THE FIRST BVP OF THERMOELASTICITY FOR A TRANSVERSALLY ISOTROPIC PLANE WITH CURVILINEAR CUTS

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Abstract: In the present paper the first boundary value problem of the theory of thermoelasticity is investigated for a transversally isotropic plane with curvilinear cuts .For solution we used the potential method and constructed the special fundamental matrices, which reduced the problem to a Fredholm integral equations of the second kind.The solvability of a system of singular integral equations is proved by using the potential method and the theory of singular integral equations. For the equation of statics of thermoelasticity we construct one particular solution and we reduce the solution of the first BVP problem of the theory of thermoelasticity to the solution of the first BVP problem for the equation of transversally-isotropic body.

Key words: Potential method, integral equation, thermoelasticity

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In this present paper the first boundary value problem (BVP) of thermoelasticity theory is investigated for a transversally-isotropic plane with curvilinear cuts. The boundary value problems of elasticity for anisotropic media with cuts were considered in [1,2]. In this paper we intend this result to BVP of thermoelasticity for a transversally-isotropic thermoelastic body. Here we shall be concerned with the plane problem of thermoelasticity (it is assumed that the second component of the three-dimensional displacement vector equals to zero and the other components u_1, u_3 and u_4 depend only on the variable x_1, x_3. In this case the basic two-dimensional equations thermoelasticity for the transversally-isotropic body can be written as follows [3]

$$C(\partial x)u = Bgradu_4, \tag{1}$$

$$\Delta_4 u_4 = a_4 \frac{\partial^2 u_4}{\partial x_1^2} + \frac{\partial^2 u_4}{\partial x_3^2} = 0, \quad j = 0, 1, \tag{2}$$

where

$$C(\partial x) = \|C_{pq}(\partial x)\|_{2x2}, \quad B = \|B_{pq}\|_{2x2}, \quad B_{11} = \beta_1, \quad B_{22} = \beta',$$
$$B_{12} = B_{21} = 0, \quad C_{11}(\partial x) = c_{11}\frac{\partial^2}{\partial x_1^2} + c_{44}\frac{\partial^2}{\partial x_3^2}, \quad C_{21}(\partial x) =$$
$$C_{12}(\partial x) = (c_{13} + c_{44})\frac{\partial^2}{\partial x_1 \partial x_3}, \quad C_{22}(\partial x) = c_{11}\frac{\partial^2}{\partial x_1^2} + c_{44}\frac{\partial^2}{\partial x_3^2},$$

c_{pq} are Hooke's coefficients, $\beta = c_{13}\alpha' + 2\alpha(c_{11} - c_{66})$, $\beta' = c_{33}\alpha' + 2\alpha c_{13}$, $a_4 = \frac{k}{k'}$, α, α' are coefficients of temperature extension, k, k' are coefficients of thermal conductivity, $u = (u_1, u_3)$ is a displacement vector, u_4 is the temperature of body.

Let the plane be weakened by curvilinear cuts $l_j = a_j b_j, j = 1, 2, .., p$. Assume that the cuts $l_j, j = 1, ..., p$, are simple nonintersecting open Lyapunov's arcs. The direction from a_j to b_j is taken as the positive one on l_j. The normal to l_j will be drawn to the right relative to motion in the positive direction. Denote by D the plane with curvilinear cuts $l_j, j = 1, 2, .., p.L = \bigcup_{j=1}^{p} l_j$. Let the domain D is filled by homogeneous transversally-isotropic material with the coefficients c_{pq}

We introduce the notations: $z = x_1 + ix_3$, $\zeta_k = y_1 + \alpha_k y_3$, $\tau_k = t_1 + \alpha_k t_3$, $\sigma_k = z_k - \varsigma_k$, $z_k = z_1 + \alpha_k z_3$, $\tau = t_1 + it_3$.

For equations (1)-(2) we pose the following first boundary value problem of static of the theory of thermoelasticity. Find in the domain D a regular solution $u(x)$, and $u_4(x)$, of equation (1)-(2), when the boundary values of the displacement vector u and the boundary value of temperature $u_4(x)$, are given on both edges of the arc l_j. Further, assume that at infinity the principal vector of external forces acting on l, stress vector, u, u_4 and the rotation are equal to zero. It is required to define the deformed state of the plane.

If we denote by $u^+(u^-)$, $u_4^+(u_4^-)$ the limits of u and u_4 on l from the left (right), then the boundary conditions of the problem will take the form

$$u^+ = f^+, \ \ u^- = f^-, \ \ u_4^+ = f_4^+, \ \ u_4^- = f_4^-, \tag{3}$$

where f^+ , f^-, f_4^+, and $f_4^-$are the known functions on l of the Holder class H, which have derivatives in the class H^* (for the definitions of the classes H and H^* see[4]) and satisfying at the ends a_j and b_j of l_j, the conditions

$$f^+(a_j) = f^-(a_j), \ \ f^+(b_j) = f^-(b_j), \ \ f_4^+(a_j) = f_4^-(a_j), \ \ f_4^+(b_j) = f_4^-(b_j).$$

It is obvious that displacement vector discontinuities along the cut l_j generate a singular stress field in the medium. Hence it is of interest for us to study the solution behavior in the neighborhood of the cuts.

Further we assume that u_4 is known, when $x \in D$. Substitute the u_4 in (1) and search the particular solution of the following equation

$$C(\partial x)u = gradu_4.$$

It is easy to prove that $u_0(x)$ is a particular solution of the equation (1)

$$u_0 = -\frac{1}{2\pi} \int\int_D \Gamma(x - y) gradu_4 dv, \tag{4}$$

where $\Gamma(x-y)$ is the basic fundamental matrix for equation $C\partial x)u=0$,

$$\Gamma(x-y)=2Im\sum_{2}^{3}\|A_{pq}^{(k)}\|_{2x2}\ln\sigma_k,$$
$$A_{11}^{(k)}=\frac{i(-1)^k(c_{44}-c_{33}a_k)}{c_{44}c_{33}(a_2-a_3)},\quad A_{12}^{(k)}=\frac{(-1)^k(c_{44}+c_{13})}{c_{44}c_{33}(a_2-a_3)}$$
$$A_{22}^{(k)}=\frac{i(-1)^k(c_{11}-c_{44}a_k)}{c_{44}c_{33}(a_2-a_3)},\quad \sigma_k=x_1-y_1+\alpha_k(x_3-y_3),\quad \alpha_k=i\sqrt{a_k},$$

$a_k,\ \ k=2,3$ are the positive roots of a characteristic equation

$$c_{44}c_{33}a_k^2-[c_{11}c_{33}+c_{44}^2-(c_{13}+c_{44})^2]a_k+c_{44}c_{11}=0.$$

In (4) $gradu_4$ is a continuous vector in D along with its first derivatives and satisfy the following condition at infinity

$$gradu_4=O(|x|^{-1-\alpha}),\alpha>0.$$

Thus the general solution of the equation (1) is $u=V+u_0$, where

$$\begin{aligned}&C(\partial x)V=0,\quad V^+(x)=f^+(z)-u_0^+(z)=F^+(z),\\&V^-(z)=f^-(z)-u_0^-(z)=F^-(z).\end{aligned}\tag{5}$$

This equation is the equation of a transversally-isotropic elastic body. i.e.. we reduce the solution of basic BVP of the theory of thermoelasticity to the solution of the basic BVP for the equation of a transversally-isotropic elastic body.

A solution of the problem (5) we seek in the form

$$V=\frac{1}{\pi}Im\int\limits_L\sum_{k=2}^{3}N^{(k)}\frac{\partial\ln\sigma_k}{\partial s}[g(s)+ih(s)]ds+\sum_{2}^{p}V_J(z)+M^{(p+1)},\tag{6}$$

where $g(t),\ \ h(t)$ are unknown densities,

$$N^{(k)}(y-x)=2Im\sum_{2}^{3}\|N_{pq}^{(k)}(\partial x)\|_{2x2}$$
$$N_{11}^{(k)}=(-1)^kb(c_{33}a_k-c_{44})\sqrt{\frac{a_2a_3}{a_k}},\quad j=0,1,$$
$$N_{21}^{(k)}=(-1)^kib(c_{13}+c_{44})\sqrt{a_2a_3}=\sqrt{a_2a_3}$$
$$N_{12}^{(k)},N_{22}^{(k)}=(-1)^kb(c_{44}a_k-c_{11})\frac{1}{\sqrt{a_k}},$$
$$b=\left(\sqrt{a_2}-\sqrt{a_3}\right)^{-1}\left(c_{33}\sqrt{a_2a_3}+c_{44}\right)^{-1},$$
$$\sigma_k=z_k-\zeta_k,\quad z_k=x_1+\alpha_kx_3,\quad \zeta_k=y_1+\alpha_ky_3,$$

$$V_j(z)=\frac{1}{\pi}Im\sum_{k=2}^{3}\|A_{pq}^{(k)}\|_{2x2}\frac{(z_k-b_j^{(k)})ln(z_k-b_j^{(k)})-(z_k-a_j^{(k)})ln(z_k-a_j^{(k)})}{b_j^{(k)}-a_j^{(k)}}M^j,$$

$$a_j^{(k)} = Rea_j + \alpha_k Ima_j, \quad b_j^{(k)} = Reb_j + \alpha_k Imb_j.$$

$M^{(j)}, j = 1, .., p+1$, are the unknown real constant vectors to be defined later on.

The vector $V_j(z)$ satisfies the following conditions:

1. $V_j(z)$ has the logarithmic singularity at infinity

$$V_j = \frac{1}{\pi} Im \sum_{k=2}^{3} \|A_{pq}^{(k)}\|_{2x2}(-lnz_k + 1)M^j + O(z_k^{-1}).$$

2. By V_j is meant a branch, which is uniquely defined on the plane cut along l_j.

3. V_j is continuously extendable on l_j from the left and right, the end points a_j and b_j inclusive, i.e., we have the equalities

$$V_j^+(a_j) = V_j^-(a_j), \quad V_j^+(b_j) = V_j^-(b_j),$$

$$V_j^+ - V_j^- = 2Re \sum_{k=2}^{3} \|A_{pq}^{(k)}\|_{2x2} \frac{\tau_k - a_j^{(k)}}{b_j^{(k)} - a_j^{(k)}} M^j, j = 1, ..p.$$

Taking into account the boundary condition , for the determination of the unknown densities, we obtain a system of singular integral equation of normal type

$$\pm g(\tau) + \frac{1}{\pi} Im \sum_{k=2}^{3} N^{(k)} \int_L \frac{\partial ln(\tau_k - \varsigma_k)}{\partial s}(g + ih)ds$$

$$+ \sum_{j=1}^{p} V_j^{\pm} + M_j^{p+1} = F^{\pm}(\tau).$$

This formula implies

$$\begin{gathered} 2g(\tau) = F^+ - F^- - Re \sum_{j=1}^{p} \sum_{k=2}^{3} \|A_{pq}^{(k)}\|_{2x2} \frac{\tau_k - a_j^{(k)}}{b_j^{(k)} - a_j^{(k)}} M^j, \\ \frac{1}{\pi} \int_L \frac{h(\varsigma)ds}{\varsigma - \tau} + \frac{1}{\pi} \int_L K(\tau, \varsigma)h(\varsigma)ds = \Omega(\tau), \end{gathered} \tag{7}$$

where

$$K(\tau,\varsigma) = -i\frac{\partial\theta}{\partial s}E + Re\sum_{k=2}^{3} N^{(k)}\frac{\partial}{\partial s}ln\left(1+\lambda_k\frac{\overline{\tau}-\overline{\varsigma}}{\tau-\varsigma}\right), \lambda_k = \frac{1+i\alpha_k}{1-i\alpha_k},$$

$$\theta = arg(\tau-\varsigma),\ \ \Omega(\tau) = \frac{1}{2}(F^{+}+F^{-}) - \frac{1}{2}\sum_{j=1}^{p}(V_j^{+}+V_j^{-})$$

$$-M^{p+1} - \frac{1}{\pi}Im\sum_{k=2}^{3} N^{(k)}\int_L \frac{\partial ln(\tau_k-\varsigma_k)}{\partial s}gds.$$

Thus we have defined the vector g on l. It is not difficult to verify that $g \in H, g' \in H^*, g(a_j) = g(b_j) = 0, \Omega \in H, \Omega' \in H^*$. Formula (7) is a system of singular integral equations of the normal type with respect to the vector h. The points a_j and b_j are nonsingular, while the total index of the class h_{2p} is equal to $-2p$ (definition of the classes H, H^* and h_{2p} can be found in [4]).

A solution of system (7), if it exists, will be expressed by a vector of the Holder class that vanishes at the points a_j, b_j, and has derivatives in the class H^*.

Next we shall prove that the homogeneous system of equations corresponding to (7) admits on a trivial solution in the class h_{2p}. Let the contrary be true. Assume $h^{(0)}$ to be nontrivial solution of the homogeneous system corresponding to (7) in the class h_{2p} and construct the potential

$$U_0(z) = \frac{1}{\pi}Re\sum_{k=2}^{3} N^{(k)}\int_L \frac{\partial ln(z_k-\varsigma_k)}{\partial s}h^{(0)}(s)ds.$$

Clearly, $U_0^{+}(z) = U_0^{-}(z) = 0$ and by the uniqueness theorem we have $U_0(z) = 0, z \in D$.Then $TU_0(z) = 0, z \in D$ and

$$(TU_0(z))^{+} - (TU_0(z))^{-} = A\frac{\partial h^0}{\partial s} = 0,$$

where A is a constant matrix.

Therefore, since $h^{(0)}(a_j) = 0$, we obtain $h^{(0)}(z) = 0$, which completes the proof. Thus the homogeneous system adjoint to (7) will have $2p$ linearly independent solutions σ_j, $j = 1,..,2p$, in the adjoint class and the condition for system (7) to be solvable will be written as

$$\int_L \Omega\sigma_j ds = 0, j = 1,..,2p. \tag{8}$$

Taking into account the latter conditions and that

$$\int_L [(TU)^{+} - (TU)^{-}]ds = 0 = -2\sum_{k=1}^{p} M^{(k)}, \tag{9}$$

we obtain a system of $2p+2$ algebraic equations with the same number of unknowns with respect to the components of the unknown vector M^j.

We shall show that system (8)-(9) is solvable. Assume that homogeneous system obtained from (8)-(9) has a nontrivial solution $M_j^0 = \left(M_{1j}^0, M_{2j}^0\right)$, $j = 1, .., p+1$, and construct the potential

$$U_0(z) = \frac{1}{\pi} Im \sum_{k=2}^{3} N^{(k)} \int_L \frac{\partial ln(z_k - \varsigma_k)}{\partial s} (g^0 + ih^0)(s)ds + \sum_{j=1}^{p} V_j^0(z) + M_{p+1}^0,$$

where

$$g_0(\tau) = -Re \sum_{j=1}^{p} \sum_{k=2}^{3} \|A_{pq}^{(k)}\|_{2x2} \frac{\tau_k - a_j^{(k)}}{b_k - a_j^{(k)}} M_j^0,$$

$$V_j^0(z) = \frac{1}{\pi} Im \sum_{k=2}^{3} \|A_{pq}^{(k)}\|_{2x2} \frac{(z_k - b_j^{(k)}) ln(z_k - b_j^{(k)}) - (z_k - a_j^{(k)}) ln(z_k - a_j^{(k)})}{b_j^{(k)} - a_j^{(k)}} M_j^0,$$

It is obvious that $U_0^+ = U_0^- = 0$. Applying the formulas $\sum\limits_{k=1}^{p} M_{(k)}^0 = 0$ and $\int\limits_{lj} [TU_0(z) - TU_0(z)]ds = -2M_j^0 = 0, j = 1, .., p, U_0(\infty) = M_j^{p+1} = 0$, we obtain $M_j^0 = 0$, which contradicts the assumption. Therefore system (8)-(9) has a unique solution.

For M_j^0 system (7) is solvable in the class h_{2p}.The solution of the problem posed is given by potential (3) constructed using the solution h of system (7) and the vector g.

Repeating word by word the above reasoning we can show that

$$u_4(z) = \frac{1}{\pi} Im \int_L \frac{\partial ln(z_4 - \varsigma_4)}{\partial s} (g_4 + ih_4)(s)ds + \sum_{j=1}^{p} V_{j4}(z) + K_{p+1},$$

where $z_4 = x_1 + \alpha_4 x_3, \ \ \varsigma_4 = y_1 + \alpha_4 y_3, \ \ \alpha_4 = i\sqrt{a_4},$

$$V_{j4}(z) = \frac{1}{\pi} Im \frac{(z_4 - b_j^{(4)}) ln(z_4 - b_j^{(4)}) - (z_4 - a_j^{(4)}) ln(z_4 - a_j^{(4)})}{b_j^{(4)} - a_j^{(4)}} K^j,$$

$$a_j^{(4)} = Rea_j + \alpha_4 Ima_j, \ \ b_j^{(4)} = Reb_j + \alpha_4 Imb_j,$$

$$2g_4(\tau) = f_4^+ - f_4^- - Im \sum_{j=1}^{p} \frac{\tau_4 - a_j^{(4)}}{b_j^{(4)} - a_j^{(4)}} K^j,$$

h_4 is a solution of the following integral equation

$$\frac{1}{\pi} \int_L \frac{h_4(\varsigma)ds}{\varsigma - \tau} + \frac{1}{\pi} \int_L K(\tau, \varsigma) h_4(\varsigma) ds = \Omega(\tau), \tag{10}$$

where

$$K(\tau,\varsigma) = -i\frac{\partial\theta}{\partial s}E + Re\frac{\partial}{\partial s}ln\left(1+\lambda_4\frac{\overline{\tau}-\overline{\varsigma}}{\tau-\varsigma}\right), \lambda_4 = \frac{1+i\alpha_4}{1-i\alpha_4},$$

$$\theta = arg(\tau-\varsigma), \ \ \Omega(\tau) = \frac{1}{2}(f_4^+ + f_4^-) - \frac{1}{2}\sum_{j=1}^{p}(V_{j4}^+ + V_{j4}^-)$$

$$-K_{p+1} - \frac{1}{\pi}Im\int\limits_L \frac{\partial ln(\tau_4-\varsigma_4)}{\partial s}g_4 ds.$$

$K^{(j)}, j = 1,..,p+1$, are the unknown real constant vectors to be defined from the equation analogously (8)-(9).

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