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A PSEUDO REPRESENTATION THEOREM FOR VARIOUS CATEGORIES OF RELATIONS

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Abstract. It is well-known that, given a Dedekind category $\mathcal{R}$ the category of (typed) matrices with coefficients from $\mathcal{R}$ is a Dedekind category with arbitrary relational sums. In this paper we show that under slightly stronger assumptions the converse is also true. Every atomic Dedekind category $\mathcal{R}$ with relational sums and subobjects is equivalent to a category of matrices over a suitable basis. This basis is the full proper subcategory induced by the integral objects of $\mathcal{R}$. Furthermore, we use our concept of a basis to extend a known result from the theory of heterogeneous relation algebras.

1. Introduction

The calculus of binary relations has played an important rôle in the interaction between algebra and logic since the middle of the nineteenth century. A first adequate development of such algebras was given by de Morgan and Peirce. Their work has been taken up and systematically extended by Schröder in [14]. More than 40 years later, Tarski started with [15] the exhaustive study of relation algebras, and more generally, Boolean algebras with operators.

The papers above deal with relational algebras presented in their classical form. Elements of such algebras might be called *quadratic* or *homogeneous*; relations over a fixed universe. Usually a relation acts between two different kinds of objects, e.g. between boys and girls. Therefore, a variant of the theory of binary relations has evolved that treats relations as *heterogeneous* or *rectangular*. A convenient framework to do so is given by category theory [1, 12, 13].

Under certain circumstances, i.e. relational products exist or the point axiom is given, a relation algebra may be represented in the algebra Rel of concrete binary relations between sets [5, 6, 11, 13]. In other words, the algebra may be seen as an algebra of Boolean matrices.

As known, not every (homogeneous or heterogeneous) relation algebra or Dedekind category need be representable and therefore need not be an algebra of Boolean matrices [1, 3, 4]. In this paper, we will show that it is possible in every Dedekind category $\mathcal{R}$ with relational sums and subobjects to characterize a full subcategory $\mathcal{B}$ such that the matrix algebra $\mathcal{B}^+$ with coefficients from $\mathcal{B}$ is equivalent to $\mathcal{R}$. This equivalence is not

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necessarily an isomorphism since isomorphic objects from $\mathcal{R}$ may be identified under this equivalence. The objects of $\mathcal{B}$ are the integral objects of $\mathcal{R}$. Integral objects are defined to be those objects A such that the set of relations $\mathcal{R}[A, A]$ on A is an integral relation algebra in the sense of [2]. They may be characterized by the fact that their identity morphism is an atom. We call $\mathcal{B}$ the basis of $\mathcal{R}$.

As shown in [1, 17], every Dedekind category may be embedded into one with relational sums and subobjects and hence into one which is equivalent to a matrix algebra. This embedding and the equivalence above is not a trivial one. We show that $\mathcal{B}$ is never isomorphic or equivalent to $\mathcal{R}$ and hence, a proper subcategory of $\mathcal{R}$.

Furthermore, we want to demonstrate the strength of our concept of a basis of a Dedekind category. Therefore, we reprove that every atomic Dedekind category may be embedded into a product of so-called simple algebras. It turns out that this product is characterized by an obvious equivalence relation on the class of objects of $\mathcal{B}$.

The paper is organized as follows. In Section 2, we briefly recall some basic definitions of various categories of relations. Section 3 is dedicated to matrix algebras with coefficients taken from a given Dedekind category. The integral objects and the basis are introduced in Section 4. Afterwards in Section 5, we prove our main theorem, i.e. a pseudo-representation theorem for atomic Dedekind categories. Finally in Section 6, we reprove the theorem mentioned above.

We assume that the reader is familiar with the basic concepts of allegories, Dedekind categories and the theory of heterogeneous relation algebras. We use the notation of [13].

2. Categories of Relations

Throughout this paper, we use the following notations. To indicate that a morphism R of a category $\mathcal{R}$ has source A and target B we write $R : A \to B$. The collection of all morphisms $R : A \to B$ is denoted by $\mathcal{R}[A, B]$ and the composition of a morphism $R : A \to B$ followed by a morphism $S : B \to C$ by $R; S$. Last but not least, the identity morphism on A is denoted by $\mathbb{I}_A$.

In this section we recall some fundamentals on Dedekind categories [7, 8]. This kind of categories is called locally complete division allegories in [1]. For further details we refer to [1, 12, 13].

2.1. Definition. *A Dedekind category $\mathcal{R}$ is a locally small category satisfying the following:*

1. *For all objects A and B the set $\mathcal{R}[A, B]$ is a complete distributive lattice. Meet, join, the induced ordering, the least and the greatest element are denoted by $\sqcap_{AB}, \sqcup_{AB}$, $\sqsubseteq_{AB}, \perp\!\!\!\perp_{AB}, \top\!\!\!\top_{AB}$, respectively.*

2. *There is a monotone operation $\breve{}$ (called conversion) such that for all relations $Q : A \to B$ and $R : B \to C$ the following holds*

$$(Q; R)^{\smile} = R^{\smile}; Q^{\smile}, \qquad (Q^{\smile})^{\smile} = Q.$$

3. *For all relations $Q : A \to B, R : B \to C$ and $S : A \to C$ the modular law*
$$Q; R \sqcap S \sqsubseteq Q; (R \sqcap Q^{\smile}; S)$$
holds.

4. *For all relations $R : B \to C$ and $S : A \to C$ there is a relation $S/R : A \to B$ (called the left residual of S and R) such that for all $Q : A \to B$ the following holds*
$$Q; R \sqsubseteq S \quad \Longleftrightarrow \quad Q \sqsubseteq S/R.$$

All the indices of elements and operations are usually omitted for brevity and can easily be reinvented.

Note that the class of complete distributive lattices is equivalent to the class of complete Heyting algebras. A Dedekind category $\mathcal{R}$ is called an atomic Dedekind category if every hom-set of $\mathcal{R}$ is an atomic lattice. If every hom-set $\mathcal{R}[A, B]$ of $\mathcal{R}$ is a Boolean algebra, $\mathcal{R}$ is called a Schröder category. We denote the complement of a relation $R : A \to B$ by $\overline{R}$. An atomic Schröder category is also called a heterogeneous relation algebra.

We use the phrase "$\mathcal{R}$ is a relational category" as a shorthand for fact that $\mathcal{R}$ is one of the structures defined above.

In the next lemma we collect some properties we will need throughout this paper. Proofs may be found in [1, 9, 10, 12, 13, 16, 17, 18].

2.2. Lemma. *Let $\mathcal{R}$ be a Dedekind category, A, B, C objects of $\mathcal{R}$ and $Q \in \mathcal{R}[A, B]$, $R_1, R_2 \in \mathcal{R}[B, C]$ and $S \in \mathcal{R}[A, C]$. Then we have*

1. $Q; R_1 \sqcap S \sqsubseteq (Q \sqcap S; R_1^{\smile}); R_1$,

2. $Q; (R_1 \sqcup R_2) = Q; R_1 \sqcup Q; R_2$,

3. $Q; (R_1 \sqcap R_2) \sqsubseteq Q; R_1 \sqcap Q; R_2$,

4. $Q \sqsubseteq Q; Q^{\smile}; Q$,

5. $\top\!\!\top_{AA}; \top\!\!\top_{AB} = \top\!\!\top_{AB}$,

6. $\top\!\!\top_{AB}; \top\!\!\top_{BB} = \top\!\!\top_{AB}$,

7. $\top\!\!\top_{AB}; \top\!\!\top_{BA}; \top\!\!\top_{AB} = \top\!\!\top_{AB}$.

An important class of relations is given by mappings.

2.3. Definition. *Let $Q \in \mathcal{R}[A, B]$ be a relation.*

1. *Q is called univalent iff $Q^{\smile}; Q \sqsubseteq \mathbb{I}_B$,*

2. *Q is called total iff $\mathbb{I}_A \sqsubseteq Q; Q^{\smile}$ or equivalently iff $Q; \top\!\!\top_{BA} = \top\!\!\top_{AA}$ (or, $Q; \top\!\!\top_{BC} = \top\!\!\top_{AC}$ for all C),*

3. *Q is called a map iff Q is univalent and total.*

In the next lemma we collect two fundamental facts concerning univalent relations. Proofs may be found in [1, 9, 10, 12, 13, 16, 17, 18].

2.4. Lemma. *Let $Q \in \mathcal{R}[A,B]$ be univalent and $R,S \in \mathcal{R}[B,C]$. Then we have*

1. $Q;(R \sqcap S) = Q;R \sqcap Q;S$,
2. *If $\mathcal{R}$ is a Schröder category and Q a mapping then* $\overline{Q;R} = Q;\overline{R}$.

We define the notion of a homomorphism between relational categories as usual.

2.5. Definition. *Let $\mathcal{R}$ and $\mathcal{S}$ be Dedekind categories and $F : \mathcal{R} \to \mathcal{S}$ a functor. Then F is called a homomorphism between Dedekind categories iff*

1. $F(\bigsqcap_{i\in I} S_i) = \bigsqcap_{i\in I} F(S_i)$,
2. $F(\bigsqcup_{i\in I} S_i) = \bigsqcup_{i\in I} F(S_i)$,
3. $F(R^{\smile}) = F(R)^{\smile}$,

hold for all relations R, S_i with $i \in I$ for some index set I. If $\mathcal{R}$ and $\mathcal{S}$ are Schröder categories and F fulfills, in addition,

4. $F(\overline{R}) = \overline{F(R)}$,

F is called a homomorphism between Schröder categories.

A pair of homomorphisms $F : \mathcal{R} \to \mathcal{S}, G : \mathcal{S} \to \mathcal{R}$ is called an equivalence iff $F \circ G$ and $G \circ F$ are naturally isomorphic to the identity functors, e.g. F and G are inverses of each other up to isomorphism.

The relational description of disjoint unions is the relational sum [13, 18]. This construction corresponds to the categorical product. By conversion, a Dedekind category is self-dual. Therefore, a product is also a coproduct. Here we want to define this concept for not necessarily finite sets of objects.

2.6. Definition. *Let $\{A_i \mid i \in I\}$ be a set of objects indexed by some set I. An object $\sum_{i\in I} A_i$, together with relations $\iota_j \in \mathcal{R}[A_j, \sum_{i\in I} A_i]$ for all $j \in I$, is called a relational sum of $\{A_i \mid i \in I\}$ iff for all $i,j \in I$ with $i \neq j$ the following holds*

$$\iota_i;\iota_i^{\smile} = \mathbb{I}_{A_i}, \qquad \iota_i;\iota_j^{\smile} = \perp\!\!\!\perp_{A_iA_j}, \qquad \bigsqcup_{i\in I} \iota_i^{\smile};\iota_i = \mathbb{I}_{\sum\limits_{i\in I} A_i}.$$

$\mathcal{R}$ "has relational sums" iff for every set of objects the relational sum does exist.

For a set of two objects $\{A, B\}$, this definition corresponds to the usual definition of a relational sum. As known, categorical products and hence relational sums, are unique up to isomorphism.

For given sets of relations $Q_i \in \mathcal{R}[A_i, C]$ and $R_i \in \mathcal{R}[A_i, B_i]$ for all $i \in I$ and relational sums $(\sum_{i\in I} A_i, \iota_i)_{i\in I}$ and $(\sum_{i\in I} B_i, \iota'_i)_{i\in I}$, we use the notation

$$\bigvee_{i\in I} Q_i := \bigsqcup_{i\in I} \iota_i^{\smile};Q_i \qquad \sum_{i\in I} R_i := \bigvee_{i\in I} R_i;\iota'_i = \bigsqcup_{i\in I} \iota_i^{\smile};R_i;\iota'_i.$$

$\bigvee_{i\in I} Q_i$ is the coproduct morphism, i.e., it is the unique relation S such that $\iota_i;S = Q_k$ for all $i \in I$.

2.7. Lemma. *Let $\sum_{i\in I} A_i$ be the relational sum of $\{A_i \mid i \in I\}$ and $\sum_{j\in J} B_j$ be the relational sum of $\{B_j \mid j \in J\}$. Then for all $R_{ij} \in \mathcal{R}[A_i, B_j]$ the following holds*

1. *If $\mathcal{R}$ is a Schröder category then $\overline{\bigsqcup_{i\in I, j\in J} \iota_i^{\smile};R_{ij};\iota_j} = \bigsqcup_{i\in I, j\in J} \iota_i^{\smile};\overline{R_{ij}};\iota_j$,*
2. *$\iota_{k_1}^{\smile};R_{k_1l_1};\iota_{l_1} \sqcap \iota_{k_2}^{\smile};R_{k_2l_2};\iota_{l_2} = \perp\!\!\!\perp_{\sum_{i\in I} A_i \sum_{j\in J} B_j}$ for all $k_1, k_2 \in I, l_1, l_2 \in J$ with $k_1 \neq k_2$ or $l_1 \neq l_2$.*

Proof. 1. By Lemma 2.4, we have

$$\begin{aligned}
\overline{\bigsqcup_{i\in I, j\in J} \iota_i^{\smile};R_{ij};\iota_j} &= (\bigsqcup_{i\in I} \iota_i^{\smile};\iota_i);\overline{\bigsqcup_{i'\in I, j'\in J} \iota_{i'}^{\smile};R_{i'j'};\iota_{j'}};(\bigsqcup_{j\in J} \iota_j^{\smile};\iota_j) \\
&= \bigsqcup_{i\in I, j\in J} \iota_i^{\smile};\iota_i;\overline{\bigsqcup_{i'\in I, j'\in J} \iota_{i'}^{\smile};R_{i'j'};\iota_{j'}};\iota_j^{\smile};\iota_j \\
&= \bigsqcup_{i\in I, j\in J} \iota_i^{\smile};\overline{\bigsqcup_{i'\in I, j'\in J} \iota_i;\iota_{i'}^{\smile};R_{i'j'};\iota_{j'};\iota_j^{\smile}};\iota_j \\
&= \bigsqcup_{i\in I, j\in J} \iota_i^{\smile};\overline{R_{ij}};\iota_j.
\end{aligned}$$

2. Suppose $k_1 \neq k_2$. Then we have

$$\begin{aligned}
\iota_{k_1}^{\smile};R_{k_1l_1};\iota_{l_1} \sqcap \iota_{k_2}^{\smile};R_{k_2l_2};\iota_{l_2} &\sqsubseteq \iota_{k_1}^{\smile};(R_{k_1l_1};\iota_{l_1} \sqcap \iota_{k_1};\iota_{k_2}^{\smile};R_{k_2l_2};\iota_{l_2}) \\
&= \perp\!\!\!\perp_{\sum_{i\in I} A_i \sum_{j\in J} B_j}.
\end{aligned}$$

The case $l_1 \neq l_2$ is shown analogously. ■

Subsets inside a Dedekind category may be represented in two different ways; by vectors (a relation v such that $v = \top\!\!\!\top;v$) or partial identities (a relation l such that $l \sqsubseteq \mathbb{I}$). These two concepts are equivalent and may both be used to characterize subobjects.

2.8. Definition. *Let $l \in \mathcal{R}[A, A]$ be a partial identity. An object B together with a relation $\psi \in \mathcal{R}[B, A]$ is called a subobject of A induced by l iff*

$$\psi;\psi^{\smile} = \mathbb{I}_B, \qquad \psi^{\smile};\psi = l.$$

A Dedekind category "has subobjects" iff for all partial identities there exists a subobject.

Notice, that we have $Q;R = Q \sqcap R$ for all partial identities Q and R (see [1, 16]).

2.9. Definition. *Let $\mathcal{R}$ be a Dedekind category. Then $\mathcal{R}_{sub}$ is defined as follows:*

1. *The class of objects of $\mathcal{R}_{sub}$ is given by the class of partial identities of $\mathcal{R}$.*
2. *The set of relations $\mathcal{R}_{sub}[l_1, l_2]$ between two partial identities $l_1 : A \to A$ and $l_2 : B \to B$ of $\mathcal{R}$ is defined as the set of relations $R : A \to B$ such that $R = l_1; R; l_2$.*

The proof of the following lemma may be found in [1, 17].

2.10. Lemma. *Let $\mathcal{R}$ be a relational category. Then $\mathcal{R}_{sub}$ is a relational category of the same kind with subobjects.*

Obviously, the functor $F : \mathcal{R} \to \mathcal{R}_{sub}$ defined by

$$F(A) := \mathbb{I}_A, \qquad F(R) := R$$

is an embedding of the corresponding relational categories.

3. Matrix Algebras

Given a Dedekind category $\mathcal{R}$, an algebra of matrices with coefficients from $\mathcal{R}$ may be defined.

3.1. Definition. *Let $\mathcal{R}$ be a Dedekind category. The algebra $\mathcal{R}^+$ of matrices with coefficients from $\mathcal{R}$ is defined by:*

1. *An object of $\mathcal{R}^+$ is a function from an arbitrary I to $\mathrm{Obj}_{\mathcal{R}}$.*
2. *For every pair $f : I \to \mathrm{Obj}_{\mathcal{R}}, g : J \to \mathrm{Obj}_{\mathcal{R}}$ of objects from $\mathcal{R}^+$, the set of morphisms $\mathcal{R}^+[f, g]$ is the set of all functions $R : I \times J \to \mathrm{Mor}_{\mathcal{R}}$ such that $R(i, j) \in \mathcal{R}[f(i), g(j)]$.*
3. *For $R \in \mathcal{R}^+[f, g]$ and $S \in \mathcal{R}^+[g, h]$, composition is defined by*
$$(R; S)(i, k) := \bigsqcup_{j \in J} R(i, j); S(j, k).$$
4. *For $R \in \mathcal{R}^+[f, g]$, conversion is defined by*
$$R^{\smile}(j, i) := (R(i, j))^{\smile}.$$
5. *For $R, S \in \mathcal{R}^+[f, g]$, union and intersection are defined by*
$$(R \sqcup S)(i, j) := R(i, j) \sqcup S(i, j), \qquad (R \sqcap S)(i, j) := R(i, j) \sqcap S(i, j).$$
6. *Identity, zero and universal elements are defined by*
$$\mathbb{I}_f(i_1, i_2) := \begin{cases} \perp\!\!\!\perp_{f(i_1)f(i_2)} & : i_1 \neq i_2 \\ \mathbb{I}_{f(i_1)} & : i_1 = i_2, \end{cases}$$
$$\perp\!\!\!\perp_{fg}(i, j) := \perp\!\!\!\perp_{f(i)g(j)}, \qquad \top\!\!\!\top_{fg}(i, j) := \top\!\!\!\top_{f(i)g(j)}.$$

7. If $\mathcal{R}$ is a Schröder category negation is defined by

$$\overline{R}(i,j) := \overline{R(i,j)}.$$

Obviously, a morphism in $\mathcal{R}^+$ may be seen as an (in general non-finite) matrix indexed by objects from $\mathcal{R}$. The proof of the following result is an easy exercise and is, therefore, omitted.

3.2. Lemma. *Let $\mathcal{R}$ be a relational category. Then $\mathcal{R}^+$ is a relational category of the same kind.*

Furthermore, the possibility to build a disjoint union $\biguplus_{i \in I} J_i$ of an arbitrary set $\{J_i \mid i \in I\}$ of sets indexed by I gives us the following lemma.

3.3. Lemma. *$\mathcal{R}^+$ has relational sums.*

Proof. Let $\{f_i : J_i \to \mathrm{Obj}_{\mathcal{R}} \mid i \in I\}$ be a set of objects of $\mathcal{R}^+$. Then the function $h : \biguplus_{i \in I} J_i \to \mathrm{Obj}_{\mathcal{R}}$ defined by $h(j) := f_i(j)$ iff $j \in J_i$ is also an object of $\mathcal{R}^+$. Now, we define

$$\iota_i(j_1, j_2) := \begin{cases} \perp\!\!\!\perp_{f_i(j_1)h(j_2)} & : j_1 \neq j_2 \\ \mathbb{I}_{f(j_1)} & : j_1 = j_2. \end{cases}$$

An easy verification shows that the above definition gives the required relational sum. ∎

Obviously, $\mathcal{R}$ may be embedded into $\mathcal{R}^+$ by sending each R to the 1×1 matrix (R). It is easily checked that $\mathcal{R}^+_{sub}$ has relational sums, too. Hence, every relational category may be faithfully embedded into a relational category of the same kind with relational sums and subobjects.

4. Integral Objects and the Basis of $\mathcal{R}$

Following concepts used in algebra, we call an object A integral if there are no zero divisors within the subalgebra $\mathcal{R}[A,A]$. In other words, the homogeneous relation algebra given by the set $\mathcal{R}[A,A]$ is an integral relation algebra in the sense of [2]. Later on, the class of integral objects will define the basis of $\mathcal{R}$.

4.1. Definition. *An object A of a Dedekind category is called integral iff $\perp\!\!\!\perp_{AA} \neq \top\!\!\!\top_{AA}$ and for all $Q, R \in \mathcal{R}[A,A]$ the equation $Q;R = \perp\!\!\!\perp_{AA}$ implies either $Q = \perp\!\!\!\perp_{AA}$ or $R = \perp\!\!\!\perp_{AA}$.*

There are two other simple properties characterizing the integral objects of a Dedekind category.

4.2. Lemma. *Let $\mathcal{R}$ be an atomic Dedekind category. Then the following properties are equivalent:*

1. A is an integral object,

2. *Every non-zero relation in $\mathcal{R}[A,A]$ is total,*

3. *$\mathbb{I}_A$ is an atom.*

Proof. 1. $\Rightarrow$ 3. : Suppose $\mathbb{I}_A$ is *not* an atom. Since $\mathcal{R}[A,A]$ is atomic there are at least two different atoms $Q \sqsubseteq \mathbb{I}_A$ and $R \sqsubseteq \mathbb{I}_A$. We have $Q;R = Q \sqcap R = \perp\!\!\!\perp_{AA}$ which contradicts A is integral.
3. $\Rightarrow$ 2. : Let $Q : A \to A$ be nonzero and suppose $\mathbb{I}_A \sqcap Q;Q^{\smile} = \perp\!\!\!\perp_{AA}$. Then we conclude a contradiction

$$\begin{aligned} Q &= \mathbb{I}_A;Q \sqcap Q \\ &\sqsubseteq (\mathbb{I}_A \sqcap Q;Q^{\smile});Q \\ &= \perp\!\!\!\perp_{AA};Q \\ &= \perp\!\!\!\perp_{AA}. \end{aligned}$$

Since $\mathbb{I}_A$ is an atom we follow $\mathbb{I}_A \sqcap Q;Q^{\smile} = \mathbb{I}_A$ and hence $\mathbb{I}_A \sqsubseteq Q;Q^{\smile}$.
2. $\Rightarrow$ 1. : Suppose $Q;R = \perp\!\!\!\perp_{AA}$ and $R \neq \perp\!\!\!\perp_{AA}$. Since R is total we conclude

$$Q = Q;\mathbb{I}_A \sqsubseteq Q;R;R^{\smile} = \perp\!\!\!\perp_{AA};R^{\smile} = \perp\!\!\!\perp_{AA},$$

which gives us the assumption. ∎

The special properties of the relations in $\mathcal{R}[A,A]$ mentioned in the last lemma may be transferred to the relation in $\mathcal{R}[A,B]$ for an arbitrary object B.

4.3. Lemma. *Let A be an integral object of an atomic Dedekind category.*

1. *If $Q;R = \perp\!\!\!\perp_{BC}$ with $Q \in \mathcal{R}[B,A]$ and $R \in \mathcal{R}[A,C]$ then either $Q = \perp\!\!\!\perp_{BA}$ or $R = \perp\!\!\!\perp_{AC}$.*

2. *If $S \neq \perp\!\!\!\perp_{AB}$ then $S;\top\!\!\!\top_{BC} = \top\!\!\!\top_{AC}$ for all C.*

Proof. 1. $Q;R = \perp\!\!\!\perp_{BC}$ implies $Q^{\smile};Q;R;R^{\smile} = Q^{\smile};\perp\!\!\!\perp_{BC};R^{\smile} = \perp\!\!\!\perp_{AA}$. Since A is integral, we have either $Q^{\smile};Q = \perp\!\!\!\perp_{AA}$ or $R;R^{\smile} = \perp\!\!\!\perp_{AA}$. In the first case we conclude using Lemma 2.2 $Q \sqsubseteq Q;Q^{\smile};Q = Q;\perp\!\!\!\perp_{AA} = \perp\!\!\!\perp_{BA}$. The other case is handled similarly.
2. Analogously to 1. $\Rightarrow$ 3. and 3. $\Rightarrow$ 2. of the last lemma by using 1. ∎

Notice, that the last lemma implies that all non-zero relations in $\mathcal{R}[A,B]$ are total if A is integral.

4.4. Definition. *Let $\mathcal{R}$ be a Dedekind category. The basis $\mathcal{B}_{\mathcal{R}}$ of $\mathcal{R}$ is defined as the full subcategory given by the class of all integral objects.*

As usual, we omit the index $\mathcal{R}$ in $\mathcal{B}_{\mathcal{R}}$ when its meaning is clear from the context.

4.5. Theorem. *Let $\mathcal{R}$ be an atomic Dedekind category with relational sums, and let $\mathcal{B}$ be the basis of $\mathcal{R}$. Then $\mathcal{B}$ is a proper subalgebra of $\mathcal{R}$.*

Proof. Let A be an object of $\mathcal{B}$ and $\iota_1 : A \to A+A$ and $\iota_2 : A \to A+A$ the relational sum. Suppose $\mathbb{I}_{A+A}$ is an atom. Then we have

$$\iota_1^{\smile};\iota_1 \sqsubseteq \iota_1^{\smile};\iota_1 \sqcup \iota_2^{\smile};\iota_2 = \mathbb{I}_{A+A}.$$

Now, we distinguish two cases:

1. $\iota_1^{\smile};\iota_1 = \perp\!\!\!\perp_{A+AA+A}$: We conclude

$$\mathbb{I}_A = \mathbb{I}_A;\mathbb{I}_A = \iota_1;\iota_1^{\smile};\iota_1;\iota_1^{\smile} = \iota_1;\perp\!\!\!\perp_{A+AA+A};\iota_1^{\smile} = \perp\!\!\!\perp_{AA},$$

   which contradicts to $\mathbb{I}_A$ being an atom.

2. $\iota_1^{\smile};\iota_1 = \mathbb{I}_{A+A}$: We conclude

$$\iota_2^{\smile};\iota_2 = \mathbb{I}_{A+A};\iota_2^{\smile};\iota_2 = \iota_1^{\smile};\iota_1;\iota_2^{\smile};\iota_2 = \iota_1^{\smile};\perp\!\!\!\perp_{AA};\iota_2 = \perp\!\!\!\perp_{A+AA+A},$$

   which also leads to a contradiction because of the symmetry of ι_1 and ι_2.

This completes the proof. ∎

The last theorem has shown that the definition of the basis of a Dedekind category is not trivial, i.e. the basis usually does not correspond to the whole algebra.

In the rest of this section we want to define an equivalence relation $\approx$ on the basis of $\mathcal{R}$. Later on, it turns out that the equivalence classes of $\approx$ characterize the simple components of the category.

4.6. Lemma. *All integral objects A of an atomic Dedekind category have at most two ideal elements, namely $\perp\!\!\!\perp_{AA}$ or $\top\!\!\!\top_{AA}$.*

Proof. Suppose $R : A \to A$ is an ideal element. Since $\mathbb{I}_A$ is an atom $R \sqcap \mathbb{I}_A$ is either $\perp\!\!\!\perp_{AA}$ or $\mathbb{I}_A$. Suppose $R \sqcap \mathbb{I}_A = \perp\!\!\!\perp_{AA}$. Then we have

$$\begin{aligned} R &= R;\mathbb{I}_A \sqcap \top\!\!\!\top_{AA} \\ &\sqsubseteq R;(\mathbb{I}_A \sqcap R^{\smile};\top\!\!\!\top_{AA}) \\ &= R;(\mathbb{I}_A \sqcap R)^{\smile} \\ &= R;\perp\!\!\!\perp_{AA} \\ &= \perp\!\!\!\perp_{AA}. \end{aligned}$$

If $R \sqcap \mathbb{I}_A = \mathbb{I}_A$ we conclude $\top\!\!\!\top_{AA} = \top\!\!\!\top_{AA};\mathbb{I}_A \sqsubseteq \top\!\!\!\top_{AA};R = R$. ∎

The last lemma leads to the following definition.

4.7. Definition. *We define a relation $\approx$ on the class of integral objects of $\mathcal{R}$ by*

$$A \approx B \quad :\Longleftrightarrow \quad \top\!\!\!\top_{AB};\top\!\!\!\top_{BA} = \top\!\!\!\top_{AA}.$$

4.8. Lemma. *$\approx$ is an equivalence relation on the basis of $\mathcal{R}$.*

Proof. By Lemma 2.2 $\approx$ is reflexive. Symmetry is implied by the following property

$$(*) \qquad \top\!\!\top_{AB};\top\!\!\top_{BA} = \top\!\!\top_{AA} \iff \top\!\!\top_{BA};\top\!\!\top_{AB} = \top\!\!\top_{BB}.$$

To prove this property, suppose $\top\!\!\top_{AB};\top\!\!\top_{BA} = \top\!\!\top_{AA}$ and $\top\!\!\top_{BA};\top\!\!\top_{AB} = \perp\!\!\perp_{BB}$. Then by using Lemma 2.2

$$\top\!\!\top_{AB} = \top\!\!\top_{AB};\top\!\!\top_{BA};\top\!\!\top_{AB} = \top\!\!\top_{AB};\perp\!\!\perp_{BB} = \perp\!\!\perp_{AB}$$

we get a contradiction. The other implication follows by duality.

To prove transitivity suppose $A \approx B$ and $B \approx C$. By definition we have $\top\!\!\top_{AB};\top\!\!\top_{BA} = \top\!\!\top_{AA}$ and $\top\!\!\top_{BC};\top\!\!\top_{CB} = \top\!\!\top_{BB}$. By $(*)$ we get $\top\!\!\top_{BA};\top\!\!\top_{AB} = \top\!\!\top_{BB}$ and $\top\!\!\top_{CB};\top\!\!\top_{BC} = \top\!\!\top_{CC}$. Using Lemma 2.2 we conclude

$$\begin{aligned}
\top\!\!\top_{CC} &= \top\!\!\top_{CB};\top\!\!\top_{BC} \\
&= \top\!\!\top_{CB};\top\!\!\top_{BB};\top\!\!\top_{BC} \\
&= \top\!\!\top_{CB};\top\!\!\top_{BA};\top\!\!\top_{AB};\top\!\!\top_{BC} \\
&\sqsubseteq \top\!\!\top_{CA};\top\!\!\top_{AC}
\end{aligned}$$

and hence $A \approx C$. ■

Between two objects of different equivalence classes of $\approx$ there exists only one relation. Later on, this property gives us the possibility to separate those components.

4.9. Lemma. *Let A and B be integral objects of an atomic Dedekind category. Then the following properties are equivalent:*

1. $A \approx B$,

2. $\top\!\!\top_{AB} \neq \perp\!\!\perp_{AB}$.

Proof. 1. $\Rightarrow$ 2. : Since $A \approx B$ we have $\top\!\!\top_{AB};\top\!\!\top_{BA} = \top\!\!\top_{AA}$. From this we conclude $\top\!\!\top_{AB} \neq \perp\!\!\perp_{AB}$ because otherwise we have $\top\!\!\top_{AB};\top\!\!\top_{BA} = \top\!\!\top_{AB};\perp\!\!\perp_{BA} = \perp\!\!\perp_{AA}$.
2. $\Rightarrow$ 1. : Lemma 4.3 gives us the assertion. ■

Notice, that the last lemma is still valid if only A is integral. Furthermore, this lemma implies that $R = \perp\!\!\perp_{AB}$ for all $R \in \mathcal{R}[A,B]$ if $A \not\approx B$.

5. A Pseudo Representation Theorem

Now, we are able to prove our main theorem.

5.1. Theorem. *Let $\mathcal{R}$ be an atomic Dedekind category with relational sums and subobjects and let $\mathcal{B}$ be the basis of $\mathcal{R}$. Then $\mathcal{R}$ and $\mathcal{B}^+$ are equivalent. Furthermore, if $\mathcal{R}$ is a relation algebra $\mathcal{R}$ and $\mathcal{B}^+$ are equivalent as relation algebras.*

Proof. First, we show that every object A of $\mathcal{R}$ is isomorphic to a relational sum $\sum_{i\in I} A_i$ of objects from $\mathcal{B}$. Let $\{l_i \mid i \in I\}$ be set of all atoms $l_i \sqsubseteq \mathbb{I}_A$. Because $\mathcal{R}$ has subobjects, this gives us a set $\{A_i \mid i \in I\}$ of objects and a set $\{\psi_i \mid i \in I\}$ of morphisms with

$$\psi_i;\psi_i^{\smile} = \mathbb{I}_{A_i}, \qquad \psi_i^{\smile};\psi_i = l_i.$$

Together with the computations

$$\begin{aligned} &\psi_i^{\smile};\psi_i;\psi_j^{\smile};\psi_j = l_i;l_j = l_i \sqcap l_j = \perp\!\!\!\perp_{AA}, \\ &\psi_i;\psi_j^{\smile} = \psi_i;\psi_i^{\smile};\psi_i;\psi_j^{\smile};\psi_j;\psi_j^{\smile} = \psi_i;\perp\!\!\!\perp_{AA};\psi_j^{\smile} = \perp\!\!\!\perp_{A_iA_j} \\ \text{and } &\bigsqcup_{i\in I}\psi_i^{\smile};\psi_i = \bigsqcup_{i\in I} l_i = \mathbb{I}_A. \end{aligned}$$

and the uniqueness of a relational sum, we have $A \cong \sum_{i\in I} A_i$ and $\psi_i = \iota_i$.

Suppose $R \sqsubseteq \mathbb{I}_{A_i}$. Then we have

$$\psi_i^{\smile};R;\psi_i \sqsubseteq \psi_i^{\smile};\psi_i = l_i.$$

Now, we distinguish two cases:

1. $\psi_i^{\smile};R;\psi_i = \perp\!\!\!\perp_{AA}$: We conclude

$$R = \psi_i;\psi_i^{\smile};R;\psi_i;\psi_i^{\smile} = \psi_i;\perp\!\!\!\perp_{AA};\psi_i^{\smile} = \perp\!\!\!\perp_{A_iA_i}.$$

2. $\psi_i^{\smile};R;\psi_i = l_i$: We conclude

$$R = \psi_i;\psi_i^{\smile};R;\psi_i;\psi_i^{\smile} = \psi_i;l_i;\psi_i^{\smile} = \psi_i;\psi_i^{\smile};\psi_i;\psi_i^{\smile} = \mathbb{I}_{A_i}.$$

This shows that $\mathbb{I}_{A_i}$ is an atom and hence A_i in $\mathcal{B}$. Now, we define the required equivalence $F : \mathcal{R} \to \mathcal{B}^+, G : \mathcal{B}^+ \to \mathcal{R}$ by

$$\begin{aligned} F(A) &:= f : I \to \mathrm{Obj}_{\mathcal{B}} \text{ with } f(i) = A_i, \\ F(R) &:= h : I_1 \times I_2 \to \mathrm{Mor}_{\mathcal{B}} \text{ with } h(i_1,i_2) = \psi_{i_1};R;\psi_{i_2}^{\smile}, \\ G(f) &:= \sum_{i\in I} f(i), \\ G(h) &:= \bigsqcup_{i\in I, j\in J} \psi_i^{\smile};h(i,j);\psi_j \end{aligned}$$

for all $R \in \mathcal{R}[A,B]$, objects $A \cong \sum_{i\in I_1} A_i, B \cong \sum_{i\in I_2} B_i$, $f \in \mathrm{Obj}_{\mathcal{B}^+}$ and $h \in \mathcal{B}^+[f,g]$. Using Lemma 2.4 and 2.7 an easy computation shows that F and G are homomorphisms of Dedekind/Schröder categories.

Moreover, we have $(G \circ F)(A) = \sum_{i \in I} A_i \cong A$ such that there is a natural isomorphism between $G \circ F$ and the identity on $\mathcal{R}$. Conversely, we have

$$\begin{aligned}(F \circ G)(f)(i,j) &= F(\bigsqcup_{i \in I, j \in J} \psi_i^{\smile}; f(i,j); \psi_j) \\ &= \psi_i; (\bigsqcup_{i \in I, j \in J} \psi_i^{\smile}; f(i,j); \psi_j); \psi_j^{\smile} \\ &= f(i,j).\end{aligned}$$

This completes the proof. ∎

The embedding theorems in Section 2 and 3 give us the following corollary.

5.2. Corollary. *Every atomic Dedekind category may be embedded into an atomic Dedekind category which is equivalent to a matrix algebra over a suitable basis.*

6. Simplicity

It is known that every homogeneous relation algebra may be embedded into a product of simple algebras. This theorem is an application of general a concept from universal algebra. In [17] it was shown that this theorem can be extended to arbitrary heterogeneous relation algebras. Furthermore, it was shown that simplicity can be characterize by just one equation, the so-called Tarski-rule

$$Q \neq \perp\!\!\!\perp_{AB} \text{ implies } \top\!\!\!\top_{CA}; Q; \top\!\!\!\top_{BD} = \top\!\!\!\top_{CD}$$

for all $Q : A \to B$ and A, B, C, D.

In this section, we want to reprove the embedding theorem above using our concept of the basis of a Dedekind category and the induced equivalence relation $\approx$.

6.1. Lemma. *Let $\mathcal{R}$ be an atomic Dedekind category with relational sums and subobjects such that all objects of basis $\mathcal{B}$ are equivalent (in resp. to $\approx$). Then $\mathcal{R}$ is simple.*

Proof. We show that $\mathcal{B}^+$ is simple. The equivalence of $\mathcal{B}^+$ and $\mathcal{R}$ then implies the assertion. Let $e : I \to \mathcal{B}, f : J \to \mathcal{B}, g : K \to \mathcal{B}$ and $h : L \to \mathcal{B}$ be objects of $\mathcal{B}^+$ and $\perp\!\!\!\perp_{fg} \neq R \in \mathcal{B}^+[f,g]$. By definition there is a $j' \in J$ and a $k' \in K$ such that $R(j',k') \neq \perp\!\!\!\perp_{f(j')g(k')}$. From Lemma 4.3 and the fact that all objects of $\mathcal{B}$ are equivalent we conclude

$$\top\!\!\!\top_{e(i)f(j')}; R(j',k'); \top\!\!\!\top_{g(k')h(l)} = \top\!\!\!\top_{e(i)f(j')}; \top\!\!\!\top_{f(j')h(l)} = \top\!\!\!\top_{e(i)h(l)}$$

for all $i \in I$ and $l \in L$. This gives us

$$(\top\!\!\!\top_{ef}; R; \top\!\!\!\top_{gh})(i,l) = \bigsqcup_{j \in J, k \in K} (\top\!\!\!\top_{ef}(i,j); R(j,k); \top\!\!\!\top_{gh}(k,l))$$

$$\begin{aligned} &= \bigsqcup_{j\in J,k\in K} (\top\!\!\top_{e(i)f(j)}; R(j,k); \top\!\!\top_{g(k)h(l)}) \\ &= \top\!\!\top_{e(i)h(l)} \\ &= \top\!\!\top_{eh}(i,l) \end{aligned}$$

and hence $\top\!\!\top_{ef}; R; \top\!\!\top_{gh} = \top\!\!\top_{eh}$. ∎

Let $B_{\approx}$ be the set of equivalence classes of $\approx$, and $\mathcal{B}_k$ be the full subcategory of $\mathcal{B}$ induced by the equivalence class k. By the last lemma $\mathcal{B}_k^+$ is simple.

6.2. Definition. *Let K be a set, and $\mathcal{R}_k$ for all $k \in K$ be Dedekind categories. The product Dedekind category $\prod_{k\in K} \mathcal{R}_k$ is defined as follows:*

1. *An object of $\prod_{k\in K} \mathcal{R}_k$ is a function $f : K \to \bigcup_{k\in K} \mathrm{Obj}_{\mathcal{R}_k}$ such that $f(k) \in \mathrm{Obj}_{\mathcal{R}_k}$.*

2. *A morphism in $(\prod_{k\in K} \mathcal{R}_k)[f, g]$ is a function $Q : K \to \bigcup_{k\in K} \mathcal{R}_k[f(k), g(k)]$ such that $Q(k) \in \mathcal{R}_k[f(k), g(k)]$.*

3. *The operations and constants are defined componentwise by*

$$\begin{aligned} (Q; S)(k) &:= Q(k); S(k), \\ (Q \sqcap R)(k) &:= Q(k) \sqcap R(k), \\ (Q \sqcup R)(k) &:= Q(k) \sqcup R(k), \\ Q^{\smile}(k) &:= Q(k)^{\smile}, \\ \mathbb{I}_f(k) &:= \mathbb{I}_{f(k)}, \\ \top\!\!\top_{fg}(k) &:= \top\!\!\top_{f(k)g(k)}, \\ \perp\!\!\perp_{fg}(k) &:= \perp\!\!\perp_{f(k)g(k)}, \end{aligned}$$

   for all $Q, R \in \prod_{k\in K} \mathcal{R}_k[f, g]$ and $S \in \prod_{k\in K} \mathcal{R}_k[g, h]$.

4. *If $\mathcal{R}$ is a Schröder category negation is defined by $\overline{Q}(k) := \overline{Q(k)}$.*

An easy verification shows that $\prod_{k\in K} \mathcal{R}_k$ is indeed a relational category of the same kind as the components $\mathcal{R}_k$.

6.3. Theorem. *Let $\mathcal{R}$ be a small atomic Dedekind category. Then $\mathcal{B}^+$ and $\prod_{k\in B_{\approx}} \mathcal{B}_k^+$ are isomorphic.*

Proof. Let $f : I \to \mathrm{Obj}_{\mathcal{B}}$ and $g : J \to \mathrm{Obj}_{\mathcal{B}}$ be objects of $\mathcal{B}^+$ and $R \in \mathcal{B}^+[f, g]$. Furthermore, let

$$\begin{aligned}
&I_k := \{i \in I \mid f(i) \text{ is a object of } \mathcal{B}_k\},\\
&J_k := \{j \in J \mid g(j) \text{ is a object of } \mathcal{B}_k\},\\
&f_k : I_k \to \mathrm{Obj}_{\mathcal{B}_k} \text{ such that } f_k(i) = f(i),\\
&R_k : I_k \times J_k \to \mathrm{Mor}_{\mathcal{B}_k} \text{ such that } R_k(i,j) = R(i,j).
\end{aligned}$$

Then we define a functor $F : \mathcal{B}^+ \to \prod\limits_{k \in B_{\approx}} \mathcal{B}_k^+$ by

$$\begin{aligned}
F(f)(k) &:= f_k,\\
F(R)(k) &:= R_k.
\end{aligned}$$

Using Lemma 4.9 we get

$$\begin{aligned}
F(R;S)(k)(i,l) &= (R;S)_k(i,l)\\
&= (R;S)(i,l)\\
&= \bigsqcup_{j \in J} R(i,j); S(j,l)\\
&= \bigsqcup_{j \in J_k} R(i,j); S(j,l)\\
&= \bigsqcup_{j \in J_k} R_k(i,j); S_k(j,l)\\
&= \bigsqcup_{j \in J_k} F(R)(k)(i,j); F(S)(k)(j,l)\\
&= (F(R)(k); F(S)(k))(i,l)
\end{aligned}$$

and hence $F(R;S) = F(R);F(S)$. An easy verification shows the other required properties of F and is, therefore, omitted. ■

Combining our two main theorems, we get the following corollary.

6.4. Corollary. *Let $\mathcal{R}$ be a small atomic Dedekind category with relational sums and subobjects. Then $\mathcal{R}$ and $\prod\limits_{k \in B_{\approx}} \mathcal{B}_k^+$ are equivalent. Furthermore, if $\mathcal{R}$ is a relation algebra $\mathcal{R}$ and $\prod\limits_{k \in B_{\approx}} \mathcal{B}_k^+$ are equivalent as relation algebras.*

Again, using the embedding theorems we get the following corollary.

6.5. Corollary. *Every small atomic Dedekind category may be embedded into an atomic Dedekind category which is equivalent to a product of simple matrix algebras.*

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