X/G$, which is the coequalizer of $\delta_X : X \times G \longrightarrow X$ and the projection, is a torsion-free generator of $\mathscr{E}_\delta$, and the representable X^*G is one for $\mathscr{B}(\mathscr{E}/X; X^*G)$ (Prop. 6.9). Moreover, the following diagram of geometric morphisms commutes.

$$\begin{tikzcd}
\mathscr{B}(\mathscr{E}/X; X^*G)/X^*G \arrow[r, "\simeq"] \arrow[d, "X^*G"'] & \mathscr{E}/X \arrow[r, "\simeq"] \arrow[d, "X"'] & \mathscr{E}_\delta/\psi_! X \arrow[d] \\
\mathscr{B}(\mathscr{E}/X; X^*G) \arrow[r, "\widehat{X}"] & \mathscr{E} \arrow[r, "\psi"] & \mathscr{E}_\delta
\end{tikzcd}$$

The equivalence in the left square is the usual canonical map $\tau = \overline{\gamma} \cdot X^*G$, which satisfies $\eta \cdot \tau \cong X^*G$, for the unit $\eta : \mathscr{E}/X \longrightarrow \mathscr{B}(\mathscr{E}/X; X^*G)$. The square commutes because

$$\widehat{X} \cdot X^*G = \delta \cdot \mathcal{I}(X) \cdot X^*G \cong \delta \cdot \mathcal{I}(X) \cdot \eta \cdot \tau \cong \delta \cdot \eta \cdot X \cdot \tau \cong X \cdot \tau \ .$$

The other square is given by the unit $X \longrightarrow \psi^* \psi_! X$. The top horizontal in this square is an equivalence when X is torsion-free.

7. Applications

We return to the motivating examples: inverse semigroups and étale groupoids. We have seen how the classifying topos of an inverse semigroup or étale groupoid carries a canonical isotropy algebra structure that we call the standard crossed topos structure (Egs. 3.2, 3.3, 3.4), and that this is an instance of an algebra structure that every topos has. We have also seen that the standard crossed topos is characterized as the terminal one, but we wish to know what this tells us about Clifford semigroups.

7.1. Algebras on inverse semigroups Consider once more the isotropy group Z associated with an inverse semigroup S , and the standard crossed topos $\theta : Z \longrightarrow \mathscr{B}(S)$ (Eg. 3.3), also denoted $\theta : Z \longrightarrow S$. We interpret the general results about crossed toposes in this particular case.

It is sometimes helpful to know that $\mathscr{B}(S)$ is equivalent to the category of presheaves on a small category $\mathbb{L}(S)$, whose objects are the idempotents of S, and whose morphisms $d \longrightarrow e$ are the elements $s \in S$ for which $s^*s = d$ and $es = s$. We picture such a morphism as

$$d = s^*s \xrightarrow[s]{\cong} ss^* \leq e \ .$$

Thus, each morphism of $\mathbb{L}(S)$ factors as an isomorphism followed by an inequality in the meet-semilattice of idempotents.

7.2. Corollary. *The standard crossed topos $\theta : Z \longrightarrow S$ on an inverse semigroup S is the terminal object of* $\mathsf{XTop}(\mathscr{B}(S))$ (Def. 3.14).

PROOF. This is a consequence of Prop. 4.11, but the following direct proof may interest the reader. We may apply our work on crossed $\mathbb{C}$-modules to $\mathscr{B}(S)$ using the equivalence of $\mathscr{B}(S)$ with presheaves on $\mathbb{L}(S)$. Under this equivalence the isotropy group $Z(E)\longrightarrow E$, which we denote simply Z, corresponds to the presheaf of groups

$$Z(e) = \{s \in Z(E) \mid s^*s = ss^* = e\} \cong \{\text{automorphisms of } \mathbb{L}(S)/e \longrightarrow \mathbb{L}(S)\}\ .$$

Furthermore, the standard crossed topos corresponds to the crossed $\mathbb{L}(S)$-module given simply by the inclusion $Z(e) \longrightarrow \mathbb{L}(S)(e, e)$. Now consider an arbitrary crossed $\mathbb{L}(S)$-module $\delta : G \longrightarrow \mathbb{L}(S)$. Then the image of δ must be contained in Z: given any $g \in G(e)$, and any idempotent d (for which we may assume without loss of generality that $d \leq e$), we have $d\delta(g_{|d}) = \delta(g)d$ by the first crossed module axiom. Since $\delta(g_{|d})$ is an automorphism of d, it follows that $\delta(g_{|d}) = d\delta(g_{|d}) = \delta(g)d$, so that $\delta(g)d$ is an automorphism of d as well. Replacing g with its inverse g^{-1}, it also follows that $d\delta(g) = d^*\delta(g^{-1})^* = (\delta(g)d)^*$, and we get $d\delta(g) = g\delta(d)$ by virtue of the fact that $d\delta(g) = \delta(g^{-1}_{|d})^* = \delta(g_{|d})$, since the restriction along d preserves inverses. ∎

We state (at the 1-categorical level) the 'external/internal' theorem (Corollary 5.10) for inverse semigroups.

7.3. COROLLARY. *Let $\theta : Z \longrightarrow S$ denote the standard crossed module on an inverse semigroup S. Then $\mathsf{XTop}(\mathscr{B}(S))$ is equivalent to the category whose objects are group homomorphisms $G \longrightarrow Z$ that are crossed modules (with θ_G) internal to $\mathscr{B}(S)$, and whose morphisms are group homomorphisms $G \longrightarrow H$ over Z.*

7.4. REMARK. We interpret Remark 5.12 for inverse semigroups. The action of the monoid of endomorphisms of Z over the center on $\mathsf{XTop}(\mathscr{B}(S), Z)$ is free with one generator, which is the standard crossed topos.

7.5. THE CLIFFORD CONSTRUCTION The inclusion $Z(E) \hookrightarrow S$ of the maximal Clifford subsemigroup of S induces a geometric morphism

$$\mathscr{B}(Z(E)) \longrightarrow \mathscr{B}(S) \tag{10}$$

between classifying toposes. Our aim is to describe the nature of this morphism, and to give a topos-theoretic description of how it arises. As always, E denotes the semilattice of idempotents of S.

We shall show that (10) is a discrete fibration on the topos category of the standard crossed topos $\theta : Z \longrightarrow \mathscr{B}(S)$. In fact, it is the representable one associated with the domain map $S \longrightarrow E$, $s \mapsto s^*s$, which is an object of $\mathscr{B}(S)$ by virtue of right multiplication by S, denoted D. (D is also known as the *Schützenberger object*, but we do not use this terminology [Funk-Steinberg '10].) As a presheaf on $\mathbb{L}(S)$, we have

$$D(e) = \{s \in S \mid s^*s = e\}\,.$$

(Unless S has a global unit, D is not a representable presheaf.) The object D is known to be a torsion-free generator of $\mathscr{B}(S)$ (§ 6.1). We have $\mathscr{B}(S)/D \simeq \mathscr{B}(E)$, where the latter is the topos of presheaves on E. Under this equivalence, the inverse image functor $D^* : \mathscr{B}(S) \longrightarrow \mathscr{B}(E)$ simply forgets the action by S.

7.6. Proposition. *Let S be an inverse semigroup, with idempotent lattice E and maximal Clifford subsemigroup $Z(E) \hookrightarrow S$. Then the geometric morphism (10) induced by this inclusion is, up to equivalence, the representable discrete fibration $\widehat{D} = \theta \cdot \mathcal{I}(D)$ over the standard crossed topos θ, where D is the canonical torsion-free generator of $\mathscr{B}(S)$. In particular,* (10) *is atomic and a Cartesian morphism of grouped toposes.*

Proof. The defining diagram for $\widehat{D}$ is as follows, extended by the equivalence $\mathscr{B}(S)/D \simeq \mathscr{B}(E)$:

$$\begin{tikzcd}
\mathscr{B}(E;Z) \arrow[r, "\simeq"] \arrow[d] & \mathscr{B}(S;Z)/\gamma^*D \arrow[r, "\mathcal{I}(D)=\widetilde{D}"'] \arrow[d, "\overline{\gamma}"'] \arrow[rr, bend left, "\widehat{D}"] & \mathscr{B}(S;Z) \arrow[r, "\theta"'] \arrow[d, "\gamma"] & \mathscr{B}(S) \\
\mathscr{B}(E) \arrow[r, "\simeq"] & \mathscr{B}(S)/D \arrow[r, "D"] & \mathscr{B}(S) &
\end{tikzcd}$$

We have written $\mathscr{B}(S;Z)$ to denote $\mathscr{B}(\mathscr{B}(S);Z)$, and the same for $\mathscr{B}(E;Z)$; we have also simply written Z for the group D^*Z in $\mathscr{B}(E)$. It is easily seen that there is an isomorphism of categories

$$\mathbb{L}(Z(E)) \cong E \rtimes Z \ ,$$

so that we have equivalences

$$\mathscr{B}(Z(E)) \simeq \mathscr{B}(E;Z) \simeq \mathscr{B}(S;Z)/\gamma^*D \ .$$

Moreover the top composite is induced by the site morphism $\mathbb{L}(Z(E)) \longrightarrow \mathbb{L}(S)$. Remaining details are left to the reader. ∎

We turn our attention to the geometric morphism

$$\mathscr{B}(S) \longrightarrow \mathscr{B}(S/\mu) \tag{11}$$

associated with the other map in the Clifford construction: the semigroup quotient map $S \longrightarrow S/\mu$.

7.7. Proposition. *The geometric morphism (11) is equivalent to the isotropy quotient*

$$\mathscr{B}(S;Z) \underset{\gamma}{\overset{\theta}{\rightrightarrows}} \mathscr{B}(S) \xrightarrow{\psi} \mathscr{B}(S)_\theta$$

of the standard crossed module $\theta : Z \longrightarrow S$ (Prop. 4.17 and Def. 4.18)*.*

Thus, $\mathscr{B}(S/\mu)$ may be thought of as the 'orbit space' of the standard crossed module on S. S/μ is a fundamental semigroup, so that its isotropy group is trivial: $\mathscr{B}(S/\mu)$ is anisotropic (Def. 4.3). The unique morphism $Z \longrightarrow \psi^*1 = 1$ may be paired with the connected, atomic ψ making a (non-Cartesian) morphism of grouped toposes. We may paraphrase Propositions 7.6 and 7.7 thus: the first map in the Clifford construction corresponds to the representable discrete fibration on the terminal crossed topos associated with the canonical torsion-free generator of $\mathscr{B}(S)$. The sequence of classifying toposes associated with the semigroup sequence (1) is equivalent to

$$\mathscr{B}(E;Z) \xrightarrow{\widehat{D}} \mathscr{B}(S) \xrightarrow{\psi} \mathscr{B}(S)_\theta \ . \tag{12}$$

Proposition 7.7 is a special case of a more general result for étale groupoids that we establish in § 7.8. As a matter of fact, all the results of § 7.1 and § 7.5 generalize to étale groupoids.

7.8. Étale groupoids Throughout, $H = (H_0, H_1)$ denotes an étale groupoid. We are working with a spatial groupoid, but localic groupoids may be treated in just the same way. We denote the Clifford sequence for the standard étale crossed module $\theta : Z \longrightarrow H$ by:

$$Z \longrightarrow H \longrightarrow G \ , \tag{13}$$

where Z is the étale groupoid $(Z_0 = H_0, Z_1 = Z)$, whose domain and codomain maps coincide. The (open) inclusion of Z in H_1 is the morphism map of the first continuous functor in (13) whose object map is the identity.

The geometric morphism $\mathscr{B}(Z) \longrightarrow \mathscr{B}(H)$ associated with $Z \longrightarrow H$ is equivalent to the discrete fibration $\widehat{D}$, where D denotes the domain map $H_1 \xrightarrow{d} H_0$ equipped with its canonical action by H.

$$\begin{tikzcd}
\mathscr{B}(H_0;Z) \arrow[r] \arrow[d, "\overline{\gamma}"'] \arrow[rr, bend left, "\widehat{D}"] & \mathscr{B}(H;Z) \arrow[r, "\theta"'] \arrow[d, "\gamma"] & \mathscr{B}(H) \\
\mathsf{Sh}(H_0) \arrow[r, "D"] & \mathscr{B}(H) &
\end{tikzcd}$$

$\mathsf{Sh}(H_0)$ denotes the topos of sheaves on the space H_0, $\mathscr{B}(H_0;Z)$ denotes $\mathscr{B}(\mathsf{Sh}(H_0);D^*Z)$, and $\mathscr{B}(H;Z)$ denotes $\mathscr{B}(\mathscr{B}(H);Z)$. We can at once define the étale groupoid G in (13) and explain how it is related to the isotropy quotient $\mathscr{B}(H) \longrightarrow \mathscr{B}(H)_\theta$. The object D is a torsion-free generator of $\mathscr{B}(H)$. Therefore, by Cor. 6.5, $\psi_! D$ is a torsion-free generator of the isotropy quotient $\mathscr{B}(H)_\theta$. Explicitly, this is the following coequalizer of spaces:

$$\begin{tikzcd}
H_1 \times_{H_0} Z \arrow[r, bend left, "\theta_D"] \arrow[r, bend right] & H_1 \arrow[r, two heads] & G_1 \ .
\end{tikzcd}$$

We have $\theta_D(g,h) = gh$, where $h \in Z$ and $d(g) = d(h)$. This defines an étale groupoid

$$G = (G_0 = H_0, G_1) \ ,$$

which we call the isotropy quotient of H. We have a continuous functor $H \longrightarrow G$ whose geometric morphism is equivalent to the isotropy quotient $\mathscr{B}(H) \longrightarrow \mathscr{B}(H)_\theta$. Just like (12), the sequence of classifying toposes associated with (13) is equivalent to

$$\mathscr{B}(H_0;Z) \xrightarrow{\widehat{D}} \mathscr{B}(H) \xrightarrow{\psi} \mathscr{B}(H)_\theta \ .$$

This sequence is an instance of (9).

We shall say that an étale groupoid H *is Clifford* if its domain and codomain maps coincide. Equivalently, H is Clifford if in (13) Z and H are equal. The étale groupoid Z associated with any étale groupoid is Clifford.

7.9. THEOREM. *An étale groupoid H is Morita equivalent to a Clifford one if and only if the representable discrete fibration on the standard crossed topos associated with a torsion-free generator of $\mathscr{B}(H)$ is an equivalence.*

PROOF. If an étale groupoid H is Morita equivalent to a Clifford one, then by the étale groupoid version of Prop. 7.6, $\widehat{D}$ is an equivalence. Conversely, if $\widehat{X}$ is an equivalence, where $X = \langle U_0 \longrightarrow H_0, \sigma\rangle$ is a torsion-free generator of $\mathscr{B}(H)$, then there is an étale groupoid $U = (U_0, U_1)$ such that $\mathscr{B}(H) \simeq \mathscr{B}(U)$. It follows that $\widehat{D}$ is an equivalence, where D is the domain object of $\mathscr{B}(U)$. In effect, what we have shown so far is that we may assume that the torsion-free object X of $\mathscr{B}(H)$ is the domain object. To conclude the argument, we have already observed that the étale groupoid Z, which is Morita equivalent to H, is Clifford. ∎

7.10. EXAMPLE. In the inverse case $H = G(S) = (E, S)$, the isotropy quotient of H coincides with the fundamental quotient S/μ.

$$\begin{tikzcd} S \times_E Z \arrow[r, bend left, "\theta"] \arrow[r, bend right] & S \arrow[r, two heads] & S/\mu \end{tikzcd}$$

Put another way, the isotropy quotient for étale groupoids generalizes the fundamental quotient for inverse semigroups. This proves Prop. 7.7.

We have the following.

7.11. PROPOSITION. *The isotropy quotient G of an étale groupoid H is effective in the sense that its isotropy group is trivial. Equivalently, the isotropy quotient $\mathscr{B}(H)_\theta \simeq \mathscr{B}(G)$ is anisotropic.*

7.12. REMARK. The meaning of 'effective' is established for étale groupoids; however, because 'effective topos' already has an established and unrelated meaning in topos theory we have chosen the term 'anisotropic' for toposes.

We conclude by stating the (1-categorical formulation of the) external/internal theorem for étale groupoids, whose topos version is Cor. 5.10.

7.13. THEOREM. *Let H be an étale groupoid, G a group in $\mathscr{B}(H)$, and $\delta : G \longrightarrow H$ an étale crossed module. Then $\mathsf{XTop}(\mathscr{B}(H))/\delta$ is equivalent to the category whose objects are crossed modules $K \longrightarrow G$* (with δ_K) *internal to $\mathscr{B}(H)$, and whose morphisms are group homomorphisms over G.*

$\mathsf{XTop}(\mathscr{B}(H))$ *is equivalent to the category whose objects are crossed modules $K \longrightarrow Z$* (with θ_K) *internal to $\mathscr{B}(H)$, and whose morphisms are group homomorphisms over Z, where $\theta : Z \longrightarrow H$ denotes the standard étale crossed module.*

8. Conclusion

We have developed some of the elementary theory of what we call crossed toposes, and how this theory is connected with the isotropy of a topos. This includes the fact that every topos has a terminal crossed topos, and an internal description of the isotropy category of a topos. We have explained the Clifford construction for inverse semigroups in terms of a general Clifford construction on crossed toposes, parameterized by objects of that topos. When applied to the classifying topos of an inverse semigroup and a torsion-free generator, the result is the classifying topos of the associated Clifford semigroup. The fundamental quotient is recovered via the isotropy quotient of the crossed topos. These constructions can be applied in a context wider than that of inverse semigroups: in particular, we have shown how they also work on the more general level of étale groupoids, and indeed for toposes.

There are various issues and questions about crossed toposes that we have not attempted to address in this paper.

1. We explained in § 6 that each crossed topos gives rise to an internal category in the category of Grothendieck toposes. Which internal categories do arise in this manner, and how may we characterize the categories of discrete fibrations on such categories?

2. We have shown how morphisms of crossed modules (on the level of sites) correspond to strict algebra morphisms. We have not explored how pseudo-maps and lax maps of algebras relate to relaxed morphisms of crossed modules.

3. A generalized notion of crossed $\mathbb{C}$-module $\mathsf{D} \longrightarrow \mathbb{C}$ may be of interest, where D is a category internal to $\mathsf{Set}^{\mathbb{C}^{op}}$. Such a structure is defined by a family of functors $\delta_C : \mathsf{D}(C) \longrightarrow \mathbb{C}(C, C)$, where we regard $\mathbb{C}(C, C)$ as a one-object category. It induces a double category of the form $\mathbb{C} \rtimes \mathsf{D} \rightrightarrows \mathbb{C}$, and hence a (pre)sheaf topos category. Although topos categories accommodate these crossed modules, without conditions on D they do not appear to be algebras for a more general isotropy monad on toposes.

4. We have not yet investigated how crossed toposes and isotropy behave under transportation along general geometric morphisms. Clearly, isotropy is Morita invariant in the sense that if two toposes $\mathscr{E}$ and $\mathscr{F}$ are equivalent, then so are their isotropy categories $\mathsf{XTop}(\mathscr{E})$ and $\mathsf{XTop}(\mathscr{F})$. It also follows immediately from the definition that when two toposes are equivalent, their isotropy groups correspond across this equivalence.

5. We have left a more thorough investigation of the isotropy factorization of a geometric morphism begun in § 4.16 for another time.

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