Bulletin of TICMI
Vol. 17, No. 1, 2013, 1–18

Boundary Stabilization for 1-d Semi-Discrete Wave Equation by Filtering Technique

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(*Received October 3, 2012; Revised February 18, 2013; Accepted March 4, 2013*)

In this paper, we consider a finite difference semi-discretization for the 1-d wave equation with a boundary feedback. First, we prove that the exponential decay of the semi-discrete energy is not uniform (with respect to the mesh size) by showing that the constant of the observability inequality blows up. This is due to the fact that spurious high frequency oscillations are present in the semi-discrete system. We prove after that a uniform exponential decay holds if the high frequencies are filtered using multiplier technique and non harmonic Fourier series. Then we compare between these two methods.

Keywords: Boundary stabilization, Finite difference method, Semi-discretization, Filtering technique, Multiplier technique, Non harmonic Fourier series.

AMS Subject Classification: 93D15, 93B07, 65N06, 65N22.

1. Introduction

Numerical controllability and stabilizability have attracted a lot of interest in recent years. Finite-difference, finite element, mixed finite element and polynomial based Galerkin approximation methods have been applied [1, 7, 9, 10, 15, 17, 18, 23–25]. J. Infante and E. Zuazua [9] showed that, when the finite difference method or the classical element method are used in the semi-discretization, the boundary observability is not uniform with respect to the mesh size. This is due to the spurious high frequency oscillations present in the semi-discrete model. Some remedies have been proposed to damp out these high-frequencies, like filtering technique [9, 23], Tychonoff regularization [9], mixed finite element method [7].

Tebou and Zuazua [17] considered a finite-difference space semi-discretization of a locally damped 1-D and 2-D wave equations in the interval and the unit square domain, respectively, and proved that adding a suitable vanishing numerical viscosity term leads to a uniform exponential decay of the energy of solutions. In [18], the authors considered a finite-difference space semi-discretization of a 1-D boundary damped wave equation and proved that the exponential decay is not uniform with respect to the net-spacing size, then they proved that a suitable vanishing numerical viscosity term leads to a uniform exponential decay.

Our purpose in this paper is to treat a finite-difference space semi-discretization of 1-D boundary damped wave equation considered in [18], using filtering technique

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ISSN: 1512-0082 print

which consists of cutting a high frequencies present in the semi-discrete model. This technique is used in [9] on the context of boundary observability for 1-D wave equation with Dirichlet boundary conditions.

Let $\Omega = (0,1)$ of $\mathbb{R}$ and consider the $1-d$ damped wave equation:

$$\begin{cases} y_{tt} - y_{xx} = 0, \ \ 0 < x < 1, \ \ t > 0 \\ y(0,t) = 0, \ y_x(1,t) + \alpha y_t(1,t) = 0, \ \ t > 0 \\ y(x,0) = y_0(x), \ \ y_t(x,0) = y_1(x), \ \ 0 < x < 1 \end{cases} \tag{1}$$

where $(y_0, y_1) \in H_0^1(0,1) \times L^2(0,1)$, and α is a positive constant.

This system arises in many important models for distributed parameter control problems. In particular, in the model of a vibrating string, where the solution $y(t,x)$ represents the transverse displacement of the string, and in models for acoustic pressure fields, the solution y(t, x) represents the fluid pressure (see, [2–4, 16] for more examples). Note that this type of problems is first studied by Banks et all. [1], where they developed a general approach based on the mixed finite element method and polynomial based Galerkin approximation that preserve uniform exponential decay rate.

The energy of system (1) is given by

$$E(t) = \frac{1}{2}\int_0^1 (|y_t(x,t)|^2 + |y_x(x,t)|^2)dx, \quad \forall t \geq 0,$$

and it obeys the following dissipation law

$$\frac{dE(t)}{dt} = -\alpha |y_t(1,t)|^2.$$

It is also known that this energy satisfies, for some $M > 0$ and $\omega > 0$ independent of the solution, the estimate (see [5, 8, 11–14, 16, 20–22])

$$E(t) \leq Me^{-\omega t}E(0), \quad \forall t \geq 0. \tag{2}$$

In this paper, we study a uniform boundary stabilizability of the finite difference semi-discretization of (1). For this purpose, we set the space step h by $h = \frac{1}{N+1}$, where $N \in \mathbb{N}$ is a given integer. Denote by y_j the approximation of the solution y of (1) at the point space $x_j = jh$ for any $j = 0, ..., N+1$. Then we introduce the following finite-difference space semi-discretization of (1)

$$\begin{cases} y_j^{''} = \frac{y_{j+1} - 2y_j + y_{j-1}}{h^2}, \ \ 0 < t < T, \ \ j = 1, ..., N \\ y_0 = 0, \ \frac{y_{N+1} - y_N}{h} + \alpha y_{N+1}^{'} = 0, \ \ 0 < t < T \\ y_j(0) = y_j^0, \ \ y_j^{'}(0) = y_j^1, \ \ j = 1, ..., N. \end{cases} \tag{3}$$

The energy of system (3) is given by

$$E_h(y,t) = \frac{h}{2}\sum_{j=0}^{N}\left[|y_j^{'}(t)|^2 + \left|\frac{y_{j+1}(t) - y_j(t)}{h}\right|^2\right],$$

which is an approximation of the continuous energy. The derivative of E_h is given by

$$E_h'(y,t) = -\alpha |y_{N+1}'|^2, \tag{4}$$

which shows that E_h is a nonincreasing function. For system (3), we prove that a decay rate of type (2) is not uniform with respect to the net-spacing h. We will show that this is equivalent to a non uniform observability for the corresponding conservative system

$$\begin{cases} u_j'' = \frac{u_{j+1}-2u_j+u_{j-1}}{h^2}, \ \ 0 < t < T, \ \ j = 1, ..., N \\ u_0 = 0, \ u_{N+1} = u_N, \ \ 0 < t < T \\ u_j(0) = u_j^0, \ \ u_j'(0) = u_j^1, \ \ j = 1, ..., N. \end{cases} \tag{5}$$

Roughly speaking, we show that the constant C in the following observability inequality, satisfied by the solutions of (5)

$$E_h(u,0) \leq C \int_0^T |u_{N+1}'|^2 dt \tag{6}$$

blows up for small h where

$$E_h(u,t) = \frac{h}{2} \sum_{j=0}^{N} \left[|u_j'(t)|^2 + \left| \frac{u_{j+1}(t) - u_j(t)}{h} \right|^2 \right]. \tag{7}$$

We prove after that a uniform exponential decay holds if the high frequencies are filtered using multiplier technique and non harmonic Fourier series.

2. Non uniform exponential decay

2.1. *The spectral analysis of the semi-discrete problem*

In this section, we give the eigenvalues and their eigenvectors of the semi-discrete problem (5). We also study some of their relationship.

Consider the eigenvalue problem associated with (5)

$$\begin{cases} -\frac{\varphi_{j+1}-2\varphi_j+\varphi_{j-1}}{h^2} = \lambda \varphi_j \ , \ \ j = 1, ..., N \\ \varphi_0 = 0, \ \varphi_{N+1} = \varphi_N. \end{cases} \tag{8}$$

The eigenvalues and eigenvectors of (8) can be given explicitly, see [18], by

$$\begin{cases} \lambda_k = \frac{4}{h^2} \sin^2 \left(\frac{(2k+1)\pi h}{2(2-h)} \right) \ , \ k = 0, ..., N-1 \\ \varphi_{k,j} = \sin(\frac{(2k+1)\pi j h}{2-h}) \ , \ j = 0, ..., N. \end{cases} \tag{9}$$

Therefore, the solution of system (5) may be expressed as

$$\vec{u}(t) = \sum_{k=0}^{N-1} \left[\alpha_k e^{i\sqrt{\lambda_k}\, t} + \beta_k e^{-i\sqrt{\lambda_k}\, t}\right] \vec{\varphi}_k,$$

with $\vec{u}(t) = (u_1(t), ..., u_N(t))$. The last formula can also be written as

$$\vec{u}(t) = \sum_k a_k e^{i\mu_k\, t} \vec{\varphi}_k,$$

with $\mu_k = \sqrt{\lambda_k}$ for $k \geq 0$ and $\mu_k = -\sqrt{\lambda_{-k}}$ for $k < 0$ and $\vec{\varphi}_{-k} = \vec{\varphi}_k$. This last form will be used in this paper.

We have the following properties of the eigenvectors of (8).

Lemma 2.1: *For any eigenvector $\vec{\varphi}$ with eigenvalue λ of system* (8) *the following identities hold*

$$\sum_{j=0}^{N} \left|\frac{\varphi_{j+1} - \varphi_j}{h}\right|^2 = \lambda \sum_{j=1}^{N} \varphi_j^2. \tag{10}$$

$$h \sum_{j=0}^{N} \left|\frac{\varphi_{j+1} - \varphi_j}{h}\right|^2 = \frac{\lambda h^2 (2-h)}{4 - \lambda h^2} \left|\frac{\varphi_N}{h}\right|^2. \tag{11}$$

Proof: Multiplying (8) by φ_j, we get

$$-\frac{1}{h^2} \sum_{j=1}^{N} (\varphi_{j+1} - 2\varphi_j + \varphi_{j-1})\, \varphi_j = \lambda \sum_{j=1}^{N} \varphi_j^2,$$

which implies that

$$-\frac{1}{h^2} \sum_{j=1}^{N} \left(\varphi_{j+1}\varphi_j - 2\varphi_j^2 + \varphi_j\varphi_{j-1}\right) = \lambda \sum_{j=1}^{N} \varphi_j^2.$$

Therefore

$$\frac{1}{h^2} \sum_{j=1}^{N} \left(2\varphi_j^2 - 2\varphi_{j+1}\varphi_j\right) + \frac{1}{h^2}\varphi_N^2 = \lambda \sum_{j=1}^{N} \varphi_j^2, \tag{12}$$

which yields

$$\frac{1}{h^2} \sum_{j=0}^{N} \left(\varphi_{j+1}^2 - 2\varphi_{j+1}\varphi_j + \varphi_j^2\right) = \lambda \sum_{j=1}^{N} \varphi_j^2.$$

This achieves the proof of (10). To show the identity (11), we multiply (8) by $j\left(\varphi_{j+1}-\varphi_{j-1}\right)$ and obtain

$$\frac{1}{h^2}\sum_{j=1}^{N} j\left(\varphi_{j+1}-2\varphi_j+\varphi_{j-1}\right)\left(\varphi_{j+1}-\varphi_{j-1}\right)=-\lambda\sum_{j=1}^{N} j\varphi_j\left(\varphi_{j+1}-\varphi_{j-1}\right).$$

Hence,

$$\frac{1}{h^2}\sum_{j=1}^{N}\left[(j-1)\varphi_j^2-2j\varphi_{j+1}\varphi_j+2(j+1)\varphi_{j+1}\varphi_j-(j+1)\varphi_j^2\right]-\frac{1}{h^2}\varphi_N^2$$

$$=-\lambda\sum_{j=1}^{N}\left[j\varphi_{j+1}\varphi_j-(j+1)\varphi_{j+1}\varphi_j\right]-\lambda(N+1)\varphi_N^2,$$

and then

$$\frac{1}{h^2}\sum_{j=1}^{N}\left[-2\varphi_j^2+2\varphi_{j+1}\varphi_j\right]-\frac{1}{h^2}\varphi_N^2=\lambda\sum_{j=1}^{N}\varphi_{j+1}\varphi_j-\frac{\lambda}{h}\varphi_N^2.$$

This implies again that

$$\left(\frac{\lambda}{h}-\frac{1}{h^2}\right)\varphi_N^2=\frac{2}{h^2}\sum_{j=1}^{N}\varphi_j^2+\left(\lambda-\frac{2}{h^2}\right)\sum_{j=1}^{N}\varphi_{j+1}\varphi_j. \tag{13}$$

Now, using (12), we derive that

$$\frac{2}{h^2}\sum_{j=1}^{N}\varphi_{j+1}\varphi_j=\left(\frac{2}{h^2}-\lambda\right)\sum_{j=1}^{N}\varphi_j^2+\frac{1}{h^2}\varphi_N^2. \tag{14}$$

Normalizing the eigenvector $\vec{\varphi}$, i.e. $h\sum_{j=1}^{N}\varphi_j^2=1$, from (13), (14) we obtain

$$\sum_{j=1}^{N}\varphi_{j+1}\varphi_j=\frac{h}{2}\left(\frac{2}{h^2}-\lambda\right)+\frac{1}{2}\varphi_N^2,$$

$$\left(\lambda-\frac{2}{h^2}\right)\sum_{j=1}^{N}\varphi_{j+1}\varphi_j=-\frac{2}{h^3}+\left(\frac{\lambda}{h}-\frac{1}{h^2}\right)\varphi_N^2.$$

Identity (10) and the last two identities provide that

$$\frac{\lambda h^2(2-h)}{4-\lambda h^2}\left|\frac{\varphi_N}{h}\right|^2 = \lambda = h\sum_{j=0}^{N}\left|\frac{\varphi_{j+1}-\varphi_j}{h}\right|^2,$$

which is exactly the claim. □

2.2. *Non uniform observability*

In this section, we show that the observability constant C in inequality (6) blows up as $h \to 0$.

Theorem 2.2: *For any $T > 0$, we have*

$$\sup_{u \text{ sol. of } (5)}\left[\frac{E_h(0)}{\int_0^T \left|u_N'(t)\right|^2 dt}\right] \longrightarrow \infty \ \text{ as } \ h \to 0.$$

Proof: Consider the particular solution of (5)

$$\vec{u}(t) = \cos\left(\sqrt{\lambda_{N-1}}\, t\right)\vec{\varphi}_{N-1}.$$

For this solution, one has

$$\begin{aligned}
E_h(0) &= \frac{h}{2}\sum_{j=0}^{N}\left|\frac{\varphi_{N-1,j+1}-\varphi_{N-1,j}}{h}\right|^2 \\
&= \frac{\lambda_{N-1}h^2(2-h)}{2(4-\lambda_{N-1}h^2)}\left|\frac{\varphi_{N-1,N}}{h}\right|^2,
\end{aligned}$$

and

$$\int_0^T \left|u_N'\right|^2 dt = \lambda_{N-1}\left|\varphi_{N-1,N}\right|^2 \int_0^T \sin^2\left(\sqrt{\lambda_{N-1}}\, t\right) dt \le T\lambda_{N-1}\left|\varphi_{N-1,N}\right|^2.$$

Then we have

$$\frac{E_h(0)}{\int_0^T \left|u_N'\right|^2 dt} \ge \frac{2-h}{2T(4-\lambda_{N-1}h^2)}. \tag{15}$$

Moreover, in view of (9), we have

$$\begin{aligned}
\lambda_{N-1}h^2 &= 4\sin^2\left(\frac{(2N-1)\pi h}{2(2-h)}\right) = 4\sin^2\left(\frac{2N\pi h}{2(2-h)} - \frac{h\pi}{2(2-h)}\right) \\
&= 4\sin^2\left(\frac{(1-h)\pi}{(2-h)} - \frac{h\pi}{2(2-h)}\right) \longrightarrow 4 \ \text{ as } \ h \to 0.
\end{aligned}$$

Thus, the result is established. □

Remark 1 :
The inequality (15) shows that the constant on the boundary observability inequality blows up as the mesh-size tends to zero. This result is in agreement with the negative observability results established in [6, 9, 15, 23–25].

In [18], the inequality similar to (15) is of the form

$$E_h(0) \geq \frac{C(T)}{h^2} \int_0^T \left|u_N'\right|^2 dt,$$

which is sufficient to prove Theorem 2.2, but in our paper we need the inequality given by (15) which is useful for filtering technique. Note that the blows up of the right side of (15) is coming from the term $4 - \lambda_{N-1} h^2$, so the idea of filtering technique is to prevent this term do not converge to 0 by choosing a number $\gamma < 4$ such that $\lambda_{N-1} h^2 \leq \gamma$.

2.3. *Non uniform exponential decay*

To show the main result of this section we need the following lemma proved in [18].

Lemma 2.3: *If there exist positive constants M and ω independent of h such that for all $y^0 = (y_j^0)_{1\leq j \leq N}$ and $y^1 = (y_j^1)_{1\leq j \leq N}$ in $\mathbb{R}^N$,*

$$E_h(y,t) \leq Me^{-\omega t} E_h(y,0)\ ,\ t \geq 0\ , \forall 0 < h < 1,$$

then there exist positive constants C and T independent of h such that for all $u^0 = (u_j^0)_{1\leq j \leq N}$ and $u^1 = (u_j^1)_{1\leq j \leq N}$ in $\mathbb{R}^N$,

$$E_h(u,0) \leq C \int_0^T |u_{N+1}'|^2 dt.$$

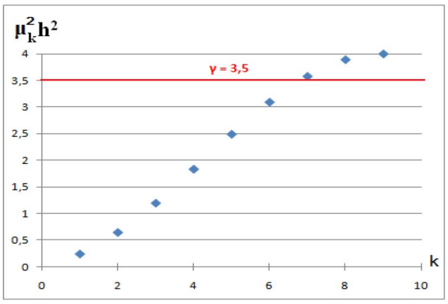
Finally, the following main result is an immediate consequence of Theorem 2.2 and the above lemma.

Theorem 2.4 : *The exponential decay of E_h to zero is not uniform with respect to the mesh size, i.e., there exist no positive constants M and ω (independent of h) such that for all y^0 and y^1 in $\mathbb{R}^N$*

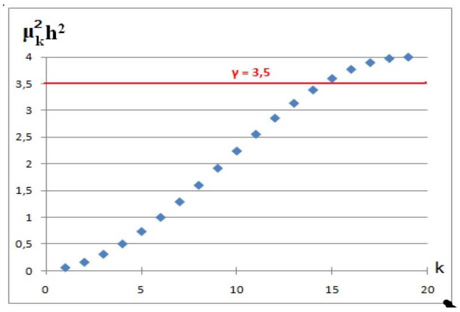
$$E_h(y,t) \leq Me^{-\omega t} E_h(y,0)\ ,\ t \geq 0, 0 < h < 1.$$

3. Uniform exponential decay by the filtering technique

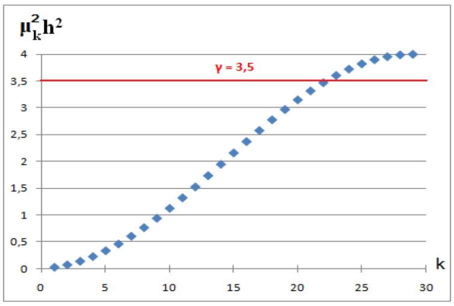
In order to obtain a positive counterpart to Theorem 2.4, we use a standard technique using in [9, 15, 23–25], which consists of filtering the high frequencies. We will adopt the non harmonic Fourier series and multiplier methods. For this, we introduce the following class $C_h(\gamma)$ of initial data of (5) and (3) generated by



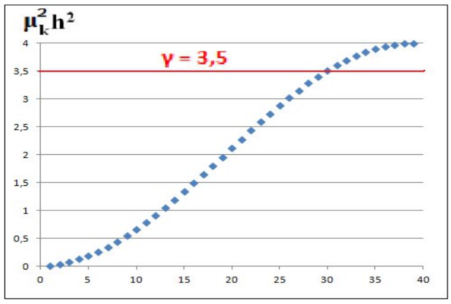
(a) $N = 9$ and $\gamma = 3.5$



(b) $N = 19$ and $\gamma = 3.5$



(c) $N = 29$ and $\gamma = 3.5$



(d) $N = 39$ and $\gamma = 3.5$

Figure 1. Number of eigenvalues to be cut for a given γ

eigenvectors of (8) associated with eigenvalues such that $\mu^2 h^2 \leq \gamma$

$$C_h(\gamma) := \left\{ \sum_{\mu_k^2 h^2 \leq \gamma} a_k \varphi_k \right\}$$

for any $0 < \gamma < 4$.

The schemes in Figure 1 show the number of eigenvalues to be cut off for a given $0 < \gamma < 4$. In figure 1(a) where $N = 9$ and $\gamma = 3.5$, are three largest eigenvalues to be cut off. In figure 1(d) where $N = 39$ and $\gamma = 3.5$, 9 are largest eigenvalues to be cut off.

3.1. *Multiplier technique*

3.1.1. Uniform observability for filtered solutions

Using the multiplier technique, we prove the uniformity of the observability constant for the filtered solutions of (5). Show first some preliminary results.

As in the continuous case, we show that the discrete energy E_h in (7) is conserved in time.

Lemma 3.1: *For any solution u of* (5)*, we have*

$$E_h(u, t) = E_h(u, 0)\ , \quad 0 \leq t \leq T.$$

Proof: By derivation of (7), we obtain

$$
\begin{aligned}
E_h'(u,t) &= h\sum_{j=1}^{N}\left[u_j'u_j'' + \left(\frac{u_{j+1}-u_j}{h}\right)\left(\frac{u_{j+1}'-u_j'}{h}\right)\right] \\
&+h\left[u_0^{'}u_0^{''} + \left(\frac{u_1-u_0}{h}\right)\left(\frac{u_1'-u_0'}{h}\right)\right] \\
&= \frac{1}{h}\sum_{j=1}^{N}\left[u_j'\left(u_{j+1}-2u_j+u_{j-1}\right) + \left(u_{j+1}-u_j\right)\left(u_{j+1}'-u_j'\right)\right] + \frac{1}{h}u_1u_1' \\
&= \frac{1}{h}\sum_{j=1}^{N}\left[u_{j+1}u_j' - 2u_ju_j' + u_{j-1}u_j' + u_{j+1}u_{j+1}' - u_{j+1}u_j' - u_ju_{j+1}' + u_ju_j'\right] \\
&+\frac{1}{h}u_1u_1' = \frac{1}{h}\left[u_0u_1' - u_Nu_{N+1}' - u_1u_1' + u_{N+1}u_{N+1}'\right] + \frac{1}{h}u_1u_1' = 0.
\end{aligned}
$$

□

To show our main result of this section, we need as well the following two lemmas.

Lemma 3.2: *For any solution u of (5) and any $h > 0$ we have*

$$
\frac{h}{2}\sum_{j=0}^{N}\int_0^T\left[u_j'u_{j+1}' + \left|\frac{u_{j+1}-u_j}{h}\right|^2\right]dt + X_h(t)|_0^T = \frac{1}{2}\int_0^T\left|u_N'\right|^2dt, \qquad (16)
$$

with

$$
X_h(t) = h\sum_{j=1}^{N} j\left(\frac{u_{j+1}-u_{j-1}}{2}\right)u_j'.
$$

Proof: Multiplying (5) by $j(\frac{u_{j+1}-u_{j-1}}{2})$ and integrating over $[0,T]$, we obtain

$$
\begin{aligned}
&\sum_{j=1}^{N}\int_0^T ju_j''\left(\frac{u_{j+1}-u_{j-1}}{2}\right)dt \qquad (17)\\
&= \frac{1}{h^2}\sum_{j=1}^{N}\int_0^T j\left(\frac{u_{j+1}-u_{j-1}}{2}\right)\left(u_{j+1}-2u_j+u_{j-1}\right)dt.
\end{aligned}
$$

On the other hand, we have

$$\sum_{j=1}^{N}\int_0^T ju_j''\left(\frac{u_{j+1}-u_{j-1}}{2}\right)dt=\frac{1}{h}X_h(t)|_0^T-\frac{1}{2}\sum_{j=1}^{N}\int_0^T ju_j'\left(u_{j+1}'-u_{j-1}'\right)dt$$

$$=\frac{1}{h}X_h(t)|_0^T-\frac{1}{2}\sum_{j=0}^{N}\int_0^T\left(ju_j'u_{j+1}'-(j+1)u_j'u_{j+1}'\right)dt$$

$$-\frac{N+1}{2}u_N'u_{N+1}'=\frac{1}{h}X_h(t)|_0^T+\frac{1}{2}\sum_{j=0}^{N}\int_0^T u_j'u_{j+1}'dt-\frac{N+1}{2}\left|u_N'\right|^2. \tag{18}$$

We see also that

$$\frac{1}{h^2}\sum_{j=1}^{N}\int_0^T j\left(\frac{u_{j+1}-u_{j-1}}{2}\right)(u_{j+1}-2u_j+u_{j-1})\,dt$$

$$=\frac{1}{2h^2}\sum_{j=1}^{N}\int_0^T\left(ju_{j+1}^2-ju_{j-1}^2-2ju_{j+1}u_j+2ju_{j-1}u_j\right)dt$$

$$=\frac{1}{2h^2}\sum_{j=1}^{N}\int_0^T\left(-2u_j^2+2u_ju_{j+1}\right)dt-\frac{1}{2h^2}\left|u_N\right|^2$$

$$=-\frac{1}{2}\sum_{j=0}^{N}\int_0^T\left|\frac{u_{j+1}-u_j}{h}\right|^2dt. \tag{19}$$

Finally, (18) and (19) in (17) yield the result. □

Lemma 3.3: *For any solution u of* (5) *and any $h>0$, we have*

$$-h\sum_{j=0}^{N}\int_0^T\left|u_j'\right|^2dt+h\sum_{j=0}^{N}\int_0^T\left|\frac{u_{j+1}-u_j}{h}\right|^2dt+Y_h(t)|_0^T=0,$$

with

$$Y_h(t)=h\sum_{j=0}^{N}u_j'u_j.$$

Proof: Multiplying equation (5) by u_j, we obtain

$$\sum_{j=1}^{N}\int_0^T u_j''u_jdt=\frac{1}{h^2}\sum_{j=1}^{N}\int_0^T u_j(u_{j+1}-2u_j+u_{j-1})dt. \tag{20}$$

Therefore,

$$\sum_{j=1}^{N}\int_0^T u_j'' u_j dt = \frac{1}{h} Y_h(t)|_0^T - \sum_{j=1}^{N}\int_0^T \left|u_j'\right|^2 dt. \tag{21}$$

In the other hand, we have

$$\begin{aligned}\frac{1}{h^2}\sum_{j=1}^{N}\int_0^T u_j(u_{j+1}-2u_j+u_{j-1})dt &= \frac{1}{h^2}\sum_{j=1}^{N}\int_0^T (u_{j+1}u_j - 2u_j^2 + u_{j-1}u_j)dt \\ &= -\frac{1}{h^2}\sum_{j=0}^{N}\int_0^T (u_{j+1}^2 - 2u_{j+1}u_j + u_j^2)dt \\ &= -\sum_{j=0}^{N}\int_0^T \left|\frac{u_{j+1}-u_j}{h}\right|^2 dt. \end{aligned}\tag{22}$$

Thus, (21), (22) and (20) allow us to conclude. □

Lemma 3.4: *We have the following inequality*

$$\left|X_h(t) - \frac{\gamma}{8}Y_h(t)\right| \leq \sqrt{1+\frac{3\gamma}{16\lambda_0}}E_h(u,0).$$

Proof: We have

$$X_h(t) - \frac{\gamma}{8}Y_h(t) = h\sum_{j=1}^{N} u_j'\left[j\frac{u_{j+1}-u_{j-1}}{2} - \frac{\gamma}{8}u_j\right].$$

Then

$$\left|X_h(t) - \frac{\gamma}{8}Y_h(t)\right| \leq \left[h\sum_{j=1}^{N}\left|u_j'\right|^2\right]^{\frac{1}{2}} \left[h\sum_{j=1}^{N}\left|j\frac{u_{j+1}-u_{j-1}}{2} - \frac{\gamma}{8}u_j\right|^2\right]^{\frac{1}{2}}. \tag{23}$$

On the other hand, we have

$$\begin{aligned}&h\sum_{j=1}^{N}\left|j\frac{u_{j+1}-u_{j-1}}{2} - \frac{\gamma}{8}u_j\right|^2 \\ &= h\sum_{j=1}^{N}\left[\frac{j^2}{4}\left|u_{j+1}-u_{j-1}\right|^2 + \frac{\gamma^2}{64}u_j^2 - \frac{\gamma j}{8}\left(u_{j+1}-u_{j-1}\right)u_j\right]\end{aligned}$$

$$\leq h\sum_{j=1}^{N}\left[\frac{j^2}{2}\left|u_{j+1}-u_j\right|^2+\frac{j^2}{2}\left|u_j-u_{j-1}\right|^2+\frac{\gamma^2}{64}u_j^2-\frac{\gamma j}{8}u_{j+1}u_j+\frac{\gamma j}{8}u_ju_{j-1}\right]$$

$$\leq h\sum_{j=0}^{N}\left|\frac{u_{j+1}-u_j}{h}\right|^2+\frac{\gamma^2 h}{64}\sum_{j=0}^{N}u_j^2+\frac{\gamma h}{8}\sum_{j=0}^{N}u_{j+1}u_j-\frac{\gamma}{8}u_N^2$$

$$\leq h\sum_{j=0}^{N}\left|\frac{u_{j+1}-u_j}{h}\right|^2+\left(\frac{\gamma^2}{64}+\frac{\gamma}{8}\right)h\sum_{j=0}^{N}u_j^2-\frac{\gamma h}{16}\sum_{j=0}^{N}\left(2u_j^2-2u_{j+1}u_j\right)-\frac{\gamma}{8}u_N^2$$

$$\leq h\sum_{j=0}^{N}\left|\frac{u_{j+1}-u_j}{h}\right|^2+\left(\frac{3\gamma}{16}\right)h\sum_{j=0}^{N}u_j^2-\frac{\gamma h}{16}\sum_{j=0}^{N}\left|u_{j+1}-u_j\right|^2+\frac{\gamma h}{16}u_{N+1}^2-\frac{\gamma}{8}u_N^2$$

$$\leq \left(1-\frac{\gamma h^2}{16}\right)h\sum_{j=0}^{N}\left|\frac{u_{j+1}-u_j}{h}\right|^2+\frac{3\gamma}{16\lambda_0}h\sum_{j=0}^{N}\left|\frac{u_{j+1}-u_j}{h}\right|^2+\left(\frac{\gamma h}{16}-\frac{\gamma}{8}\right)u_N^2$$

$$\leq \left(1-\frac{\gamma h^2}{16}+\frac{3\gamma}{16\lambda_0}\right)h\sum_{j=0}^{N}\left|\frac{u_{j+1}-u_j}{h}\right|^2. \tag{24}$$

Combining (23) and (24) we deduce, by Young inequality, that

$$\left|X_h(t)-\frac{\gamma}{8}Y_h(t)\right|\leq\sqrt{1-\frac{\gamma h^2}{16}+\frac{3\gamma}{16\lambda_0}}\left[h\sum_{j=1}^{N}\left|u_j'\right|^2\right]^{\frac{1}{2}}\left[h\sum_{j=0}^{N}\left|\frac{u_{j+1}-u_j}{h}\right|^2\right]^{\frac{1}{2}}$$

$$\leq\sqrt{1+\frac{3\gamma}{16\lambda_0}}\;E_h(0).$$

□

Now, we can announce our main result in this subsection.

Theorem 3.5: *Assume that $\gamma < 4$. Then there exists $T_1(\gamma) > 2$ such that for all $T > T_1(\gamma)$, there exists $C_1(T,\gamma)$ such that*

$$E_h(u,0)\leq C_1(T,\gamma)\int_0^T\left|u_N'(t)\right|^2dt,$$

for every solution, with u^0 and u^1 in the class $C_h(\gamma)$, and all h.

Proof: Let u be a solution of (5) where u^0 and u^1 in the class $C_h(\gamma)$. Using Lemma 3.1, equality (16) may be written as

$$TE_h(u,0)+\frac{h}{2}\sum_{j=0}^{N}\int_0^T\left[u_j'u_{j+1}'-\left|u_j'\right|^2\right]dt+X_h(t)|_0^T=\frac{1}{2}\int_0^T\left|u_N'\right|^2dt. \tag{25}$$

For the second term of (25), we have

$$\sum_{j=0}^{N}\left[u_j'u_{j+1}'-\left|u_j'\right|^2\right] = -\frac{1}{2}\sum_{j=0}^{N}\left|u_{j+1}'-u_j'\right|^2+\frac{1}{2}\left|u_N'\right|^2$$

$$= -\frac{1}{2}\sum_{\mu_k^2h^2\leq\gamma}|a_k|^2\mu_k^4h^2\sum_{j=1}^{N}|\varphi_{k,j}|^2+\frac{1}{2}\left|u_N'\right|^2$$

$$\geq -\frac{1}{2}\gamma\sum_{\mu_k^2h^2\leq\gamma}|a_k|^2\mu_k^2\sum_{j=1}^{N}|\varphi_{k,j}|^2+\frac{1}{2}\left|u_N'\right|^2.$$

Hence,

$$\sum_{j=0}^{N}\left[u_j'u_{j+1}'-\left|u_j'\right|^2\right]\geq -\frac{1}{2}\gamma\sum_{j=0}^{N}\left|u_j'\right|^2+\frac{1}{2}\left|u_N'\right|^2.$$

From (25) and the last estimate, we deduce that

$$TE_h(u,0)-\frac{\gamma}{4}h\sum_{j=0}^{N}\int_0^T\left|u_j'\right|^2dt+\frac{h}{4}\int_0^T\left|u_N'\right|^2dt+X_h(t)|_0^T\leq\frac{1}{2}\int_0^T\left|u_N'\right|^2dt. \quad (26)$$

Lemma 3.3 implies that

$$h\sum_{j=1}^{N}\int_0^T\left|u_j'\right|^2dt=TE_h(u,0)+\frac{1}{2}Y_h(t)|_0^T. \quad (27)$$

Reporting (27) in (26) we get

$$T\left(1-\frac{\gamma}{4}\right)E_h(u,0)-\frac{\gamma}{8}Y_h(t)|_0^T+X_h(t)|_0^T\leq\frac{2-h}{4}\int_0^T\left|u_N'\right|^2dt. \quad (28)$$

Combining (28) and Lemma 3.4 we deduce that

$$\left[T\left(1-\frac{\gamma}{4}\right)-2\sqrt{1+\frac{3\gamma}{16\lambda_0}}\right]E_h(u,0)\leq\frac{1}{2}\int_0^T\left|u_N'\right|^2dt, \quad (29)$$

which implies that

$$E_h(u,0)\leq\frac{1}{2\left(T\left(1-\frac{\gamma}{4}\right)-2\sqrt{1+\frac{3\gamma}{16\lambda_0}}\right)}\int_0^T\left|u_N'\right|^2dt,$$

for

$$T > \frac{2\sqrt{1+\frac{3\gamma}{16\lambda_0}}}{1-\frac{\gamma}{4}}.$$

Thus Theorem 3.5 holds with

$$T_1(\gamma) = \frac{2\sqrt{1+\frac{3\gamma}{16\lambda_0}}}{1-\frac{\gamma}{4}}, \tag{30}$$

and

$$C_1(T,\gamma) = \frac{1}{2\left(T\left(1-\frac{\gamma}{4}\right) - 2\sqrt{1+\frac{3\gamma}{16\lambda_0}}\right)}. \tag{31}$$

□

3.1.2. Uniform exponential decay for filtered solutions

We set $y = u + z$ with $u_j^0 = y_j^0$ and $u_j^1 = y_j^1$ where $y^0 \in C_h(\gamma)$, $y^1 \in C_h(\gamma)$ and z solves the problem

$$\begin{cases} z_j^{''} - \frac{z_{j+1}-2z_j+z_{j-1}}{h^2} = 0 \ , j = 1, ..., N \\ z_0 = 0 \ , \ \frac{z_{N+1}-z_N}{h} = -\alpha y_{N+1}^{'} \ , \ j = 0, ..., N \\ z_j(0) = 0, \ \ v_j^{'}(0) = 0, \ \ j = 1, ..., N. \end{cases} \tag{32}$$

We have the following Lemma

Lemma 3.6: *Let $T > 0$. There exists $C > 0$ and $K > 0$ such that for every, $0 < h < 1$, we have*

$$\int_0^T |z_{N+1}^{'}|^2 dt \leq C \int_0^T |y_{N+1}^{'}|^2 dt + K E_h(y, 0).$$

Proof:

The energy of system (32) is given by

$$E_h(z,t) = \frac{h}{2} \sum_{j=0}^{N} \left[|z_j^{'}(t)|^2 + \left| \frac{z_{j+1}(t) - z_j(t)}{h} \right|^2 \right]$$

and its derivative is given by

$$E_h^{'}(z,t) = -\alpha z_{N+1}^{'} y_{N+1}^{'}.$$

Applying Young's inequality, we get

$$E_h(z,t) \leq \frac{\alpha^2}{4\varepsilon} \int_0^T |y_{N+1}^{'}|^2 dt + \varepsilon \int_0^T |z_{N+1}^{'}|^2 dt.$$

Multiplying (32) by $j\frac{z_{j+1}-z_{j-1}}{2}$ and integrating over $[0,T]$, we obtain

$$h\sum_{j=1}^{N} jz'_j\left(\frac{z_{j+1}-z_{j-1}}{2}\right)|_0^T - h\sum_{j=1}^{N}\int_0^T jz'_j\left(\frac{z'_{j+1}-z'_{j-1}}{2}\right)dt$$
$$-h\sum_{j=1}^{N}\int_0^T j\left(\frac{z_{j+1}-z_{j-1}}{2}\right)\left(\frac{z_{j+1}-2z_j+z_{j-1}}{h^2}\right)dt = 0. \qquad (33)$$

We have

$$h\sum_{j=1}^{N} jz'_j\left(\frac{z_{j+1}-z_{j-1}}{2}\right)|_0^T \qquad (34)$$
$$= h^2\sum_{j=1}^{N} jz'_j\left(\frac{z_{j+1}-z_j}{2h}\right)|_0^T + h^2\sum_{j=1}^{N-1}(j+1)z'_{j+1}\left(\frac{z_{j+1}-z_j}{2h}\right)|_0^T,$$

$$-h\sum_{j=1}^{N}\int_0^T jz'_j\left(\frac{z'_{j+1}-z'_{j-1}}{2}\right)dt = \frac{h}{2}\sum_{j=0}^{N}\int_0^T |z'_j|^2dt + \frac{h}{4}\int_0^T |z'_{N+1}|^2dt$$
$$-\frac{h^3}{4}\sum_{j=0}^{N}\int_0^T |\frac{z'_{j+1}-z'_j}{h}|^2dt + \frac{\alpha^2h^2}{4}\int_0^T |y''_{N+1}|^2dt \qquad (35)$$
$$-\frac{1}{4}\int_0^T (|z'_{N+1}|^2+|z'_N|^2)dt,$$

and

$$-h\sum_{j=1}^{N}\int_0^T j(\frac{z_{j+1}-z_{j-1}}{2})(\frac{z_{j+1}-2z_j+z_{j-1}}{h^2})dt \qquad (36)$$
$$= \frac{h}{2}\sum_{j=0}^{N}\int_0^T \left|\frac{z'_{j+1}-z'_j}{h}\right|^2 dt - \frac{\alpha^2}{2}\int_0^T |y'_{N+1}|^2dt.$$

Reporting (34), (35) and (36) in (33) we get

$$\frac{h^3}{4}\sum_{j=0}^{N}\int_0^T |\frac{z'_{j+1}-z'_j}{h}|^2dt + \frac{1-h}{4}\int_0^T |z'_{N+1}|^2dt + \frac{1}{4}\int_0^T |z'_N|^2dt$$
$$= h^2\sum_{j=0}^{N} jz'_j\left(\frac{z_{j+1}-z_j}{2h}\right)|_0^T + h^2\sum_{j=0}^{N-1}(j+1)z'_{j+1}\left(\frac{z_{j+1}-z_j}{2h}\right)|_0^T$$
$$+\int_0^T E(z,t)dt - \frac{\alpha^2}{2}\int_0^T |y'_{N+1}|^2dt + \frac{\alpha^2h^2}{4}\int_0^T |y''_{N+1}|^2dt.$$

Using (34), we obtain

$$\frac{h^3}{4}\sum_{j=0}^{N}\int_0^T |\frac{z'_{j+1}-z'_j}{h}|^2 dt + \frac{3-2h}{8}\int_0^T |z'_{N+1}|^2 dt$$
$$\leq \frac{\alpha^2}{4\varepsilon}(1+T)\int_0^T |y'_{N+1}|^2 dt + \varepsilon(1+T)\int_0^T |z'_{N+1}|^2 dt + \frac{\alpha^2 h^2}{4}\int_0^T |y''_{N+1}|^2 dt.$$

We choose $\varepsilon = \frac{3-2h}{16(1+T)}$, so that

$$\int_0^T |z'_{N+1}|^2 dt \leq 64\alpha^2(1+T)^2\int_0^T |y'_{N+1}|^2 dt + 4\alpha^2 h^2\int_0^T |y''_{N+1}|^2 dt.$$

On the other hand, it is easy to check that (see [18])

$$4\alpha^2 h^2\int_0^T |y''_{N+1}|^2 dt \leq KE_h(y,0).$$

Finally, we get

$$\int_0^T |z'_{N+1}|^2 dt \leq 64\alpha^2(1+T)^2\int_0^T |y'_{N+1}|^2 dt + KE_h(y,0).$$

which gives the proof with $C = 64\alpha^2(1+T)^2$. □

Now, we can announce our main result in this subsection.

Theorem 3.7: *The exponential decay of E_h to zero is uniform with respect to the mesh size in the range $C_h(\gamma)$, i.e., there exist positive constants M_1 and ω_1 independent of h such that for all y^0 and y^1 in the class $C_h(\gamma)$,*

$$E_h(y,t) \leq M_1 e^{-\omega t} E_h(y,0)\ ,\ t \geq 0\ , 0 < h < 1.$$

Proof: From (29) and $y = u + z$, we get

$$\left[T\left(1-\frac{\gamma}{4}\right) - 2\sqrt{1+\frac{3\gamma}{16\lambda_0}}\right] E_h(u,0) \leq \frac{1}{2}\int_0^T \left|u'_{N+1}\right|^2 dt$$
$$\leq \int_0^T \left|z'_{N+1}\right|^2 dt + \int_0^T \left|y'_{N+1}\right|^2 dt.$$

By Lemma 3.6, we have

$$\left[T\left(1-\frac{\gamma}{4}\right) - 2\sqrt{1+\frac{3\gamma}{16\lambda_0}} - K\right] E_h(u,0) \leq (C+1)\int_0^T \left|y'_{N+1}\right|^2 dt.$$

On the other hand, we have $E_h(y,0) = E_h(u,0)$. Therefore, for

$T > \left(2\sqrt{1+\frac{3\gamma}{16\lambda_0}}+K\right) / \left(1-\frac{\gamma}{4}\right)$, we get

$$E_h(y,0) \leq (C+1)\left[T\left(1-\frac{\gamma}{4}\right)-2\sqrt{1+\frac{3\gamma}{16\lambda_0}}-K\right]^{-1}\int_0^T \left|y'_{N+1}\right|^2 dt.$$

Using (4), we obtain

$$E_h(y,T) \leq E_h(y,0) \leq \alpha^{-1}(C+1)\left[T\left(1-\frac{\gamma}{4}\right)-2\sqrt{1+\frac{3\gamma}{16\lambda_0}}-K\right]^{-1}\int_0^T -E_h^{'}(y,t)dt.$$

Then, we obtain

$$E_h(y,T) \leq \frac{C'}{C'+1}E_h(y,0),$$

with $C' = \alpha^{-1}(C+1)\left[T\left(1-\frac{\gamma}{4}\right)-2\sqrt{1+\frac{3\gamma}{16\lambda_0}}-K\right]^{-1}$.

As the system (3) is invariant by translation, we can deduce that for all $n \in \mathbb{N}$

$$E_h(y,(n+1)T) \leq \frac{C'}{C'+1}E_h(y,nT).$$

By iteration, we get

$$E_h(y,(n+1)T) \leq \left(\frac{C'}{C'+1}\right)^{n+1} E_h(y,0).$$

Therefore

$$E_h(y,(n+1)T) \leq e^{-\omega_1(n+1)T}E_h(y,0),$$

with $w_1 = \frac{1}{T}ln\left(\frac{C'+1}{C'}\right)$.

For $t > 0$, there exists $n \in \mathbb{N}$ such that $nT \leq t \leq (n+1)T$. Using (4), we get

$$E_h(y,t) \leq E_h(y,nT).$$

which implies that

$$E_h(y,t) \leq e^{-\omega_1 nT}E_h(y,0).$$

Hence

$$E_h(y,t) \leq \frac{C'+1}{C'}e^{-\omega_1(n+1)T}E_h(y,0).$$

Using the inequality $t < (n+1)T$, we get

$$E_h(y,t) \leq \frac{C'+1}{C'}e^{-\omega_1 t}E_h(y,0).$$

This establishes the result with $M_1 = \frac{C'+1}{C'}$. □

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