# THE MIXED PROBLEM FOR A DEGENERATE OPERATOR EQUATION

**Liparit Tepoyan**

*Yerevan State University*
*tepoyan@yahoo.com*

*Abstract:* We consider a mixed problem for a degenerate differentialoperator equation of higher order. We establish some embedding theorems in weighted Sobolev spaces and show existence and uniqueness of the generalized solution of this problem. We also give a description of the spectrum for the corresponding operator.

*Key words:* Differential equations in abstract spaces, boundary value problems, weighted Sobolev spaces, spectral theory of linear operators

*MSC 2000: 34B05, 34L05, 46E35*

## 1 Introduction

In this article we consider the mixed problem for the operator equation

$$Pu \equiv (-1)^m D_t^m (t^\alpha D_t^m) u + t^\alpha A u = f, \tag{1}$$

where $t \in (0, b)$, $\alpha \geq 0$, $D_t \equiv d/dt$, $f \in L_{2,-\alpha}((0, b), \mathcal{H})$, and $A : \mathcal{H} \to \mathcal{H}$ is a linear operator in a separable Hilbert space $\mathcal{H}$.

Our approach, similar to that used in [3], [15] for the case $m = 1$ and in [12] for $m = 2$, is based on the consideration of the one-dimensional equation (1), i.e., when $A$ is the operator of multiplication by a number $a$ (see [8]). Note that in [3], [12] and [13] have been considered Dirichlet problem for the equation (1). It has been proved in [14] that the spectrum of the operator $\mathbb{L}u \equiv (-1)^m t^{2m-\alpha} D_t^m (t^\alpha D_t^m) u$, $\mathbb{L} : L_{\alpha-2m} \to L_{\alpha-2m}$, $\alpha \neq 1, 3, \ldots, 2m-1$ is purely continuous and coincides with the ray $[4^{-m}(1-\alpha)^2(3-\alpha)^2 \cdots (2m-1-\alpha)^2; +\infty)$.

In Section 2 we define the weighted Sobolev spaces $\dot{W}_\alpha^m$, $W_\alpha^m(b)$, $W_\alpha^m(0)$, $W_\alpha^m$, describe the behaviour of the functions from these spaces close to $t = 0$ and prove some embedding and compactness theorems. Furthermore we define the generalized solution for the one-dimensional equation (1) and give a sufficient

condition on the number $a$ which guarantees existence and uniqueness of the generalized solution for every $f \in L_{2,-\alpha}$.
In Section 3 under some conditions on the spectrum of the operator $A$ we prove unique solvability of the operator equation (1) for every
$f \in L_{2,-\alpha}((0,b),\mathcal{H})$ and give the description of the spectrum for the corresponding operator $\mathbb{P} = t^{-\alpha}P$.
Note that the operator $A$ in the equation (1) in general is unbounded in $\mathcal{H}$.

# 2 The One-dimensional Case

## 2.1 Spaces $\dot{W}_\alpha^m$, $W_\alpha^m(b)$, $W_\alpha^m(0)$ and $W_\alpha^m$

Let $\dot{C}^m[0,b]$ be the set of $m$–times continuously differentiable functions $u(t)$ defined on $[0,b]$ and satisfying the conditions

$$u^{(k)}(t)\big|_{t=0} = u^{(k)}(t)\big|_{t=b} = 0, \quad k = 0,1,\ldots,m-1. \tag{2}$$

Let $\dot{W}_\alpha^m$, $\alpha \geq 0$, be the completion of $\dot{C}^m[0,b]$ in the norm

$$|u, W_\alpha^m|^2 = \int_0^b t^\alpha\,|u^{(m)}(t)|^2\,dt. \tag{3}$$

Denote $L_{2,\beta} = \{f,\ |f, L_{2,\beta}|^2 = \int_0^b t^\beta |f(t)|^2\,dt < \infty\}$.

**Proposition 2.1.** *For every $u \in \dot{W}_\alpha^m$ close to $t=0$ we have*

$$|u^{(j)}(t)|^2 \leq C_j t^{2m-2j-1-\alpha} |u, \dot{W}_\alpha^m|^2 \tag{4}$$

*where $\alpha \neq 1,3,\ldots,2m-1, j = 0,1,\ldots,m-1$ and in the case $\alpha = 2n+1$, $n = 0,1,\ldots,m-1$, $t^{2m-2j-2n-2}|\ln t|$ instead of $t^{2m-2j-1-\alpha}$ in the expression (4), $j = 0,1,\ldots,m-n-1$* (see [13]).

**Proposition 2.2.** *For every $\alpha \neq 1,3,\ldots,2m-1$ we have a continuous embedding*

$$\dot{W}_\alpha^m \subset L_{2,\alpha-2m} \tag{5}$$

*which is not compact. Moreover for every $\beta > \alpha - 2m$ the embedding $\dot{W}_\alpha^m \subset L_{2,\beta}$ is compact.*

For the proof of these propositions see [5], [9] and [13]. Note also that here for $\alpha \neq 1, 3, \ldots, 2m-1$ we have used Hardy's inequality (see [7])

$$\int_0^b t^{\beta-2}|u(t)|^2\, dt \leq \frac{4}{(\beta-1)^2}\int_0^b t^{\beta}|u'(t)|^2\, dt, \quad \beta \neq 1 \tag{6}$$

where the number $4(\beta-1)^{-2}$ is the best possible constant. Note that for $\alpha = 1, 3, \ldots, 2m-1$ the embedding (5) fails.

**Remark 2.3.** *In the space $\dot{W}_\alpha^m$ for $\alpha \neq 1, 3, \ldots, 2m-1$ we can define an equivalent norm*

$$||u||_\alpha^2 \equiv \int_0^b \left|\left(t^{\alpha/2}u\right)^{(m)}\right|^2\, dt. \tag{7}$$

Indeed, to estimate the norm (7) by the norm (3) we first apply the Leibniz rule for differentiation of the expression $(t^{\alpha/2}u)^{(m)}$, then use Hardy's inequality (6) for the derivatives $u^{(k)}(t), k = 0, 1, \ldots, m-1$, and estimate all terms by $\int_0^b t^\alpha |u^{(m)}(t)|^2\, dt$. To estimate (3) by the norm (7) we first set $u(t) = t^{-\alpha/2}v(t)$ and after differentiation again use Hardy's inequality for the derivatives $v^{(k)}(t), k = 0, 1, \ldots, m-1$, estimate all terms by $\int_0^b |v^{(m)}(t)|^2\, dt$ and then return to the function $u(t)$.
Note that Remark 2.3 for the case $\alpha < 1$ was proved in [6].
Remark 2.3 allows us to write

$$\dot{W}_\alpha^m = t^{-\alpha/2}\dot{W}^m(0, b).$$

Denote by $W_\alpha^m$ the completion of $C^m[0, b]$ in the norm

$$|u, W_\alpha^m|^2 = \int_0^b \left(t^\alpha\, |u^{(m)}(t)|^2 + t^\alpha |u(t)|^2\right)\, dt. \tag{8}$$

**Proposition 2.4.** *For every $u \in W_\alpha^m$ close to $t = 0$ we have*

$$|u^{(j)}(t)|^2 \leq (B_j + C_j t^{2m-2j-1-\alpha})|u, W_\alpha^m|^2, \tag{9}$$

*where $\alpha \neq 1, 3, \ldots, 2m-1, j = 0, 1, \ldots, m-1$ and in the case $\alpha = 2n+1$, $n = 0, 1, \ldots, m-1$, $t^{2m-2j-2n-2}|\ln t|$ instead of $t^{2m-2j-1-\alpha}$ in the expression (9), $j = 0, 1, \ldots, m-n-1$.*

*Proof.* Let $0 < h < b/2m$. Then there are constants $a_{jk}$ and $b_{jk}$, $j = 1, 2, \ldots, m-1, k = 0, 1, \ldots, m-1$, such that for every $t \in (0, b/2)$ we have the following equalities (see [9])

$$u^{(j)}(t) = \sum_{k=0}^{m-1} \left[ a_{jk} u(t+kh) + b_{jk} \int_t^{t+kh} (t+kh-x)^{m-1} u^{(m)}(x)\, dx \right]. \tag{10}$$

For $x \in [t, t+kh]$ we have $0 \leq t+kh-x \leq (m-1)h$ and, therefore,

$$\left| \int_t^{t+kh} (t+kh-x)^{m-1} u^{(m)}(x)\, dx \right| \leq c \int_t^b |u^{(m)}(x)|\, dx. \tag{11}$$

From (10) together with the inequality (11) we have

$$|u^{(j)}(t)|^2 \leq c \left( \sum_{k=0}^{m-1} |u(t+kh)|^2 + \left( \int_t^b |u^{(m)}(x)|\, dx \right)^2 \right). \tag{12}$$

Multiplying both sides of (12) by $t^\alpha$ and integrating over $(0, b/2)$ for $j = 1, 2, \ldots, m-1$ we get

$$|u^{(j)}, L_{2,\alpha}(0, b/2)|^2 \leq c \left( |u, L_{2,\alpha}|^2 + \int_0^{b/2} t^\alpha \left( \int_t^b |u^{(m)}(x)|\, dx \right)^2 dt \right). \tag{13}$$

Now for $\alpha \neq 1$ we can write

$$\int_0^{b/2} t^\alpha \left( \int_t^b |u^{(m)}(x)|\, dx \right)^2 dt \leq$$

$$\leq c \int_0^{b/2} t^\alpha (b^{1-\alpha} - t^{1-\alpha}) \left( \int_t^b x^\alpha |u^{(m)}(x)|^2\, dx \right) dt.$$

In the case $\alpha = 1$ we proceed in the same way. Therefore, from the inequality (13) we get the following estimate for the intermediate derivatives $u^{(j)}(t)$, $j = 1, 2, \ldots, m-1$ for every $\alpha \geq 0$

$$|u^{(j)}, L_{2,\alpha}(0, b/2)| \leq c |u, W_\alpha^m|. \tag{14}$$

Inequality (14) is true also for the case $t \in (b/2, b)$, because then we indeed do not have a weight (see [2]).

Let $\alpha \neq 1, 3, \ldots, 2m-1$ and $t_0 \in (0, b]$. Then we can write

$$u^{(m-1)}(t) - u^{(m-1)}(t_0) = \int_{t_0}^t u^{(m)}(x)\, dx. \tag{15}$$

Therefore, we get

$$\begin{aligned} |u^{(m-1)}(t) - u^{(m-1)}(t_0)|^2 &\le \left| \int_{t_0}^{t} x^{-\alpha}\, dx \int_{t_0}^{t} x^{\alpha} |u^{(m)}(x)|^2\, dx \right| \le \\ &\le c|t^{1-\alpha} - t_0^{1-\alpha}| |u, W_\alpha^m|^2. \end{aligned}$$

From the equality (15) it follows that

$$\left|u^{(m-1)}(t)\right|^2 \le 2\left( \left|u^{(m-1)}(t_0)\right|^2 + \left| \int_{t_0}^{t} u^{(m)}(\tau)\, d\tau \right|^2 \right). \tag{16}$$

The second term of the right-hand side of (16) is already estimated. Now we estimate $|u^{(m-1)}(t_0)|^2$. Multiplying both sides of (16) by $t^\alpha$ (after interchanging $t$ and $t_0$) and integrating over $(0, b)$ we get

$$\left|u^{(m-1)}(t_0)\right|^2 \int_0^b t^\alpha\, dt \le 2\left|u^{(m-1)}, L_{2,\alpha}\right|^2 + 2\int_0^b t^\alpha \left| \int_{t_0}^{t} u^{(m)}(\tau)\, d\tau \right|^2 dt,$$

which together with (14) proves that $|u^{(m-1)}(t_0)|^2 \le c|u, W_\alpha^m|^2$. From (16) now we conclude that $|u^{(m-1)}(t)|^2 \le (c_1 + c_2 t^{1-\alpha})|u, W_\alpha^m|^2$, i.e., the inequality (9) for $j = m-1$ is proved. To show (9) for $j = m-2$ we employ the inequality (16) (for $m-1$ instead of $m$), then using (14) we estimate $|u^{(m-2)}(t_0)|^2$ through $|u, W_\alpha^m|$. Then, using (9) for $j = m-1$ we can write

$$\left|u^{(m-2)}(t)\right|^2 \le 2\left( c|u, W_\alpha^m\|^2 + |t - t_0||u, W_\alpha^m|^2 \left| \int_{t_0}^{t} (c_1 + c_2 t^{1-\alpha})\, dt \right| \right),$$

i.e., we get (9) for $j = m-2$. In similar way we prove the inequality for $1 \le j \le m-3$. If $\alpha = 2n+1$ for some $n = 0, 1, \ldots, m-1$ then by succesively estimating of $|u^{(j)}(t)|, j = 0, 1, \ldots, m-1$, starting with $|u^{(m-n-1)}(t)|$ up to $|u(t)|$ instead of $t^{1-\alpha}$ we get $|\ln t|$ and then continue the proof in the same way. Note also that the numbers $B_j$ and $C_j$ for $j = 0, 1, \ldots, m-1$ do not depend on $u \in W_\alpha^m$.

*The proof is complete.*

Denote by $s_\alpha$ (the number of the "maintained conditions") $s_\alpha = m - [\frac{\alpha+1}{2}]$ for $0 \le \alpha < 2m-1$ and $s_\alpha = 0$ for $\alpha \ge 2m-1$, where $[a]$ is the integer part of the number $a$. From Proposition 2.2 it follows that in the case $\alpha < 1$ (weak degeneracy) $u^{(j)}(0)$ exist for all $j = 0, 1, \ldots, m-1$, while for $\alpha \ge 1$ (strong degeneracy) not all $u^{(j)}(0)$ exist. More precisely, for $0 \le \alpha < 2m-1$ the derivatives at zero $u^{(j)}(0)$ exist only for $j = 0, 1, \ldots, s_\alpha - 1$, while for $\alpha \ge 2m-1$ all $u^{(j)}(0),\ j = 0, 1, \ldots, m-1$ in general may be infinite.

**Proposition 2.5.** *The embedding*

$$W_\alpha^m \subset L_{2,\alpha} \tag{17}$$

*is compact for every $\alpha \geq 0$.*

*Proof.* To show the compactness of the embedding (17) we first observe that this embedding is compact for every interval $(1/n, b)$, where $n \in \mathbb{N}$, $n \geq n_0$, $1/n_0 < b$ (see [2]). Let now $\{u_k\}_{k=1}^\infty$ be some bounded sequence in $W_\alpha^m$ and $|u_k, W_\alpha^m| \leq M$. We have to prove that there is some convergent in $L_{2,\alpha}$ subsequence. For every $n \geq n_0$ from $\{u_k\}_{k=1}^\infty$ we can choose a subsequence $\{u_{nk}\}_{k=1}^\infty$ which is convergent in $L_{2,\alpha}(1/n, b)$. Let us prove that the diagonal–sequence $\{u_{nn}\}_{n=n_0}^\infty$ is convergent in $L_{2,\alpha}$. We only need to verify that $|u_{nn} - u_{mm}, L_{2,\alpha}(0,\varepsilon)| \to 0$ as $\varepsilon \to +0$ for every $m, n \geq n_0$. Indeed, using the inequality (9) for $j = 0$ we have (for $\alpha \neq 1, 3, \ldots, 2m-1$)

$$|u_{nn} - u_{mm}, L_{2,\alpha}(0,\varepsilon)|^2 \leq 2M^2 \int_0^\varepsilon t^\alpha \left(B_0 + C_0 t^{2m-1-\alpha}\right) dt \leq C\varepsilon.$$

For the case $\alpha = 2n+1, n = 0, 1, \ldots, m-1$ we use the inequality
$|u(t)|^2 \leq (B_0 + C_0 t^{2m-2n-2} |\ln t|) |u, W_\alpha^m|^2$ .
*The proof is complete.*

**Remark 2.6.** *The embedding*

$$W_\alpha^m \subset L_{2,\beta} \tag{18}$$

*is compact for every $\alpha > 2m-1$ and $\beta > \alpha - 2m$.*

Indeed, the embedding (18) follows from the inequality (9) for $j = 0$. To prove the compactness of the embedding (18) it is enough to show (see the proof of Proposition 2.5) that

$$|u_{nn} - u_{mm}, L_{2,\beta}(0,\varepsilon)|^2 \leq 2M^2 \int_0^\varepsilon t^\beta \left(B_0 + C_0 t^{2m-1-\alpha}\right) dt \leq C\varepsilon^{\beta+2m-\alpha},$$

because $\beta + 1 > \alpha - 2m + 1 > 2m - 1 - 2m + 1 = 0$, $\beta + 2m - 1 - \alpha + 1 > \alpha - 2m + 2m - \alpha = 0$ and $\beta > \beta + 2m - 1 - \alpha$.

Observe that in the case $\beta = \alpha - 2m$ and $\alpha \leq 2m - 1$ in contrast to the embedding (5) the embedding (18) fails (see [9]). In this case we only have

$W_\alpha^m \subset L_{2,\beta},\ \beta > -1$. However, for $\alpha > 2m-1$ we have the embedding $W_\alpha^m \subset L_{2,\alpha-2m}$ (not compact) which can be proved by using of Hardy's inequality (6) (see [5] and [9]).

Note also that with the help of the diagonal sequence we obtain the compactness of the embedding

$$H^{s,\delta}(\mathbb{R}^n) \subset H^{s_1,\delta_1}(\mathbb{R}^n), \quad s > s_1,\ \delta > \delta_1,\ s,\delta \in \mathbb{R},$$

where $H^{s,\delta}(\mathbb{R}^n) = (1+|x|^2)^{-\delta/2}H^s(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$ is the Sobolev space, because this embedding is compact on $B_r = \{x \in \mathbb{R}^n,\ |x| \le r\}$ for every $r > 0$. Another proof is given in [10].

Denote by $W_\alpha^m(0)$ the completion of $\{u \in C^m[0,b],\ u^{(k)}(t)\big|_{t=0} = 0,$ $k = 0,1,\dots,m-1\}$ in the norm (8). The definition of the space $W_\alpha^m(0)$ implies that the functions $u \in W_\alpha^m(0)$ near to $t=0$ have the same estimate as the functions $u \in \dot{W}_\alpha^m$, while in contrast to the space $\dot{W}_\alpha^m$ for the functions $u \in W_\alpha^m(0)$ the conditions $u^{(k)}(t)\big|_{t=b} = 0, k = 0,1,\dots,m-1$ in general fail. Note also that the codimension of the space $W_\alpha^m(0)$ in $W_\alpha^m$ is equal to $\mathrm{codim W_\alpha^m(0) = dim(W_\alpha^m/W_\alpha^m(0)) = s_\alpha}$, where $W_\alpha^m/W_\alpha^m(0)$ is the quotient space. Therefore, using the definition of the numbers $s_\alpha$ it follows that for $\alpha \ge 2m-1$ the spaces $W_\alpha^m(0)$ and $W_\alpha^m$ coincide.

Denote by $W_\alpha^m(b)$ the completion of $\{u \in C^m[0,b],\ u^{(k)}(t)\big|_{t=b} = 0,$ $k = 0,1,\dots,m-1\}$ in the norm (3). Now for the functions $u \in W_\alpha^m(b)$ near to $t=0$ we have the inequalities (9) and $u^{(k)}(t)\big|_{t=b} = 0, k = 0,1,\dots,m-1$. The codimension of the space $\dot{W}_\alpha^m$ in $W_\alpha^m(b)$ is equal to

$$\mathrm{codim}\dot{\mathrm{W}}_\alpha^{\mathrm{m}} = \mathrm{dim(W_\alpha^m(b)/\dot{W}_\alpha^m) = s_\alpha},$$

therefore, for $\alpha \ge 2m-1$ we have $\dot{W}_\alpha^m = W_\alpha^m(b)$.

From the definition of the space $W_\alpha^m(0)$ immediately follows that we have a continuous embedding $W_\alpha^m(0) \subset L_{2,\alpha-2m}, \alpha \ne 1,3,\dots,2m-1$, which is not compact. As a consequence of Proposition 2.5 we have the compact embeddings $W_\alpha^m(0), W_\alpha^m(b) \subset L_{2,\alpha}$ for every $\alpha \ge 0$.

## 2.2 Mixed Problem of First Type

Now we consider the mixed problem of first type for the following special case of the one-dimensional equation (1)

$$Bu \equiv (-1)^m(t^\alpha u^{(m)})^{(m)} + t^\alpha u = f, \quad f \in L_{2,-\alpha}\,. \tag{19}$$

**Definition 2.7.** *A function $u \in W_\alpha^m(0)$ is called a generalized solution of the mixed problem of first type for the equation* (19) *if for every $v \in W_\alpha^m(0)$ we have*

$$(t^\alpha u^{(m)}, v^{(m)}) + (t^\alpha u, v) = (f, v) \tag{20}$$

*where $(\cdot\,,\cdot)$ is the scalar product in $L_2(0,b)$* (see [11]).

**Proposition 2.8.** *The generalized solution of the mixed problem of first type for the equation* (19) *exists and is unique for every $f \in L_{2,-\alpha}$.*

The uniqueness of the generalized solution immediately follows from Definition 2.7. To prove the existence first we note that the linear functional $l_f(v) \equiv (f, v)$ is continuous in $W_\alpha^m(0)$ because

$$|l_f(v)| \le |f, L_{2,-\alpha}||v, L_{2,\alpha}| \le |f, L_{2,-\alpha}||v, W_\alpha^m(0)|.$$

Using Riesz's lemma on the representation of linear continuous functionals we can write it in the form

$$l_f(v) \equiv (f, v) = \{u, v\}_\alpha, \quad v \in W_\alpha^m(0),$$

where $\{u, v\}_\alpha$ is the scalar product in $W_\alpha^m(0)$. The element $u \in W_\alpha^m(0)$ realizing the linear functional $l_f(v)$ gives us the generalized solution.

If the generalized solution $u(t)$ is classical then from (20) we conclude that for $\alpha = 0$ the function $u(t)$ fulfills the following conditions (see [11])

$$u^{(k)}(t)\big|_{t=0} = u^{(2m-k-1)}(t)\big|_{t=b} = 0, \quad k = 0, 1, \ldots, m-1, \tag{21}$$

i.e., we have Dirichlet conditions at the left endpoint of the segment $[0, b]$ and Neumann conditions at the right endpoint. Note that the conditions (21) are of Sturm type and, therefore, regular (see [4]).

**Definition 2.9.** *We say that $u \in W_\alpha^m(0)$ belongs to $D(B)$, if the equality* (20) *is satisfied for some $f \in L_{2,-\alpha}$. In this case we will write $Bu = f$.*

According to Definition 2.9 we have an operator

$$B : D(B) \subset W_\alpha^m(0) \subset L_{2,\alpha} \to L_{2,-\alpha}\,.$$

To get an operator in the same space we set $g(t) = t^{-\alpha} f(t)$. It is evident that $g(t)$ belongs to $L_{2,\alpha}$ and $|f, L_{2,-\alpha}| = |g, L_{2,\alpha}|$. Therefore, we get an operator $\mathbb{B} \equiv t^{-\alpha} B : D(\mathbb{B}) = D(B) \subset W_\alpha^m \subset L_{2,\alpha} \to L_{2,\alpha}$ with $\mathbb{B}u = g$ in $L_{2,\alpha}$.

**Proposition 2.10.** *The operator* $\mathbb{B} : L_{2,\alpha} \to L_{2,\alpha}$ *is selfadjoint and positive. Moreover, the inverse operator* $\mathbb{B}^{-1} : L_{2,\alpha} \to L_{2,\alpha}$ *is compact.*

*Proof.* Symmetry and positivity of the operator $\mathbb{B}$ follow from Definition 2.9. Let $v \in \mathcal{D}(\mathbb{B}^*)$ and $\mathbb{B}^* v = 0$. Then for every $u \in \mathcal{D}(\mathbb{B})$ we have $(\mathbb{B}u, v) = (u, \mathbb{B}^* v)$. From Proposition 2.8 it follows that $v = 0$, i.e., the operator $\mathbb{B}^*$ is invertible. Now using (8) and the equality (20) with $v = u$ we get

$$|u, W^m_\alpha|^2 = (f, u) \leq |f, L_{2,-\alpha}||u, L_{2,\alpha}| \leq |g, L_{2,\alpha}||u, W^m_\alpha|.$$

Therefore, we have

$$|u, L_{2,\alpha}| \leq |\mathbb{B}u, L_{2,\alpha}|, \tag{22}$$

i.e., the operator $\mathbb{B}^{-1}$ is bounded. Thus the operator $(\mathbb{B}^*)^{-1}$ is defined on $L_{2,\alpha}$ (see [4]). Now the self–adjointness of the operator $\mathbb{B}$ is a consequence of its symmetry. The compactness of the operator $\mathbb{B}^{-1} : L_{2,\alpha} \to L_{2,\alpha}$ follows from the inequality (22) and from the compactness of the embedding $W^m_\alpha(0) \subset W^m_\alpha$ (see Proposition 2.5).
*The proof is complete.*

**Corollary 2.11.** *The operator* $\mathbb{B}$ *has a discrete spectrum, and the system of the corresponding eigenfunctions is dense in* $L_{2,\alpha}$.

This follows from the connection of the spectra of the operators $\mathbb{B}$ and $\mathbb{B}^{-1}$ and from the properties of compact selfadjoint operators (see [4]).

Note that if $\lambda$ is an eigenvalue and $u(t)$ a corresponding eigenfunction of the operator $\mathbb{B}$ then we have

$$(-1)^m (t^\alpha u^{(m)})^{(m)} + t^\alpha u = \lambda t^\alpha u. \tag{23}$$

From the inequality (23) and Definition 2.7 it follows that $\lambda \geq 1$. For $0 \leq \alpha < 1$ the number $\lambda = 1 \notin \sigma\mathbb{B}$ ($\sigma\mathbb{B}$ is the spectrum of the operator $\mathbb{B}$) and for $\alpha \geq 1$ it is an eigenvalue for the operator $\mathbb{B}$ with the multiplicity $m - s_\alpha$ (for the definition of the number $s_\alpha$ see Subsection 2.1), since $t^{s_\alpha} P_{m-s_\alpha-1}(t)$ for every polynomial $P_{m-s_\alpha-1}(t)$ of order $m - s_\alpha - 1$ is an eigenfunction (see Proposition 2.1). Therefore, for the solvability of the equation

$$(-1)^m (t^\alpha u^{(m)})^{(m)} = f, \quad f \in L_{2,-\alpha}\,, \tag{24}$$

we get the following result:

**Proposition 2.12.** *The generalized solution of the mixed problem of first type for the equation* (24) *for* $\alpha \geq 1$ *exists if and only if* $(f, P_{m-s_\alpha-1}(t)) = 0$ *for any polynomial* $P_{m-s_\alpha-1}(t)$ *of order* $m - s_\alpha - 1$.

Here we have used both $(g,\, P_{m-s_\alpha-1}(t))_\alpha = (f,\, P_{m-s_\alpha-1}(t))$ since $t^\alpha g(t) = f(t)$ ($(\cdot,\cdot)_\alpha$ is the scalar product in $L_{2,\alpha}$) and the definition of the operator $\mathbb{B}$. Note that the generalized solution of the mixed problem of first type for the equation (24) for $\alpha \geq 1$ is unique up to an arbitrary additive polynomial of order $m - s_\alpha - 1$.
Now we can consider the general case of the one-dimensional equation (1)

$$Pu \equiv (-1)^m (t^\alpha u^{(m)})^{(m)} + a t^\alpha u = f, \quad f \in L_{2,-\alpha}\,, \tag{25}$$

because the number $1 - a$ can be regarded as a spectral parameter for the operator $\mathbb{B}$. Therefore, if $1 - a \notin \sigma\mathbb{B}$ then the equation (25) is uniquely solvable for every $f \in L_{2,-\alpha}$.

## 2.3 Mixed Problem of Second Type

Now we consider the mixed problem of second type for the another special case of the one-dimensional equation (1)

$$Su \equiv (-1)^m (t^\alpha u^{(m)})^{(m)} = f, \quad f \in L_{2,-\alpha}\,. \tag{26}$$

**Definition 2.13.** *A function* $u \in W^m_\alpha(b)$ *is called a generalized solution of the mixed problem of second type for the equation* (26) *if for every* $v \in W^m_\alpha(b)$ *we have*

$$(t^\alpha u^{(m)}, v^{(m)}) + (t^\alpha u, v) = (f, v). \tag{27}$$

In the same way as in Subsection 2.2 we prove the existence and uniqueness of the generalized solution for every $f \in L_{2,-\alpha}$ and define a corresponding operator

$$S : D(S) \subset W^m_\alpha(b) \subset L_{2,\alpha} \to L_{2,-\alpha}\,.$$

For the classical solution $u(t)$ in the case $\alpha = 0$ from (27) we get the conditions (see [11])

$$u^{(2m-k-1)}(t)\big|_{t=0} = u^{(k)}(t)\big|_{t=b} = 0, \quad k = 0, 1, \ldots, m-1,$$

i.e., we have Neumann conditions at the left endpoint of the segment $[0, b]$ and Dirichlet conditions at the right endpoint.

Define $\mathbb{S} \equiv t^{-\alpha} S$, $\mathbb{S} : L_{2,\alpha} \to L_{2,\alpha}$. In similar way we prove that the operator $\mathbb{S} : L_{2,\alpha} \to L_{2,\alpha}$ is selfadjoint, positive and the inverse operator $\mathbb{S}^{-1}$ is compact. Therefore, the operator $\mathbb{S}$ has a discrete spectrum.

Now we can consider the general equation (25) because the number $-a$ can be regarded as a spectral parameter for the operator $\mathbb{S}$. Hence, if $-a \notin \sigma\mathbb{S}$ then the mixed problem of second type for the equation (25) is uniquely solvable for every $f \in L_{2,-\alpha}$.

# 3 The Operator Equation

In this section we consider the operator equation (1). Suppose that the operator $A : \mathcal{H} \to \mathcal{H}$ commutes with $D_t$ and has a complete system of eigenfunctions $\{\varphi_k\}_{k=1}^{\infty}$, forming a Riesz basis (see [4]) in $\mathcal{H}$. Hence $A\varphi_k = a_k \varphi_k$, $k \in \mathbb{N}$, for every $x \in \mathcal{H}$ we have $x = \sum_{k=1}^{\infty} x_k \varphi_k$, and there are some positive constants $c_1$ and $c_2$ such that

$$c_1 \sum_{k=1}^{\infty} |x_k|^2 \le ||x||^2 \le c_2 \sum_{k=1}^{\infty} |x_k|^2.$$

Hence for every $u \in L_{2,\alpha}((0, b), \mathcal{H})$, $f \in L_{2,-\alpha}((0, b), \mathcal{H})$ we have

$$u = \sum_{k=1}^{\infty} u_k(t)\varphi_k, \quad f = \sum_{k=1}^{\infty} f_k(t)\varphi_k, \quad k \in \mathbb{N}. \tag{28}$$

Therefore, the operator equation (1) can be decomposed into an infinite chain of ordinary differential equations

$$P_k u_k \equiv (-1)^m (t^\alpha u_k^{(m)})^{(m)} + a_k t^\alpha u_k = f_k, \quad f_k \in L_{2,-\alpha}, \quad k \in \mathbb{N}. \tag{29}$$

For the equations (29) we can define the generalized solutions $u_k(t), k \in \mathbb{N}$, of the mixed problem of first or second type (see Section 2).

**Definition 3.1.** *A function $u \in L_{2,\alpha}((0, b), \mathcal{H})$ is called a generalized solution of the mixed problem of first or second type for the equation* (1) *if the functions $u_k(t), k \in \mathbb{N}$, in the representation* (28) *are generalized solutions of the mixed problem of first or second type for the equations* (29).

**Proposition 3.2.** *The operator equation* (1) *is uniquely solvable for every* $f \in L_{2,-\alpha}((0,b),\mathcal{H})$ *if and only if the equations* (29) *are uniquely solvable for every* $f_k \in L_{2,-\alpha}$, $k \in \mathbb{N}$, *and the inequalities*

$$|u_k, L_{2,\alpha}| \leq c|f_k, L_{2,-\alpha}| \tag{30}$$

*are satisfied uniformly with respect to* $k \in \mathbb{N}$.

For the proof of Proposition 3.2 see [4].
Let the numbers $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\mu_k\}_{k=1}^{\infty}$ are the eigenvalues of the operators $\mathbb{B}$ and $\mathbb{S}$ (see Section 2). Suppose that

$$\rho(1 - a_k, \lambda_m) > \varepsilon, \quad k, m \in \mathbb{N}, \tag{31}$$

$$\rho(-a_k, \mu_m) > \varepsilon, \quad k, m \in \mathbb{N}, \tag{32}$$

where $\varepsilon > 0$ and $\rho$ is the distance in the complex plane.

**Theorem 3.3.** *Under the condition* (31) ((32)) *the generalized solution of the mixed problem of first type* (*second type*) *for the operator equation* (1) *exists and is unique for every* $f \in L_{2,-\alpha}((0,b),\mathcal{H})$.

First note that under the condition (31) ((32)) the equations (29) are uniquely solvable for every $f_k \in L_{2,-\alpha}$, $k \in \mathbb{N}$ and the inequalities (30) are satisfied. Now the proof of Theorem 3.3 follows from Proposition 3.2.

Let $g = t^{-\alpha} f$, $f \in L_{2,-\alpha}((0,b),\mathcal{H})$. Then $g \in L_{2,\alpha}((0,b),\mathcal{H})$ and we define an operator

$$\mathbb{P} \equiv t^{-\alpha} P : D(\mathbb{P}) = D(P) \subset L_{2,\alpha}((0,b),\mathcal{H}) \to L_{2,\alpha}((0,b),\mathcal{H}),$$

with $\mathbb{P}u = g$ in $L_{2,\alpha}((0,b),\mathcal{H})$. It follows from the condition (31) ((32)) that for the generalized solution $u \in W_{\alpha}^{m}(0)$ ($u \in W_{\alpha}^{m}(b)$) of the mixed problem of first type (second type) we have

$$|u, L_{2,\alpha}((0,b),\mathcal{H})| \leq c|g, L_{2,\alpha}((0,b),\mathcal{H})|.$$

The operator $\mathbb{P}^{-1} : L_{2,\alpha}((0,b),\mathcal{H}) \to L_{2,\alpha}((0,b),\mathcal{H})$ in general is not compact in contrast to Proposition 2.10 (it will be compact only in the case when the space $\mathcal{H}$ is finite-dimensional). If the operator $A : \mathcal{H} \to \mathcal{H}$ is selfadjoint we can describe the spectrum of the operator $\mathbb{P}$.

**Proposition 3.4.** *The spectrum of the operator* $\mathbb{P}$ *for the mixed problem of first type* (*of second type*) *is the subset of the direct sum of the spectra* $\sigma\mathbb{B}$ *and* $\sigma(A-I)$ ($\sigma\mathbb{S}$ *and* $\sigma A$)*, i.e.,*

$$\sigma\mathbb{P} \subset \sigma\mathbb{B} + \sigma(A-I) \equiv \{\lambda_1 + \lambda_2 - 1 : \lambda_1 \in \sigma\mathbb{B},\ \lambda_2 \in \sigma A\}, (\sigma\mathbb{P} \subset \sigma\mathbb{S} + \sigma A).$$

The proof of Proposition 3.4 immediately follows from the equality $\mathbb{P} = \mathbb{B} \otimes I_{\mathcal{H}} + I_{L_{2,\alpha}} \otimes (A-I)$ ($\mathbb{P} = \mathbb{S} \otimes I_{\mathcal{H}} + I_{L_{2,\alpha}} \otimes A$) (see [1]).

# References


[1] J. M. Berezanski, "Expansion in Eigenfunctions of Selfadjoint Operators", Transl. Math. Monographs 17, American Mathematical Soc., Providence, 1968.

[2] V. I. Burenkov, "Sobolev Spaces on Domains", Teubner, 1999.

[3] A. A. Dezin, "Degenerate operator equations", Math. USSR Sbornik, vol. 43, no. 3, p.p. 287–298, 1982.

[4] A. A. Dezin, "Partial Differential Equations" (An Introduction to a General Theory of Linear Boundary Value Problems), Springer, 1987.

[5] P. A. Djarov, "Compactness of embeddings in some spaces with power weight", Izvestya VUZ–ov, matematika, vol. 8, p.p. 82–85, 1988 (Russian).

[6] R. Duduchava, D. Elliott and W. L. Wendland, "The spline collocation method for Mellin convolution equations", Univ. Stuttgart, Sonderforsch. 404, Mehrfeldprobleme in der Kontinuumsmechanik, Bericht 96/04, 1996.

[7] G. H. Hardy, J. E. Littlewood, G. Polya, "Inequalities", Cambridge Univ. Press, Cambridge, 1964.

[8] L. D. Kudryavtzev, "On a variational method of determination of generalized solution of differential equations in the function spaces with power weight", Differ. Urav., vol. 19, no. 10, p.p. 1723–1740, 1983 (Russian).

[9] L. D. Kudryavtzev, "On equivalent norms in the weight spaces", Trudy Mat. Inst. AN SSSR, vol. 170, p.p. 161–190, 1984 (Russian).

[10] B.–W. Schulze, "Boundary Value Problems and Singular Pseudo–Differential Operators", J. Wiley, Chichester, 1998.

[11] R. E. Showalter, "Hilbert Space Methods for Partial Differential Equations", Electronic Journal of Differential Equations, Monograph 01, 1994.

[12] L. Tepoyan, "Degenerate fourth–order diffeential–operator equations", Differ. Urav., vol. 23, no. 8, p.p. 1366–1376, 1987; English Transl. in Amer. Math. Soc., no. 8, 1988.

[13] L. Tepoyan, "On a degenerate differential–operator equation of higher order", Izvestiya Natsionalnoi Akademii Nauk Armenii. Matematika, vol. 34, no. 5, p.p. 48–56, 1999.

[14] L. Tepoyan, “On the spectrum of a degenerate operator", Izvestiya Natsionalnoi Akademii Nauk Armenii. Matematika, vol. 38, no. 5, p.p. 53–57, 2003.

[15] M. I. Višik, “Boundary–value problems for elliptic equations degenerate on the boundary of a region", Mat. Sb., 35(77), 1954, p.p. 513–568; English transl. in Amer. Math. Soc. Transl., vol. 35, no. 2, 1964.

*Received July, 18, 2008; accepted November, 26, 2008.*