**LECTURE NOTES of TICMI**

# THE $L^p$-DISSIPATIVITY OF PARTIAL DIFFERENTIAL OPERATORS

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**Summary.** After giving some classical results concerning the dissipativity of linear operators on Banach spaces and the generation of contractive semigroups, the course will focus on the $L^p$-dissipativity of partial differential operators. Some recent results, obtained in joint papers with Vladimir Maz'ya, will be discussed. The main one is an algebraic necessary and sufficient condition for the $L^p$-dissipativity of the scalar operator $\nabla^t(A\nabla)$, where $A$ is a matrix whose entries are complex measures and whose imaginary part is symmetric. We survey several other results connected to this condition and obtained mainly by V. Maz'ya and his co-authors. They concern operators with lower order terms, operators with constant complex coefficients, the angle of dissipativity, systems of partial differential operators, higher order operators.

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# Preface

These Notes are an enlarged version of an Advanced Course given at the Tbilisi International Centre of Mathematics and Informatics in April 2010.

They are intended to introduce to the concept of $L^p$-dissipativity of partial differential operators and to present some recent results obtained in joint papers with Vladimir Maz'ya.

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Alberto Cialdea

# Introduction

Let us consider the classical Cauchy-Dirichlet problem for the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u, & \text{for } t > 0, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $\varphi$ is a given function in $C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

It is well known that the unique solution of problem (1) in the class of smooth bounded solutions is given by the formula

$$u(x,t) = \frac{1}{\sqrt{(4\pi t)^n}} \int_{\mathbb{R}^n} \varphi(y)\, e^{-\frac{|x-y|^2}{4t}} dy, \quad x \in \mathbb{R}^n, t > 0. \tag{2}$$

From (2) it follows immediately

$$|u(x,t)| \leqslant \|\varphi\|_\infty, \quad t > 0, \tag{3}$$

since

$$\frac{1}{\sqrt{(4\pi t)^n}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} dy = 1 \quad (t > 0). \tag{4}$$

Inequality (3) leads to

$$\|u(\cdot,t)\|_\infty \leqslant \|\varphi\|_\infty, \quad t > 0,$$

and this in turn implies that the norm $\|u(\cdot,t)\|_\infty$ is a decreasing function of $t$. In fact, fix $t_0 > 0$ and consider the problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v, & \text{for } t > t_0, \\ v(x,t_0) = u(x,t_0), & x \in \mathbb{R}^n. \end{cases} \tag{5}$$

It is clear that the unique solution of (5) is given by $v(x,t) = u(x,t)$ $(t > t_0)$ and we have

$$\|v(\cdot,t)\|_\infty \leqslant \|u(\cdot,t_0)\|_\infty, \quad t > t_0,$$

i.e.

$$\|u(\cdot,t)\|_\infty \leqslant \|u(\cdot,t_0)\|_\infty, \quad t > t_0.$$

But the $L^\infty$ norm is not the only norm for which we have this kind of dissipativity. Let us consider the $L^p$-norm with $1 < p < \infty$. By Cauchy-Hölder inequality, from (2) we get

$$|u(x,t)| \leqslant \left( \frac{1}{\sqrt{(4\pi t)^n}} \int_{\mathbb{R}^n} |\varphi(y)|^p e^{-\frac{|x-y|^2}{4t}} dy \right)^{1/p} \left( \frac{1}{\sqrt{(4\pi t)^n}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} dy \right)^{1/p'}$$

$(1/p + 1/p' = 1)$ and then, keeping in mind (4),

$$|u(x,t)|^p \leqslant \frac{1}{\sqrt{(4\pi t)^n}} \int_{\mathbb{R}^n} |\varphi(y)|^p e^{-\frac{|x-y|^2}{4t}} dy.$$

Integrating over $\mathbb{R}^n$ and applying Tonelli's Theorem we find

$$\int_{\mathbb{R}^n} |u(x,t)|^p dx \leqslant \frac{1}{\sqrt{(4\pi t)^n}} \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} |\varphi(y)|^p e^{-\frac{|x-y|^2}{4t}} dy =$$
$$\frac{1}{\sqrt{(4\pi t)^n}} \int_{\mathbb{R}^n} |\varphi(y)|^p dy \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} dx = \int_{\mathbb{R}^n} |\varphi(y)|^p dy$$

and we have proved that

$$\|u(\cdot,t)\|_p \leqslant \|\varphi\|_p. \tag{6}$$

As before, this inequality implies that the norm $\|u(\cdot,t)\|_p$ is a decreasing function of $t$.

Let us consider now the more general problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = Au, & \text{for } t > 0, \\ u(x,0) = \varphi(x), & x \in \Omega, \end{cases} \tag{7}$$

where $\Omega$ is a domain in $R^n$ and $A$ is an elliptic partial differential operator of order two

$$Au = \sum_{|\alpha| \leqslant 2} a_\alpha(x)\, D^\alpha u\,. \tag{8}$$

A natural question arises: under which conditions for the operator $A$ the solution $u(x,t)$ of the problem (7) satisfies the inequality (6) ?

The aim of this short course is to answer to this question.

In order to precise our goal, let us make a simple remark. As we know already, (6) implies that $\|u(\cdot,t)\|_p$ is a decreasing function of $t$ and then

$$\frac{d}{dt}\|u(\cdot,t)\|_p \leqslant 0. \tag{9}$$

On the other hand, at least formally, we have [(1)]

$$\frac{d}{dt}\|u(\cdot,t)\|_p^{\,p} = \frac{d}{dt}\int_\Omega |u(x,t)|^p dx = p\,\mathscr{R}e \int_\Omega \langle \partial_t u, u\rangle |u|^{p-2} dx, \tag{10}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $\mathbb{C}$.

[(1)]Note that $\partial_t|u| = \partial_t\sqrt{u\,\overline{u}} = (u_t\overline{u} + u\overline{u}_t)/(2\sqrt{u\,\overline{u}}) = \mathscr{R}e(u_t\overline{u}/(2|u|))$.

Since $u$ is the solution of the problem (7), keeping in mind (10), we have that (9) holds if and only if

$$\mathscr{R}e \int_\Omega \langle Au, u\rangle |u|^{p-2} dx \leqslant 0.$$

This leads to the following definition: *let $A$ a linear operator from $D(A) \subset L^p(\Omega)$ to $L^p(\Omega)$; $A$ is said to be $L^p$-dissipative if*

$$\mathscr{R}e \int_\Omega \langle Au, u\rangle |u|^{p-2} dx \leqslant 0 \quad \forall\, u \in D(A). \tag{11}$$

From what we have said before, if $A$ is $L^p$-dissipative and if the problem (7) has solution, then (9) holds.

Of course there are several details we have to precise: at first, we have to understand what condition (11) means when $1 \leqslant p < 2$ and moreover we have to justify in a rigorous way all the procedure for obtaining it.

This can be done in an abstract and very general setting, by means of the Functional Analysis and Section 1.2 is devoted to such a purpose.

We end this Introduction with a well known fact (see e.g. [34, p.215]). If the operator (8) has real smooth coefficients and it can be written in divergence form, then we have the $L^p$-dissipativity for any $p$. If $2 \leqslant p < \infty$ this can be deduced easily by integration by parts. If $A = \partial_i(a_{ij}\partial_j)$ ($a_{ji} = a_{ij} \in C^1(\overline{\Omega})$), we can write

$$\int_\Omega \langle Au, u\rangle |u|^{p-2} dx = -\int_\Omega a_{ij}\partial_j u\, \partial_i(\overline{u}\, |u|^{p-2})\, dx\,.$$

If we suppose that $a_{ij}\xi_i\xi_j \geqslant 0$ for any $\xi \in \mathbb{R}^n$, an easy calculation shows that

$$\mathscr{R}e \int_\Omega a_{ij}\partial_j u\, \partial_i(\overline{u}\, |u|^{p-2})\, dx \geqslant 0$$

and the $L^p$-dissipativity of $A$ follows. Some extra arguments are necessary for the case $1 \leqslant p < 2$.

If the operator (8) has complex coefficients and they are not smooth, the investigation is not so simple.

During the last half a century various aspects of the $L^p$-theory of semigroups generated by linear differential operators were studied in [4, 8, 2, 37, 9, 16, 35, 17, 10, 11, 21, 22, 19, 20, 3, 7, 15, 23, 36, 32, 5, 30, 6] and others. An account of the subject can be found in the book [33], which contains also an extensive bibliography.

These notes are divided in three parts. In the first one we provide the basic facts of the general theory. For us the crucial result in this part is the

Lumer-Phillips Theorem. In [29] V. G. Maz'ya and P. E. Sobolevskiĭ obtained independently of Lumer and Phillips the same result under the assumption that the norm of the Banach space is Gâteaux-differentiable. In the same paper some applications to second order elliptic operators are given. It is interesting to remark that this paper was sent to the journal in 1960, before the Lumer-Phillips paper [24] appeared.

Much of the material of this part, containing classical results, is taken from the books [14, 34].

In the second chapter we give a detailed description of the results obtained in [5] and [6]. They concern the $L^p$-dissipativity of scalar second order partial differential operators with complex coefficients.

In these two parts, we have tried to provide a self-contained exposition, giving all the proofs of the needed results.

In the last chapter we survey, without proofs, several connected results concerning the $L^p$-dissipativity of systems of partial differential operators and the nondissipativity of higher order operators. These results were mostly obtained by Vladimir Maz'ya and his co-authors G. Kresin, M. Langer and myself.

# Chapter 1

# A short introduction to Semigroup Theory

## 1.1 The Hille-Yosida Theorem

### 1.1.1 Uniformly continuous semigroups

Let $X$ be a Banach space. A *semigroup of linear operators on* $X$ is a family of linear and continuous operators $T(t)$ $(0 \leqslant t < \infty)$ from $X$ into itself such that

$T(0) = I$,

$T(t+s) = T(t)T(s) \qquad (s, t \geqslant 0)$.

The linear operator

$$Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t} \tag{1.1}$$

is the *infinitesimal generator of the semigroup* $T(t)$.

The domain $D(A)$ of the operator $A$ is the set of $x \in X$ such that the following limit does exist

$$\lim_{t \to 0^+} \frac{T(t)x - x}{t}.$$

We remark that the linear operator $A$ does not need to be continuous.

The semigroup $T(t)$ *is said to be uniformly continuous* if

$$\lim_{t \to 0^+} \|T(t) - I\| = 0. \tag{1.2}$$

The uniformly continuous semigroups satisfy a lot of nice properties. However, as the next result shows, these semigroups are very particular. This is shown by the following theorem, where $B(X)$ denotes the class of linear and continuous operators from $X$ into itself.

**Theorem 1** *The operator $A$ is the generator of a uniformly continuous semigroup if and only if $A \in B(X)$. Moreover*

$$T(t) = e^{tA}, \tag{1.3}$$

*where*

$$e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}. \tag{1.4}$$

We mention two important properties of a uniformly continuous semigroup $T(t)$:

(i) there exists a constant $\omega \geqslant 0$ such that

$$\|T(t)\| \leqslant e^{\omega t},$$

(ii) the function $T(t)$ is differentiable in norm and

$$\frac{dT(t)}{dt} = AT(t) = T(t)A \quad (t > 0). \tag{1.5}$$

Formula (1.5) shows that the function $u(t) = T(t)u_0$ is the solution of the evolution problem

$$\begin{cases} \dfrac{du}{dt} = Au \qquad (t > 0), \\ \\ u(0) = u_0, \end{cases} \tag{1.6}$$

where $u_0$ is a given element of $X$.

### 1.1.2 Strongly continuous semigroups

Unfortunately, if $A$ is a partial differential operator, usually it is an unbounded operator, i.e. it does not belong to $B(X)$.

Then it is necessary to weaker the condition (1.2) and consider more general semigroups. We say that $T(t)$ is a strongly continuous semigroup (briefly, a $C^0$-semigroup) if

$$\lim_{t \to 0^+} T(t)x = x \qquad \forall\, x \in X.$$

The operator $A$ is said to be the generator of the $C^0$-semigroup if (1.1) holds for any $x \in D(A)$. We remark that in this case the operator $A$ may be unbounded.

We have seen that if $T(t)$ is a uniformly continuous semigroup, then $\|T(t)\| \leqslant e^{\omega\, t}$. In case of a $C^0$-semigroup, we have

**Theorem 2** *Let $T(t)$ be a $C_0$ semigroup. There exist two constants $\omega \geqslant 0$, $M \geqslant 1$ such hat*

$$\|T(t)\| \leqslant M\, e^{\omega t}, \qquad 0 \leqslant t < \infty. \tag{1.7}$$

**Proof.** First let us show that there exist $M,\ \eta > 0$ such that

$$\|T(t)\| \leqslant M \qquad \forall\, t \in [0, \eta]. \tag{1.8}$$

If (1.8) is false, we can find a sequence of real numbers $t_n > 0$ such that $\|T(t_n)\| > n$, $t_n \to 0$. It follows that there exists $x \in X$ such that

$$\sup_{n \in \mathbb{N}} \|T(t_n)x\| = \infty.$$

If not, we would have

$$\sup_{n \in \mathbb{N}} \|T(t_n)x\| < \infty \qquad \forall\, x \in X;$$

in view of the Banach-Steinhaus Theorem, this implies

$$\sup_{n \in \mathbb{N}} \|T(t_n)\| < \infty$$

and this is absurd. Formula (1.8) is proved.

Since $\|T(0)\| = 1$, we have $M \geqslant 1$. Let now $t$ be a nonnegative number; we can write $t = n\eta + \delta$, where $n$ is a natural number and $0 \leqslant \delta < \eta$. Therefore

$$\|T(t)\| = \|T(\delta)T(\eta)^n\| \leqslant M^{n+1} = M^{1+\frac{t-\delta}{\eta}} \leqslant M^{1+\frac{t}{\eta}} = M\, e^{\omega t},$$

where $\omega = (\log M)/\eta$. $\square$

A first consequence is that $T(t)x$ is continuous.

**Theorem 3** *Let $T(t)$ be a $C_0$ semigroup. For any $x \in X$, $T(t)x$ is a continuos function on $X$ of the real variable $t \geqslant 0$.*

**Proof.** The continuity from the right at $t = 0$ is obvious. Let us fix $t > 0$ and take $h \geqslant 0$; we have

$$\|T(t+h)x - T(t)x\| \leqslant \|T(t)\|\|T(h)x - x\| \leqslant Me^{\omega\, t}\|T(h)x - x\|$$

and then

$$\lim_{h \to 0^+} \|T(t+h)x - T(t)x\| = 0.$$

On the other hand, if $t - h \geqslant 0$, we have also

$$\|T(t-h)x - T(t)x\| \leqslant \|T(t-h)\|\|x - T(h)x\| \leqslant Me^{\omega\,(t-h)}\|\|x - T(h)x\|\,.$$

It follows

$$\lim_{h\to 0^-} \|T(t+h)x - T(t)x\| = 0$$

and the result is proved. □

The next theorem shows some interesting properties of $C^0$-semigroups.

**Theorem 4** *Let $T(t)$ be a $C_0$-semigroup and $A$ its generator. Then*

*a)* $\displaystyle\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} T(s)x\,ds = T(t)x \qquad \forall\ x \in X.$

*b)* $x \in X \implies \displaystyle\int_0^t T(s)x\,ds \in D(A)$ *and*

$$A\left(\int_0^t T(s)x\,ds\right) = T(t)x - x. \tag{1.9}$$

*c)* $x \in D(A) \implies T(t)x \in D(A)$ *and*

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax. \tag{1.10}$$

*d) for any $x \in D(A)$ we have*

$$T(t)x - T(s)x = \int_s^t T(\tau)Ax\,d\tau = \int_s^t AT(\tau)x\,d\tau. \tag{1.11}$$

**Proof.** Fix $x \in X$ and $t > 0$; given $\varepsilon > 0$, in view of the previous theorem, there exists $\delta_\varepsilon > 0$ such that

$$\|T(s)x - T(t)x\| < \varepsilon \qquad |s-t| < \delta_\varepsilon.$$

It follows that, if $|s-t| < \delta_\varepsilon$, then

$$\left\|\frac{1}{h}\int_t^{t+h}(T(s)x - T(t)x)\,ds\right\| \leqslant \frac{1}{|h|}\left|\int_t^{t+h}\|T(s)x - T(t)x\|ds\right| < \varepsilon,$$

i.e. a) (it is obvious how to change the proof for $t = 0$).

As far b) is concerned, fix $x \in X$ and $h > 0$. One has

$$\frac{T(h)-I}{h}\int_0^t T(s)x\,ds = \frac{1}{h}\int_0^t (T(s+h)x - T(s)x)ds = \frac{1}{h}\int_t^{t+h} T(s)x\,ds - \frac{1}{h}\int_0^h T(s)x\,ds.$$

The last term tends to $T(t)x - x$ as $h \to 0^+$ because of a), the integral in b) belongs to $D(A)$ and b) holds.

Let now $x \in D(A)$ and $h > 0$; we have

$$\frac{T(h) - I}{h} T(t)x = T(t) \left( \frac{T(h) - I}{h} \right) x \;\to\; T(t)Ax.$$

This shows that $T(t)x$ belongs to $D(A)$ and moreover $AT(t)x = T(t)Ax$.

We have also proved that

$$\frac{d^+}{dt} T(t)x = AT(t)x = T(t)Ax.$$

Let us consider now the left derivative. We can write

$$\frac{T(t-h)x - T(t)x}{-h} - T(t)Ax = T(t-h) \left[ \frac{x - T(h)x}{-h} \right] - T(t)Ax =$$
$$T(t-h) \left[ \frac{T(h)x - x}{h} - Ax \right] + [T(t-h) - T(t)]Ax.$$

Since $x \in D(A)$, we have

$$\lim_{h \to 0^+} \frac{T(h)x - x}{h} = Ax\,.$$

The norm $\|T(s)\|$ being bounded on the compact sets, in view of Theorem 2 (note the, differently from the uniformly continuous semigroups, $T(t)$ does not need to be continuous !), we find

$$\lim_{h \to 0^+} T(t-h) \left[ \frac{T(h)x - x}{h} - Ax \right] = 0.$$

Moreover

$$\lim_{h \to 0^+} [T(t-h) - T(t)]Ax = 0$$

and thus

$$\frac{d^-}{dt} T(t)x = T(t)Ax.$$

This proves the statement c).

Finally, (1.11) is obtained by integrating (1.10). □

We recall that the operator $A$ is closed if its graph is closed, i.e. if

$$\begin{cases} x_n \in D(A) \\ x_n \to x \\ Ax_n \to y \end{cases} \quad \Longrightarrow \quad \begin{cases} x \in D(A) \\ Ax = y. \end{cases} \tag{1.12}$$

**Theorem 5** *Let $A$ be the generator of the $C_0$-semigroup $T(t)$. Then $A$ is a densely defined closed operator.*

**Proof.** We start by proving that $D(A)$ is dense in $X$. Let $x \in X$ and define

$$x_t = \frac{1}{t}\int_0^t T(s)x\,ds.$$

From b) of previous theorem, $x_t \in D(A)$ and from a) $x_t \to x$. This means that $\overline{D(A)} = X$.

In order to prove that $A$ is a closed operator, we have to show that (1.12) holds. Since $x_n \in D(A)$, (1.11) implies

$$T(t)x_n - x_n = \int_0^t T(s)Ax_n ds.$$

Letting $n \to \infty$, one has

$$T(t)x - x = \int_0^t T(s)y\,ds$$

from which

$$\frac{T(t)x - x}{t} = \frac{1}{t}\int_0^t T(s)y\,ds.$$

As $t \to 0^+$, the right hand side tends to $y$ and thus $x \in D(A)$, $Ax = y$.

□

The next result shows that a $C_0$-semigroup is uniquely determined by its generator.

**Theorem 6** *Let $A$ and $B$ two generators of the $C_0$-semigroups $T(t)$ and $S(t)$ respectively. If $A = B$ then the two semigroups coincide, i.e. $T(t) = S(t)$ for any $t \geqslant 0$.*

**Proof.** Let $x \in D(A) = D(B)$. From (1.10) it follows

$$\begin{aligned}\frac{d}{ds}T(t-s)S(s)x &= -AT(t-s)S(s)x + T(t-s)BS(s)x = \\ &-T(t-s)AS(s)x + T(t-s)BS(s)x = 0 \qquad (0 < s < t)\end{aligned}$$

and then the function $T(t-s)S(s)x$ of the real variable $s$ is constant. In particular $T(t)x = S(t)x$, i.e. $T(t) = S(t)$ on $D(A)$. The domain $D(A)$ being dense in $X$ (see Theorem 5), it follows that $T(t) = S(t)$. □

Properties (1.7) and (1.10) imply that, as in the case of uniformly continuous semigroups, for any given $u_0 \in D(A)$ the function $u(t) = T(t)u_0$ is the only solution of the abstract Cauchy problem (1.6).

**Remark 1** It is still possible to solve the Cauchy problem (1.6) where $u_0$ is an arbitrary element of $X$. In order to do that, it is necessary to introduce a concept of generalized solution. For this we refer to [34, Ch.4].

**Example 1** An example of $C_0$-semigroup.

Let $X = C^0([0,\infty])$, where this symbol means the space of the complex valued functions defined in $[0,\infty)$ such that there exists the limit

$$\lim_{x\to+\infty} f(x).$$

The space $X$, equipped with the norm

$$\|f\|_\infty = \sup_{x\in[0,+\infty)} |f(x)| \ ,$$

becomes a Banach space (prove it !). Define the family of operators $T(t)$ $(t \geqslant 0)$ by

$$[T(t)f](x) = f(x+t).$$

Obviously, for any $t \geqslant 0$, it makes sense to consider $f(x+t)$. Moreover, $T(t)f$ is a continuous function and since

$$\lim_{x\to+\infty}[T(t)f](x) = \lim_{x\to+\infty} f(x).$$

$T(t)$ maps $X$ into itself. Let us remark that

$$\|T(t)f\|_\infty \leqslant \|f\|_\infty \,.$$

It is clear that $T(t)$ is a semigroup. Let us prove that it is a $C_0$-semigroup, i.e. that

$$\lim_{t\to 0^+} \|T(t)f - f\|_\infty = 0\,. \tag{1.13}$$

By hypothesis, there exists $\alpha \in \mathbb{C}$ to which $f(x)$ tends as $x \to +\infty$. Given $\varepsilon > 0$, there exists $K_\varepsilon > 0$ such that

$$|f(x) - \alpha| < \varepsilon \qquad \forall\ x \geqslant K_\varepsilon.$$

This implies

$$|f(x+t)-f(x)| \leqslant |f(x+t)-\alpha|+|\alpha-f(x)| < 2\,\varepsilon \qquad \forall\ x \geqslant K_\varepsilon,\ t \geqslant 0. \tag{1.14}$$

On the other hand $f$ is uniformly continuous on $[0, K_\varepsilon + 1]$ and then there exists $\delta_\varepsilon > 0$ (which can be supposed to be less than 1) such that

$$|f(x+t) - f(x)| < \varepsilon \qquad \forall\ x \in [0, K_\varepsilon],\ 0 \leqslant t < \delta_\varepsilon\,.$$

Keeping in mind (1.14), we find

$$|f(x+t) - f(x)| < 2\,\varepsilon \qquad \forall\, x \in [0,\infty),\, 0 \leqslant t < \delta_\varepsilon$$

and (1.13) is proved (note that basically we have considered a compactification of $[0,\infty)$ and showed that $f$ has to be uniformly continuous).

What is the generator $A$ of $T(t)$ and its domain $D(A)$ ?

The function $f$ belongs to $D(A)$ if and only if there exists in $X$ the limit

$$Af = \lim_{t\to 0^+} \frac{T(t)f - f}{t} = \lim_{t\to 0^+} \frac{f(t+\cdot) - f(\cdot)}{t}\,.$$

In particular,

$$Af(x) = \lim_{t\to 0^+} \frac{f(t+x) - f(x)}{t} \qquad \forall\, x \in [0,\infty)$$

and then $f$ admits the right derivative for any $x \geqslant 0$ and the right derivative, $Af$, is continuous everywhere. But then, in view of a well known result in the theory of functions of one real variable (see, e.g.., [34], p.42–43), $f$ is differentiable for any $x > 0$.

Moreover, since $Af \in X$, there exists also

$$\lim_{x\to+\infty} f'(x).$$

Therefore $D(A)$ is contained in the space of the functions $f \in C^1([0,\infty))$ such that $f' \in X$ and $Af = f'$.

Vice versa, if $f \in C^1([0,\infty))$ and $f' \in X$, then $f \in D(A)$. In fact, we have

$$\frac{f(x+t) - f(x)}{t} - f'(x) = \frac{1}{t}\int_x^{x+t} [f'(u) - f'(x)]\, du\,.$$

But, since $f' \in X$, $f'$ is uniformly continuous and then $|f'(u) - f'(x)| < \varepsilon$ for $|x-u|$ less than a certain $\delta_\varepsilon$. Thus

$$\left|\frac{f(x+t) - f(x)}{t} - f'(x)\right| \leqslant \frac{1}{t}\int_x^{x+t} |f'(u) - f'(x)|\, du < \varepsilon \qquad 0 < t < \delta_\varepsilon$$

and this shows that

$$\lim_{t\to 0^+} \left\|\frac{f(\cdot+t) - f}{t} - f'\right\|_\infty = 0,$$

i.e. $f \in D(A)$, $Af = f'$. We have thus proved that

$$D(A) = \{f \in X \mid \exists\, f',\ f' \in X\}, \qquad Af = f'.$$

### 1.1.3 The Hille-Yosida Theorem

For us it will be particular important the case in which we can choose $\omega = 0$ and $M = 1$ in the inequality (1.7). In this case we have

$$\|T(t)\| \leqslant 1$$

and the semigroup is said to be a contraction semigroup or a semigroup of contractions. If the operator $A$ is the generator of a $C^0$-semigroup of contractions, the solution of the Cauchy problem (1.6) satisfies the estimates

$$\|u(t)\| \leqslant \|u_0\| . \tag{1.15}$$

If $X = L^p(\Omega)$, (1.15) coincides with (6). This why we would like to have conditions under which $A$ is such a generator. A first answer is given by the famous Hille-Yosida Theorem. For the proof we need some lemmas.

We recall that the resolvent of a linear operator $A$, $\varrho(A)$, is the set of the complex numbers $\lambda$ such that there exists the resolvent operator $(\lambda I - A)^{-1}$ and it is continuous. The spectrum $\sigma(A)$ of the operator $A$ is defined as $\mathbb{C} \setminus \varrho(A)$. By $R(\lambda : A)$ ($\lambda \in \varrho(A)$), or shortly $R_\lambda$, we denote the operator $(\lambda I - A)^{-1}$.

**Lemma 1** *Let $A$ be a linear operator. If $\varrho(A) \neq \emptyset$ then $A$ is closed.*

**Proof.** Suppose that the sequence $\{x_n\}$, contained in the domain of $A$, is such that $x_n \to x$ and $Ax_n \to y$.

Given $\lambda \in \varrho(A)$, we get

$$\lambda x_n - Ax_n \to \lambda x - y$$

and then

$$x_n \to R_\lambda(\lambda x - y).$$

Because of the uniqueness of the limit, we find

$$x = R_\lambda(\lambda x - y).$$

This shows that $x \in D(\lambda I - A) = D(A)$ and $(\lambda I - A)x = \lambda x - y$, i.e. $Ax = y$ and the lemma is proved (see (1.12)). □

**Lemma 2** *If $\lambda, \mu \in \varrho(A)$ we have the resolvent identity* [(1)]

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu . \tag{1.16}$$

*Moreover the operators $R_\lambda$ and $R_\mu$ commute: $R_\lambda R_\mu = R_\mu R_\lambda$.*

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(1) Note that if $\lambda, \mu, A$ were numbers, we would have

$$\frac{1}{\lambda - A} - \frac{1}{\mu - A} = \frac{\mu - \lambda}{(\lambda - A)(\mu - A)} .$$

**Proof.** We have

$$(\lambda I - A)[R_\lambda - R_\mu](\mu I - A) = [I - (\lambda I - A)R_\mu](\mu I - A) =$$
$$(\mu I - A) - (\lambda I - A) = (\mu - \lambda)I$$

and (1.16) follows. By exchanging $\lambda$ and $\mu$ we prove the commutativity. □

Let $A$ be an operator such that[(2)] $\mathbb{R}^+ \subset \varrho(A)$; we can then consider $R_\lambda$ for any $\lambda > 0$. The operator $A_\lambda = \lambda A R_\lambda$ is called the Yosida approximation of $A$. Even if $A$ is unbounded, the operator $A_\lambda$ is a linear and continuous operator defined all over $X$. The linearity is obvious, while $A_\lambda$ is continuous, because

$$(\lambda I - A)R_\lambda = I \qquad \Longleftrightarrow \qquad AR_\lambda = \lambda R_\lambda - I$$

and then

$$A_\lambda = \lambda^2 R_\lambda - \lambda I. \tag{1.17}$$

The next lemma shows in which sense $A_\lambda$ is an approximation of $A$.

**Lemma 3** *Let $A$ be a densely defined operator such that $\mathbb{R}^+ \subset \varrho(A)$ and*

$$\|R_\lambda\| \leqslant \frac{1}{\lambda} \qquad \forall\, \lambda > 0.$$

*Then*

$$\lim_{\lambda\to 0^+} \lambda R_\lambda x = x \qquad \forall\, x \in X \tag{1.18}$$

$$\lim_{\lambda\to 0^+} A_\lambda x = Ax \qquad \forall\, x \in D(A). \tag{1.19}$$

**Proof.** Suppose first $x \in D(A)$; since

$$R_\lambda(\lambda I - A)x = x$$

we get

$$\lambda R_\lambda x = x + R_\lambda A x.$$

Limit (1.18) for any $x \in D(A)$ follows from the inequality

$$\|R_\lambda A x\| \leqslant \frac{1}{\lambda} \|Ax\|.$$

Let now $x \in X$; given $\varepsilon > 0$, by hypothesis there exists $y \in D(A)$ such that $\|x - y\| < \varepsilon$. Since

$$\|\lambda R_\lambda x - x\| \leqslant \|\lambda R_\lambda x - \lambda R_\lambda y\| + \|\lambda R_\lambda y - y\| + \|y - x\| \leqslant 2\|x - y\| + \|\lambda R_\lambda y - y\|$$

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[(2)] By $\mathbb{R}^+$ we denote the set $\{\lambda \in \mathbb{R} \mid \lambda > 0\}$.

we have

$$\limsup_{\lambda\to\infty} \|\lambda R_\lambda x - x\| \leqslant 2\varepsilon\,.$$

Because of the arbitrariness of $\varepsilon$, (1.18) is proved for any $x \in X$.

As far as (1.19) is concerned, formula (1.18) clearly implies

$$\lim_{\lambda\to 0^+} \lambda R_\lambda A x = Ax \qquad \forall\, x \in D(A)$$

and (1.19) follows, because $R_\lambda$ commute with on the domain of $A$ [(3)] . □

**Lemma 4** *Let $U(t)$ and $V(t)$ be two contraction semigroups whose generators are $C$ and $D$ respectively. Suppose that $U(t)$ and $V(t)$ commute, i.e. $U(t)V(s) = V(s)U(t)$ for any $s, t \geqslant 0$. Then*

$$\|U(t)x - V(t)x\| \leqslant t\,\|Cx - Dx\| \qquad \forall\, x \in D(C) \cap D(D). \tag{1.20}$$

**Proof.** First observe that from the commutativity of $U(t)$ and $V(t)$ it follows that also the generator of $U(t)$, $C$, commute with $V(t)$. Specifically, let $x \in D(C)$; we can write

$$\frac{U(h) - I}{h} V(t)x = V(t)\frac{U(h)x - x}{h}\,;$$

therefore $x \in D(C) \Rightarrow V(t)x \in D(C)$ and

$$CV(t)x = V(t)Cx$$

(for any $t \geqslant 0$). Keeping in mind (1.10), we have for any $x \in D(C) \cap D(D)$

$$U(t)x - V(t)x = \int_0^t \frac{d}{ds}[U(s)V(t-s)x]\, ds =$$
$$\int_0^t [U(s)CV(t-s)x - U(s)V(t-s)Dx]\, ds$$

and then

$$U(t)x - V(t)x = \int_0^t [U(s)V(t-s)Cx - U(s)V(t-s)Dx]\, ds\,.$$

This implies

$$\|U(t)x - V(t)x\| \leqslant \int_0^t \|U(s)\|\,\|V(t-s)\|\,\|Cx - Dx\|\, ds \leqslant t\,\|Cx - Dx\|\,.$$

---

[(3)]In fact $(\lambda I - A)R_\lambda = R_\lambda(\lambda I - A) = I$ on $D(A)$ and then $AR_\lambda x = R_\lambda A x$ for any $x \in D(A)$.

□

We are now in a position to prove the famous Hille-Yosida theorem.

**Theorem 7 (Hille-Yosida)** *A linear operator $A$ generates a $C^0$ semigroup of contractions $T(t)$ if, and only if,*

*(i) $A$ is closed and $D(A)$ is dense in $X$;*

*(ii) $\varrho(A) \supset \varrho^+$ and*

$$\|R_\lambda\| \leqslant \frac{1}{\lambda}, \quad \forall\, \lambda > 0. \tag{1.21}$$

**Proof.** Suppose that $A$ is the generator of a contraction semigroups. We know already that $A$ is a densely defined and closed operator (see theorem 5).

In order to prove b), observe that for any $\lambda > 0$, $e^{-\lambda t}T(t)$ is a contraction semigroup, because

$$\|e^{-\lambda t}T(t)\| = e^{-\lambda t}\|T(t)\| \leqslant 1.$$

The generator of $e^{-\lambda t}T(t)$ is $A - \lambda I$; in fact

$$\frac{e^{-\lambda t}T(t)x - x}{t} = e^{-\lambda t}\frac{T(t)x - x}{t} + \frac{e^{-\lambda t} - 1}{t}T(t)x$$

and then

$$\lim_{t\to 0^+} \frac{e^{-\lambda t}T(t)x - x}{t} = Ax - \lambda x, \qquad \forall\, x \in D(A) = D(A - \lambda I).$$

We can apply (1.9) and (1.11) to $A - \lambda I$, obtaining

$$e^{-\lambda t}T(t)x - x = (A - \lambda I)\left(\int_0^t e^{-\lambda s}T(s)x\,ds\right) \qquad \forall\, x \in X;$$

$$e^{-\lambda t}T(t)x - x = \int_0^t e^{-\lambda s}T(s)(A - \lambda I)x\,ds \qquad \forall\, x \in D(A).$$

Letting $t \to +\infty$ we find

$$x = (\lambda I - A)\left(\int_0^\infty e^{-\lambda s}T(s)x\,ds\right) \qquad \forall\, x \in X$$

$$x = \int_0^\infty e^{-\lambda s}T(s)(\lambda I - A)x\,ds \qquad \forall\, x \in D(A).$$

The first equality shows that the range of $\lambda I - A$ is all of $X$, while the second one implies that $\lambda I - A$ is injective. Then there exists $(\lambda I - A)^{-1}$ and, setting $y = (\lambda I - A)x$,

$$(\lambda I - A)^{-1}y = \int_0^\infty e^{-\lambda s}T(s)y\,ds\,.$$

This leads to

$$\|(\lambda I-A)^{-1}y\| \leqslant \int_0^\infty e^{-\lambda s}\|T(s)\|\,\|y\|\,ds \leqslant \|y\|\int_0^\infty e^{-\lambda s}ds = \frac{\|y\|}{\lambda}.$$

Then we have proved that $(\lambda I-A)^{-1}$ is continuous (i.e. $\lambda \in \varrho(A)$) and (1.21) holds.

Vice versa, let $A$ satisfy a) and b). For any $\lambda > 0$ we can consider the Yosida approximation $A_\lambda$ and the limit (1.19) holds. The operator $A_\lambda$, being linear and continuous, is the generator of a semigroup uniformly continuous $e^{tA_\lambda}$. This is a contractive semigroup, because, keeping in mind (1.17), we have

$$\|e^{tA_\lambda}\| = \|e^{-\lambda t I}e^{\lambda^2 t R_\lambda}\| \leqslant e^{-\lambda t}e^{\lambda^2 t\|R_\lambda\|} \leqslant e^{-\lambda t}e^{\lambda t} = 1$$

(here we used that $\lambda\|R_\lambda\| \leqslant 1$).

Define

$$T(t)x = \lim_{\lambda\to\infty} e^{tA_\lambda}x \qquad \forall\, x \in X. \tag{1.22}$$

To see that this definition makes sense, we have to show that this limit does exist for any $x \in X$.

Let us start by first showing that this limit does exist for any $x \in D(A)$: given $\lambda, \mu > 0$, it is easy to show that the contraction semigroups $e^{tA_\lambda}$ and $e^{tA_\mu}$ commute and then we can apply Lemma 4. From (1.20) it follows

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\| \leqslant t\,\|A_\lambda x - A_\mu x\| \qquad \forall\, x \in D(A).$$

But, if $x \in D(A)$, (1.19) shows that $A_\lambda x \to Ax$ and then, given $\varepsilon > 0$, there exists $\lambda_\varepsilon > 0$ such that, for $\lambda, \mu > \lambda_\varepsilon$, one has $\|A_\lambda x - A_\mu x\| < \varepsilon$. Thus

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\| \leqslant t\,\varepsilon.$$

This shows that the limit (1.22) does exist for any $x \in D(A)$. Let now $x \in X$; given $\varepsilon > 0$ let $y \in D(A)$such that $\|x-y\| < \varepsilon$. We have

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\| \leqslant \|e^{tA_\lambda}x - e^{tA_\lambda}y\| + \|e^{tA_\lambda}y - e^{tA_\mu}y\| + \|e^{tA_\mu}y - e^{tA_\mu}x\| \leqslant \\ 2\|x-y\| + \|e^{tA_\lambda}y - e^{tA_\mu}y\|$$

and then there exists $\lambda_\varepsilon$ such that, for $\mu > \lambda_\varepsilon$, one has

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\| \leqslant 3\varepsilon'.$$

Thus there exists the limit (1.22) for any $x \in X$.

(1.22) implies also

$$\lim_{\lambda\to\infty}\|T(t)x\| = \lim_{\lambda\to\infty}\|e^{tA_\lambda}x\| \leqslant \|x\| \qquad \forall\, x \in X$$

and then $\|T(t)\| \leqslant 1$.

From (1.22) easily follows that $T(0) = I$, $T(t+s) = T(t)T(s)$, i.e. $T(t)$ is a semigroup. Let us show that $T(t)$ is continuous:

$$\lim_{t\to 0^+} T(t)x = x \qquad \forall\, x \in X \tag{1.23}$$

We prove first (1.23) when $x \in D(A)$. In fact, if $x \in D(A)$, we have

$$e^{tA_\lambda}x - x = \int_0^t \frac{d}{ds}[e^{sA_\lambda}x]\, ds = \int_0^t e^{sA_\lambda}A_\lambda x\, ds\,,$$

from which, letting $\lambda \to \infty$ and keeping in mind (1.19), it follows

$$T(t)x - x = \int_0^t T(s)Ax\, ds \tag{1.24}$$

(invoking the dominated convergence theorem, we can pass the limit under the integral sign, because $\|T(s)Ax\| \leqslant \|Ax\|$). We have then

$$\|T(t)x - x\| \leqslant \int_0^t \|T(s)Ax\|\, ds \leqslant t\|Ax\|$$

and (1.23) follows for any $x \in D(A)$. The density of $D(A)$ in $X$ implies the result for any $x \in X$.

We have shown that is a contractive $C^0$-semigroup. To complete the proof, we have to show that $A$ is the generator of $T(t)$.

Let $B$ be the generator of $T(t)$; we have to show that $A = B$.

Dividing (1.24) by $t$ we get

$$\frac{T(t)x - x}{t} = \frac{1}{t}\int_0^t T(s)Ax\, ds \qquad \forall\, x \in D(A)$$

and then the limit (1.1) exists and

$$\lim_{t\to 0^+} \frac{T(t)x - x}{t} = T(t)Ax \qquad \forall\, x \in D(A).$$

This shows that $D(A) \subset D(B)$ and $Bx = Ax$ on $D(A)$. On the other hand $I \in \varrho(A)$ (by hypothesis) and $I \in \varrho(B)$ ( because of the necessity part of the present theorem). Therefore $(I - A)^{-1}$ and $(I - B)^{-1}$ do exist and are continuous. In particular, $I - A$ and $I - B$ are surjective operators. Thus

$$(I - B)D(A) = (I - A)D(A) = X = (I - B)D(B)$$

from which it follows: $D(A) = D(B)$ and then $A = B$. □

Since $A$ can be unbounded, the semigroup generated by $A$ cannot be given by (1.3), because the series in (1.4) does not converge. Nevertheless, we can still prove that the semigroup is given by $e^{tA}$, provided this exponential is understood in a generalized sense. This is shown by the next result.

**Lemma 5** *If $A$ is the generator of a $C^0$-semigroup, then $T(t)$ is given by* (1.3), *where this exponential is understood as*

$$e^{tA}x = \lim_{\lambda\to\infty} e^{tA_\lambda}x, \qquad x \in X.$$

**Proof.** In the proof of the Hille-Yosida Theorem, we have seen that there exists the limit

$$\lim_{\lambda\to\infty} e^{tA_\lambda}x, \qquad \forall\, x \in X$$

and that it defines a semigroup $S(t)$, whose generator is $A$.

Both the semigroups $T(t)$ and $S(t)$ being generated by $A$, Theorem 6 implies $T(t) \equiv S(t)$. □

Another interesting formula is the following one, which shows that the resolvent can be considered as the Laplace transform of the semigroup

$$R_\lambda u = \int_0^\infty e^{-\lambda t}[T(t)u]\, dt \qquad (\mathscr{R}e\, \lambda > \omega).$$

From the Hille-Yosida Theorem, one can obtain also the following characterization of the generators of $C^0$-semigroups, where $M$ and $\omega$ are the constants appearing in (1.7)

**Theorem 8** *A linear operator $A$ generates a $C^0$ semigroup $T(t)$ if, and only if,*

*(i) $A$ is closed and $D(A)$ is dense in $X$;*

*(ii) $\varrho(A) \supset \{\lambda \in \varrho \,|\, \lambda > \omega\}$ and*

$$\|R_\lambda^n\| \leqslant \frac{M}{(\lambda-\omega)^n} \quad \forall\, \lambda > \omega,\ n = 1, 2, \ldots.$$

## 1.2 The dissipativity in an abstract setting

### 1.2.1 Dissipative operators on Banach spaces

Let $X$ be a (complex) Banach space and denote by $X^*$ its (topological) dual space.

Given $x \in X$, denote by $\mathscr{I}(x)$ the set

$$\mathscr{I}(x) = \{x^* \in X^* \mid \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}.$$

The set $\mathscr{I}(x)$ is called *the dual set of* $x$.

It is not difficult to prove that, for any $x \in X$, $\mathscr{I}(x)$ is not empty. In fact, if $x = 0$, $\mathscr{I}(0) = \{0\}$. If $x \neq 0$, in view of the Hahn-Banach Theorem, there exists $f \in X^*$ such that

$$\langle f, x\rangle = \|x\|, \qquad \|f\| = 1.$$

Thus $x^* = \|x\|\, f$ belongs to $\mathscr{I}(x)$, since

$$\langle x^*, x\rangle = \|x\|\langle f, x\rangle = \|x\|^2, \qquad \|x^*\| = \|x\|\,\|f\| = \|x\|.$$

Generally speaking, the set $\mathscr{I}(x)$ can contain more than one element. This does not happen if $X^*$ is strictly convex, in particular if $X$ is a Hilbert space.

**Lemma 6** *If $X^*$ is strictly convex, for any $x \in X$ the set $\mathscr{I}(x)$ contains only one element.*

**Proof.** Since $\mathscr{I}(0) = \{0\}$, the result is true if $x = 0$.

Let now $x \neq 0$ and let $f, g$ be in $\mathscr{I}(x)$; let us prove that

$$\frac{f+g}{2} \in \mathscr{I}(x). \tag{1.25}$$

In fact, we have

$$\langle f, x\rangle = \|x\|^2 = \|f\|^2, \qquad \langle g, x\rangle = \|x\|^2 = \|g\|^2$$

and therefore

$$\left\langle \frac{f+g}{2}, x \right\rangle = \frac{1}{2}\langle f, x\rangle + \frac{1}{2}\langle g, x\rangle = \|x\|^2.$$

This implies

$$\|x\|^2 \leqslant \frac{1}{2}\|f+g\|\,\|x\|$$

from which

$$\|x\| \leqslant \frac{1}{2}\|f+g\|\,.$$

On the other hand, since $\|f+g\| \leqslant \|f\| + \|g\| = 2\,\|x\|$, we have

$$\frac{1}{2}\|f+g\| = \|x\|$$

and this proves (1.25).

We have thus

$$\|f+g\| = 2\,\|x\| = \|f\| + \|g\|\,.$$

Because of the strictly convexity of $X^*$, this implies that $f$ and $g$ are linearly dependent, i.e. there exists $(a,b) \neq (0,0)$ such that $af + bg = 0$ a.e..

Supposing $a \neq 0$, we find $f = \lambda\, g$. Since

$$\langle g, x\rangle = \|x\|^2 = \langle f, x\rangle = \lambda\langle g, x\rangle$$

we have $\lambda = 1$ (note that $x \neq 0$ and then $\langle g, x\rangle \neq 0$), i.e. $f = g$. □

Let us determine the dual set $\mathscr{I}(x)$ in the particular case of the $L^p$ spaces. Since the spaces $L^p(\Omega)$ $(1 < p < \infty)$ are strictly convex, the dual set $\mathscr{I}(f)$ contains only one element $f^*$. Let us look for $f^*$ in the following form

$$f^*(x) \begin{cases} = c_f \overline{f(x)} |f(x)|^\alpha & \text{if } f(x) \neq 0 \\ = 0 & \text{if } f(x) = 0 \end{cases}$$

where $c_f$ and $\alpha$ are to be determined.

Since $f^*$ has to belong to $L^q$, and since

$$|f^*(x)|^q = c_f^q |f(x)|^{q(\alpha+1)}$$

where $f \neq 0$, we must have $q(\alpha+1) = p$, i.e. $\alpha = \frac{p}{q} - 1 = p - 2$.

Imposing the condition $\langle f^*, f\rangle = \|f\|_p^2$ leads to

$$\|f\|_p^2 = \langle f^*, f\rangle = c_f \int_{f\neq 0} |f|^{\alpha+2} dx = c_f \int_\Omega |f|^p dx = c_f \|f\|_p^p$$

and then

$$c_f = \|f\|_p^{2-p}.$$

Let us prove that we have also $\|f^*\|_q = \|f\|_p$. In fact, since $|f^*|^q = c_f^q |f|^{q(\alpha+1)} = c_f^q |f|^p$ (where $f \neq 0$), we have

$$\int_\Omega |f^*|^q dx = c_f^q \int_{f\neq 0} |f|^p dx = \|f\|_p^{q(2-p)+p} = \|f\|_p^q.$$

We have then proved

**Lemma 7** *Let $X = L^p(\Omega)$ $(1 < p < \infty)$. The dual set $\mathscr{I}(f)$ contains only the element $f^*$, where*

$$f^*(x) \begin{cases} = \|f\|_p^{2-p} \overline{f(x)}\, |f(x)|^{p-2} & \textit{if } f(x) \neq 0 \\ = 0 & \textit{if } f(x) = 0. \end{cases}$$

**Remark 2** If $p = 1$, we can take

$$f^*(x) \begin{cases} = \|f\|_1 \dfrac{\overline{f(x)}}{|f(x)|} & \text{if } f(x) \neq 0 \\ = \psi(x) & \text{if } f(x) = 0, \end{cases}$$

where $\psi$ is any measurable function such that $|\psi(x)| \leqslant \|f\|_1$ a.e..

We leave the proof to the reader. This shows that there are infinite functions $f^*$ in $\mathscr{I}(f)$, provided that the set $\{x \in \Omega \,|\, f(x) = 0\}$ has positive measure and $\|f\|_1 > 0$.

Let $A : D(A) \subset X \to X$ a linear operator, $X$ being a (complex) Banach space. *$A$ is said to be dissipative if, for any $x \in D(A)$, there exists $x^* \in \mathscr{I}(x)$ such that*

$$\mathscr{R}e\, \langle x^*, x\rangle \leqslant 0.$$

**Remark 3** Let $A$ be a linear operator defined on a subspace $D(A)$ contained in $L^p(\Omega)$, $\Omega$ being a domain in $\mathbb{R}^n$ $(1 < p < \infty)$. Thanks to Lemma 7, the operator $A$ is dissipative with respect to the $L^p$-norm, briefly is $L^p$-dissipative, if, and only if,

$$\mathscr{R}e \int_\Omega \langle Au, u\rangle |u|^{p-2} dx \leqslant 0, \quad \forall\, u \in D(A),$$

where the integral is extended on the set $\{x \in \Omega \mid u(x) \neq 0\}$.

A lemma which plays a key role is the following one.

**Lemma 8** *Let $x, y \in X$. The inequality*

$$\|x\| \leqslant \|x - \alpha y\| \tag{1.26}$$

*holds for any $\alpha > 0$ if, and only if, there exists $\varphi \in \mathscr{I}(x)$ such that*

$$\mathscr{R}e\langle \varphi, y\rangle \leqslant 0. \tag{1.27}$$

**Proof.** If $x = 0$ the result is trivial, since $\mathscr{I}(0) = \{0\}$. Let $x \neq 0$.

If (1.27) is true, for any $\alpha > 0$ we may write

$$\|x\|^2 = \langle \varphi, x\rangle \leqslant \langle \varphi, x\rangle - \alpha\, \mathscr{R}e\langle \varphi, y\rangle = \mathscr{R}e\langle \varphi, x - \alpha y\rangle \leqslant \|x\|\, \|x - \alpha y\|$$

and (1.26) follows.

Conversely, let us suppose that (1.26) holds for any $\alpha > 0$. Let $\varphi_\alpha$ be an element of $\mathscr{I}(x - \alpha y)$ and define $\psi_\alpha = \varphi_\alpha / \|\varphi_\alpha\|$. Note that $\varphi_\alpha \neq 0$, because $\|x\| \leqslant \|x - \alpha y\| = \|\varphi_\alpha\|$ and we are assuming $x \neq 0$.

Moreover

$$\langle\psi_\alpha, x-\alpha y\rangle = \langle\varphi_\alpha, x-\alpha y\rangle/\|\varphi_\alpha\| = \|x-\alpha y\| \geqslant \|x\|\,. \tag{1.28}$$

Because of the Banach-Alaoglu Theorem, we can find a sequence $\{\alpha_n\}$ of positive numbers such that $\alpha_n \to 0$ and $\psi_{\alpha_n} \stackrel{*}{\rightharpoonup} \psi_0$, with

$$\|\psi_0\| \leqslant 1. \tag{1.29}$$

Putting $\alpha = \alpha_n$ in (1.28) and letting $n \to \infty$, we find [(4)]

$$\|x\| = \langle\psi_o, x\rangle \leqslant \|\psi_0\|\,\|x\|$$

from which, keeping in mind (1.29), we find

$$\|\psi_o\| = 1.$$

Define $\varphi = \|x\|\psi_0$ . Since

$$\langle\varphi, x\rangle = \|x\|\,\langle\psi_o, x\rangle = \|x\|^2 = \|\varphi\|^2$$

we have $\varphi \in \mathscr{I}(x)$. Moreover, the inequality

$$\|x\| \leqslant \|x-\alpha y\| = \mathscr{R}e\langle\psi_a, x\rangle - \alpha\,\mathscr{R}e\langle\psi_a, y\rangle \leqslant \|x\| - \alpha\,\mathscr{R}e\langle\psi_a, y\rangle$$

shows that

$$\mathscr{R}e\langle\psi_a, y\rangle \leqslant 0\,.$$

Putting $\alpha = \alpha_n$ and letting $n \to \infty$, we find (1.27). □

This lemma has some interesting consequences.

**Corollary 1** *The linear operator $A$ is dissipative if and only if, for any $x \in D(A)$, we have*

$$\|x\| \leqslant \|x - \alpha A x\| \tag{1.30}$$

*for any $\alpha > 0$.*

**Proof.** The operator $A$ is dissipative if, and only if, for any $x \in D(A)$, there exists $\varphi \in \mathscr{I}(x)$ such that $\mathscr{R}e\langle\varphi, Ax\rangle \leqslant 0$. Fixed $x \in D(A)$, Lemma 8 (where $y = Ax$) shows that this happens if and only if (1.30) holds for any $\alpha > 0$. □

The operator $A$ is said to be $m$-dissipative if $A$ is dissipative and $\varrho(A) \cap \mathbb{R}^+ \neq \emptyset$.

(4) Note that, if $x_n \to x_0$ and $\psi_n \stackrel{*}{\rightharpoonup} \psi_0$, then $\langle\psi_n, x_n\rangle \to \langle\psi, x\rangle$.

**Corollary 2** *The operator $A$ is $m$-dissipative if, and only if, $A$ is dissipative and there exists $\lambda > 0$ such that $\mathscr{R}(\lambda I - A) = X$.*

**Proof.** If $A$ is dissipative and $\mathscr{R}(\lambda I - A) = X$, then $(\lambda I - A)^{-1}$ does exist and is continuous, in view of (1.30). This shows that $A$ is $m$-dissipative. The converse is obvious. □

**Corollary 3** *If $A$ is closed and dissipative, then for any $\lambda > 0$ the range $\mathscr{R}(\lambda I - A)$ is closed.*

**Proof.** Let $y_n$ be a sequence in $\mathscr{R}(\lambda I - A)$ such that $y_n \to y_0$. We can write $y_n = \lambda x_n - A x_n$, for some $x_n \in D(\lambda I - A) = D(A)$.

Because of Corollary 1, we have

$$\|x_{n+p} - x_n\| \leqslant \|(\lambda x_{n+p} - A x_{n+p}) - (\lambda x_n - A x_n)\| = \|y_{n+p} - y_n\|$$

and then $\{x_n\}$ is a Cauchy sequence in $X$. Let $x_0$ be its limit. We have $A x_n = \lambda x_n - y_n \to \lambda x_0 - y_0$. Since $A$ is a closed operator, $x_0$ belongs to $D(A)$ and $A x_0 = \lambda x_0 - y_0$, i.e. $y_0 = \lambda x_0 - A x_0$. This shows that $y_0$ belongs to $\mathscr{R}(\lambda I - A)$, i.e. that $\mathscr{R}(\lambda I - A)$ is closed. □

**Lemma 9** *Let $A$ be a linear operator and let $\mu \in \varrho(A)$. Then $\lambda \in \varrho(A)$ if and only if $U^{-1}$ belongs to $B(X)$, where*

$$U = I + (\lambda - \mu)(\mu I - A)^{-1}.$$

*In this case* (5)

$$(\lambda I - A)^{-1} = (\mu I - A)^{-1} U^{-1}.$$

**Proof.** If $U^{-1}$ does exist and is continuous, we have

$$(\lambda I - A)(\mu I - A)^{-1} U^{-1} = [(\lambda - \mu) I + (\mu I - A)](\mu I - A)^{-1} U^{-1} = \\ [(\lambda - \mu)(\mu I - A)^{-1} + I]\, U^{-1} = U\, U^{-1} = I.$$

In the same way

$$(\mu I - A)^{-1} U^{-1} (\lambda I - A) = (\mu I - A)^{-1} U^{-1} [(\lambda - \mu) I + \mu I - A] = \\ (\mu I - A)^{-1} U^{-1} [(\lambda - \mu)(\mu I - A)^{-1} + I](\mu I - A) = (\mu I - A)^{-1}(\mu I - A) = I_{D(A)}.$$

(5) Note that if $\lambda, \mu, A$ were numbers, we would have

$$\frac{1}{\lambda - A} = \frac{1}{\lambda - \mu + \mu - A} = \frac{1}{\mu - A} \, \frac{1}{1 + \frac{\lambda - \mu}{\mu - A}} \, .$$

This means that $(\lambda I - A)^{-1}$ exists, is given by $(\mu I - A)^{-1}U^{-1}$ and thus it belongs to $B(X)$.

The proof of the converse is similar. □

### 1.2.2 The Lumer-Phillips Theorem

The next theorems provides new necessary and sufficient conditions under which $A$ generates a contraction semigroup.

**Theorem 9 (Lumer-Phillips)** *If $A$ generates a $C^0$ semigroup of contractions, then*
*(i) $\overline{D(A)} = X$;*
*(ii) $A$ is dissipative. More precisely, for any $x \in D(A)$, we have*

$$\mathscr{R}e\langle x^*, Ax\rangle \leqslant 0, \forall\ x^* \in \mathscr{I}(x);$$

*(iii) $\varrho(A) \supset \varrho^+$.*
*Conversely, if*
*(i') $\overline{D(A)} = X$;*
*(ii') $A$ is dissipative;*
*(iii') $\varrho(A) \cap \varrho^+ \neq \emptyset$,*
*then $A$ generates a $C^0$ semigroup of contractions.*

**Proof.** Because of Hille-Yosida Theorem 7, the operator $A$ generates a $C^0$-semigroup of contractions if and only if the following conditions are satisfied:
(a) $A$ is closed;
(b) $\overline{D(A)} = X$;
(c) $\varrho(A) \supset \mathbb{R}^+$;
(d) $\|R_\lambda\| \leqslant \dfrac{1}{\lambda} \qquad \forall\ \lambda > 0$.

Let us suppose that $A$ generates the $C^0$ semigroup of contractions $T(t)$. Since (a)-(d) hold true, conditions (i) and (iii) are certainly satisfied. In order to prove (ii), let $x^*$ denote any element in $\mathscr{I}(x)$. We have

$$\langle x^*, T(t)x - x\rangle = \langle x^*, T(t)x\rangle - \|x\|^2$$

and since

$$|\langle x^*, T(t)x\rangle| \leqslant \|x^*\|\,\|T(t)x\| \leqslant \|x\|^2,$$

we find

$$\mathscr{R}e\langle x^*, T(t)x - x\rangle = \mathscr{R}e\langle x^*, T(t)x\rangle - \|x\|^2 \leqslant 0.$$

Supposing $x \in D(A)$, dividing by $t$ and letting $t \to 0^+$, we get

$$\mathscr{R}e\langle x^*, Ax\rangle \leqslant 0$$

and (ii) is proved.

Conversely, let $A$ be an operator satisfying (i')-(iii'). Condition (b) is obviously true.

Condition (a) follows from the fact that $\varrho(A) \neq \emptyset$ (see Lemma 1).

Let now $\mu \in \varrho(A) \cap \mathbb{R}^+$ and $\alpha = \frac{1}{\mu}$; since

$$I - \alpha A = I - \frac{1}{\mu}A = \frac{1}{\mu}(\mu I - A)$$

the existence of $(\mu I - A)^{-1}$ implies that $(I - \alpha A)^{-1}$ does exist and

$$(I - \alpha A)^{-1} = \mu\,(\mu I - A)^{-1}.$$

Because of the dissipativity of $A$ we have (see Corollary 1) $\|(I-\alpha A)^{-1}\| \leqslant 1$, i.e.

$$\|(\mu I - A)^{-1}\| \leqslant \frac{1}{\mu} \tag{1.31}$$

If we choose $\lambda$ such that $|\lambda - \mu| < \mu$, we get

$$\|(\lambda - \mu)(\mu I - A)^{-1}\| \leqslant \frac{|\lambda - \mu|}{\mu} < 1$$

and then the operator $I + (\lambda - \mu)(\mu I - A)^{-1}$ is invertible [6].

---

[6] If $B$ is a linear and continuous operator such that $\|I - B\| < 1$, it is invertible and

$$B^{-1} = \sum_{n=0}^{\infty}(I - B)^n.$$

This follows from a general result holding in an algebra $\mathscr{B}$ with unity $e$: *if $x \in \mathscr{B}$ is such that $\|x\| < 1$, then $e - x$ is invertible and*

$$(e - x)^{-1} = \sum_{n=0}^{\infty} x^n.$$

Indeed, for any integer $s$ we have

$$(e - x)\sum_{n=0}^{s} x^n = e - x^{s+1} = \sum_{n=0}^{s} x^n (e - x).$$

Letting $s \to \infty$,

$$(e - x)\sum_{n=0}^{\infty} x^n = e = \sum_{n=0}^{\infty} x^n (e - x).$$

Lemma 9 assures that $\lambda \in \varrho(A)$. We have shown that $\mu \in \varrho(A) \cap \mathbb{R}^+$ implies that all the interval $(0, 2\mu)$ [7] is contained in $\varrho(A)$. Replacing $\mu$ by $\frac{3}{2}\mu$ we find that $\varrho(A)$ contains also every $\lambda > 0$ such that

$$\left|\lambda - \frac{3}{2}\mu\right| < \frac{3}{2}\mu,$$

i.e. $(0, 3\mu) \subset \varrho(A)$. By iterating the argument it follows that $\mathbb{R}^+ \subset \varrho(A)$ and (c) is proved.

Assertion (d) follows from (1.31), taking into account that $\mathbb{R}^+ \subset \varrho(A)$. □

Lumer-Phillips Theorem can be stated in an equivalent form by using the concept of $m$-dissipativity.

**Theorem 10 (Lumer-Phillips)** *The operator $A$ generates a $C^0$ semigroup of contractions if, and only if, $A$ is m-dissipative and $\overline{D(A)} = X$.*

**Proof.** The proof follows immediately from Theorem 9 and the definition of $m$-dissipative operator. □

The following theorem provides a useful sufficient condition for the generation of a semigroup of contractions.

**Theorem 11** *Let $A$ be a closed operator with $\overline{D(A)} = X$. If $A$ and $A^*$ are dissipative, then $A$ generates a $C^0$ semigroup of contractions.*

**Proof.** Because of Lumer-Phillips Theorem 10, we have to prove that $A$ is $m$-dissipative. Since $A$ is dissipative by hypothesis, all we have to show is that there exists $\lambda > 0$ such that $\mathscr{R}(\lambda I - A) = X$ (see Corollary 2).

Let $\lambda$ be a positive number. In view of Corollary 3, $\mathscr{R}(\lambda I - A)$ is closed. If $\mathscr{R}(\lambda I - A) \neq X$, we can find $\varphi \in X^*$ such that $\varphi \neq 0$ and

$$\langle \varphi, \lambda x - Ax \rangle = 0, \qquad \forall\, x \in D(\lambda I - A) = D(A). \tag{1.32}$$

Condition (1.32) can be written as

$$\langle \lambda\varphi - A^*\varphi, x \rangle = 0, \qquad \forall\, x \in D(A).$$

From the density of $D(A)$ it follows $\lambda\varphi - A^*\varphi = 0$.

On the other hand, in view of the dissipativity of $A^*$ and Corollary 1, we have $\|\varphi\| \leqslant \|\lambda\varphi - A^*\varphi\|$. Then $\varphi = 0$ and this is absurd. □

[7] The inequality $|\lambda - \mu| < \mu$ ($\mu > 0$, $\lambda \in \mathbb{C}$), means that $\lambda$ belongs to the open disc with radius $\mu$ and center $\mu$. Therefore, if we have $\lambda \in \mathbb{R}$ – as in the text – the inequality means $0 < \lambda < 2\mu$.

**Example 2 (The Convection Equation)** Let us consider in $L^2(\mathbb{R}^n)$ the operator $A$ defined as

$$D(A) = \{f \in L^2(\mathbb{R}^n) \mid V \cdot \nabla f \in L^2(\mathbb{R}^n)\}$$
$$Af = -V \cdot \nabla f,$$

where $V = (v_1, \ldots, v_n)$ is a real constant vector. The operator $A$ is $L^2$-dissipative. In fact,

$$(Af, f) = -\int_{\mathbb{R}^n} f\, V \cdot \nabla f\, dx$$

and since[8]

$$-\int_{\mathbb{R}^n} f\, V \cdot \nabla f\, dx = \int_{\mathbb{R}^n} (V \cdot \nabla f)\, f\, dx\,,$$

we find

$$(Af, f) = 0 \qquad \forall\ f \in D(A).$$

Let us show that $A$ is $m$-dissipative: $\varrho(A) \cap \mathbb{R}^+ \neq \emptyset$. Indeed, we shall prove more: $\varrho(A) \supset \mathbb{R}^+$.

Let $\lambda > 0$; given $g \in L^2(\mathbb{R}^n)$, consider the equation

$$\lambda f - Af = g\,,$$

i.e.

$$\lambda f + V \cdot \nabla f = g\,. \tag{1.33}$$

This equation has one and only one solution in $D(A)$. In fact, let

$$f(x) = \int_0^\infty e^{-\lambda s} g(x - sV)\, ds\,.$$

Suppose first that $g \in \mathring{C}^\infty(\mathbb{R}^n)$; we have

$$V \cdot \nabla f(x) = \int_0^\infty e^{-\lambda s} V \cdot \nabla_x [g(x - sV)]\, ds = -\int_0^\infty e^{-\lambda s} \frac{d}{ds}[g(x - sV)]\, ds =$$
$$-\left[e^{-\lambda s} g(x - sV)\right]_{s=0}^{s=\infty} - \lambda \int_0^\infty e^{-\lambda s} g(x - sV)\, ds = g(x) - \lambda f(x)$$

---

(8) The integration by parts can be justified in the following way. It is well known that $\mathring{C}^\infty(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$; in a similar way, one can prove that, given $f \in D(A)$, there exists a sequence $f_m \in \mathring{C}^\infty(\mathbb{R}^n)$ such that $f_m \to f$, $V \cdot \nabla f_m \to V \cdot \nabla f$ in $L^2(\mathbb{R}^n)$. Therefore

$$\int_{\mathbb{R}^n} f\, V \cdot \nabla f\, dx = \lim_{m\to\infty} \int_{\mathbb{R}^n} f_m\, V \cdot \nabla f_m\, dx =$$
$$-\lim_{m\to\infty} \int_{\mathbb{R}^n} (V \cdot \nabla f_m)\, f_m\, dx = -\int_{\mathbb{R}^n} (V \cdot \nabla f)\, f\, dx.$$

and then $f$ is solution of (1.33).

To check that $f \in D(A)$, we have to prove that $f$ and $V \cdot \nabla f$ belong to $L^2(\mathbb{R}^n)$. One has

$$|f(x)| \leqslant \int_0^\infty e^{-\frac{\lambda s}{2}} e^{-\frac{\lambda s}{2}} |g(x-sV)|\, ds \leqslant$$
$$\left(\int_0^\infty e^{-\lambda s} ds\right)^{\frac{1}{2}} \left(\int_0^\infty e^{-\lambda s} |g(x-sV)|^2 ds\right)^{\frac{1}{2}} =$$
$$\left(\frac{1}{\lambda}\right)^{\frac{1}{2}} \left(\int_0^\infty e^{-\lambda s} |g(x-sV)|^2 ds\right)^{\frac{1}{2}},$$

from which

$$\int_{\mathbb{R}^n} |f(x)|^2 dx \leqslant \frac{1}{\lambda} \int_{\mathbb{R}^n} dx \int_0^\infty e^{-\lambda s} |g(x-sV)|^2 ds\,.$$

Invoking Tonelli's Theorem

$$\int_{\mathbb{R}^n} |f(x)|^2 dx \leqslant \frac{1}{\lambda} \int_0^\infty e^{-\lambda s} ds \int_{\mathbb{R}^n} |g(x-sV)|^2 dx =$$
$$\frac{1}{\lambda} \int_0^\infty e^{-\lambda s} ds\ \|g\|^2_{L^2(\mathbb{R}^n)} = \frac{1}{\lambda^2} \|g\|^2_{L^2(\mathbb{R}^n)}$$

i.e.

$$\|f\|_{L^2(\mathbb{R}^n)} \leqslant \frac{1}{\lambda} \|g\|_{L^2(\mathbb{R}^n)}. \tag{1.34}$$

Inequality (1.34) shows not only that $f \in L^2(\mathbb{R}^n)$, but also that $V \cdot \nabla f \in L^2(\mathbb{R}^n)$. In fact, we know that $V \cdot \nabla f = g - \lambda f$ and then

$$\|V \cdot \nabla f\|_{L^2(\mathbb{R}^n)} \leqslant \|g\|_{L^2(\mathbb{R}^n)} + \lambda \|f\|_{L^2(\mathbb{R}^n)} \leqslant 2\|g\|_{L^2(\mathbb{R}^n)}. \tag{1.35}$$

Let now $g \in L^2(\mathbb{R}^n)$; there exists a sequence $\{g_n\}$ in $\mathring{C}^\infty(\Omega)$ such that $\|g_n - g\|_{L^2(\mathbb{R}^n)} \to 0$. Inequalities (1.34), (1.35) imply that, setting

$$f_n(x) = \int_0^\infty e^{-\lambda s} g_n(x-sV)\, ds\,,$$

we have $f_n \in D(A)$ and

$$\|f_n - f\|_{L^2(\mathbb{R}^n)} \to 0, \qquad \|V \cdot \nabla f_n - V \cdot \nabla f\|_{L^2(\mathbb{R}^n)} \to 0.$$

This shows that $f \in D(A)$. Moreover, from what we have just seen, $\lambda f_n + V \cdot f_n = g_n$. Letting $n \to \infty$, we get (1.33).

We have then proved that, given $g \in L^2(\mathbb{R}^n)$, there exists a solution $f \in D(A)$ of (1.33). In view of Corollary 2, $A$ is $m$-dissipative.

Because of Lumer-Phillips Theorem 10, $A$ is the generator of a contraction semigroup.

The theory previously developed provides an existence and uniqueness result for the Cauchy problem for the convection equation

$$\begin{cases} u_t(x,t) = -V \cdot \nabla_x u(x,t) \\ u(x,0) = u_0(x) \end{cases}$$

and the solution satisfies the inequality

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leqslant \|u_0\|_{L^2(\mathbb{R}^n)}.$$

# Chapter 2

# $L^p$-dissipativity of scalar second order operators

## 2.1 General results

### 2.1.1 The main lemma

By $\mathring{C}(\Omega)$ we denote the space of complex valued continuous functions having compact support in $\Omega$. Let $\mathring{C}^1(\Omega)$ consist of all the functions in $\mathring{C}(\Omega)$ having continuos partial derivatives of the first order.

In what follows, $\mathscr{A}$ is a $n \times n$ matrix function with complex valued entries $a^{hk} \in (\mathring{C}(\Omega))^*$, $\mathscr{A}^t$ is its transposed matrix and $\mathscr{A}^*$ is its adjoint matrix, i.e. $\mathscr{A}^* = \overline{\mathscr{A}}^t$.

Let $\mathbf{b} = (b_1, \dots, b_n)$ and $\mathbf{c} = (c_1, \dots, c_n)$ stand for complex valued vectors with $b_j, c_j \in (\mathring{C}(\Omega))^*$. By $a$ we mean a complex valued scalar distribution in $(\mathring{C}^1(\Omega))^*$.

We denote by $\mathscr{L}(u,v)$ the sesquilinear form

$$\mathscr{L}(u,v) = \int_\Omega (\langle \mathscr{A}\,\nabla u, \nabla v\rangle - \langle \mathbf{b}\nabla u, v\rangle + \langle u, \overline{\mathbf{c}}\nabla v\rangle - a\langle u, v\rangle)$$

defined on $\mathring{C}^1(\Omega) \times \mathring{C}^1(\Omega)$.

The integrals appearing in this definition have to be understood in a proper way. The entries $a^{hk}$ being measures, the meaning of the first term is

$$\int_\Omega \langle \mathscr{A}\,\nabla u, \nabla v\rangle = \int_\Omega \partial_k u\, \partial_h \overline{v}\, da^{hk}.$$

Similar meanings have the terms involving $\mathbf{b}$ and $\mathbf{c}$. Finally, the last term is the action of the distribution $a \in (\mathring{C}^1(\Omega))^*$ on the function $\langle u, v\rangle$ belonging to $\mathring{C}^1(\Omega)$.

The form $\mathscr{L}$ is related to the operator

$$Au = \operatorname{div}(\mathscr{A}\,\nabla u) + \mathbf{b}\nabla u + \operatorname{div}(\mathbf{c}u) + au. \tag{2.1}$$

where div denotes the divergence operator.

The operator $A$ acts from $\mathring{C}^1(\Omega)$ to $(\mathring{C}^1(\Omega))^*$ through the relation

$$\mathscr{L}(u,v) = \int_\Omega \langle Au, v\rangle$$

for any $u, v \in \mathring{C}^1(\Omega)$.

Instead of studying the dissipativity of the operator $A$, we start with the dissipativity of the form $\mathscr{L}$. Such a concept was given in [5].

The form $\mathscr{L}$ is called $L^p$*-dissipative* if for all $u \in \mathring{C}^1(\Omega)$

$$\mathscr{R}e\,\mathscr{L}(u, |u|^{p-2}u) \geqslant 0 \qquad \text{if } p \geqslant 2; \tag{2.2}$$

$$\mathscr{R}e\,\mathscr{L}(|u|^{p'-2}u, u) \geqslant 0 \qquad \text{if } 1 < p < 2. \tag{2.3}$$

The necessity of differentiating the case $1 < p < 2$ from $p \geqslant 2$ is due to the fact that $|u|^{q-2}u \in \mathring{C}^1(\Omega)$ for $q \geqslant 2$ and $u \in \mathring{C}^1(\Omega)$.

The following lemma will play a key role.

**Lemma 10** *The form $\mathscr{L}$ is $L^p$-dissipative if and only if for all $v \in \mathring{C}^1(\Omega)$*

$$\begin{gathered}\mathscr{R}e \int_\Omega \Big[\langle \mathscr{A}\,\nabla v, \nabla v\rangle - (1-2/p)\langle(\mathscr{A} - \mathscr{A}^*)\nabla(|v|), |v|^{-1}\overline{v}\nabla v\rangle - \\ (1-2/p)^2\langle \mathscr{A}\,\nabla(|v|), \nabla(|v|)\rangle\Big] + \int_\Omega \langle \mathscr{I}m(\mathbf{b}+\mathbf{c}), \mathscr{I}m(\overline{v}\nabla v)\rangle + \\ \int_\Omega \mathscr{R}e(\operatorname{div}(\mathbf{b}/p - \mathbf{c}/p') - a)|v|^2 \geqslant 0.\end{gathered} \tag{2.4}$$

*Here and in the sequel the integrand is extended by zero on the set where $v$ vanishes.*

**Proof.** The proof of this Lemma is quite technical.

Suppose that $p \geqslant 2$ and that (2.4) holds. Take $u \in \mathring{C}^1(\Omega)$ and set

$$v = |u|^{(p-2)/2}u. \tag{2.5}$$

The function $v$ belongs to $\mathring{C}^1(\Omega)$ and $|v| = |u|^{p/2}$, i.e. $|u| = |v|^{2/p}$. From 2.5 it follows also that $u = |v|^{(2-p)/p}v$.

A direct calculation shows that

$$\begin{gathered}\langle \mathscr{A}\,\nabla u, \nabla(|u|^{p-2}u)\rangle = \langle \mathscr{A}\,\nabla(|v|^{\frac{2-p}{p}}v), \nabla(|v|^{\frac{p-2}{p}}v)\rangle = \\ \Big\langle \mathscr{A}\,(\nabla v - (1-2/p)|v|^{-1}v\nabla|v|), \nabla v + (1-2/p)|v|^{-1}v\nabla|v|\Big\rangle = \\ \langle \mathscr{A}\,\nabla v, \nabla v\rangle - (1-2/p)\left(\langle |v|^{-1}v\,\mathscr{A}\,\nabla|v|, \nabla v\rangle - \langle \mathscr{A}\,\nabla v, |v|^{-1}v\nabla|v|\rangle\right) - \\ -(1-2/p)^2\left\langle \mathscr{A}\,\nabla|v|, \nabla|v|\right\rangle.\end{gathered}$$

Since

$$\mathscr{R}e(\langle v\,\mathscr{A}\,\nabla|v|,\nabla v\rangle-\langle\mathscr{A}\,\nabla v,v\nabla|v|\rangle)=$$
$$\mathscr{R}e(v\langle\mathscr{A}\,\nabla|v|,\nabla v\rangle-\overline{\langle v\,\mathscr{A}^*\,\nabla|v|,\nabla v\rangle})=\mathscr{R}e(\langle v(\mathscr{A}-\mathscr{A}^*)\nabla|v|,\nabla v\rangle)$$

we have

$$\mathscr{R}e\langle\mathscr{A}\,\nabla u,\nabla(|u|^{p-2}u)\rangle=\mathscr{R}e\,\Big[\langle\mathscr{A}\,\nabla v,\nabla v\rangle-$$
$$(1-2/p)\langle(\mathscr{A}-\mathscr{A}^*)\nabla(|v|),|v|^{-1}\overline{v}\nabla v\rangle-(1-2/p)^2\langle\mathscr{A}\,\nabla(|v|),\nabla(|v|)\rangle\Big].$$

Moreover, we have

$$\langle\mathbf{b}\nabla u,|u|^{p-2}u\rangle=(1-2/p)\,|v|\,\mathbf{b}\nabla|v|+\overline{v}\,\mathbf{b}\nabla v$$

and then

$$\mathscr{R}e\langle\mathbf{b}\nabla u,|u|^{p-2}u\rangle=2\,\mathscr{R}e(\mathbf{b}/p)\,\mathscr{R}e(\overline{v}\nabla v)-(\mathscr{I}m\,\mathbf{b})\,\mathscr{I}m(\overline{v}\nabla v)=$$
$$\mathscr{R}e(\mathbf{b}/p)\nabla(|v|^2)-(\mathscr{I}m\,\mathbf{b})\,\mathscr{I}m(\overline{v}\nabla v).$$

An integration by parts gives

$$\int_\Omega\mathscr{R}e\langle\mathbf{b}\nabla u,|u|^{p-2}u\rangle=-\int_\Omega\mathscr{R}e(\nabla^t(\mathbf{b}/p))|v|^2-\int_\Omega\langle\mathscr{I}m\,\mathbf{b},\mathscr{I}m(\overline{v}\nabla v)\rangle\,. \quad (2.6)$$

In the same way we find

$$\mathscr{R}e\langle u,\overline{\mathbf{c}}\nabla(|u|^{p-2}u)\rangle=\mathscr{R}e\,((1-2/p)\,|v|\mathbf{c}\nabla|v|+v\,\mathbf{c}\nabla\overline{v})=$$
$$2\,\mathscr{R}e(\mathbf{c}/p')\,\mathscr{R}e(\overline{v}\nabla v)+(\mathscr{I}m\,\mathbf{c})\,\mathscr{I}m(\overline{v}\nabla v)=$$
$$\mathscr{R}e(\mathbf{c}/p')\nabla(|v|^2)+(\mathscr{I}m\,\mathbf{c})\,\mathscr{I}m(\overline{v}\nabla v)$$

and then

$$\int_\Omega\mathscr{R}e\langle u,\overline{\mathbf{c}}\nabla(|u|^{p-2}u)\rangle=-\int_\Omega\mathscr{R}e(\nabla^t(\mathbf{c}/p')|v|^2+\int_\Omega\langle\mathscr{I}m\,\mathbf{c},\mathscr{I}m(\overline{v}\nabla v)\rangle. \quad (2.7)$$

Finally, since we have also

$$\mathscr{R}e(a\langle u,|u|^{p-2}u\rangle=(\mathscr{R}e\,a)|u|^p=(\mathscr{R}e\,a)|v|^2,$$

the left-hand side in (2.4) is equal to $\mathscr{R}e\,\mathscr{L}(u,|u|^{p-2}u)$ and (2.2) follows from (2.4).

Viceversa, let us suppose (2.2) holds and let $v\in\mathring{C}^1(\Omega)$. Since the function $u=|v|^{\frac{2-p}{p}}v$ does not need to belong to $\mathring{C}^1(\Omega)$, we cannot proceed as for the Sufficiency. In order to overcome this difficulty, set

$$g_\varepsilon=(|v|^2+\varepsilon^2)^{\frac{1}{2}},\quad u_\varepsilon=g_\varepsilon^{\frac{2}{p}-1}v. \quad (2.8)$$

We have

$$\langle \mathscr{A}\,\nabla u_\varepsilon, \nabla(|u_\varepsilon|^{p-2}u_\varepsilon)\rangle =$$
$$|u_\varepsilon|^{p-2}\langle \mathscr{A}\,\nabla u_\varepsilon, \nabla u_\varepsilon\rangle + (p-2)|u_\varepsilon|^{p-3}\langle \mathscr{A}\,\nabla u_\varepsilon, u_\varepsilon \nabla |u_\varepsilon|\rangle$$

On the other hand, since $\partial_h g_\varepsilon = g_\varepsilon^{-1}|v|\partial_h|v|$, we can write

$$|u_\varepsilon|^{p-2}\partial_h u_\varepsilon \partial_k \overline{u}_\varepsilon = g_\varepsilon^{2-p}|v|^{p-2}\left[(1-2/p)^2\, g_\varepsilon^{-2}|v|^2\partial_h g_\varepsilon \partial_k g_\varepsilon - \right.$$
$$\left.(1-2/p)\, g_\varepsilon^{-1}(v\partial_h g_\varepsilon \partial_k \overline{v} + \overline{v}\partial_h v \partial_k g_\varepsilon) + \partial_h v \partial_k \overline{v}\right] =$$
$$(1-2/p)^2\, g_\varepsilon^{-(p+2)}|v|^{p+2}\partial_h|v|\partial_k|v| -$$
$$(1-2/p)\, g_\varepsilon^{-p}|v|^{p-1}(v\partial_h|v|\partial_k\overline{v} + \overline{v}\partial_h v\partial_k|v|) + g_\varepsilon^{2-p}|v|^{p-2}\partial_h v \partial_k \overline{v}.$$

This leads to

$$|u_\varepsilon|^{p-2}\langle \mathscr{A}\,\nabla u_\varepsilon, \nabla u_\varepsilon\rangle = (1-2/p)^2\, g_\varepsilon^{-(p+2)}|v|^{p+2}\langle \mathscr{A}\,\nabla|v|, \nabla|v|\rangle -$$
$$(1-2/p)\, g_\varepsilon^{-p}|v|^{p-1}(\langle \mathscr{A}\, v\nabla|v|, \nabla v\rangle + \langle \mathscr{A}\,\nabla v, v\,\nabla|v|) + g_\varepsilon^{2-p}|v|^{p-2}\langle \mathscr{A}\,\nabla v, \nabla v\rangle\,.$$

In the same way

$$|u_\varepsilon|^{p-3}\langle \mathscr{A}\,\nabla u_\varepsilon, u_\varepsilon\nabla|u_\varepsilon|\rangle =$$
$$\left[(1-2/p)^2\, g_\varepsilon^{-(p+2)}|v|^{p+2} - (1-2/p)\, g_\varepsilon^{-p}|v|^p\right]\langle \mathscr{A}\,\nabla|v|, \nabla|v|\rangle +$$
$$\left[-(1-2/p)\, g_\varepsilon^{-p}|v|^{p-1} + g_\varepsilon^{-p+2}|v|^{p-3}\right]\langle \mathscr{A}\,\nabla v, v\nabla|v|\rangle.$$

Observing that $g_\varepsilon$ tends to $|v|$ as $\varepsilon \to 0$ and that $g_\varepsilon^{-1}|v| \leqslant 1$, referring to Lebesgue's dominated convergence theorem we find

$$\begin{gathered}
\lim_{\varepsilon\to 0}\int_\Omega \langle \mathscr{A}\,\nabla u_\varepsilon, \nabla(|u_\varepsilon|^{p-2}u_\varepsilon)\rangle = \\
\int_\Omega \langle \mathscr{A}\,\nabla v, \nabla v\rangle - \\
(1-2/p)\int_\Omega \frac{1}{|v|}\left(\langle v\,\mathscr{A}\,\nabla|v|, \nabla v\rangle - \langle \mathscr{A}\,\nabla v, v\,\nabla|v|\rangle\right) - \\
-(1-2/p)^2\int_\Omega \langle \mathscr{A}\,\nabla|v|, \nabla|v|\rangle\ \ .
\end{gathered} \tag{2.9}$$

Similar computations show that

$$\langle \mathbf{b}\nabla u_\varepsilon, |u_\varepsilon|^{p-2}u_\varepsilon\rangle = -(1-2/p)g_\varepsilon^{-p}|v|^{p+1}\mathbf{b}\nabla|v| + g_\varepsilon^{2-p}|v|^{p-2}\overline{v}\mathbf{b}\nabla v,$$
$$\langle u_\varepsilon, \overline{\mathbf{c}}\nabla(|u_\varepsilon|^{p-2}u_\varepsilon)\rangle = g_\varepsilon^{2-p}|v|^{p-2}\mathbf{c}\Big[(1-p)\,(1-2/p)\, g_\varepsilon^{-2}|v|^3\nabla|v| +$$
$$+(p-2)|v|\nabla|v| + v\nabla\overline{v}\Big],$$
$$a\langle u_\varepsilon, |u_\varepsilon|^{p-2}u_\varepsilon\rangle = a g_\varepsilon^{2-p}|v|^p,$$

from which follows

$$\lim_{\varepsilon\to 0}\int_\Omega \langle \mathbf{b}\nabla u_\varepsilon, |u_\varepsilon|^{p-2}u_\varepsilon\rangle = \int_\Omega (-(1-2/p)\,|v|\,\mathbf{b}\nabla|v| + \overline{v}\,\mathbf{b}\nabla v)\,, \tag{2.10}$$

$$\lim_{\varepsilon\to 0}\int_\Omega \langle u_\varepsilon, \overline{\mathbf{c}}\nabla(|u_\varepsilon|^{p-2}u_\varepsilon)\rangle = \int_\Omega ((1-2/p)\,|v|\mathbf{c}\nabla|v| + v\,\mathbf{c}\nabla\overline{v})\,, \tag{2.11}$$

$$\lim_{\varepsilon\to 0}\int_\Omega a\langle u_\varepsilon, |u_\varepsilon|^{p-2}u_\varepsilon\rangle = \int_\Omega a|v|^2. \tag{2.12}$$

From (2.9)–(2.12) we obtain that

$$\lim_{\varepsilon\to 0}\mathscr{R}e\,\mathscr{L}(u_\varepsilon, |u_\varepsilon|^{p-2}u_\varepsilon)$$

exists and is equal to the left-hand side of (2.4). This shows that (2.2) implies (2.4) and the necessity is proved for $p \geqslant 2$.

The proof for $1 < p < 2$ hinges on the remark that (2.3) can be written as

$$\mathscr{R}e\int_\Omega (\langle \mathscr{A}^*\,\nabla u, \nabla(|u|^{p'-2}u)\rangle + \langle \overline{\mathbf{c}}\nabla u, |u|^{p'-2}u\rangle - \langle \nabla u, \mathbf{b}\nabla(|u|^{p'-2}u)\rangle - \\ -a\langle u, |u|^{p'-2}u\rangle)\ \geqslant 0.$$

We omit the details. □

The interest of this Lemma is that it transform conditions (2.2)-(2.3) in condition (2.4). Even if the last one seems to be much more complicated, it has the big advantage that it does not contain the term $|u|^{p-2}$.

### 2.1.2 A necessary condition and a sufficient condition

Lemma 10 has several consequences. A first one is that an $L^p$-dissipative operator has to be degenerate elliptic:

**Corollary 4** *If the form $\mathscr{L}$ is $L^p$-dissipative, we have*

$$\langle \mathscr{R}e\,\mathscr{A}\,\xi, \xi\rangle \geqslant 0 \tag{2.13}$$

*for any $\xi \in \mathbb{R}^n$.*

**Proof.** We remark that condition (2.13) has to be understood in the sense of measures, i.e. it means that, for any $\xi \in \mathbb{R}^n$,

$$\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\xi, \xi\rangle\, v \geqslant 0$$

for any nonnegative $v \in \overset{\circ}{C}(\Omega)$.

Given a function $v$, let us set

$$X = \mathscr{R}e(|v|^{-1}\overline{v}\,\nabla v), \quad Y = \mathscr{I}m(|v|^{-1}\overline{v}\,\nabla v),$$

on the set $\{x \in \Omega \mid v \neq 0\}$. We have

$$\begin{gathered}\mathscr{R}e\langle \mathscr{A}\,\nabla v, \nabla v\rangle = \mathscr{R}e\left\langle \mathscr{A}(|v|^{-1}\overline{v}\,\nabla v), |v|^{-1}\overline{v}\,\nabla v\right\rangle = \\ \langle \mathscr{R}e\,\mathscr{A}\,X, X\rangle + \langle \mathscr{R}e\,\mathscr{A}\,Y, Y\rangle + \langle \mathscr{I}m(\mathscr{A} - \mathscr{A}^t)X, Y\rangle, \\ \mathscr{R}e\langle (\mathscr{A} - \mathscr{A}^*)\nabla(|v|), \nabla v\rangle |v|^{-1} v = \mathscr{R}e\langle (\mathscr{A} - \mathscr{A}^*)X, X + iY\rangle = \\ \langle \mathscr{I}m(\mathscr{A} - \mathscr{A}^*)X, Y\rangle, \\ \mathscr{R}e\langle \mathscr{A}\,\nabla |v|, \nabla |v|\rangle = \langle \mathscr{R}e\,\mathscr{A}\,X, X\rangle.\end{gathered}$$

Since $\mathscr{L}$ is $L^p$-dissipative, (2.4) holds. Hence, keeping in mind that the next integral is extended on the set where $v$ does not vanish,

$$\begin{gathered}\int_\Omega \Big\{ \frac{4}{p\,p'} \langle \mathscr{R}e\,\mathscr{A}\,X, X\rangle + \langle \mathscr{R}e\,\mathscr{A}\,Y, Y\rangle + \\ 2\langle (p^{-1}\,\mathscr{I}m\,\mathscr{A} + p'^{-1}\,\mathscr{I}m\,\mathscr{A}^*)X, Y\rangle + \langle \mathscr{I}m(\mathbf{b} + \mathbf{c}), Y\rangle |v| + \\ \mathscr{R}e\,[\mathrm{div}\,(\mathbf{b}/p - \mathbf{c}/p') - a]\,|v|^2 \Big\} \geqslant 0.\end{gathered} \tag{2.14}$$

We define the function

$$v(x) = \varrho(x)\, e^{i\varphi(x)}$$

where $\varrho$ and $\varphi$ are real functions with $\varrho \in \overset{\circ}{C}{}^1(\Omega)$ and $\varphi \in C^1(\Omega)$. Since

$$|v|^{-1}\overline{v}\,\nabla v = |\varrho|^{-1}(\varrho\, e^{-i\varphi}\,(\nabla \varrho + i\varrho \nabla \varphi)\, e^{i\varphi}) = |\varrho|^{-1}\varrho \nabla \varrho + i|\varrho|\nabla \varphi$$

on the set $\{x \in \Omega \mid \varrho(x) \neq 0\}$, it follows from (2.14) that

$$\begin{gathered}\frac{4}{p\,p'} \int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\nabla \varrho, \nabla \varrho\rangle + \int_\Omega \varrho^2 \langle \mathscr{R}e\,\mathscr{A}\,\nabla \varphi, \nabla \varphi\rangle + \\ 2\int_\Omega \varrho \langle (p^{-1}\,\mathscr{I}m\,\mathscr{A} + p'^{-1}\,\mathscr{I}m\,\mathscr{A}^*)\nabla \varrho, \nabla \varphi\rangle + \\ \int_\Omega \varrho \langle \mathscr{I}m(\mathbf{b} + \mathbf{c}), \nabla \varphi\rangle + \int_\Omega \mathscr{R}e\,[\mathrm{div}\,(\mathbf{b}/p - \mathbf{c}/p') - a]\,\varrho^2 \geqslant 0\end{gathered} \tag{2.15}$$

for any $\varrho \in \overset{\circ}{C}{}^1(\Omega)$, $\varphi \in C^1(\Omega)$.

We choose $\varphi$ by the equality

$$\varphi = \frac{\mu}{2}\,\log(\varrho^2 + \varepsilon),$$

where $\mu \in \mathbb{R}$ and $\varepsilon > 0$. Then (2.15) takes the form

$$\begin{gathered}\frac{4}{p\,p'}\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\nabla\varrho,\nabla\varrho\rangle + \mu^2 \int_\Omega \frac{\varrho^4}{(\varrho^2+\varepsilon)^2}\langle \mathscr{R}e\,\mathscr{A}\,\nabla\varrho,\nabla\varrho\rangle + \\ 2\mu\int_\Omega \frac{\varrho^2}{\varrho^2+\varepsilon}\langle (p^{-1}\,\mathscr{I}m\,\mathscr{A} + p'^{-1}\,\mathscr{I}m\,\mathscr{A}^*)\nabla\varrho,\nabla\varrho\rangle + \\ \mu\int_\Omega \frac{\varrho^3}{\varrho^2+\varepsilon}\langle \mathscr{I}m(\mathbf{b}+\mathbf{c}),\nabla\varrho\rangle + \int_\Omega \mathscr{R}e\,[\mathrm{div}\,(\mathbf{b}/p - \mathbf{c}/p') - a]\,\varrho^2 \geqslant 0\end{gathered} \tag{2.16}$$

Letting $\varepsilon \to 0^+$ in (2.16) leads to

$$\begin{gathered}\frac{4}{p\,p'}\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\nabla\varrho,\nabla\varrho\rangle + \mu^2 \int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\nabla\varrho,\nabla\varrho\rangle + \\ 2\mu\int_\Omega \langle (p^{-1}\,\mathscr{I}m\,\mathscr{A} + p'^{-1}\,\mathscr{I}m\,\mathscr{A}^*)\nabla\varrho,\nabla\varrho\rangle + \\ \mu\int_\Omega \varrho\langle \mathscr{I}m(\mathbf{b}+\mathbf{c}),\nabla\varrho\rangle + \int_\Omega \mathscr{R}e\,[\mathrm{div}\,(\mathbf{b}/p - \mathbf{c}/p') - a]\,\varrho^2 \geqslant 0.\end{gathered} \tag{2.17}$$

Since this holds for any $\mu \in \mathbb{R}$, we have

$$\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\nabla\varrho,\nabla\varrho\rangle \geqslant 0 \tag{2.18}$$

for any $\varrho \in \mathring{C}^1(\Omega)$.

Taking $\varrho(x) = \psi(x)\cos\langle\xi,x\rangle$ with a real $\psi \in \mathring{C}^1(\Omega)$ and $\xi \in \mathbb{R}^n$, we find

$$\begin{gathered}\int_\Omega \{\langle \mathscr{R}e\,\mathscr{A}\,\nabla\psi,\nabla\psi\rangle\cos^2\langle\xi,x\rangle - [\langle \mathscr{R}e\,\mathscr{A}\,\xi,\nabla\psi\rangle + \\ \langle \mathscr{R}e\,\mathscr{A}\,\nabla\psi,\xi\rangle]\,\sin\langle\xi,x\rangle\cos\langle\xi,x\rangle + \langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle\psi^2(x)\sin^2\langle\xi,x\rangle\} \geqslant 0.\end{gathered}$$

On the other hand, taking $\varrho(x) = \psi(x)\sin\langle\xi,x\rangle$,

$$\begin{gathered}\int_\Omega \{\langle \mathscr{R}e\,\mathscr{A}\,\nabla\psi,\nabla\psi\rangle\sin^2\langle\xi,x\rangle + [\langle \mathscr{R}e\,\mathscr{A}\,\xi,\nabla\psi\rangle + \\ \langle \mathscr{R}e\,\mathscr{A}\,\nabla\psi,\xi\rangle]\,\sin\langle\xi,x\rangle\cos\langle\xi,x\rangle + \langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle\psi^2(x)\cos^2\langle\xi,x\rangle\} \geqslant 0.\end{gathered}$$

The two inequalities we have obtained lead to

$$\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\nabla\psi,\nabla\psi\rangle + \int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle\psi^2 \geqslant 0.$$

Because of the arbitrariness of $\xi$, we find

$$\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle\psi^2 \geqslant 0.$$

On the other hand, any nonnegative function $v \in \mathring{C}(\Omega)$ can be approximated in the uniform norm in $\Omega$ by a sequence $\psi_n^2$, with $\psi_n \in \mathring{C}^\infty(\Omega)$, and then $\langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi\rangle$ is a nonnegative measure. □

It will be clear later that (2.13) is not sufficient for the$L^p$-dissipativity.

The next corollary provides a sufficient condition. It shows that the $L^p$-dissipativity of $A$ follows from the nonnegativity of a certain polynomial (whose coefficients are measures) in $2n$ real variables. This polynomial depend on the real parameters $\alpha, \beta$, which can be arbitrarily fixed.

**Corollary 5** *Let $\alpha, \beta$ two real constants. If*

$$\begin{gathered}\frac{4}{p\,p'}\langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi\rangle + \langle \mathscr{R}e\, \mathscr{A}\, \eta, \eta\rangle + 2\langle (p^{-1}\, \mathscr{I}m\, \mathscr{A} + p'^{-1}\, \mathscr{I}m\, \mathscr{A}^*)\xi, \eta\rangle + \\ \langle \mathscr{I}m(\mathbf{b}+\mathbf{c}), \eta\rangle - 2\langle \mathscr{R}e(\alpha \mathbf{b}/p - \beta \mathbf{c}/p'), \xi\rangle + \\ \mathscr{R}e\,[\operatorname{div}\,((1-\alpha)\mathbf{b}/p - (1-\beta)\mathbf{c}/p') - a] \geqslant 0\end{gathered} \tag{2.19}$$

*for any $\xi, \eta \in \mathbb{R}^n$, the form $\mathscr{L}$ is $L^p$-dissipative.*

**Proof.** In the proof of Lemma 10 we have integrated by parts in (2.6) and (2.7). More generally, we have

$$\begin{gathered}2/p \int_\Omega \langle \mathscr{R}e\, \mathbf{b}, \mathscr{R}e(\overline{v}\nabla v)\rangle = 2\alpha/p \int_\Omega \langle \mathscr{R}e\, \mathbf{b}, \mathscr{R}e(\overline{v}\nabla v)\rangle - \\ (1-\alpha)/p \int_\Omega \mathscr{R}e(\nabla^t \mathbf{b})|v|^2\,; \\ 2/p' \int_\Omega \langle \mathscr{R}e\, \mathbf{c}, \mathscr{R}e(\overline{v}\nabla v)\rangle = 2\beta/p' \int_\Omega \langle \mathscr{R}e\, \mathbf{c}, \mathscr{R}e(\overline{v}\nabla v)\rangle - \\ (1-\beta)/p' \int_\Omega \mathscr{R}e(\nabla^t \mathbf{c})|v|^2\,.\end{gathered}$$

This leads to write conditions (2.4) in a slightly different form:

$$\begin{gathered}\mathscr{R}e \int_\Omega \Big[\langle \mathscr{A}\, \nabla v, \nabla v\rangle - (1-2/p)\langle (\mathscr{A} - \mathscr{A}^*)\nabla(|v|), |v|^{-1}\overline{v}\nabla v\rangle - \\ (1-2/p)^2\langle \mathscr{A}\, \nabla(|v|), \nabla(|v|)\rangle\Big] + \int_\Omega \langle \mathscr{I}m(\mathbf{b}+\mathbf{c}), \mathscr{I}m(\overline{v}\nabla v)\rangle - \\ 2\int_\Omega \langle \mathscr{R}e(\alpha \mathbf{b}/p - \beta \mathbf{c}/p'), \mathscr{R}e(\overline{v}\nabla v)\rangle + \\ \int_\Omega \mathscr{R}e(\operatorname{div}((1-\alpha)\mathbf{b}/p - (1-\beta)\mathbf{c}/p') - a)|v|^2 \geqslant 0.\end{gathered}$$

By using the functions $X$ and $Y$ introduced in Corollary 4, the left-hand side of the last inequality can be written as

$$\int_\Omega Q(X,Y),$$

where $Q$ denotes the polynomial (2.19).

The result follows from Lemma 10. $\square$

Generally speaking, conditions (2.19) are not necessary for $L^p$-dissipativity. We show this by the following example, where $\mathscr{I}m\,\mathscr{A}$ is not symmetric. Later we give another example showing that, even for symmetric matrices $\mathscr{I}m\,\mathscr{A}$, conditions (2.19) are not necessary for $L^p$-dissipativity (see Example 7). Nevertheless in the next section we show that the conditions are necessary for the $L^p$-dissipativity, provided the operator $A$ has no lower order terms and the matrix $\mathscr{I}m\,\mathscr{A}$ is symmetric (see Theorem 12 and Remark 4).

**Example 3** Let $n = 2$ and

$$\mathscr{A} = \begin{pmatrix} 1 & i\gamma \\ -i\gamma & 1 \end{pmatrix},$$

where $\gamma$ is a real constant, $\mathbf{b} = \mathbf{c} = a = 0$. In this case polynomial (2.19) is given by

$$(\eta_1 + \gamma\xi_2)^2 + (\eta_2 - \gamma\xi_1)^2 - (\gamma^2 - 4/(pp'))|\xi|^2.$$

Taking $\gamma^2 > 4/(pp')$, condition (2.19) is not satisfied, while we have the $L^p$-dissipativity, because the corresponding operator $A$ is nothing but the Laplacian.

### 2.1.3 Some other consequences of the main lemma

The next Corollary is an interpolation result

**Corollary 6** *If the form $\mathscr{L}$ is both $L^p$- and $L^{p'}$-dissipative, it is also $L^r$-dissipative for any $r$ between $p$ and $p'$, i.e. for any $r$ given by*

$$1/r = t/p + (1-t)/p' \qquad (0 \leqslant t \leqslant 1). \tag{2.20}$$

**Proof.** From the proof of Corollary 4 we know that (2.14) holds. In the same way, we find

$$\begin{gathered}\int_\Omega \Big\{ \frac{4}{p'\,p} \langle \mathscr{R}e\,\mathscr{A}\, X, X\rangle + \langle \mathscr{R}e\,\mathscr{A}\, Y, Y\rangle - \\ 2\langle (p'^{-1}\,\mathscr{I}m\,\mathscr{A} + p^{-1}\,\mathscr{I}m\,\mathscr{A}^*) X, Y\rangle + \langle \mathscr{I}m(\mathbf{b}+\mathbf{c}), Y\rangle |v| + \\ \mathscr{R}e\,[\mathrm{div}\,(\mathbf{b}/p' - \mathbf{c}/p) - a]\,|v|^2 \Big\} \geqslant 0.\end{gathered} \tag{2.21}$$

We multiply (2.14) by $t$, (2.21) by $(1-t)$ and sum up. Since

$$t/p' + (1-t)/p = 1/r' \quad \text{and} \quad r\,r' \leqslant p\,p'\,,$$

we find, keeping in mind Corollary 4,

$$\int_\Omega \Big\{ \frac{4}{r\,r'} \langle \mathscr{R}e\, \mathscr{A}\, X, X\rangle + \langle \mathscr{R}e\, \mathscr{A}\, Y, Y\rangle -$$
$$2\langle (r^{-1}\, \mathscr{I}m\, \mathscr{A} + r'^{-1}\, \mathscr{I}m\, \mathscr{A}^*)X, Y\rangle + \langle \mathscr{I}m(\mathbf{b}+\mathbf{c}), Y\rangle |v| +$$
$$+ \mathscr{R}e\, [\operatorname{div}(\mathbf{b}/r - \mathbf{c}/r') - a]\, |v|^2 \Big\} \geqslant 0$$

and $\mathscr{L}$ is $L^r$-dissipative by Lemma 10 . □

**Corollary 7** *Suppose that either*

$$\mathscr{I}m\, \mathscr{A} = 0, \qquad \mathscr{R}e \operatorname{div} \mathbf{b} = \mathscr{R}e \operatorname{div} \mathbf{c} = 0 \tag{2.22}$$

*or*

$$\mathscr{I}m\, \mathscr{A} = \mathscr{I}m\, \mathscr{A}^t, \quad \mathscr{I}m(\mathbf{b}+\mathbf{c}) = 0, \quad \mathscr{R}e \operatorname{div} \mathbf{b} = \mathscr{R}e \operatorname{div} \mathbf{c} = 0. \tag{2.23}$$

*If $\mathscr{L}$ is $L^p$-dissipative, it is also $L^r$-dissipative for any $r$ given by* (2.20).

**Proof.** Assume that (2.22) holds. With the notation introduced in Corollary 4, inequality (2.4) reads as

$$\int_\Omega \Big( \frac{4}{p\,p'} \langle \mathscr{R}e\, \mathscr{A}\, X, X\rangle + \langle \mathscr{R}e\, \mathscr{A}\, Y, Y\rangle +$$
$$\langle \mathscr{I}m(\mathbf{b}+\mathbf{c}), Y\rangle |v| - \mathscr{R}e\, a |v|^2 \Big) \geqslant 0.$$

Since the left-hand side does not change after replacing $p$ by $p'$, Lemma 10 gives the result.

Let (2.23) holds. Using the formula

$$\begin{gathered} p^{-1}\, \mathscr{I}m\, \mathscr{A} + p'^{-1}\, \mathscr{I}m\, \mathscr{A}^* = \\ p^{-1}\, \mathscr{I}m\, \mathscr{A} - p'^{-1}\, \mathscr{I}m\, \mathscr{A}^t = -(1-2/p)\, \mathscr{I}m\, \mathscr{A}\,, \end{gathered} \tag{2.24}$$

we obtain

$$\int_\Omega \Big( \frac{4}{p\,p'} \langle \mathscr{R}e\, \mathscr{A}\, x, x\rangle + \langle \mathscr{R}e\, \mathscr{A}\, Y, Y\rangle -$$
$$2\,(1-2/p) \langle \mathscr{I}m\, \mathscr{A}\, X, Y\rangle - \mathscr{R}e\, a |v|^2 \Big) \geqslant 0.$$

Replacing $v$ by $\overline{v}$, we find

$$\int_\Omega \Big(\frac{4}{p\,p'}\langle \mathscr{R}e\,\mathscr{A}\,x,x\rangle + \langle \mathscr{R}e\,\mathscr{A}\,Y,Y\rangle +$$
$$2\,(1-2/p)\langle \mathscr{I}m\,\mathscr{A}\,X,Y\rangle - \mathscr{R}e\,a|v|^2\Big) \geqslant 0$$

and we have the $L^{p'}$-dissipativity by $1-2/p=-1+2/p'$. The reference to Corollary 6 completes the proof. □

## 2.2 The operator $\operatorname{div}(\mathscr{A}\,\nabla u)$. The main theorem

In this section we consider operator (2.1) without lower order terms:

$$Au = \operatorname{div}(\mathscr{A}\,\nabla u) \tag{2.25}$$

with the coefficients $a^{hk} \in (\mathring{C}(\Omega))^*$. The following theorem contains an algebraic necessary and sufficient condition for the $L^p$-dissipativity.

This result is new even for smooth coefficients, when it implies a criterion for the $L^p$-contractivity of the corresponding semigroup (see Theorem 15 below).

**Theorem 12** *Let the matrix $\mathscr{I}m\,\mathscr{A}$ be symmetric, i.e. $\mathscr{I}m\,\mathscr{A}^t=\mathscr{I}m\,\mathscr{A}$. The form*

$$\mathscr{L}(u,v)=\int_\Omega \langle \mathscr{A}\,\nabla u,\nabla v\rangle$$

*is $L^p$-dissipative if and only if*

$$|p-2|\,|\langle \mathscr{I}m\,\mathscr{A}\,\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle \tag{2.26}$$

*for any $\xi\in\mathbb{R}^n$, where $|\cdot|$ denotes the total variation.*

**Proof.**

*Sufficiency.* In view of Corollary 5 the form $\mathscr{L}$ is $L^p$-dissipative if

$$\frac{4}{p\,p'}\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle + \langle \mathscr{R}e\,\mathscr{A}\,\eta,\eta\rangle - 2(1-2/p)\langle \mathscr{I}m\,\mathscr{A}\,\xi,\eta\rangle \geqslant 0 \tag{2.27}$$

for any $\xi,\eta\in\mathbb{R}^n$.

By putting

$$\lambda = \frac{2\sqrt{p-1}}{p}\;\xi$$

we write (2.27) in the form

$$\langle \mathscr{R}e\, \mathscr{A}\, \lambda, \lambda \rangle + \langle \mathscr{R}e\, \mathscr{A}\, \eta, \eta \rangle - \frac{p-2}{\sqrt{p-1}} \langle \mathscr{I}m\, \mathscr{A}\, \lambda, \eta \rangle \geqslant 0.$$

Then (2.27) is equivalent to

$$\mathscr{S}(\xi, \eta) := \langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi \rangle + \langle \mathscr{R}e\, \mathscr{A}\, \eta, \eta \rangle - \frac{p-2}{\sqrt{p-1}} \langle \mathscr{I}m\, \mathscr{A}\, \xi, \eta \rangle \geqslant 0$$

for any $\xi, \eta \in \mathbb{R}^n$.

For any nonnegative $\varphi \in \mathring{C}(\Omega)$, define

$$\lambda_\varphi = \min_{|\xi|^2 + |\eta|^2 = 1} \int_\Omega \mathscr{S}(\xi, \eta)\, \varphi\, .$$

Let us fix $\xi_0, \eta_0$ such that $|\xi_0|^2 + |\eta_0|^2 = 1$ and

$$\lambda_\varphi = \int_\Omega \mathscr{S}(\xi_0, \eta_0)\, \varphi\, .$$

We have the algebraic system

$$\begin{cases} \displaystyle\int_\Omega \left( 2\, \mathscr{R}e\, \mathscr{A}\, \xi_0 - \frac{p-2}{2\sqrt{p-1}}\ \mathscr{I}m(\mathscr{A} - \mathscr{A}^*) \eta_0 \right) \varphi\ = 2\, \lambda_\varphi\, \xi_0 \\ \displaystyle\int_\Omega \left( 2\, \mathscr{R}e\, \mathscr{A}\, \eta_0 - \frac{p-2}{2\sqrt{p-1}}\ \mathscr{I}m(\mathscr{A} - \mathscr{A}^*) \xi_0 \right) \varphi\ = 2\, \lambda_\varphi\, \eta_0\, . \end{cases}$$

This implies

$$\int_\Omega \left( 2\, \mathscr{R}e\, \mathscr{A}(\xi_0 - \eta_0) + \frac{p-2}{2\sqrt{p-1}}\ \mathscr{I}m(\mathscr{A} - \mathscr{A}^*)(\xi_0 - \eta_0) \right) \varphi\ = \\ 2\, \lambda_\varphi\, (\xi_0 - \eta_0)$$

and therefore

$$\int_\Omega \left( 2 \langle \mathscr{R}e\, \mathscr{A}(\xi_0 - \eta_0), \xi_0 - \eta_0 \rangle + \frac{p-2}{\sqrt{p-1}} \langle \mathscr{I}m\, \mathscr{A}(\xi_0 - \eta_0), \xi_0 - \eta_0 \rangle \right) \varphi\ = \\ 2\, \lambda_\varphi\, |\xi_0 - \eta_0|^2.$$

The left-hand side is nonnegative because of (2.26). Hence, if $\lambda_\varphi < 0$, we find $\xi_0 = \eta_0$. On the other hand we have

$$\lambda_\varphi = \int_\Omega \mathscr{S}(\xi_0, \xi_0)\, \varphi\ = \\ \int_\Omega \left( 2 \langle \mathscr{R}e\, \mathscr{A}\, \xi_0, \xi_0 \rangle - \frac{p-2}{\sqrt{p-1}} \langle \mathscr{I}m\, \mathscr{A}\, \xi_0, \xi_0 \rangle \right) \varphi\ \geqslant 0.$$

This shows that $\lambda_\varphi \geqslant 0$ for any nonnegative $\varphi$ and the sufficiency is proved.

*Necessity.* We know from the proof of Corollary 4 that if $\mathscr{L}$ is $L^p$-dissipative, then (2.17) holds for any $\varrho \in \mathring{C}^1(\Omega)$, $\mu \in \mathbb{R}$. In the present case, keeping in mind (2.24), (2.17) can be written as

$$\int_\Omega \langle \mathscr{B}\, \nabla \varrho, \nabla \varrho \rangle \geqslant 0,$$

where

$$\mathscr{B} = \frac{4}{p\,p'}\, \mathscr{R}e\, \mathscr{A} + \mu^2\, \mathscr{R}e\, \mathscr{A} - 2\, \mu\, (1-2/p)\, \mathscr{I}m\, \mathscr{A}\, .$$

In the proof of Corollary 4, we have also seen that from (2.18) for any $\varrho \in \mathring{C}^1(\Omega)$, (2.13) follows. In the same way, the last relation implies $\langle \mathscr{B}\, \xi, \xi \rangle \geqslant 0$, i.e.

$$\frac{4}{p\,p'}\, \langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi \rangle + \mu^2 \langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi \rangle - 2\, \mu\, (1-2/p) \langle \mathscr{I}m\, \mathscr{A}\, \xi, \xi \rangle \geqslant 0$$

for any $\xi \in \mathbb{R}^n$, $\mu \in \mathbb{R}$.

Because of the arbitrariness of $\mu$ we have

$$\int_\Omega \langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi \rangle\, \varphi \, \geqslant 0$$

$$(1-2/p)^2 \left( \int_\Omega \langle \mathscr{I}m\, \mathscr{A}\, \xi, \xi \rangle\, \varphi \right)^2 \leqslant \frac{4}{p\,p'} \left( \int_\Omega \langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi \rangle\, \varphi \right)^2 ,$$

i.e.

$$|p-2| \left| \int_\Omega \langle \mathscr{I}m\, \mathscr{A}\, \xi, \xi \rangle\, \varphi \right| \leqslant 2\, \sqrt{p-1}\, \int_\Omega \langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi \rangle\, \varphi$$

for any $\xi \in \mathbb{R}^n$ and for any nonnegative $\varphi \in \mathring{C}(\Omega)$.

We have

$$|p-2| \left| \int_\Omega \langle \mathscr{I}m\, \mathscr{A}\, \xi, \xi \rangle \varphi \right| \leqslant 2\sqrt{p-1}\, \int_\Omega \langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi \rangle |\varphi|$$

for any $\varphi \in \mathring{C}(\Omega)$ and this implies (2.26), because

$$|p-2| \int_\Omega |\langle \mathscr{I}m\, \mathscr{A}\, \xi, \xi \rangle|\, g = |p-2| \sup_{\substack{\varphi \in \mathring{C}(\Omega) \\ |\varphi| \leqslant g}} \left| \int_\Omega \langle \mathscr{I}m\, \mathscr{A}\, \xi, \xi \rangle \varphi \right| \leqslant$$

$$2\sqrt{p-1} \sup_{\substack{\varphi \in \mathring{C}(\Omega) \\ |\varphi| \leqslant g}} \int_\Omega \langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi \rangle |\varphi| \leqslant 2\sqrt{p-1} \int_\Omega \langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi \rangle g$$

for any nonnegative $g \in \mathring{C}(\Omega)$. □

**Remark 4** From the proof of Theorem 12 we see that condition (2.26) holds if and only if

$$\frac{4}{p\,p'}\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle + \langle \mathscr{R}e\,\mathscr{A}\,\eta,\eta\rangle - 2(1-2/p)\langle \mathscr{I}m\,\mathscr{A}\,\xi,\eta\rangle \geqslant 0$$

for any $\xi,\eta \in \mathbb{R}^n$. This means that conditions (2.19) are necessary and sufficient for the operators considered in Theorem 12.

**Remark 5** Let us assume that either $A$ has lower order terms or they are absent and $\mathscr{I}m\,\mathscr{A}$ is not symmetric. Using the same arguments as in Theorem 12, one could prove that (2.26) is still a necessary condition for $A$ to be $L^p$-dissipative. However, in general, it is not sufficient. This is shown by the next example (see also Theorem 13 below for the particular case of constant coefficients).

**Example 4** Let $n = 2$ and let $\Omega$ be a bounded domain. Denote by $\sigma$ a not identically vanishing real function in $\mathring{C}^2(\Omega)$ and let $\lambda \in \mathbb{R}$. Consider operator (2.25) with

$$\mathscr{A} = \begin{pmatrix} 1 & i\lambda\partial_1(\sigma^2) \\ -i\lambda\partial_1(\sigma^2) & 1 \end{pmatrix},$$

i.e.

$$Au = \partial_1(\partial_1 u + i\lambda\partial_1(\sigma^2)\,\partial_2 u) + \partial_2(-i\lambda\partial_1(\sigma^2)\,\partial_1 u + \partial_2 u),$$

where $\partial_i = \partial/\partial x_i$ $(i = 1, 2)$.

By definition, we have $L^2$-dissipativity if and only if

$$\mathscr{R}e\int_\Omega ((\partial_1 u + i\lambda\partial_1(\sigma^2)\,\partial_2 u)\partial_1\overline{u} + (-i\lambda\partial_1(\sigma^2)\,\partial_1 u + \partial_2 u)\partial_2\overline{u})\,dx \geqslant 0$$

for any $u \in \mathring{C}^1(\Omega)$, i.e. if and only if

$$\int_\Omega |\nabla u|^2 dx - 2\lambda\int_\Omega \partial_1(\sigma^2)\,\mathscr{I}m(\partial_1\overline{u}\,\partial_2 u)\,dx \geqslant 0$$

for any $u \in \mathring{C}^1(\Omega)$. Taking $u = \sigma\,\exp(itx_2)$ $(t \in \mathbb{R})$, we obtain, in particular,

$$t^2\int_\Omega \sigma^2 dx - t\lambda\int_\Omega (\partial_1(\sigma^2))^2 dx + \int_\Omega |\nabla\sigma|^2 dx \geqslant 0. \tag{2.28}$$

Since

$$\int_\Omega (\partial_1(\sigma^2))^2 dx > 0,$$

we can choose $\lambda \in \mathbb{R}$ so that (2.28) is impossible for all $t \in \mathbb{R}$. Thus $A$ is not $L^2$-dissipative, although (2.26) is satisfied.

Since $A$ can be written as

$$Au = \Delta u - i\lambda(\partial_{21}(\sigma^2)\,\partial_1 u - \partial_{11}(\sigma^2)\,\partial_2 u),$$

the same example shows that (2.26) is not sufficient for the $L^2$-dissipativity in the presence of lower order terms, even if $\mathscr{I}m\,\mathscr{A}$ is symmetric.

**Remark 6** It is nice to remark that from (2.26) we can immediately deduce the following facts: let $A$ be the differential operator (2.25) satisfying the hypothesis of Theorem 12. Let us suppose that $A$ is a degenerate elliptic operator (i.e. it satisfies (2.13)). Then

(i) the corresponding form $\mathscr{L}$ if $L^2$-dissipative;

(ii) if the operator $A$ has real coefficients ($\mathscr{I}m\,\mathscr{A} = 0$), then the corresponding form $\mathscr{L}$ if $L^p$-dissipative for any $p$.

**Remark 7** In view of Theorem 12, it is now clear why condition (2.13) cannot be sufficient for the $L^p$-dissipativity when $p \neq 2$.

## 2.3 Operators with lower order terms

We know from Remark 5 that, if the partial differential operator $A$ contains lower order terms, the algebraic condition 2.26 is not necessary and sufficient for the $L^p$-dissipativity. One could ask if there are other algebraic necessary and sufficient conditions for these more general operators.

Generally speaking, this is not possible. We can convince ourselves of that by means of the following examples.

**Example 5** Let $A$ be the operator

$$Au = \Delta u + a(x)u$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where $a(x)$ is a real smooth function. Denote by $\lambda_1$ the first eigenvalue of the Dirichlet problem for Laplace equation in $\Omega$. A sufficient condition for $A$ to be $L^2$-dissipative is $\mathscr{R}e\,a \leqslant \lambda_1$ and we cannot give an algebraic characterization of $\lambda_1$.

**Example 6** Let $A$ be the operator

$$Au = \Delta u + \mu u$$

in a domain $\Omega \subset \mathbb{R}^n$, where $\mu$ is a nonnegative measure. Lemma 10 shows that $A$ is $L^p$-dissipative if and only if

$$\int_\Omega |w|^2 d\mu \leqslant \frac{4}{pp'} \int_\Omega |\nabla w|^2 dx \qquad \forall\, w \in \mathring{C}^\infty(\Omega). \tag{2.29}$$

It is easy to show that, if (2.29) holds, then

$$\frac{\mu(F)}{\mathrm{cap}_\Omega(F)} \leqslant \frac{4}{pp'} \tag{2.30}$$

for any compact set $F \subset \Omega$, where $\mathrm{cap}_\Omega(F)$ is the relative capacity of $F$

$$\mathrm{cap}_\Omega(F) = \inf\Big\{ \int_\Omega |\nabla u|^2 dx \ :\ u \in \mathring{C}^\infty(\Omega),\ u \geqslant 1 \text{ on } F \Big\}.$$

In fact, if $u \in \mathring{C}^\infty(\Omega)$, with $u \geqslant 1$ on $F$, (2.29) implies that

$$\mu(F) \leqslant \int_F u^2 d\mu \leqslant \int_\Omega u^2 d\mu \leqslant \frac{4}{pp'} \int_\Omega |\nabla u|^2 dx$$

and then

$$\mu(F) \leqslant \frac{4}{pp'} \inf_{\substack{u \in \mathring{C}^\infty(\Omega) \\ u \geqslant 1 \text{ on } F}} \int_\Omega |\nabla u|^2 dx,$$

i.e. (2.30).

On the other hand, if

$$\frac{\mu(F)}{\mathrm{cap}_\Omega(F)} \leqslant \frac{1}{pp'} \tag{2.31}$$

for any compact set $F \subset \Omega$, then (2.29) holds. This is a deep result and it is due to V. Maz'ya (see [25, 26, 27]). One can show that the necessary condition (2.30) is not sufficient and the sufficient condition (2.31) is not necessary.

However, if the operator has constant coefficients, then one can still give necessary and sufficient conditions. This is the subject of the following subsection.

### 2.3.1 Operators with constant coefficients

In this section we characterize the $L^p$-dissipativity for the operator (2.1) with constant complex coefficients. Without loss of generality we can write $A$ as

$$Au = \nabla^t(\mathscr{A}\,\nabla u) + \mathbf{b}\nabla u + au, \tag{2.32}$$

assuming that the matrix $\mathscr{A}$ is symmetric.

**Theorem 13** *Let $\Omega$ be an open set in $\mathbb{R}^n$ which contains balls of arbitrarily large radius. The operator $A$ is $L^p$-dissipative if and only if there exists a real constant vector $V$ such that*

$$2\,\mathscr{R}e\,\mathscr{A}\,V + \mathscr{I}m\,\mathbf{b} = 0, \tag{2.33}$$

$$\mathscr{R}e\,a + \langle \mathscr{R}e\,\mathscr{A}\,V, V\rangle \leqslant 0 \tag{2.34}$$

*and the inequality*

$$|p-2|\,|\langle \mathscr{I}m\, \mathscr{A}\,\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle \tag{2.35}$$

*holds for any* $\xi\in\mathbb{R}^n$*.*

**Proof.** First, let us prove the theorem for the special case $\mathbf{b}=0$, i.e. for the operator

$$A=\nabla^t(\mathscr{A}\,\nabla u)+au.$$

If $A$ is $L^p$-dissipative, (2.4) holds for any $v\in \mathring{C}^1(\Omega)$. We find, by repeating the arguments used in the proof of Theorem 12, that

$$\begin{gathered}\frac{4}{p\,p'}\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\nabla\varrho,\nabla\varrho\rangle\,dx+\mu^2\int_\Omega\langle \mathscr{R}e\,\mathscr{A}\,\nabla\varrho,\nabla\varrho\rangle\,dx-\\ 2\,\mu\,(1-2/p)\int_\Omega\langle \mathscr{I}m\,\mathscr{A}\,\nabla\varrho,\nabla\varrho\rangle\,dx-(\mathscr{R}e\,a)\int_\Omega\varrho^2dx\geqslant 0\end{gathered} \tag{2.36}$$

for any $\varrho\in\mathring{C}^\infty(\Omega)$ and for any $\mu\in\mathbb{R}$. As in the proof of Theorem 12 this implies (2.35). On the other hand, we can find a sequence of balls contained in $\Omega$ with centers $x_m$ and radii $m$. Set

$$\varrho_m(x)=m^{-n/2}\sigma\left((x-x_m)/m\right),$$

where $\sigma\in\mathring{C}^\infty(\mathbb{R}^n)$, $\operatorname{spt}\sigma\subset B_1(0)$ and

$$\int_{B_1(0)}\sigma^2(x)\,dx=1.$$

Putting in (2.36) $\mu=1$ and $\varrho=\varrho_m$, we obtain

$$\begin{gathered}\frac{4}{p\,p'}\int_{B_1(0)}\langle \mathscr{R}e\,\mathscr{A}\,\nabla\sigma,\nabla\sigma\rangle\,dy+\int_{B_1(0)}\langle \mathscr{R}e\,\mathscr{A}\,\nabla\sigma,\nabla\sigma\rangle\,dy\,-\\ 2\,(1-2/p)\int_{B_1(0)}\langle \mathscr{I}m\,\mathscr{A}\,\nabla\sigma,\nabla\sigma\rangle\,dy-m^2(\mathscr{R}e\,a)\geqslant 0\end{gathered}$$

for any $m\in\mathbb{N}$. This implies $\mathscr{R}e\,a\leqslant 0$. Note that in this case the algebraic system (2.33) has always the trivial solution and that for any eigensolution $V$ (if they exist) we have $\langle \mathscr{R}e\,\mathscr{A}\,V,V\rangle=0$. Then (2.34) is satisfied.

Conversely, if (2.35) is satisfied, we have (see Remark 4)

$$\frac{4}{p\,p'}\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle+\langle \mathscr{R}e\,\mathscr{A}\,\eta,\eta\rangle-2\,(1-2/p)\langle \mathscr{I}m\,\mathscr{A}\,\xi,\xi\rangle\geqslant 0$$

for any $\xi,\eta\in\mathbb{R}^n$. If also (2.34) is satisfied (i.e. if $\mathscr{R}e\,a\leqslant 0$), $A$ is $L^p$-dissipative in view of Corollary 5.

Let us consider the operator in the general form (2.32). If $A$ is $L^p$-dissipative, we find, by repeating the arguments employed in the proof of Theorem 12, that

$$
\begin{gathered}
\frac{4}{p\,p'}\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\nabla\varrho,\nabla\varrho\rangle\,dx+\int_\Omega \varrho^2\langle \mathscr{R}e\,\mathscr{A}\,\nabla\varphi,\nabla\varphi\rangle\,dx- \\
2\,(1-2/p)\int_\Omega \varrho\,\langle \mathscr{I}m\,\mathscr{A}\,\nabla\varrho,\nabla\varphi\rangle\,dx+ \\
\int_\Omega \varrho^2\langle \mathscr{I}m\,\mathbf{b},\nabla\varphi\rangle\,dx-\mathscr{R}e\,a\int_\Omega \varrho^2 dx\geqslant 0
\end{gathered}
$$

for any $\varrho\in \mathring{C}^1(\Omega)$, $\varphi\in C^1(\Omega)$. By fixing $\varrho$ and choosing $\varphi=t\langle\eta,x\rangle$ ($t\in\mathbb{R}$, $\eta\in\mathbb{R}^n$) we get

$$
\frac{4}{p\,p'}\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\nabla\varrho,\nabla\varrho\rangle\,dx+(t^2\langle \mathscr{R}e\,\mathscr{A}\,\eta,\eta\rangle+t\,\langle \mathscr{I}m\,\mathbf{b},\eta\rangle-\mathscr{R}e\,a)\int_\Omega \varrho^2\,dx\geqslant 0
$$

for any $t\in\mathbb{R}$. This leads to

$$
|\langle \mathscr{I}m\,\mathbf{b},\eta\rangle|^2\leqslant K\,\langle \mathscr{R}e\,\mathscr{A}\,\eta,\eta\rangle
$$

for any $\eta\in\mathbb{R}^n$ and this inequality shows that system (2.33) is solvable. Let $V$ be a solution of this system and let

$$
z=e^{-i\langle V,x\rangle}u.
$$

One checks directly that

$$
Au=(\nabla^t(\mathscr{A}\,\nabla z)+\langle \mathbf{c},\nabla z\rangle+\alpha z)e^{i\langle V,x\rangle}
$$

where

$$
\mathbf{c}=2i\,\mathscr{A}\,V+\mathbf{b},\qquad \alpha=a+i\langle \mathbf{b},V\rangle-\langle \mathscr{A}\,V,V\rangle.
$$

Since we have

$$
\int_\Omega \langle Au,u\rangle|u|^{p-2}dx=\int_\Omega \langle\nabla^t(\mathscr{A}\,\nabla z)+\langle \mathbf{c},\nabla z\rangle+\alpha z,z\rangle|z|^{p-2}dx\,,
$$

the $L^p$-dissipativity of $A$ is equivalent to the $L^p$-dissipativity of the operator

$$
\nabla^t(\mathscr{A}\,\nabla z)+\langle \mathbf{c},\nabla z\rangle+\alpha z\,.
$$

On the other hand Lemma 10 shows that, as far as the first order terms are concerned, the $\mathscr{R}e\,\mathbf{b}$ does not play any role. Since $\mathscr{I}m\,\mathbf{c}=\mathbf{0}$ because of (2.33), the $L^p$-dissipativity of $A$ is equivalent to the $L^p$-dissipativity of the operator

$$\nabla^t(\mathscr{A}\,\nabla z) + \alpha z\,. \tag{2.37}$$

By what we have already proved above, the last operator is $L^p$-dissipative if and only if (2.35) is satisfied and $\mathscr{R}e\,\alpha \leqslant 0$. From (2.33) it follows that $\mathscr{R}e\,\alpha$ is equal to the left-hand side of (2.34).

Conversely, if there exists a solution $V$ of (2.33), (2.34), and if (2.35) is satisfied, operator (2.37) is $L^p$-dissipative. Since this is equivalent to the $L^p$-dissipativity of $A$, the proof is complete. □

**Corollary 8** *Let $\Omega$ be an open set in $\mathbb{R}^n$ which contains balls of arbitrarily large radius. Let us suppose that the matrix $\mathscr{R}e\,\mathscr{A}$ is not singular. The operator $A$ is $L^p$-dissipative if and only if* (2.35) *holds and*

$$4\,\mathscr{R}e\,a \leqslant -\langle(\mathscr{R}e\,\mathscr{A})^{-1}\,\mathscr{I}m\,\mathbf{b}, \mathscr{I}m\,\mathbf{b}\rangle. \tag{2.38}$$

**Proof.** If $\mathscr{R}e\,\mathscr{A}$ is not singular, the only vector $V$ satisfying (2.33) is

$$V = -(1/2)(\mathscr{R}e\,\mathscr{A})^{-1}\,\mathscr{I}m\,\mathbf{b}$$

and (2.34) is satisfied if and only if (2.38) holds. The result follows from Theorem 13. □

**Remark 8** The Corollary 8 was proved in this way in [5]. The same result, obtained with a different approach, can be found also in [17].

**Example 7** Let $n = 1$ and $\Omega = \mathbb{R}^1$. Consider the operator

$$\left(1 + 2\,\frac{\sqrt{p-1}}{p-2}\,i\right) u'' + 2iu' - u,$$

where $p \neq 2$ is fixed. Conditions (2.35) and (2.38) are satisfied and this operator is $L^p$-dissipative, in view of Corollary 8.

On the other hand, the polynomial considered in Corollary 5 (with $\alpha = \beta = 0$) is

$$Q(\xi,\eta) = \left(2\,\frac{\sqrt{p-1}}{p}\,\xi - \eta\right)^2 + 2\eta + 1$$

which is not nonnegative for any $\xi, \eta \in \mathbb{R}$. This shows that, in general, condition (2.19) is not necessary for the $L^p$-dissipativity, even if the matrix $\mathscr{I}m\,\mathscr{A}$ is symmetric.

## 2.4 Dissipativity and semigroups. Operators with smooth coefficients

In this Section we want to investigate the relations between the concept of dissipativity of the form $\mathscr{L}$ (introduced in Section 2.1) and the usual concept of dissipativity of the operator $A$, as considered in Section 1.2. We consider the generation of the corresponding semigroups as well.

We shall do that for operators with smooth coefficients. In all this section $A$ will be the operator

$$Au = \mathrm{div}(\mathscr{A}\,\nabla u) + \mathbf{b}\nabla u + a\,u \tag{2.39}$$

with the coefficients $a^{hk}, b^h \in C^1(\overline{\Omega})$, $a \in C^0(\overline{\Omega})$. Here $\Omega$ is a bounded domain in $\mathbb{R}^n$, whose boundary is in the class $C^{2,\alpha}$ for some $\alpha \in [0,1)$ (this regularity assumption could be weakened, but we prefer to avoid the technicalities related to such generalizations).

We consider $A$ as an operator defined on the set

$$D(A) = W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega). \tag{2.40}$$

### 2.4.1 The dissipativity of the form $\mathscr{L}$ and the dissipativity of the operator $A$

We recall that the operator $A$ is $L^p$-dissipative if

$$\mathscr{R}e \int_\Omega \langle Au, u\rangle |u|^{p-2} dx \leqslant 0 \tag{2.41}$$

for any $u \in D(A)$.

The aim of this Subsection is to show that the $L^p$-dissipativity of $A$ is equivalent to the $L^p$-dissipativity of the sesquilinear form

$$\mathscr{L}(u,v) = \int_\Omega (\langle \mathscr{A}\,\nabla u, \nabla v\rangle - \langle \mathbf{b}\nabla u, v\rangle - a\langle u, v\rangle)\,.$$

In order to obtain that, we need some lemmas.

**Lemma 11** *The form $\mathscr{L}$ is $L^p$-dissipative if and only if*

$$\begin{gathered}\mathscr{R}e \int_\Omega \Big[\langle \mathscr{A}\,\nabla v, \nabla v\rangle - (1-2/p)\langle(\mathscr{A}-\mathscr{A}^*)\nabla(|v|), |v|^{-1}\overline{v}\nabla v\rangle - \\ (1-2/p)^2\langle \mathscr{A}\,\nabla(|v|), \nabla(|v|)\rangle\Big] dx + \\ \int_\Omega \langle \mathscr{I}m\,\mathbf{b}, \mathscr{I}m(\overline{v}\nabla v)\rangle dx + \int_\Omega \mathscr{R}e(\nabla^t(\mathbf{b}/p) - a)|v|^2 dx \geqslant 0\end{gathered} \tag{2.42}$$

*for any $v \in H_0^1(\Omega)$.*

**Proof.**

*Sufficiency.* We know from Lemma 10 that $\mathscr{L}$ is $L^p$-dissipative if and only if (2.42) holds for any $v \in \mathring{C}^1(\Omega)$. Since $\mathring{C}^1(\Omega) \subset H_0^1(\Omega)$, the sufficiency follows.

*Necessity.* Given $v \in H_0^1(\Omega)$, we can find a sequence $\{v_n\} \subset \mathring{C}^1(\Omega)$ such that $v_n \to v$ in $H_0^1(\Omega)$. Let us show that

$$\chi_{E_n} |v_n|^{-1} \overline{v}_n \nabla v_n \to \chi_E |v|^{-1} \overline{v} \nabla v \quad \text{in } L^2(\Omega), \tag{2.43}$$

where $E_n = \{x \in \Omega \mid v_n(x) \neq 0\}$, $E = \{x \in \Omega \mid v(x) \neq 0\}$. We may assume $v_n(x) \to v(x)$, $\nabla v_n(x) \to \nabla v(x)$ almost everywhere in $\Omega$. We see that

$$\chi_{E_n} |v_n|^{-1} \overline{v}_n \nabla v_n \to \chi_E |v|^{-1} \overline{v} \nabla v \tag{2.44}$$

almost everywhere on the set $E \cup \{x \in \Omega \setminus E \mid \nabla v(x) = 0\}$. Since the set $\{x \in \Omega \setminus E \mid \nabla v(x) \neq 0\}$ has zero measure, we can say that (2.44) holds almost everywhere in $\Omega$.

Moreover, since

$$\int_G |\chi_{E_n} |v_n|^{-1} \overline{v}_n \nabla v_n|^2 dx \leqslant \int_G |\nabla v_n|^2 dx$$

for any measurable set $G \subset \Omega$ and $\{\nabla v_n\}$ is convergent in $L^2(\Omega)$, the sequence $\{|\, \chi_{E_n} |v_n|^{-1} \overline{v}_n \nabla v_n - \chi_E |v|^{-1} \overline{v} \nabla v\, |^2\}$ has uniformly absolutely continuos integrals. Now we may appeal to Vitali's Theorem to obtain (2.43).

From this it follows that (2.42) for any $v \in H_0^1(\Omega)$ implies (2.42) for any $v \in \mathring{C}^1(\Omega)$. Lemma 10 shows that $\mathscr{L}$ is $L^p$-dissipative. □

**Lemma 12** *The form $\mathscr{L}$ is $L^p$-dissipative if and only if*

$$\mathscr{R}e \int_\Omega (\langle \mathscr{A} \nabla u, \nabla(|u|^{p-2} u)\rangle - \langle \mathbf{b} \nabla u, |u|^{p-2} u\rangle - a\, |u|^p) dx \geqslant 0 \tag{2.45}$$

*for any $u \in \Xi$, where $\Xi$ denotes the space $\{u \in C^2(\overline{\Omega}) \mid u|_{\partial\Omega} = 0\}$.*

**Proof.**

*Necessity.* Since $\mathscr{L}$ is $L^p$-dissipative, (2.42) holds for any $v \in H_0^1(\Omega)$. Let $u \in \Xi$. We introduce the function

$$\varrho_\varepsilon(s) = \begin{cases} \varepsilon^{\frac{p-2}{2}} & \text{if } 0 \leqslant s \leqslant \varepsilon \\ s^{\frac{p-2}{2}} & \text{if } s > \varepsilon. \end{cases}$$

Setting

$$v_\varepsilon = \varrho_\varepsilon(|u|)\, u$$

a direct computation shows that $u = \sigma_\varepsilon(|v_\varepsilon|)\, v_\varepsilon$ and $\varrho^2_\varepsilon(|u|)\, u = [\sigma_\varepsilon(|v_\varepsilon|)]^{-1}\, v_\varepsilon$, where

$$\sigma_\varepsilon(s) = \begin{cases} \varepsilon^{\frac{2-p}{2}} & \text{if } 0 \leqslant s \leqslant \varepsilon^{\frac{p}{2}} \\ s^{\frac{2-p}{p}} & \text{if } s > \varepsilon^{\frac{p}{2}}. \end{cases}$$

Therefore

$$\langle \mathscr{A}\, \nabla u, \nabla[\varrho^2_\varepsilon(|u|)\, u]\rangle = \langle \mathscr{A}\, \nabla[\sigma_\varepsilon(|v_\varepsilon|)\, v_\varepsilon], \nabla[(\sigma_\varepsilon(|v_\varepsilon|))^{-1} v_\varepsilon]\rangle =$$
$$\langle \mathscr{A}\, [\sigma_\varepsilon(|v_\varepsilon|)\, \nabla v_\varepsilon + \sigma'_\varepsilon(|v_\varepsilon|)\, v_\varepsilon\, \nabla|v_\varepsilon|]\, , \sigma_\varepsilon(|v_\varepsilon|)^{-1}\nabla v_\varepsilon -$$
$$\sigma'_\varepsilon(|v_\varepsilon|)\sigma_\varepsilon^{-2}(|v_\varepsilon|) v_\varepsilon\, \nabla|v_\varepsilon|\rangle =$$
$$\langle \mathscr{A}\, \nabla v_\varepsilon, \nabla v_\varepsilon\rangle + \sigma'_\varepsilon(|v_\varepsilon|)\sigma_\varepsilon(|v_\varepsilon|)^{-1}\left(\langle v_\varepsilon\, \mathscr{A}\, \nabla|v_\varepsilon|, \nabla v_\varepsilon\rangle - \langle \mathscr{A}\, \nabla v_\varepsilon, v_\varepsilon\, \nabla|v_\varepsilon|\rangle\right) -$$
$$-\sigma'_\varepsilon(|v_\varepsilon|)^2\sigma_\varepsilon(|v_\varepsilon|)^{-2}\langle v_\varepsilon\, \mathscr{A}\, \nabla|v_\varepsilon|, v_\varepsilon \nabla|v_\varepsilon|\rangle\, .$$

Since

$$\frac{\sigma'_\varepsilon(|v_\varepsilon|)}{\sigma_\varepsilon(|v_\varepsilon|)} = \begin{cases} 0 & \text{if } 0 < |u| < \varepsilon \\ -(1-2/p)\ |v_\varepsilon|^{-1} & \text{if } |u| > \varepsilon \end{cases}$$

we may write

$$\int_\Omega \langle \mathscr{A}\, \nabla u, \nabla[\varrho^2_\varepsilon(|u|)\, u]\rangle\, dx = \int_\Omega \langle \mathscr{A}\, \nabla v_\varepsilon, \nabla v_\varepsilon\rangle\, dx -$$
$$-(1-2/p)\int_{E_\varepsilon} \frac{1}{|v_\varepsilon|}\left(\langle v_\varepsilon\, \mathscr{A}\, \nabla|v_\varepsilon|, \nabla v_\varepsilon\rangle - \langle \mathscr{A}\, \nabla v_\varepsilon, v_\varepsilon\, \nabla|v_\varepsilon|\rangle\right)\, dx -$$
$$-(1-2/p)^2\int_{E_\varepsilon} \langle \mathscr{A}\, \nabla|v_\varepsilon|, \partial_h \nabla|v_\varepsilon|\rangle\, dx,$$

where $E_\varepsilon = \{x \in \Omega \mid |u(x)| > \varepsilon\}$. Then

$$\int_\Omega \langle \mathscr{A}\, \nabla u, \nabla[\varrho^2_\varepsilon(|u|)\, u]\rangle\, dx = \int_\Omega \langle \mathscr{A}\, \nabla v_\varepsilon, \nabla v_\varepsilon\rangle\, dx -$$
$$(1-2/p)\int_\Omega \frac{1}{|v_\varepsilon|}\left(\langle v_\varepsilon\, \mathscr{A}\, \nabla|v_\varepsilon|, \nabla v_\varepsilon\rangle - \langle \mathscr{A}\, \nabla v_\varepsilon, v_\varepsilon\, \nabla|v_\varepsilon|\rangle\right)\, dx -$$
$$(1-2/p)^2\int_\Omega \langle \mathscr{A}\, \nabla|v_\varepsilon|, \nabla|v_\varepsilon|\rangle\, dx + R(\varepsilon),$$

where

$$R(\varepsilon) = (1-2/p)\int_{\Omega\setminus E_\varepsilon} \frac{1}{|v_\varepsilon|}\left(v_\varepsilon\langle \mathscr{A}\, \nabla v_\varepsilon|, \nabla v_\varepsilon\rangle - \langle \mathscr{A}\, \nabla v_\varepsilon, v_\varepsilon\, \nabla|v_\varepsilon|\rangle\right)\, dx -$$
$$(1-2/p)^2\int_{\Omega\setminus E_\varepsilon} \langle \mathscr{A}\, \nabla|v_\varepsilon|, \nabla|v_\varepsilon|\rangle\, dx.$$

It is proved in [19] that if $u \in C^2(\overline{\Omega})$ and $u|_{\partial\Omega} = 0$, then

$$\lim_{\varepsilon\to 0} \varepsilon^r \int_{\Omega\setminus E_\varepsilon} |\nabla u|^2 dx = 0 \tag{2.46}$$

for any $r > -1$. Since

$$|\nabla |v_\varepsilon|| = \left| \mathscr{R}e \left( \frac{\overline{v}_\varepsilon \nabla v_\varepsilon}{|v_\varepsilon|} \chi_{E_0} \right) \right| \leqslant |\nabla v_\varepsilon| = \varepsilon^{\frac{p-2}{2}} |\nabla u|$$

in $E_0 \setminus E_\varepsilon$, we obtain

$$\left| \int_{\Omega\setminus E_\varepsilon} \langle \mathscr{A} \nabla |v_\varepsilon|, \nabla |v_\varepsilon| \rangle \, dx \right| \leqslant K \varepsilon^{p-2} \int_{\Omega\setminus E_\varepsilon} |\nabla u|^2 dx \to 0$$

as $\varepsilon \to 0$. We have also

$$|v_\varepsilon|^{-1} |\langle v_\varepsilon \mathscr{A} \nabla |v_\varepsilon|, \nabla v_\varepsilon \rangle - \langle \mathscr{A} \nabla v_\varepsilon, v_\varepsilon \nabla |v_\varepsilon| \rangle| \leqslant K \varepsilon^{p-2} |\nabla u|^2$$

and thus $R(\varepsilon) = o(1)$ as $\varepsilon \to 0$.

We have proved that

$$\begin{gathered} \mathscr{R}e \int_\Omega \langle \mathscr{A} \nabla u, \nabla [\varrho_\varepsilon^2(|u|)\, u] \rangle \, dx = \mathscr{R}e \Big[ \int_\Omega \langle \mathscr{A} \nabla v_\varepsilon, \nabla v_\varepsilon \rangle \, dx - \\ (1-2/p) \int_\Omega \langle (\mathscr{A} - \mathscr{A}^*) \nabla |v_\varepsilon|, |v_\varepsilon|^{-1} \overline{v}_\varepsilon \nabla v_\varepsilon \rangle dx - \\ (1-2/p)^2 \int_\Omega \langle \mathscr{A} \nabla |v_\varepsilon|, \nabla |v_\varepsilon| \rangle \, dx \Big] + o(1). \end{gathered} \tag{2.47}$$

By means of similar computations, we find by the identity

$$\begin{gathered} \int_\Omega \langle \mathbf{b} \nabla u, |u|^{p-2} u \rangle dx = \int_{\Omega\setminus E_\varepsilon} \langle \mathbf{b} \nabla u, |u|^{p-2} u \rangle dx - \\ (1-2/p) \int_{E_\varepsilon} \langle \mathbf{b}, |v_\varepsilon| \nabla (|v_\varepsilon|) \rangle dx + \int_{E_\varepsilon} \langle \mathbf{b} \nabla v_\varepsilon, v_\varepsilon \rangle dx \end{gathered}$$

that

$$\begin{gathered} \mathscr{R}e \int_\Omega \langle \mathbf{b} \nabla u, |u|^{p-2} u \rangle dx = \\ \int_\Omega \langle \mathscr{R}e(\mathbf{b}/p), \nabla (|v_\varepsilon|^2) \rangle dx - \int_\Omega \langle \mathscr{I}m\, \mathbf{b}, \mathscr{I}m(\overline{v}_\varepsilon \nabla v) \rangle dx + o(1). \end{gathered} \tag{2.48}$$

Moreover

$$\begin{gathered} \int_\Omega |u|^p dx = \int_{E_\varepsilon} |u|^p dx + \int_{\Omega\setminus E_\varepsilon} |u|^p dx = \\ \int_{E_\varepsilon} |v_\varepsilon|^2 dx + \int_{\Omega\setminus E_\varepsilon} |u|^p dx = \int_\Omega |v_\varepsilon|^2 dx + o(1). \end{gathered} \tag{2.49}$$

Equalities (2.47), (2.48) and (2.49) lead to

$$\begin{gathered}
\mathscr{R}e\int_\Omega (\langle \mathscr{A}\,\nabla u, \nabla[\varrho_\varepsilon^2(|u|)\,u]\rangle - \langle \mathbf{b}\nabla u, |u|^{p-2}u\rangle - a|u|^p)dx = \\
\mathscr{R}e\Big[\int_\Omega \langle \mathscr{A}\,\nabla v_\varepsilon, \nabla v_\varepsilon\rangle\, dx - \\
-(1-2/p)\int_\Omega \langle(\mathscr{A}-\mathscr{A}^*)\,\nabla |v_\varepsilon|, \nabla v_\varepsilon\rangle) v_\varepsilon |v_\varepsilon|^{-1} dx - \\
-(1-2/p)^2 \int_\Omega \langle \mathscr{A}\,\nabla |v_\varepsilon|, \nabla |v_\varepsilon|\rangle\, dx\Big] + \\
\int_\Omega \mathscr{R}e(\nabla^t(\mathbf{b}/p)|v_\varepsilon|^2 dx + \int_\Omega \langle \mathscr{I}m\,\mathbf{b}, \mathscr{I}m(\overline{v}_\varepsilon \nabla v)\rangle dx - \\
\int_\Omega \mathscr{R}e\, a\,|v_\varepsilon|^2 dx + o(1).
\end{gathered} \tag{2.50}$$

As far as the left-hand side of (2.50) is concerned, we have

$$\begin{gathered}
\int_\Omega \langle \mathscr{A}\,\nabla u, \nabla[\varrho_\varepsilon^2(|u|)\,u]\rangle\, dx = \\
\varepsilon^{p-2}\int_{\Omega\setminus E_\varepsilon} \langle \mathscr{A}\,\nabla u, \nabla u\rangle\, dx + \int_{E_\varepsilon} \langle \mathscr{A}\,\nabla u, \nabla(|u|^{p-2}u)\rangle\, dx
\end{gathered}$$

and then

$$\begin{gathered}
\lim_{\varepsilon\to 0}\mathscr{R}e\int_\Omega (\langle \mathscr{A}\,\nabla u, \nabla[\varrho_\varepsilon^2(|u|)\,u]\rangle - \langle \mathbf{b}\nabla u, |u|^{p-2}u\rangle - a|u|^p)dx = \\
\int_\Omega \langle \nabla u, \nabla(|u|^{p-2}u)\rangle - \langle \mathbf{b}\nabla u, |u|^{p-2}u\rangle - a|u|^p)dx.
\end{gathered}$$

Letting $\varepsilon \to 0$ in (2.50), we complete the proof of the necessity.

*Sufficiency.* Suppose that (2.45) holds. Let $v \in \Xi$ and let $u_\varepsilon$ be defined by (2.8). We have $u_\varepsilon \in \Xi$ and arguing as in the necessity part of Lemma 10, we find (2.9), (2.10) and (2.12). These limit relations lead to (2.42) for any $v \in \Xi$ and thus (2.42) is true for any $v \in H_0^1(\Omega)$ (see the proof of Lemma 11). In view of Lemma 11, the form $\mathscr{L}$ is $L^p$-dissipative. □

**Theorem 14** *The operator $A$ is $L^p$-dissipative if and only if the form $\mathscr{L}$ is $L^p$-dissipative.*

**Proof.**

*Necessity.* Let $u \in \Xi$ and $g_\varepsilon = (|u|^2+\varepsilon^2)^{\frac{1}{2}}$. Since $g_\varepsilon^{p-2}\overline{u} \in \Xi$ we have

$$-\int_\Omega \langle \nabla^t(\mathscr{A}\,\nabla u), u\rangle g_\varepsilon^{p-2} dx = \int_\Omega \langle \mathscr{A}\,\nabla u, \nabla(g_\varepsilon^{p-2}u)\rangle dx$$

and since

$$\partial_h(g_\varepsilon^{p-2}\overline{u}) = (p-2)g_\varepsilon^{p-4}\,\mathscr{R}e(\langle\partial_h u, u\rangle)\,\overline{u} + g_\varepsilon^{p-2}\partial_h\overline{u}$$

we have also

$$\partial_h(g_\varepsilon^{p-2}\overline{u}) =$$

$$\begin{cases} (p-2)|u|^{p-4}\,\mathscr{R}e(\langle\partial_h u, u\rangle)\,\overline{u} + |u|^{p-2}\partial_h\overline{u} = \partial_h(|u|^{p-2}\overline{u}) & \text{if } x \in F_0 \\ \varepsilon^{p-2}\partial_h\overline{u} & \text{if } x \in \Omega\setminus F_0. \end{cases}$$

We find, keeping in mind (2.46), that

$$\lim_{\varepsilon\to 0}\int_\Omega \langle\mathscr{A}\,\nabla u, \nabla(g_\varepsilon^{p-2}u)\rangle dx = \int_\Omega \langle\mathscr{A}\,\nabla u, \nabla(|u|^{p-2}u)\rangle dx\,.$$

On the other hand, using Lemma 3.3 in [20], we see that

$$\lim_{\varepsilon\to 0}\int_\Omega \langle\nabla^t(\mathscr{A}\,\nabla u), u\rangle g_\varepsilon^{p-2} dx = \int_\Omega \langle\nabla^t(\mathscr{A}\,\nabla u), u\rangle |u|^{p-2} dx.$$

Then

$$-\int_\Omega \langle\nabla^t(\mathscr{A}\,\nabla u), u\rangle |u|^{p-2} dx = \int_\Omega \langle\mathscr{A}\,\nabla u, \nabla(|u|^{p-2}u)\rangle dx \tag{2.51}$$

for any $u \in \Xi$. Hence

$$-\int_\Omega \langle Au, u\rangle |u|^{p-2} dx =$$

$$\int_\Omega (\langle\mathscr{A}\,\nabla u, \nabla(|u|^{p-2}u)\rangle - \langle\mathbf{b}\nabla u, |u|^{p-2}u\rangle - a\,|u|^p) dx\,.$$

Therefore (2.45) holds. We can conclude now that the form $\mathscr{L}$ is $L^p$-dissipative, because of Lemma 12.

*Sufficiency.* Given $u \in D(A)$, we can find a sequence $\{u_n\} \subset \Xi$ such that $u_n \to u$ in $W^{2,p}(\Omega)$. Keeping in mind (2.51), we have

$$-\int_\Omega \langle Au, u\rangle |u|^{p-2} dx = -\lim_{n\to\infty}\int_\Omega \langle Au_n, u_n\rangle |u_n|^{p-2} dx =$$

$$\lim_{n\to\infty}\int_\Omega \langle\mathscr{A}\,\nabla u_n, \nabla(|u_n|^{p-2}u_n)\rangle - \langle\mathbf{b}\nabla u_n, |u_n|^{p-2}u_n\rangle - a\,|u_n|^p) dx.$$

Since $\mathscr{L}$ is $L^p$-dissipative, (2.45) holds for any $u \in \Xi$ and (2.41) is true for any $u \in D(A)$.

□

### 2.4.2 Intervals of dissipativity

The next result permits to determine the best interval of $p$'s for which the operator

$$Au = \nabla^t(\mathscr{A}\,\nabla u) \tag{2.52}$$

is $L^p$-dissipative. We set

$$\lambda = \inf_{(\xi,x)\in\mathcal{M}} \frac{\langle \mathscr{R}e\,\mathscr{A}(x)\xi,\xi\rangle}{|\langle \mathscr{I}m\,\mathscr{A}(x)\xi,\xi\rangle|},$$

where $\mathcal{M}$ is the set of $(\xi, x)$ with $\xi \in \mathbb{R}^n$, $x \in \Omega$ such that $\langle \mathscr{I}m\,\mathscr{A}(x)\xi,\xi\rangle \neq 0$.

**Corollary 9** *Let $A$ be the operator* (2.52)*. Let us suppose that the matrix $\mathscr{I}m\,\mathscr{A}$ is symmetric and that*

$$\langle \mathscr{R}e\,\mathscr{A}(x)\xi,\xi\rangle \geqslant 0 \tag{2.53}$$

*for any $x \in \Omega$, $\xi \in \mathbb{R}^n$. If $\mathscr{I}m\,\mathscr{A}(x) = 0$ for any $x \in \Omega$, $A$ is $L^p$-dissipative for any $p > 1$. If $\mathscr{I}m\,\mathscr{A}$ does not vanish identically on $\Omega$, $A$ is $L^p$-dissipative if and only if*

$$2 + 2\lambda(\lambda - \sqrt{\lambda^2+1}) \leqslant p \leqslant 2 + 2\lambda(\lambda + \sqrt{\lambda^2+1}). \tag{2.54}$$

**Proof.**
When $\mathscr{I}m\,\mathscr{A}(x) = 0$ for any $x \in \Omega$, the statement follows from Theorem 12. Let us assume that $\mathscr{I}m\,\mathscr{A}$ does not vanish identically; note that this implies $\mathcal{M} \neq \emptyset$.

*Necessity.* If the operator (2.52) is $L^p$-dissipative, Theorem 12 shows that

$$|p-2|\,|\langle \mathscr{I}m\,\mathscr{A}(x)\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathscr{R}e\,\mathscr{A}(x)\xi,\xi\rangle \tag{2.55}$$

for any $x \in \Omega$, $\xi \in \mathbb{R}^n$. In particular we have

$$\frac{|p-2|}{2\sqrt{p-1}} \leqslant \frac{\langle \mathscr{R}e\,\mathscr{A}(x)\xi,\xi\rangle}{|\langle \mathscr{I}m\,\mathscr{A}(x)\xi,\xi\rangle|}$$

for any $(\xi, x) \in \mathcal{M}$ and then

$$\frac{|p-2|}{2\sqrt{p-1}} \leqslant \lambda.$$

This inequality is equivalent to (2.54).

*Sufficiency.* If (2.54) holds, we have $(p-2)^2 \leqslant 4(p-1)\lambda^2$. Note that $p > 1$, because $2 + 2\lambda(\lambda - \sqrt{\lambda^2+1}) > 1$.

Since $\lambda \geqslant 0$ in view of (2.53), we find $|p-2| \leqslant 2\sqrt{p-1}\lambda$ and (2.55) is true for any $(\xi, x) \in \mathcal{M}$. On the other hand, if $x \in \Omega$ and $\xi \in \mathbb{R}^n$ with $(\xi, x) \notin \mathcal{M}$, (2.55) is trivially satisfied and then it holds for any $x \in \Omega$, $\xi \in \mathbb{R}^n$. Theorem 12 gives the result. □

The next Corollary provides a characterization of operators which are $L^p$-dissipative only for $p = 2$.

**Corollary 10** *Let $A$ be as in Corollary 9. The operator $A$ is $L^p$-dissipative only for $p = 2$ if and only if $\mathscr{I}m\, \mathscr{A}$ does not vanish identically and $\lambda = 0$.*

**Proof.** Inequalities (2.54) are satisfied only for $p = 2$ if and only if $\lambda(\lambda - \sqrt{\lambda^2 - 1}) = \lambda(\lambda + \sqrt{\lambda^2 + 1})$ and this happens if and only if $\lambda = 0$. Thus the result is a consequence of Corollary 9. □

### 2.4.3 Contractive semigroups generated by the operator $\mathrm{div}(\mathscr{A}\, \nabla u)$

Let $A$ be the operator $\mathrm{div}(\mathscr{A}\, \nabla u)$ with smooth coefficients. In this subsection we want to investigate when $A$ generates a contraction semigroup.

In the next Theorem we suppose that $A$ is strongly elliptic, i.e.

$$\langle \mathscr{R}e\, \mathscr{A}(x)\xi, \xi \rangle > 0$$

for any $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^n \setminus \{0\}$.

**Theorem 15** *Let $A$ be the strongly elliptic operator* (2.52) *with $\mathscr{I}m\, \mathscr{A} = \mathscr{I}m\, \mathscr{A}^t$. The operator $A$ generates a contraction semigroup on $L^p$ if and only if*

$$|p-2|\, |\langle \mathscr{I}m\, \mathscr{A}(x)\xi, \xi \rangle| \leqslant 2\sqrt{p-1}\, \langle \mathscr{R}e\, \mathscr{A}(x)\xi, \xi \rangle \tag{2.56}$$

*for any $x \in \Omega$, $\xi \in \mathbb{R}^n$.*

**Proof.**

*Sufficiency.* It is a classical result that the operator $A$ defined on (2.40) and acting in $L^p(\Omega)$ is a densely defined closed operator (see [1], [28, Theorem 1, p.302]).

From Theorem 12 we know that the form $\mathscr{L}$ is $L^p$-dissipative and Theorem 14 shows that $A$ is $L^p$-dissipative. Finally the formal adjoint operator

$$A^* u = \nabla^t (\mathscr{A}^*\, \nabla u)$$

with $D(A^*) = W^{2,p'}(\Omega) \cap \mathring{W}^{1,p'}(\Omega)$, is the adjoint operator of $A$ and since $\mathscr{I}m\,\mathscr{A}^* = \mathscr{I}m(\mathscr{A}^*)^t$ and (2.56) can be written as

$$|p'-2|\,|\langle \mathscr{I}m\,\mathscr{A}^*(x)\xi,\xi\rangle| \leqslant 2\sqrt{p'-1}\,\langle \mathscr{R}e\,\mathscr{A}^*(x)\xi,\xi\rangle, \tag{2.57}$$

we have also the $L^{p'}$-dissipativity of $A^*$.

The result is a consequence of Theorem 11.

*Necessity.* If $A$ generates a contraction semigroup on $L^p$, it is $L^p$-dissipative. Therefore (2.56) holds because of Theorem 12. □

### 2.4.4 Quasi-dissipativity and quasi-contractivity

We know that, in case either A has lower order terms or they are absent and $\mathscr{I}m\,\mathscr{A}$ is not symmetric, condition (2.56) is not sufficient for the $L^p$-dissipativity. As we shall see now, it turns out that, for these more general operators, (2.56) is necessary and sufficient for the so called quasi-dissipativity of $A$, i.e. the dissipativity of $A-\omega I$ for a suitable $\omega>0$. In other words, $A$ is $L^p$-quasi-dissipative if there exists $\omega \geqslant 0$ such that

$$\mathscr{R}e\int_\Omega \langle Au,u\rangle |u|^{p-2}dx \leqslant \omega\,\|u\|_p^p$$

for any $u\in D(A)$.

As a consequence, condition (2.56) is necessary and sufficient for the quasi-contractivity of the semigroup generated by $A$ (see Theorem 17 below).

**Lemma 13** *The operator* (2.39) *is $L^p$-quasi-dissipative if and only if there exists $\omega\geqslant 0$ such that*

$$\begin{gathered}\mathscr{R}e\int_\Omega \Big[\langle \mathscr{A}\,\nabla v,\nabla v\rangle - (1-2/p)\langle(\mathscr{A}-\mathscr{A}^*)\nabla(|v|),|v|^{-1}\overline{v}\nabla v\rangle - \\ (1-2/p)^2\langle \mathscr{A}\,\nabla(|v|),\nabla(|v|)\rangle\Big]dx + \int_\Omega \langle \mathscr{I}m\,\mathbf{b},\mathscr{I}m(\overline{v}\nabla v)\rangle\,dx + \\ \int_\Omega \mathscr{R}e(\operatorname{div}(\mathbf{b}/p)-a)|v|^2dx \geqslant -\omega\int_\Omega |v|^2dx\end{gathered} \tag{2.58}$$

*for any $v\in H_0^1(\Omega)$.*

**Proof.** The result follows from Lemma 11. □

**Theorem 16** *The strongly elliptic operator* (2.39) *is* $L^p$*-quasi-dissipative if and only if*

$$|p-2|\,|\langle \mathscr{I}m\,\mathscr{A}(x)\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathscr{R}e\,\mathscr{A}(x)\xi,\xi\rangle \tag{2.59}$$

*for any* $x\in\Omega$, $\xi\in\mathbb{R}^n$.

**Proof.**

*Necessity.* By using the functions $X$, $Y$ introduced in Corollary 4, we write condition (2.58) in the form

$$\int_\Omega \Big\{ \frac{4}{p\,p'}\langle \mathscr{R}e\,\mathscr{A}\,X,X\rangle + \langle \mathscr{R}e\,\mathscr{A}\,Y,Y\rangle +$$
$$2\langle (p^{-1}\,\mathscr{I}m\,\mathscr{A} + p'^{-1}\,\mathscr{I}m\,\mathscr{A}^*)X,Y\rangle + \langle \mathscr{I}m\,\mathbf{b},Y\rangle |v| +$$
$$\mathscr{R}e\,[\mathrm{div}(\mathbf{b}/p)-a+\omega]\,|v|^2 \Big\} dx \geqslant 0\,.$$

As in the proof of Corollary 4, this inequality implies

$$\frac{4}{p\,p'}\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\nabla\varrho,\nabla\varrho\rangle dx + \mu^2\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\nabla\varrho,\nabla\varrho\rangle dx +$$
$$2\mu\int_\Omega \langle (p^{-1}\,\mathscr{I}m\,\mathscr{A} + p'^{-1}\,\mathscr{I}m\,\mathscr{A}^*)\nabla\varrho,\nabla\varrho\rangle dx +$$
$$\mu\int_\Omega \varrho\langle \mathscr{I}m\,\mathbf{b},\nabla\varrho\rangle dx + \int_\Omega \mathscr{R}e\,[\mathrm{div}\,(\mathbf{b}/p)-a+\omega]\,\varrho^2 dx \geqslant 0$$

for any $\varrho\in\mathring{C}^1(\Omega)$, $\mu\in\mathbb{R}$. Since

$$\langle \mathscr{I}m\,\mathscr{A}^*\,\nabla\varrho,\nabla\varrho\rangle = -\langle \mathscr{I}m\,\mathscr{A}^t\,\nabla\varrho,\nabla\varrho\rangle = -\langle \mathscr{I}m\,\mathscr{A}\,\nabla\varrho,\nabla\varrho\rangle$$

we have

$$\frac{4}{p\,p'}\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\nabla\varrho,\nabla\varrho\rangle dx + \mu^2\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\nabla\varrho,\nabla\varrho\rangle dx -$$
$$2(1-2/p)\mu\int_\Omega \langle \mathscr{I}m\,\mathscr{A}\,\nabla\varrho,\nabla\varrho\rangle dx +$$
$$\mu\int_\Omega \varrho\langle \mathscr{I}m\,\mathbf{b},\nabla\varrho\rangle dx + \int_\Omega \mathscr{R}e\,[\mathrm{div}\,(\mathbf{b}/p)-a+\omega]\,\varrho^2 dx \geqslant 0$$

for any $\varrho\in\mathring{C}^1(\Omega)$, $\mu\in\mathbb{R}$.

Taking $\varrho(x)=\psi(x)\cos\langle\xi,x\rangle$ and $\varrho(x)=\psi(x)\sin\langle\xi,x\rangle$ with $\psi\in\mathring{C}^1(\Omega)$ and arguing as in the proof of Corollary 4, we find

$$\int_\Omega \langle \mathscr{B}\,\nabla\psi,\nabla\psi\rangle dx + \int_\Omega \langle \mathscr{B}\,\xi,\xi\rangle \psi^2 dx +$$
$$\mu\int_\Omega \langle \mathscr{I}m\,\mathbf{b},\nabla\psi\rangle\psi\, dx + \int_\Omega \mathscr{R}e\,[\mathrm{div}\,(\mathbf{b}/p)-a+\omega]\,\psi^2 dx \geqslant 0\,,$$

where $\mu \in \mathbb{R}$ and

$$\mathscr{B} = \frac{4}{p\,p'}\,\mathscr{R}e\,\mathscr{A} + \mu^2\,\mathscr{R}e\,\mathscr{A} - 2(1-2/p)\mu\,\mathscr{I}m\,\mathscr{A}\ .$$

Because of the arbitrariness of $\xi$ we see that

$$\int_\Omega \langle \mathscr{B}\,\xi,\xi\rangle \psi^2 dx \geqslant 0$$

for any $\psi \in \mathring{C}^1(\Omega)$. Hence $\langle \mathscr{B}\,\xi,\xi\rangle \geqslant 0$, i.e.

$$\frac{4}{p\,p'}\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle + \mu^2\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle - 2(1-2/p)\mu\langle \mathscr{I}m\,\mathscr{A}\,\xi,\xi\rangle \geqslant 0$$

for any $x \in \Omega$, $\xi \in \mathbb{R}^n$, $\mu \in \mathbb{R}$. Inequality (2.59) follows from the arbitrariness of $\mu$.

*Sufficiency.* Assume first that $\mathscr{I}m\,\mathscr{A}$ is symmetric. By repeating the first part of the proof of sufficiency of Theorem 12, we find that (2.59) implies

$$\frac{4}{p\,p'}\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle + \langle \mathscr{R}e\,\mathscr{A}\,\eta,\eta\rangle - 2(1-p/2)\langle \mathscr{I}m\,\mathscr{A}\,\xi,\eta\rangle \geqslant 0 \tag{2.60}$$

for any $x \in \Omega$, $\xi,\eta \in \mathbb{R}^n$.

In order to prove (2.58), it is not restrictive to suppose

$$\mathscr{R}e(\mathrm{div}(\mathbf{b}/p) - a) = 0.$$

Since $A$ is strongly elliptic, there exists a non singular real matrix $\mathscr{C} \in C^1(\overline{\Omega})$ such that

$$\langle \mathscr{R}e\,\mathscr{A}\,\eta,\eta\rangle = \langle \mathscr{C}\,\eta,\mathscr{C}\,\eta\rangle$$

for any $\eta \in \mathbb{R}^n$. Setting

$$\mathscr{S} = (1-2/p)(\mathscr{C}^t)^{-1}\,\mathscr{I}m\,\mathscr{A},$$

we have

$$|\,\mathscr{C}\,\eta - \mathscr{S}\,\xi|^2 = \langle \mathscr{R}e\,\mathscr{A}\,\eta,\eta\rangle - 2(1-p/2)\langle \mathscr{I}m\,\mathscr{A}\,\xi,\eta\rangle + |\,\mathscr{S}\,\xi|^2.$$

This leads to the identity

$$\begin{gathered}\frac{4}{p\,p'}\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle + \langle \mathscr{R}e\,\mathscr{A}\,\eta,\eta\rangle - 2(1-p/2)\langle \mathscr{I}m\,\mathscr{A}\,\xi,\eta\rangle = \\ |\,\mathscr{C}\,\eta - \mathscr{S}\,\xi|^2 + \frac{4}{p\,p'}\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle - |\,\mathscr{S}\,\xi|^2\end{gathered} \tag{2.61}$$

for any $\xi, \eta \in \mathbb{R}^n$. In view of (2.60), putting $\eta = \mathscr{C}^{-1}\mathscr{S}\xi$ in (2.61), we obtain

$$\frac{4}{p\,p'}\langle \mathscr{R}e\,\mathscr{A}\,\xi, \xi\rangle - |\,\mathscr{S}\,\xi|^2 \geqslant 0 \tag{2.62}$$

for any $\xi \in \mathbb{R}^n$.

On the other hand, we may write

$$\langle \mathscr{I}m\,\mathbf{b}, Y\rangle = \langle (\mathscr{C}^{-1})^t\,\mathscr{I}m\,\mathbf{b}, \mathscr{C}\,Y\rangle =$$
$$\langle (\mathscr{C}^{-1})^t\,\mathscr{I}m\,\mathbf{b}, \mathscr{C}\,Y - \mathscr{S}\,X\rangle + \langle (\mathscr{C}^{-1})^t\,\mathscr{I}m\,\mathbf{b}, \mathscr{S}\,X\rangle\,.$$

By the Cauchy inequality

$$\int_\Omega \langle (\mathscr{C}^{-1})^t\,\mathscr{I}m\,\mathbf{b}, \mathscr{C}\,Y - \mathscr{S}\,X\rangle |v|\,dx \geqslant$$
$$-\int_\Omega |\,\mathscr{C}\,Y - \mathscr{S}\,X|^2 dx - \frac{1}{4}\int_\Omega |(\mathscr{C}^{-1})^t\,\mathscr{I}m\,\mathbf{b}|^2 |v|^2 dx$$

and, integrating by parts,

$$\int_\Omega \langle (\mathscr{C}^{-1})^t\,\mathscr{I}m\,\mathbf{b}, \mathscr{S}\,X\rangle |v|\,dx = \frac{1}{2}\int_\Omega \langle (\mathscr{C}^{-1}\,\mathscr{S})^t\,\mathscr{I}m\,\mathbf{b}, \nabla(|v|^2)\rangle\,dx =$$
$$-\frac{1}{2}\int_\Omega \nabla^t((\mathscr{C}^{-1}\,\mathscr{S})^t\,\mathscr{I}m\,\mathbf{b})\,|v|^2\,dx\,.$$

This implies that there exists $\omega \geqslant 0$ such that

$$\int_\Omega \langle \mathscr{I}m\,\mathbf{b}, Y\rangle\,|v|\,dx \geqslant -\int_\Omega |\,\mathscr{C}\,Y - \mathscr{S}\,X|^2 dx - \omega\int_\Omega |v|^2 dx$$

and then, in view of (2.61),

$$\int_\Omega \Big\{ \frac{4}{p\,p'}\langle \mathscr{R}e\,\mathscr{A}\,X, X\rangle + \langle \mathscr{R}e\,\mathscr{A}\,Y, Y\rangle +$$
$$2(1-p/2)\langle \mathscr{I}m\,\mathscr{A}\,X, Y\rangle + \langle \mathscr{I}m\,\mathbf{b}, Y\rangle |v| \Big\} dx \geqslant$$
$$\int_\Omega \left( \frac{4}{p\,p'}\langle \mathscr{R}e\,\mathscr{A}\,X, X\rangle - |\,\mathscr{S}\,X|^2 \right) dx - \omega\int_\Omega |v|^2 dx\ .$$

Inequality (2.62) gives the result.

We have proved the sufficiency under the assumption $\mathscr{I}m\,\mathscr{A}^t = \mathscr{I}m\,\mathscr{A}$. In the general case, the operator $A$ can be written in the form

$$Au = \nabla^t((\mathscr{A} + \mathscr{A}^t)\nabla u)/2 + \mathbf{c}\nabla u + au,$$

where

$$\mathbf{c} = \nabla^t(\mathscr{A} - \mathscr{A}^t)/2 + \mathbf{b}.$$

Since $(\mathscr{A} + \mathscr{A}^t)$ is symmetric, we know that $A$ is $L^p$-quasi-dissipative if and only if

$$|p-2|\,|\langle \mathscr{I}m(\mathscr{A} + \mathscr{A}^t)\xi, \xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathscr{R}e(\mathscr{A} + \mathscr{A}^t)\xi, \xi\rangle$$

for any $\xi \in \mathbb{R}^n$, which is exactly condition (2.59). □

With Theorem 16 in hand, we may obtain the following corollary.

**Corollary 11** *Let $A$ be the strongly elliptic operator* (2.39). *If $\mathscr{I}m\,\mathscr{A}(x) = 0$ for any $x \in \Omega$, $A$ is $L^p$-quasi-dissipative for any $p > 1$. If $\mathscr{I}m\,\mathscr{A}$ does not vanish identically on $\Omega$, $A$ is $L^p$-quasi-dissipative if and only if* (2.54) *holds.*

**Proof.** The proof is similar to that of Corollary 9, the role of Theorem 12 being played by Theorem 16. □

The next theorem gives a criterion for the $L^p$-quasi-contractivity of the semigroup generated by $A$ (i.e. the $L^p$-contractivity of the semigroup generated by $A - \omega I$).

**Theorem 17** *Let $A$ be the strongly elliptic operator* (2.39). *The operator $A$ generates a quasi-contraction semigroup on $L^p$ if and only if* (2.56) *holds for any $x \in \Omega$, $\xi \in \mathbb{R}^n$.*

**Proof.**

*Sufficiency.* Let us consider $A$ as an operator defined on (2.40) and acting in $L^p(\Omega)$. As in the proof of Theorem 15, one can see that $A$ is a densely defined closed operator and that the formal adjoint coincides with the adjoint $A^*$. Theorem 16 shows that $A$ is $L^p$-quasi-dissipative. On the other hand, condition (2.57) holds and then $A^*$ is $L^{p'}$-quasi-dissipative. As in Theorem 15, this implies that $A$ generates a quasi-contraction semigroup on $L^p$.

*Necessity.* If $A$ generates a quasi-contraction semigroup on $L^p$, $A$ is $L^p$-quasi-dissipative and (2.56) holds. □

## 2.5 The angle of dissipativity

In this section we want to determine the angle of dissipativity of the operator

$$A = \nabla^t(\mathscr{A}(x)\nabla),$$

where $\mathscr{A} = \{a_{ij}(x)\}$ $(i,j = 1,\dots,n)$ is a matrix with complex locally integrable entries defined in a domain $\Omega \subset \mathbb{R}^n$.

This means that we want to find the complex values $z$ such that the operator $zA$ is $L^p$-dissipative, provided $A$ itself is $L^p$-dissipative.

It is known that, if $\mathscr{A}$ is a real matrix, then $zA$ is dissipative if and only if $z$ belongs to a certain angle, which does not depend on the operator $A$ (see Remark 9 below). We shall find that for a complex matrix $\mathscr{A}$ the situation is quite different: $zA$ is $L^p$-dissipative if and only if $z$ belongs to a certain angle, which depends on the operator $A$.

We know that, if $\mathscr{I}m\,\mathscr{A}$ is symmetric, there is the $L^p$-dissipativity of the Dirichlet problem for the differential operator $A$ if and only if

$$|p-2|\,|\langle \mathscr{I}m\,\mathscr{A}(x)\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\langle \mathscr{R}e\,\mathscr{A}(x)\xi,\xi\rangle \tag{2.63}$$

for almost every $x \in \Omega$ and for any $\xi \in \mathbb{R}^n$.

For the sake of completeness we give a proof of the following elementary lemma

**Lemma 14** *Let $P$ and $Q$ two real measurable functions defined on a set $\Omega \subset \mathbb{R}^n$. Let us suppose that $P(x) \geqslant 0$ almost everywhere. The inequality*

$$P(x)\,\cos\vartheta - Q(x)\,\sin\vartheta \geqslant 0 \qquad (\vartheta \in [-\pi,\pi]) \tag{2.64}$$

*holds for almost every $x \in \Omega$ if and only if*

$$\operatorname{arccot}\,[\operatorname*{ess\,inf}_{x\in\Xi}\,(Q(x)/P(x))] - \pi \leqslant \vartheta \leqslant \operatorname{arccot}\,[\operatorname*{ess\,sup}_{x\in\Xi}\,(Q(x)/P(x))], \tag{2.65}$$

*where $\Xi = \{x \in \Omega \mid P^2(x) + Q^2(x) > 0\}$ and we set*

$$Q(x)/P(x) = \begin{cases} +\infty & \text{if } P(x) = 0,\ Q(x) > 0 \\ -\infty & \text{if } P(x) = 0,\ Q(x) < 0. \end{cases}$$

*Here $0 < \operatorname{arccot} y < \pi$, $\operatorname{arccot}(+\infty) = 0$, $\operatorname{arccot}(-\infty) = \pi$ and*

$$\operatorname*{ess\,inf}_{x\in\Xi}\,(Q(x)/P(x)) = +\infty, \quad \operatorname*{ess\,sup}_{x\in\Xi}\,(Q(x)/P(x)) = -\infty$$

*if $\Xi$ has zero measure.*

**Proof.** If $\Xi$ has positive measure and $P(x) > 0$, inequality (2.64) means

$$\cos\vartheta - (Q(x)/P(x))\,\sin\vartheta \geqslant 0$$

and this is true if and only if

$$\operatorname{arccot}\,(Q(x)/P(x)) - \pi \leqslant \vartheta \leqslant \operatorname{arccot}\,(Q(x)/P(x)). \tag{2.66}$$

If $x \in \Xi$ and $P(x) = 0$, (2.64) means

$$-\pi \leqslant \vartheta \leqslant 0, \text{ if } Q(x) > 0, \quad 0 \leqslant \vartheta \leqslant \pi, \text{ if } Q(x) < 0.$$

This shows that (2.64) is equivalent to (2.66) provided that $x \in \Xi$. On the other hand, if $x \notin \Xi$, $P(x) = Q(x) = 0$ almost everywhere and (2.64) is always satisfied. Therefore, if $\Xi$ has positive measure, (2.64) and (2.65) are equivalent.

If $\Xi$ has zero measure, the result is trivial. □

The next theorem provides a necessary and sufficient condition for the $L^p$-dissipativity of the Dirichlet problem for the differential operator $zA$, where $z \in \mathbb{C}$.

**Theorem 18** *Let the matrix $\mathscr{A}$ be symmetric. Let us suppose that the operator $A$ is $L^p$-dissipative. Set*

$$\Lambda_1 = \operatorname*{ess\,inf}_{(x,\xi)\in\Xi} \frac{\langle \mathscr{I}m\, \mathscr{A}(x)\xi, \xi\rangle}{\langle \mathscr{R}e\, \mathscr{A}(x)\xi, \xi\rangle}, \quad \Lambda_2 = \operatorname*{ess\,sup}_{(x,\xi)\in\Xi} \frac{\langle \mathscr{I}m\, \mathscr{A}(x)\xi, \xi\rangle}{\langle \mathscr{R}e\, \mathscr{A}(x)\xi, \xi\rangle},$$

*where*

$$\Xi = \{(x,\xi) \in \Omega \times \mathbb{R}^n \mid \langle \mathscr{R}e\, \mathscr{A}(x)\xi, \xi\rangle > 0\}. \tag{2.67}$$

*The operator $zA$ is $L^p$-dissipative if and only if*

$$\vartheta_- \leqslant \arg z \leqslant \vartheta_+\,, \tag{2.68}$$

*where*

$$\vartheta_- = \begin{cases} \operatorname{arccot}\left(\frac{2\sqrt{p-1}}{|p-2|} - \frac{p^2}{|p-2|}\,\frac{1}{2\sqrt{p-1}+|p-2|\Lambda_1}\right) - \pi & \text{if } p \neq 2 \\ \operatorname{arccot}(\Lambda_1) - \pi & \text{if } p = 2 \end{cases}$$

$$\vartheta_+ = \begin{cases} \operatorname{arccot}\left(-\frac{2\sqrt{p-1}}{|p-2|} + \frac{p^2}{|p-2|}\,\frac{1}{2\sqrt{p-1}-|p-2|\Lambda_2}\right) & \text{if } p \neq 2 \\ \operatorname{arccot}(\Lambda_2) & \text{if } p = 2. \end{cases}$$

**Proof.** The matrix $\mathscr{A}$ being symmetric, $\mathscr{I}m(e^{i\vartheta}A)$ is symmetric and in view of (2.63), the operator $e^{i\vartheta}A$ (with $\vartheta \in [-\pi,\pi]$) is $L^p$-dissipative if and only if

$$\begin{aligned} &|p-2|\,|\langle \mathscr{R}e\, \mathscr{A}(x)\xi,\xi\rangle \sin\vartheta + \langle \mathscr{I}m\, \mathscr{A}(x)\xi,\xi\rangle\cos\vartheta| \leqslant \\ &2\sqrt{p-1}(\langle \mathscr{R}e\, \mathscr{A}(x)\xi,\xi\rangle\cos\vartheta - \langle \mathscr{I}m\, \mathscr{A}(x)\xi,\xi\rangle \sin\vartheta) \end{aligned} \tag{2.69}$$

for almost every $x \in \Omega$ and for any $\xi \in \mathbb{R}^n$. Suppose $p \neq 2$. Setting

$$\begin{aligned} a(x,\xi) &= |p-2|\,\langle \mathscr{R}e\, \mathscr{A}(x)\xi,\xi\rangle, \quad b(x,\xi) = |p-2|\,\langle \mathscr{I}m\, \mathscr{A}(x)\xi,\xi\rangle, \\ c(x,\xi) &= 2\sqrt{p-1}\,\langle \mathscr{R}e\, \mathscr{A}(x)\xi,\xi\rangle, \quad d(x,\xi) = 2\sqrt{p-1}\,\langle \mathscr{I}m\, \mathscr{A}(x)\xi,\xi\rangle, \end{aligned}$$

the inequality in (2.69) can be written as the system

$$\begin{cases} (c(x,\xi)-b(x,\xi))\cos\vartheta-(a(x,\xi)+d(x,\xi))\sin\vartheta\geqslant 0,\\ (c(x,\xi)+b(x,\xi))\cos\vartheta+(a(x,\xi)-d(x,\xi))\sin\vartheta\geqslant 0. \end{cases} \tag{2.70}$$

Noting that $c(x,\xi)\pm b(x,\xi)\geqslant 0$ because of (2.63), the solutions of the inequalities in (2.70) are given by the $\vartheta$'s satisfying both of the following conditions (see Lemma 14)

$$\begin{cases} \operatorname{arccot}\left(\operatorname*{ess\,inf}_{(x,\xi)\in\Xi_1}\frac{a(x,\xi)+d(x,\xi)}{c(x,\xi)-b(x,\xi)}\right)-\pi\leqslant\vartheta\leqslant\operatorname{arccot}\left(\operatorname*{ess\,sup}_{(x,\xi)\in\Xi_1}\frac{a(x,\xi)+d(x,\xi)}{c(x,\xi)-b(x,\xi)}\right)\\ \operatorname{arccot}\left(\operatorname*{ess\,inf}_{(x,\xi)\in\Xi_2}\frac{d(x,\xi)-a(x,\xi)}{c(x,\xi)+b(x,\xi)}\right)-\pi\leqslant\vartheta\leqslant\operatorname{arccot}\left(\operatorname*{ess\,sup}_{(x,\xi)\in\Xi_2}\frac{d(x,\xi)-a(x,\xi)}{c(x,\xi)+b(x,\xi)}\right), \end{cases} \tag{2.71}$$

where

$$\Xi_1=\{(x,\xi)\in\Omega\times\mathbb{R}^n\mid (a(x,\xi)+d(x,\xi))^2+(c(x,\xi)-b(x,\xi))^2>0\},$$
$$\Xi_2=\{(x,\xi)\in\Omega\times\mathbb{R}^n\mid (a(x,\xi)-d(x,\xi))^2+(b(x,\xi)+c(x,\xi))^2>0\}.$$

We have

$$a(x,\xi)\,d(x,\xi)=b(x,\xi)\,c(x,\xi),$$
$$a^2(x,\xi)+b^2(x,\xi)+c^2(x,\xi)+d^2(x,\xi)=p^2(\langle\mathscr{Re}\,\mathscr{A}(x)\xi,\xi\rangle^2+\langle\mathscr{Im}\,\mathscr{A}(x)\xi,\xi\rangle^2)$$

and then, keeping in mind (2.63), we may write $\Xi_1=\Xi_2=\Xi$, where $\Xi$ is given by (2.67).

Moreover
$$\frac{a(x,\xi)+d(x,\xi)}{c(x,\xi)-b(x,\xi)}\geqslant\frac{d(x,\xi)-a(x,\xi)}{c(x,\xi)+b(x,\xi)}$$
and then $\vartheta$ satisfies all of the inequalities in (2.71) if and only if

$$\operatorname{arccot}\left(\operatorname*{ess\,inf}_{(x,\xi)\in\Xi}\frac{d(x,\xi)-a(x,\xi)}{c(x,\xi)+b(x,\xi)}\right)-\pi\leqslant\vartheta\leqslant\operatorname{arccot}\left(\operatorname*{ess\,sup}_{(x,\xi)\in\Xi}\frac{a(x,\xi)+d(x,\xi)}{c(x,\xi)-b(x,\xi)}\right) \tag{2.72}$$

A direct computation shows that

$$\frac{d(x,\xi)-a(x,\xi)}{c(x,\xi)+b(x,\xi)}=\frac{2\sqrt{p-1}}{|p-2|}-\frac{p^2}{|p-2|}\,\frac{1}{2\sqrt{p-1}+|p-2|\Lambda(x,\xi)},$$
$$\frac{a(x,\xi)+d(x,\xi)}{c(x,\xi)-b(x,\xi)}=-\frac{2\sqrt{p-1}}{|p-2|}+\frac{p^2}{|p-2|}\,\frac{1}{2\sqrt{p-1}-|p-2|\Lambda(x,\xi)}$$

where

$$\Lambda(x,\xi) = \frac{\langle \mathscr{I}m\, \mathscr{A}(x)\xi,\xi\rangle}{\langle \mathscr{R}e\, \mathscr{A}(x)\xi,\xi\rangle}\,.$$

Hence condition (2.72) is satisfied if and only if (2.68) holds.

If $p=2$, (2.69) is simply

$$\langle \mathscr{R}e\, \mathscr{A}(x)\xi,\xi\rangle \cos\vartheta - \langle \mathscr{I}m\, \mathscr{A}(x)\xi,\xi\rangle \sin\vartheta \geqslant 0$$

and the result follows directly from Lemma 14. □

**Remark 9** If $\mathscr{A}$ is a real matrix, then $\Lambda_1 = \Lambda_2 = 0$ and the angle of dissipativity does not depend on the operator. In fact we have

$$\frac{2\sqrt{p-1}}{|p-2|} - \frac{p^2}{2\sqrt{p-1}|p-2|} = -\frac{|p-2|}{2\sqrt{p-1}}$$

and Theorem 18 shows that $zA$ is dissipative if and only if

$$\operatorname{arccot}\left(-\frac{|p-2|}{2\sqrt{p-1}}\right) - \pi \leqslant \arg z \leqslant \operatorname{arccot}\left(\frac{|p-2|}{2\sqrt{p-1}}\right),$$

i.e.

$$|\arg z| \leqslant \arctan\left(\frac{2\sqrt{p-1}}{|p-2|}\right).$$

This is a well known result (see, e.g., [12, 13, 31]).

# Chapter 3

# Systems and higher order operators

In this Chapter we survey several results concerning the $L^p$-dissipativity of systems of partial differential operators and the problem of the dissipativity of higher order operators.

Such results are connected to the ones contained in the previous Chapter.

We are not going to give the proofs, but for each result, we indicate the paper containing them.

## 3.1 Systems of partial differential operators

### 3.1.1 $L^p$-contractivity for weakly coupled systems

In this Section we consider the operator

$$\mathcal{A}_p u = \partial_i(a_{ij}\partial_j u) + a_i \partial_i u + Au, \qquad u \in (W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))^N, \tag{3.1}$$

where $a_{ij}, a_i$ are $C^1(\overline{\Omega})$ real functions and $A$ is a $N \times N$-matrix with complex $C^0(\overline{\Omega})$ entries. The matrix $\{a_{ij}\}$ is supposed to be pointwise symmetric.

We introduce also the following operator which is associated to (3.1)

$$\mathcal{A}u = \frac{4}{pp'}\partial_i(a_{ij}\partial_j u) + \frac{1}{2p}(p(A + A^*) - 2\partial_i a_i I)u, \tag{3.2}$$

where $u \in (H^2(\Omega) \cap H_0^1(\Omega))^N$.

The following result obtained by Langer and Maz'ya shows that the $L^p$-contractivity of the operator $\mathcal{A}_p$ is related to the $L^2$-contractivity of the operator $\mathcal{A}$.

**Theorem 19 ([19])** *Let $\Omega \subset \mathbb{R}^n$ a bounded domain with a $C^{2,\alpha}$ boundary ($0 < \alpha \leqslant 1$) and let $\mathcal{A}_p$ the operator* (3.1) *which is supposed to be elliptic. If the operator* (3.2) *generates a contraction semigroup on $(L^2(\Omega))^N$, then $\mathcal{A}_p$ generates a contraction semigroup on $(L^p(\Omega))^N$. Conversely, if there is a basis of constant eigenvectors to $A+A^*$, then $\mathcal{A}$ generates a contraction semigroup on $(L^2(\Omega))^N$ if $\mathcal{A}_p$ generates a contraction semigroup on $(L^p(\Omega))^N$. In particular, the converse holds in the scalar case.*

This result was obtained by considering the functionals

$$\begin{aligned} J(w) &= \int_\Omega \Big( \frac{4}{pp'} a_{ij} \langle \partial_i w, \partial_j w \rangle + \mathscr{R}e \langle (p^{-1} \partial_i a_i I - A) w, w \rangle \Big)\, dx, \\ J_p(w) &= \int_\Omega \Big( a_{ij} \langle \partial_i w, \partial_j w \rangle + \mathscr{R}e \langle (p^{-1} \partial_i a_i I - A) w, w \rangle \Big)\, dx \\ &\quad - \frac{(p-2)^2}{p^2} \int_{\{w \neq 0\}} a_{ij}\, \mathscr{R}e \langle \partial_i w, w \rangle\, \mathscr{R}e \langle \partial_j w, w \rangle |w|^{-2}\, dx, \end{aligned}$$

and the related constants

$$\begin{aligned} \mu &= \inf\{ J(w) : w \in (H_0^1(\Omega))^N, \|w\|_2 = 1 \}, \\ \mu_p &= \inf\{ J_p(w) : w \in (H_0^1(\Omega))^N, \|w\|_2 = 1 \}. \end{aligned}$$

**Lemma 15 ([19])** *The operator $\mathcal{A}_p$ is dissipative in $(L^p(\Omega))^N$ if and only if $\mu_p \geqslant 0$.*

We have also

**Lemma 16 ([19])** *Let $1 < p < \infty$ and suppose that the principal part of $\mathcal{A}_p$ is positive. Then $\mathcal{A}$ and $\mathcal{A}_p$ generate the semigroups $T$ on $(L^2(\Omega))^N$ and $T_p$ on $(L^p(\Omega))^N$, respectively, fulfilling the inequalities*

$$\|T(t)\| \leqslant e^{-\mu t}, \quad \|T_p(t)\| \leqslant e^{-\mu_p t}, \quad t \geqslant 0.$$

*The constants $\mu$ and $\mu_p$ are the best possible.*

Lemma 16 implies that if $\mu = \mu_p$, then $\mathcal{A}_p$ generates a contraction semigroup on $(L^p)^N$ if and only if $\mathcal{A}$ generates a contraction semigroup on $(L^2)^N$. It is therefore interesting to understand the relation between $\mu$ and $\mu_p$.

**Lemma 17 ([19])** *Suppose that $1 < p < \infty$ and that the principal part of $\mathcal{A}_p$ is positive. Then $\mu = \mu_p$ if and only if at least one of the nonzero generalized solutions of the equation*

$$-\frac{4}{pp'} \partial_i (a_{ij} \partial_j w) + \frac{1}{2p} (2 \partial_i a_i I - p(A + A^*)) w = \mu w, \quad w \in (H_0^1(\Omega))^N,$$

*is of the form $w = fc$ for some real-valued scalar function $f$ and some $c \in \mathbb{C}^N$. Moreover, $\mu$ is the least eigenvalue of the left-hand side of the equation.*

**Corollary 12 ([19])** *Suppose that the principal part of $\mathcal{A}_p$ is positive. Then $\mu \leqslant \mu_p$. If $\mu = \mu_p$, there is a constant eigenvector to $A + A^*$ on $\overline{\Omega}$.*

With these results at hand Langer and Maz'ya proved Theorem 19 formulated above.

The following interesting example concerns the equality $\mu = \mu_p$, which is always satisfied if $N = 1$, in view of Lemma 17.

**Example 8** Let $A$ be the matrix

$$A(x) = \begin{pmatrix} 1 & |x| \\ |x| & -1 \end{pmatrix}, \quad x \in \overline{\Omega}.$$

Since $A$ has no constant eigenvectors, we have $\mu < \mu_p$ (see Corollary 12). Therefore we can choose a suitable constant $c$ such that $\mathcal{A}_p + cI$ generates a contraction semigroup on $(L^p(\Omega))^2$, while $\mathcal{A}$ does not generate a contraction semigroup on $(L^2(\Omega))^2$.

## 3.1.2 Parabolic systems

### The maximum modulus principle for a parabolic system

In this subsection we discuss the $L^\infty$ case. This subject has been widely investigated by Kresin and Maz'ya (see [18] for a general survey of their result). In particular they proved that for a system which is uniformly parabolic in the sense of Petrovskii and in which the coefficients do not depend on $t$, the maximum modulus principle holds if and only if the principal part of the system is scalar and the coefficients of the system satisfy a certain algebraic inequality (see Theorem 20 below).

They have considered the case in which the coefficients depend on $t$ as well, where they found necessary and, separately, sufficient conditions for the validity of the maximum modulus principle.

They studied also maximum principles in which the norm is understood in a generalized sense, i.e. as the Minkowski functional of a compact convex body in $\mathbb{R}^n$ containing the origin. Also in this general case they give necessary and (separately if the coefficients of the system depend on $t$) sufficient conditions for the validity of the maximum norm principle.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a $C^{2,\alpha}$ boundary ($0 < \alpha \leqslant 1$) and let $Q_T$ be the cylinder $\Omega \times (0, T)$.

Let $\mathcal{A}$ be the differential operator

$$\mathcal{A}u = \partial_i(A_{ij}\partial_j u) + A_i\partial_i u + Au$$

where $A_{ij}, A_i$ and $A$ are $N \times N$ matrices whose entries are complex valued functions. The elements of $A_{ij}$, $A_i$ and $A$ belong to $C^{2,\alpha}(\overline{\Omega})$, $C^{1,\alpha}(\overline{\Omega})$ and $C^{0,\alpha}(\overline{\Omega})$ respectively.

Moreover $A_{ij} = A_{ji}$ and there exists $\delta > 0$ such that for every $x \in \overline{\Omega}$ and every $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$, the zeros of the polynomial

$$\lambda \mapsto \det(\xi_i\xi_j A_{ij} + \lambda I)$$

satisfy the inequality $\mathscr{R}e\,\lambda \leqslant -\delta|\xi|^2$.

$$\begin{cases} \partial_t u - \mathcal{A}u = 0, & \text{on } Q_T, \\ u(\cdot, 0) = \varphi, & \text{on } \Omega, \\ u|_{\partial\Omega\times[0,T]} = 0, \end{cases} \tag{3.3}$$

where $\varphi \in (C^{2,\alpha}(\overline{\Omega}))^N$ and vanishes on $\partial\Omega$.

**Theorem 20 ([17])** *Let $u$ be the solution of (3.3). In order that*

$$\|u(\cdot,t)\|_\infty \leqslant \|\varphi\|_\infty \quad \forall t \in [0,T],$$

*for all $\varphi \in (C^{2,\alpha}(\overline{\Omega}))^N$ vanishing on $\partial\Omega$, it is necessary and sufficient that*

(a) *there are real-valued scalar functions $a_{ij}$ on $\overline{\Omega}$ such that for every $i, j$, $A_{ij} = a_{ij}I$ and the $n \times n$-matrix $\{a_{ij}\}$ is positive definite;*

(b) *for all $\eta_i, \zeta \in \mathbb{C}^N$, $i = 1, \dots, n$, with $\mathscr{R}e\langle\eta_i, \zeta\rangle = 0$, the inequality*

$$\mathscr{R}e\,\{a_{ij}\langle\eta_i, \eta_j\rangle - \langle A_i\eta_i, \zeta\rangle - \langle A\zeta, \zeta\rangle\} \geqslant 0$$

*holds on $\Omega$.*

In the scalar case $n = 1$, condition (b) is reduced to the requirement that the inequality

$$-4\,\mathscr{R}e\,A \geqslant b_{ij}\,\mathscr{I}m\,A_i\,\mathscr{I}m\,A_j$$

holds on $\Omega$, where $\{b_{ij}\} = \{a_{ij}\}^{-1}$ (cfr. (2.38)).

Let us consider now the problem

$$\begin{cases} \partial_t w + \mathcal{A}^* w = 0, & \text{on } Q_T, \\ w(\cdot, T) = \psi, & \text{on } \Omega, \\ w|_{\partial\Omega\times[0,T]} = 0, \end{cases} \tag{3.4}$$

where $\mathcal{A}^*$ is the formally adjoint operator of $\mathcal{A}$

$$\mathcal{A}^* w = \partial_i(A^*_{ij}\partial_j w) - A^*_i\partial_i w + (A^* - \partial_i A^*_i)w.$$

Theorem 20 implies

**Corollary 13** *Let $w$ be the solution of* (3.4). *In order that*

$$\|w(\cdot,t)\|_\infty \leqslant \|\psi\|_\infty \quad \forall t\in[0,T],$$

*for all $\psi\in(C^{2,\alpha}(\overline{\Omega}))^N$ vanishing on $\partial\Omega$, it is necessary and sufficient that*

(a) *there are real-valued scalar functions $a_{ij}$ on $\overline{\Omega}$ such that for every $i,j$, $A_{ij}=a_{ij}I$ and the $n\times n$-matrix $\{a_{ij}\}$ is positive definite;*

(b) *for all $\eta_i,\zeta\in\mathbb{C}^N$, $i=1,\dots,n$, with $\mathscr{R}e\langle\eta_i,\zeta\rangle=0$, the inequality*

$$\mathscr{R}e\,\{a_{ij}\langle\eta_i,\eta_j\rangle+\langle A_i\zeta,\eta_i\rangle-\langle(A-\partial_iA_i)\zeta,\zeta\rangle\}\geqslant 0$$

*holds on $\Omega$.*

Hinging on Theorem 20 and its Corollary 13 and using interpolation, one arrives to the necessary and sufficient conditions for the validity of the $L^p$ maximum principle for all $p\in[1,\infty]$ simultaneously:

**Corollary 14 ([19])** *Let $u$ be the solution of* (3.3). *In order that*

$$\|u(\cdot,t)\|_p \leqslant \|\varphi\|_p \quad \forall t\in[0,T],$$

*for all $\varphi\in(C^{2,\alpha}(\overline{\Omega}))^N$ vanishing on $\partial\Omega$ and for all $p\in[1,\infty]$, it is necessary and sufficient that*

(a) *there are real-valued scalar functions $a_{ij}$ on $\overline{\Omega}$ such that for every $i,j$, $A_{ij}=a_{ij}I$ and the $n\times n$-matrix $\{a_{ij}\}$ is positive definite;*

(b) *for all $\eta_i,\zeta\in\mathbb{C}^N$, $i=1,\dots,n$, with $\mathscr{R}e\langle\eta_i,\zeta\rangle=0$, the inequalities*

$$\begin{aligned}\mathscr{R}e\,\{a_{ij}\langle\eta_i,\eta_j\rangle-\langle A_i\eta_i,\zeta\rangle-\langle A\zeta,\zeta\rangle\} &\geqslant 0,\\ \mathscr{R}e\,\{a_{ij}\langle\eta_i,\eta_j\rangle+\langle A_i\zeta,\eta_i\rangle-\langle(A-\partial_iA_i)\zeta,\zeta\rangle\} &\geqslant 0\end{aligned}$$

*hold on $\Omega$.*

### Applications to semigroup theory

Theorem 14 implies the following result about the contractivity property of the semigroup generated by $\mathcal{A}_p$, where $\mathcal{A}_p$ is the extension of the operator $\mathcal{A}$ to the space

$$D(\mathcal{A}_p)=(W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))^N,\quad 1<p<\infty$$

and $\mathcal{A}_1$ is the closure in $(L^1(\Omega))^N$ of the operator $\mathcal{A}$, whose domain is the space of the functions in $(C^{2,\alpha}(\overline{\Omega}))^N$ vanishing on $\partial\Omega$.

**Theorem 21 ([19])** *The operators $\mathcal{A}_p$ generate contraction semigroups on $(L^p(\Omega))^N$ for all $p \in [1, \infty)$ and on $(C_0(\overline{\Omega}))^N$ for $p = \infty$ simultaneously if and only if*

(a) *there are real-valued scalar functions $a_{ij}$ on $\overline{\Omega}$ such that for every $i, j$, $A_{ij} = a_{ij} I$ and the $n \times n$-matrix $\{a_{ij}\}$ is positive definite;*

(b) *for all $\eta_i, \zeta \in \mathbb{C}^N$, $i = 1, \dots, n$, with $\mathscr{R}e\langle \eta_i, \zeta\rangle = 0$, the inequalities*

$$\begin{aligned} \mathscr{R}e\, \{a_{ij}\langle \eta_i, \eta_j\rangle - \langle A_i \eta_i, \zeta\rangle - \langle A\zeta, \zeta\rangle\} &\geqslant 0, \\ \mathscr{R}e\, \{a_{ij}\langle \eta_i, \eta_j\rangle + \langle A_i \zeta, \eta_i\rangle - \langle (A - \partial_i A_i)\zeta, \zeta\rangle\} &\geqslant 0 \end{aligned}$$

*hold on $\Omega$.*

**Example 9** Let us consider the Schrödinger operator with magnetic field

$$-(i\nabla + m)^t(i\nabla + m) - V,$$

where $m$ is an $\mathbb{R}^n$-valued function on $\Omega$ and $V$ is complex-valued. Theorem 21 shows that this operator generates contraction semigroups on $L^p(\Omega)$ for all $p \in [1, \infty]$ simultaneously if and only if

$$-4\,\mathscr{R}e\, A \geqslant \sum_{j=1}^{n} (\mathscr{I}m\, A_j)^2, \qquad -4\,\mathscr{R}e(\overline{A - \partial_j A_j}) \geqslant \sum_{j=1}^{n} (-\,\mathscr{I}m\, \overline{A_j})^2.$$

This is equivalent to the condition $\mathscr{R}e\, V \geqslant 0$ on $\Omega$.

### 3.1.3 Two-dimensional elasticity

Let us consider the classical operator of two-dimensional elasticity

$$Eu = \Delta u + (1 - 2\nu)^{-1} \nabla \operatorname{div} u, \tag{3.5}$$

where $\nu$ is the Poisson ratio. It is well known that $E$ is strongly elliptic if and only if either $\nu > 1$ or $\nu < 1/2$.

In order to obtain a necessary and sufficient condition for the $L^p$-dissipativity of the elasticity system, we start with some results concerning systems of partial differential equations of the form

$$A = \partial_h(\mathscr{A}^{hk}(x)\partial_k), \tag{3.6}$$

where $\mathscr{A}^{hk}(x) = \{a_{ij}^{hk}(x)\}$ are $m \times m$ matrices whose elements are complex locally integrable functions defined in an arbitrary domain $\Omega$ of $\mathbb{R}^n$ ($1 \leqslant i, j \leqslant m$, $1 \leqslant h, k \leqslant n$).

**Lemma 18 ([6])** *The operator* (3.6) *is $L^p$-dissipative in the domain $\Omega \subset \mathbb{R}^n$ if and only if*

$$\int_\Omega \Big( \mathscr{R}e\langle \mathscr{A}^{hk}\, \partial_k w, \partial_h w\rangle$$

$$-(1-2/p)^2 |w|^{-4}\, \mathscr{R}e\langle \mathscr{A}^{hk}\, w, w\rangle\, \mathscr{R}e\langle w, \partial_k w\rangle\, \mathscr{R}e\langle w, \partial_h w\rangle$$

$$-(1-2/p)|w|^{-2}\, \mathscr{R}e(\langle \mathscr{A}^{hk}\, w, \partial_h w\rangle\, \mathscr{R}e\langle w, \partial_k w\rangle$$

$$-\langle \mathscr{A}^{hk}\, \partial_k w, w\rangle\, \mathscr{R}e\langle w, \partial_h w\rangle)\Big) dx \geqslant 0$$

*for any $w \in (\mathring{C}^1(\Omega))^m$.*

In the particular case $n = 2$ we can deduce from Lemma 18 a necessary algebraic condition:

**Theorem 22 ([6])** *Let $\Omega$ be a domain of $\mathbb{R}^2$. If the operator* (3.6) *is $L^p$-dissipative, we have*

$$\begin{gathered}\mathscr{R}e\langle (\mathscr{A}^{hk}(x)\xi_h\xi_k)\lambda, \lambda\rangle - (1-2/p)^2\, \mathscr{R}e\langle (\mathscr{A}^{hk}(x)\xi_h\xi_k)\omega, \omega\rangle (\mathscr{R}e\langle \lambda, \omega\rangle)^2 \\ -(1-2/p)\, \mathscr{R}e(\langle (\mathscr{A}^{hk}(x)\xi_h\xi_k)\omega, \lambda\rangle - \langle (\mathscr{A}^{hk}(x)\xi_h\xi_k)\lambda, \omega\rangle)\, \mathscr{R}e\langle \lambda, \omega\rangle \\ \geqslant 0\end{gathered}$$

*for almost every $x \in \Omega$ and for any $\xi \in \mathbb{R}^2$, $\lambda$, $\omega \in \mathbb{C}^m$, $|\omega| = 1$.*

By means of Lemma 18 and Theorem 22 it is possible to prove the following criterion for the $L^p$-dissipativity of the two-dimensional elasticity:

**Theorem 23 ([6])** *The operator* (3.5) *is $L^p$-dissipative if and only if*

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 \leqslant \frac{2(\nu-1)(2\nu-1)}{(3-4\nu)^2}.$$

By two last theorems one can make easily a comparison between $E$ and $\Delta$ from the point of view of the $L^p$-dissipativity.

**Corollary 15 ([6])** *There exists $k > 0$ such that $E - k\Delta$ is $L^p$-dissipative if and only if*

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 < \frac{2(\nu-1)(2\nu-1)}{(3-4\nu)^2}\, .$$

*There exists $k < 2$ such that $k\Delta - E$ is $L^p$-dissipative if and only if*

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 < \frac{2\nu(2\nu-1)}{(1-4\nu)^2}\, .$$

### 3.1.4 A class of systems of partial differential operators

In this Section we consider a system of partial differential operators of the form

$$Au = \partial_h(\mathscr{A}^h(x)\partial_h u), \tag{3.7}$$

where $\mathscr{A}^h(x) = \{a^h_{ij}(x)\}$ $(i, j = 1, \ldots, m)$ are matrices with complex locally integrable entries defined in a domain $\Omega \subset \mathbb{R}^n$ $(h = 1, \ldots, n)$. We note that elasticity system is not a system of this kind.

One can characterize the $L^p$-dissipativity of such operators by reducing the study to one dimensional case. The next two Subsections are devoted to auxiliary results for systems of ordinary differential equations.

#### Dissipativity for systems of ordinary differential equations

The results of this Subsection concern the operator

$$Au = (\mathscr{A}(x)u')', \tag{3.8}$$

where $\mathscr{A}(x) = \{a_{ij}(x)\}$ $(i, j = 1, \ldots, m)$ is a matrix with complex locally integrable entries defined in the bounded or unbounded interval $(a, b)$.

The corresponding sesquilinear form $\mathscr{L}(u, w)$ is given by

$$\mathscr{L}(u, w) = \int_a^b \langle \mathscr{A}\, u', w' \rangle \, dx.$$

**Theorem 24 ([6])** *The operator $A$ is $L^p$-dissipative if and only if*

$$\begin{aligned} &\mathscr{R}e\langle \mathscr{A}(x)\lambda, \lambda\rangle - (1-2/p)^2\, \mathscr{R}e\langle \mathscr{A}(x)\omega, \omega\rangle (\mathscr{R}e\langle \lambda, \omega\rangle)^2 \\ &-(1-2/p)\, \mathscr{R}e(\langle \mathscr{A}(x)\omega, \lambda\rangle - \langle \mathscr{A}(x)\lambda, \omega\rangle)\, \mathscr{R}e\langle \lambda, \omega\rangle \geqslant 0 \end{aligned}$$

*for almost every $x \in (a, b)$ and for any $\lambda, \omega \in \mathbb{C}^m$, $|\omega| = 1$.*

A consequence of this Theorem is

**Corollary 16 ([6])** *If the operator $A$ is $L^p$-dissipative, then*

$$\mathscr{R}e\langle \mathscr{A}(x)\lambda, \lambda\rangle \geqslant 0$$

*for almost every $x \in (a, b)$ and for any $\lambda \in \mathbb{C}^m$.*

We can precisely determine the angle of dissipativity of the matrix ordinary differential operator (3.8) with complex coefficients.

**Theorem 25 ([6])** *Let the operator* (3.8) *be $L^p$-dissipative. The operator $zA$ is $L^p$-dissipative if and only if*

$$\vartheta_- \leqslant \arg z \leqslant \vartheta_+,$$

*where*

$$\vartheta_- = \operatorname{arccot}\left(\operatorname*{ess\,inf}_{(x,\lambda,\omega)\in\Xi} (Q(x,\lambda,\omega)/P(x,\lambda,\omega))\right) - \pi,$$
$$\vartheta_+ = \operatorname{arccot}\left(\operatorname*{ess\,sup}_{(x,\lambda,\omega)\in\Xi} (Q(x,\lambda,\omega)/P(x,\lambda,\omega))\right),$$

$$\begin{aligned} P(x,\lambda,\omega) = \mathscr{R}e\langle\mathscr{A}(x)\lambda,\lambda\rangle - (1-2/p)^2\,\mathscr{R}e\langle\mathscr{A}(x)\omega,\omega\rangle(\mathscr{R}e\langle\lambda,\omega\rangle)^2 \\ -(1-2/p)\,\mathscr{R}e(\langle\mathscr{A}(x)\omega,\lambda\rangle - \langle\mathscr{A}(x)\lambda,\omega\rangle)\,\mathscr{R}e\langle\lambda,\omega\rangle, \\ Q(x,\lambda,\omega) = \mathscr{I}m\langle\mathscr{A}(x)\lambda,\lambda\rangle - (1-2/p)^2\,\mathscr{I}m\langle\mathscr{A}(x)\omega,\omega\rangle(\mathscr{R}e\langle\lambda,\omega\rangle)^2 \\ -(1-2/p)\,\mathscr{I}m(\langle\mathscr{A}(x)\omega,\lambda\rangle - \langle\mathscr{A}(x)\lambda,\omega\rangle)\,\mathscr{R}e\langle\lambda,\omega\rangle \end{aligned}$$

*and $\Xi$ is the set*

$$\Xi = \{(x,\lambda,\omega)\in(a,b)\times\mathbb{C}^m\times\mathbb{C}^m \mid |\omega|=1,\ P^2(x,\lambda,\omega)+Q^2(x,\lambda,\omega)>0\}.$$

Another consequence of Theorem 24 is the possibility of making a comparison between $A$ and the operator $I(d^2/dx^2)$.

**Corollary 17 ([6])** *There exists $k>0$ such that $A - kI(d^2/dx^2)$ is $L^p$-dissipative if and only if*

$$\operatorname*{ess\,inf}_{\substack{(x,\lambda,\omega)\in(a,b)\times\mathbb{C}^m\times\mathbb{C}^m \\ |\lambda|=|\omega|=1}} P(x,\lambda,\omega) > 0.$$

*There exists $k>0$ such that $kI(d^2/dx^2) - A$ is $L^p$-dissipative if and only if*

$$\operatorname*{ess\,sup}_{\substack{(x,\lambda,\omega)\in(a,b)\times\mathbb{C}^m\times\mathbb{C}^m \\ |\lambda|=|\omega|=1}} P(x,\lambda,\omega) < \infty.$$

*There exists $k\in\mathbb{R}$ such that $A - kI(d^2/dx^2)$ is $L^p$-dissipative if and only if*

$$\operatorname*{ess\,inf}_{\substack{(x,\lambda,\omega)\in(a,b)\times\mathbb{C}^m\times\mathbb{C}^m \\ |\lambda|=|\omega|=1}} P(x,\lambda,\omega) > -\infty.$$

**Criteria formulated in terms of eigenvalues of the matrix $\mathscr{A}(x)$**

In the particular case in which the coefficients $a_{ij}$ of operator (3.8) are real, we can give a necessary and sufficient condition for the $L^p$-dissipativity of $A$ in terms of eigenvalues of the matrix $\mathscr{A}$.

**Theorem 26 ([6])** *Let $\mathscr{A}$ be a real matrix $\{a_{hk}\}$ with $h, k = 1, \ldots, m$. Let us suppose $\mathscr{A} = \mathscr{A}^t$ and $\mathscr{A} \geqslant 0$ (in the sense $\langle \mathscr{A}(x)\xi, \xi\rangle \geqslant 0$, for almost every $x \in (a, b)$ and for any $\xi \in \mathbb{R}^m$). The operator $A$ is $L^p$-dissipative if and only if*

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 (\mu_1(x) + \mu_m(x))^2 \leqslant \mu_1(x)\mu_m(x)$$

*almost everywhere, where $\mu_1(x)$ and $\mu_m(x)$ are the smallest and the largest eigenvalues of the matrix $\mathscr{A}(x)$ respectively. In the particular case $m = 2$ this condition is equivalent to*

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 (\operatorname{tr} \mathscr{A}(x))^2 \leqslant \det \mathscr{A}(x)$$

*almost everywhere.*

We have also:

**Corollary 18 ([6])** *Let $\mathscr{A}$ be a real and symmetric matrix. Denote by $\mu_1(x)$ and $\mu_m(x)$ the smallest and the largest eigenvalues of $\mathscr{A}(x)$ respectively. There exists $k > 0$ such that $A - kI(d^2/dx^2)$ is $L^p$-dissipative if and only if*

$$\operatorname*{ess\,inf}_{x\in(a,b)} \left[(1 + \sqrt{p\,p'}/2)\,\mu_1(x) + (1 - \sqrt{p\,p'}/2)\,\mu_m(x)\right] > 0. \tag{3.9}$$

*In the particular case $m = 2$ condition (3.9) is equivalent to*

$$\operatorname*{ess\,inf}_{x\in(a,b)} \left[\operatorname{tr} \mathscr{A}(x) - \frac{\sqrt{p\,p'}}{2} \sqrt{(\operatorname{tr} \mathscr{A}(x))^2 - 4 \det \mathscr{A}(x)}\right] > 0.$$

If we require something more about the matrix $\mathscr{A}$ we have also

**Corollary 19 ([6])** *Let $\mathscr{A}$ be a real and symmetric matrix. Suppose $\mathscr{A} \geqslant 0$ almost everywhere. Denote by $\mu_1(x)$ and $\mu_m(x)$ the smallest and the largest eigenvalues of $\mathscr{A}(x)$ respectively. If there exists $k > 0$ such that $A - kI(d^2/dx^2)$ is $L^p$-dissipative, then*

$$\operatorname*{ess\,inf}_{x\in(a,b)} \left[\mu_1(x)\mu_m(x) - \left(\frac{1}{2} - \frac{1}{p}\right)^2 (\mu_1(x) + \mu_m(x))^2\right] > 0. \tag{3.10}$$

*If, in addition, there exists $C$ such that*

$$\langle \mathscr{A}(x)\xi, \xi\rangle \leqslant C|\xi|^2 \tag{3.11}$$

*for almost every $x \in (a,b)$ and for any $\xi \in \mathbb{R}^m$, the converse is also true. In the particular case $m = 2$ condition* (3.10) *is equivalent to*

$$\operatorname*{ess\,inf}_{x\in(a,b)} \left[\det \mathscr{A}(x) - \left(\frac{1}{2} - \frac{1}{p}\right)^2 (\operatorname{tr} \mathscr{A}(x))^2\right] > 0.$$

Generally speaking, assumption (3.11) cannot be omitted even if $\mathscr{A} \geqslant 0$.

**Example 10** Consider $(a,b) = (1,\infty)$, $m = 2$, $\mathscr{A}(x) = \{a_{ij}(x)\}$ where

$$a_{11}(x) = (1 - 2/\sqrt{pp'})x + x^{-1}, \quad a_{12}(x) = a_{21}(x) = 0,$$
$$a_{22}(x) = (1 + 2/\sqrt{pp'})x + x^{-1}.$$

We have

$$\mu_1(x)\mu_2(x) - \left(\frac{1}{2} - \frac{1}{p}\right)^2 (\mu_1(x) + \mu_2(x))^2 = (8 + 4x^{-2})/(p\,p')$$

and (3.10) holds. But (3.9) is not satisfied, because

$$(1 + \sqrt{p\,p'}/2)\,\mu_1(x) + (1 - \sqrt{p\,p'}/2)\,\mu_2(x) = 2x^{-1}.$$

**Corollary 20 ([6])** *Let $\mathscr{A}$ be a real and symmetric matrix. Denote by $\mu_1(x)$ and $\mu_m(x)$ the smallest and the largest eigenvalues of $\mathscr{A}(x)$ respectively. There exists $k > 0$ such that $kI(d^2/dx^2) - A$ is $L^p$-dissipative if and only if*

$$\operatorname*{ess\,sup}_{x\in(a,b)} \left[(1 - \sqrt{p\,p'}/2)\,\mu_1(x) + (1 + \sqrt{p\,p'}/2)\,\mu_m(x)\right] < \infty. \tag{3.12}$$

*In the particular case $m = 2$ condition* (3.12) *is equivalent to*

$$\operatorname*{ess\,sup}_{x\in(a,b)} \left[\operatorname{tr} \mathscr{A}(x) + \frac{\sqrt{p\,p'}}{2}\sqrt{(\operatorname{tr} \mathscr{A}(x))^2 - 4\det \mathscr{A}(x)}\right] < \infty.$$

In the case of a positive matrix $\mathscr{A}$, we have

**Corollary 21 ([6])** *Let $\mathscr{A}$ be a real and symmetric matrix. Suppose $\mathscr{A} \geqslant 0$ almost everywhere. Denote by $\mu_1(x)$ and $\mu_m(x)$ the smallest and the largest eigenvalues of $\mathscr{A}(x)$ respectively. There exists $k > 0$ such that $kI(d^2/dx^2) - A$ is $L^p$-dissipative if and only if*

$$\operatorname*{ess\,sup}_{x\in(a,b)} \mu_m(x) < \infty.$$

**$L^p$-dissipativity of the operator** (3.7)

We describe now necessary and sufficient conditions for the $L^p$-dissipativity of the system of partial differential operators (3.7).

By $y_h$ we denote the $(n-1)$-dimensional vector $(x_1, \dots, x_{h-1}, x_{h+1}, \dots, x_n)$ and we set $\omega(y_h) = \{x_h \in \mathbb{R} \mid x \in \Omega\}$.

**Lemma 19 ([6])** *The operator* (3.7) *is $L^p$-dissipative if and only if the ordinary differential operators*

$$A(y_h)[u(x_h)] = d(\mathscr{A}^h(x)du/dx_h)/dx_h$$

*are $L^p$-dissipative in $\omega(y_h)$ for almost every $y_h \in \mathbb{R}^{n-1}$ $(h = 1, \dots, n)$. This condition is void if $\omega(y_h) = \emptyset$.*

**Theorem 27 ([6])** *The operator* (3.7) *is $L^p$-dissipative if and only if*

$$\begin{aligned}&\mathscr{R}e\langle \mathscr{A}^h(x_0)\lambda, \lambda\rangle - (1-2/p)^2\, \mathscr{R}e\langle \mathscr{A}^h(x_0)\omega, \omega\rangle (\mathscr{R}e\langle\lambda, \omega\rangle)^2 \\ &-(1-2/p)\, \mathscr{R}e(\langle \mathscr{A}^h(x_0)\omega, \lambda\rangle - \langle \mathscr{A}^h(x_0)\lambda, \omega\rangle)\, \mathscr{R}e\langle\lambda, \omega\rangle \geqslant 0\end{aligned} \tag{3.13}$$

*for almost every $x_0 \in \Omega$ and for any $\lambda, \omega \in \mathbb{C}^m$, $|\omega| = 1$, $h = 1, \dots, n$.*

**Remark 10** In the scalar case $(m = 1)$, operator (3.7) can be considered as an operator we dealt with in Section 2.2.

In fact, if $Au = \sum_{h=1}^n \partial_h(a^h \partial_h u)$, $a^h$ being a scalar function, $A$ can be written in the form (2.25) with $\mathscr{A} = \{c_{hk}\}$, $c_{hh} = a^h$, $c_{hk} = 0$ if $h \neq k$. The conditions obtained in Section 2.2 can be directly compared with (3.13). We know that operator $A$ is $L^p$-dissipative if and only if (2.27) holds. In this particular case (2.27) is clearly equivalent to the following $n$ conditions

$$\frac{4}{p\,p'}\, (\mathscr{R}e\, a^h)\, \xi^2 + (\mathscr{R}e\, a^h)\, \eta^2 - 2(1-2/p)(\mathscr{I}m\, a^h)\, \xi\eta \geqslant 0 \tag{3.14}$$

almost everywhere and for any $\xi, \eta \in \mathbb{R}$, $h = 1, \dots, n$. On the other hand, in this case, (3.13) reads as

$$\begin{aligned}&(\mathscr{R}e\, a^h)|\lambda|^2 - (1-2/p)^2 (\mathscr{R}e\, a^h)(\mathscr{R}e(\lambda\overline{\omega})^2 \\ &-2(1-2/p)(\mathscr{I}m\, a^h)\, \mathscr{R}e(\lambda\overline{\omega})\, \mathscr{I}m(\lambda\overline{\omega}) \geqslant 0\end{aligned} \tag{3.15}$$

almost everywhere and for any $\lambda, \omega \in \mathbb{C}$, $|\omega| = 1$, $h = 1, \dots, n$. Setting $\xi + i\eta = \lambda\overline{\omega}$ and observing that $|\lambda|^2 = |\lambda\overline{\omega}|^2 = (\mathscr{R}e(\lambda\overline{\omega}))^2 + (\mathscr{I}m(\lambda\overline{\omega}))^2$, we see that conditions (3.14) (and then (2.27)) are equivalent to (3.15).

Theorem 27 permits to determine the angle of dissipativity of the operator (3.7):

**Theorem 28 ([6])** *Let $A$ be $L^p$-.dissipative. The operator $zA$ is $L^p$-dissipative if and only if $\vartheta_- \leqslant \arg z \leqslant \vartheta_+$, where*

$$\vartheta_- = \max_{h=1,\dots,n} \operatorname{arccot}\left( \operatorname*{ess\,inf}_{(x,\lambda,\omega)\in\Xi_h} (Q_h(x,\lambda,\omega)/P_h(x,\lambda,\omega)) \right) - \pi,$$
$$\vartheta_+ = \min_{h=1,\dots,n} \operatorname{arccot}\left( \operatorname*{ess\,sup}_{(x,\lambda,\omega)\in\Xi_h} (Q_h(x,\lambda,\omega)/P_h(x,\lambda,\omega)) \right),$$

*and*

$$\begin{gathered} P_h(x,\lambda,\omega) = \mathscr{R}e\langle \mathscr{A}^h(x)\lambda,\lambda\rangle - (1-2/p)^2\,\mathscr{R}e\langle \mathscr{A}^h(x)\omega,\omega\rangle(\mathscr{R}e\langle\lambda,\omega\rangle)^2 \\ -(1-2/p)\,\mathscr{R}e(\langle \mathscr{A}^h(x)\omega,\lambda\rangle - \langle \mathscr{A}^h(x)\lambda,\omega\rangle)\,\mathscr{R}e\langle\lambda,\omega\rangle, \\ Q_h(x,\lambda,\omega)=\mathscr{I}m\langle \mathscr{A}^h(x)\lambda,\lambda\rangle - (1-2/p)^2\mathscr{I}m\langle \mathscr{A}^h(x)\omega,\omega\rangle(\mathscr{R}e\langle\lambda,\omega\rangle)^2 \\ -(1-2/p)\,\mathscr{I}m(\langle \mathscr{A}^h(x)\omega,\lambda\rangle - \langle \mathscr{A}^h(x)\lambda,\omega\rangle)\,\mathscr{R}e\langle\lambda,\omega\rangle, \\ \Xi_h = \{(x,\lambda,\omega)\in\Omega\times\mathbb{C}^m\times\mathbb{C}^m \mid |\omega|=1,\ P_h^2(x,\lambda,\omega)+Q_h^2(x,\lambda,\omega)>0\}. \end{gathered}$$

If $A$ has real coefficients, we can characterize the $L^p$-dissipativity in terms of the eigenvalues of the matrices $\mathscr{A}^h(x)$:

**Theorem 29 ([6])** *Let $A$ be the operator* (3.7)*, where $\mathscr{A}^h$ are real matrices $\{a_{ij}^h\}$ with $i,j=1,\dots,m$. Let us suppose $\mathscr{A}^h = (\mathscr{A}^h)^t$ and $\mathscr{A}^h \geqslant 0$ ($h=1,\dots,n$). The operator $A$ is $L^p$-dissipative if and only if*

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2 (\mu_1^h(x)+\mu_m^h(x))^2 \leqslant \mu_1^h(x)\,\mu_m^h(x)$$

*for almost every $x\in\Omega$, $h=1,\dots,n$, where $\mu_1^h(x)$ and $\mu_m^h(x)$ are the smallest and the largest eigenvalues of the matrix $\mathscr{A}^h(x)$ respectively. In the particular case $m=2$ this condition is equivalent to*

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2 (\operatorname{tr}\mathscr{A}^h(x))^2 \leqslant \det\mathscr{A}^h(x)$$

*for almost every $x\in\Omega$, $h=1,\dots,n$.*

## 3.2 Higher order differential operators

There are many papers dealing with the contractivity of semigroups generated by scalar or vector second order partial differential operators, but Langer and Maz'ya [20] are the only ones for the time being to consider similar questions for higher order differential operators. As we shall see in this Section, higher order case has some peculiarities.

### 3.2.1 Noncontractivity of higher order operators

The following simple example suggests that we cannot have $L^1$-contractivity for higher order operators in one dimension:

$$\frac{\partial}{\partial t}u(x,t) + (-1)^m \frac{\partial^{2m}}{\partial x^{2m}}u(x,t) = 0, \qquad x \in \mathbb{R}, t \geqslant 0.$$

The solution $u$ is given by

$$u(x,t) = \int_{\mathbb{R}} K_t(x-y)\, u(y,0)\, dy\,,$$

where the kernel $K_t$ is such that

$$\widehat{K}_t(\xi) = e^{-\xi^{2m}t}, \qquad \xi \in \mathbb{R}, t \geqslant 0,$$

$\widehat{K}_t$ being the Fourier transform of $K_t$.

Since for $m > 1$ we have

$$1 = \widehat{K}_t(0) = \int_{\mathbb{R}} K_t(x)\, dx, \qquad 0 = \widehat{K}_t''(0) = -\int_{\mathbb{R}} x^2 K_t(x)\, dx,$$

the $L^1$-norm $\|K_t\|$ has to be $> 1$ and therefore the semigroup generated by the operator

$$(-1)^{m+1}\frac{d^{2m}}{dx^{2m}}$$

can not be contractive.

Maz'ya and Langer considered multi-dimensional operators with locally integrable coefficients and they found that if $1 \leqslant p < \infty$, $p \neq 2$, no linear partial differential operator of order higher than two which contains $(\mathring{C}^\infty(\Omega))^N$ in its domain of definition can generate a contraction semigroup on $(L^p(\Omega))^N$.

This result is obtained at first by a deep study of the one-dimensional case, where the following necessary and sufficient conditions can be proved:

**Theorem 30 ([20])** *Let $k \in \mathbb{N}$ and $p \in [1, \infty)$. The integral*

$$\int w^{(k)} |w|^{p-1} \operatorname{sgn} w\, dx$$

*preserves sign as $w$ ranges over real-valued elements of $\mathring{C}^\infty(\Omega)$ if and only if $p = 2$ or $k \in \{0, 1, 2\}$.*

Suppose now we have a linear partial differential operator $A$

$$A = \sum_{|\alpha| \leqslant k} a_\alpha D^\alpha, \tag{3.16}$$

where the coefficients $a_\alpha$ are in $L^1_{\mathrm{loc}}(\Omega)$, $\Omega$ being a domain in $\mathbb{R}^n$.

**Theorem 31 ([20])** *Suppose that* $p \in [1, \infty)$, $p \neq 2$. *If*

$$\mathscr{R}e \int_{\Omega} \langle Au, u \rangle \, |u|^{p-2} \, dx$$

*does not change sign as* $u$ *ranges over* $\mathring{C}^{\infty}(\Omega)$, *then* $A$ *is of order* 0, 1 *or* 2.

If $u$ does not range over $\mathring{C}^{\infty}(\Omega)$, but only over $(\mathring{C}^{\infty}(\Omega))^{+}$ (the class of nonnegative functions of $\mathring{C}^{\infty}(\Omega)$), then we have a quite different result, provided the operator has real-valued coefficients:

**Theorem 32 ([20])** *Suppose that* $p \in (1, \infty)$, $p \neq 2$ *and that* $A$ *is a linear partial differential operator with real-valued coefficient functions. Assume that*

$$\int_{\Omega} (Au) u^{p-1} \, dx$$

*does not change sign as* $u$ *ranges over* $(\mathring{C}^{\infty}(\Omega))^{+}$. *Then either* $A$ *is of order* 0, 1 *or* 2, *or* $A$ *is of order* 4 *and* $\frac{3}{2} \leqslant p \leqslant 3$.

From Theorem 31 it follows the non contractivity of higher order operators of the form (3.16), where $a_\alpha$ are $N \times N$-matrices whose entries belong to $L^1_{\text{loc}}(\Omega)$ ($N$ being a positive integer). In fact for such operators we have:

**Theorem 33 ([20])** *If* $1 \leqslant p < \infty$, $p \neq 2$, *no linear partial differential operator of order higher than two which contains* $(\mathring{C}^{\infty}(\Omega))^{N}$ *in its domain of definition can generate a contraction semigroup on* $(L^p(\Omega))^N$.

### 3.2.2 The cone of nonnegative functions

Sometimes it is known that the solutions of the Cauchy problem

$$\begin{cases} s'(t) = A[s(t)] \\ s(0) = s_0 \end{cases} \tag{3.17}$$

are nonnegative on some interval. This is why the problem of contractivity on the cone of nonnegative functions arises.

It is well known that the solutions of a Cauchy problem are norm decreasing, provided that the related semigroup is contractive. The next lemma can be considered as a parallel result in the cone of nonnegative functions, when the theory of semigroups cannot be applied anymore.

In this Section the spaces $L^p$ are real.

**Lemma 20 ([20])** *Suppose that the Cauchy problem* (3.17) *has a unique solution of class* $C^1(\mathbb{R}^+, L^p)$ *for every* $s_0$ *in*$D(A)$.

*If* $1 < p < \infty$, *then*

$$\frac{d}{dt}\|s(t)\|_p\Big|_{t=0^+} \leqslant 0$$

*for every* $s(0) \in (D(A))^+$ *if and only if*

$$\int_\Omega (Au)u^{p-1}\,dx \leqslant 0 \tag{3.18}$$

*for every* $u \in (D(A))^+$. *In the case* $p = 1$, (3.18) *holds for every* $u \in (D(A))^+$ *if*

$$\liminf_{t\to 0^+} t^{-1}(\|s(t)\|_1 - \|s(0)\|_1) \leqslant 0 \tag{3.19}$$

*for every* $s(0) \in (D(A))^+$.

The following result follows from Lemma 20 and Theorem 32

**Theorem 34 ([20])** *Let* $1 < p < \infty$, $p \neq 2$ *and suppose that* $\mathring{C}^\infty(\Omega)$ *is a subset of the domain* $D(A)$ *of the linear partial differential operator* $A$. *Assume furthermore that the coefficients of* $A$ *belong to* $L^1_{\text{loc}}(\Omega)$ *and that the Cauchy problem* (3.17) *has a unique solution for all nonnegative initial data in* $D(A)$. *If*

$$\frac{d}{dt}\|s(t)\|_p\Big|_{t=0^+} \leqslant 0$$

*for every* $s(0) \in (D(A))^+$, *then either* $A$ *is of order* 0, 1 *or* 2, *or* $A$ *is of order* 4 *and* $\frac{3}{2} \leqslant p \leqslant 3$.

This theorem does not consider the case $p = 1$. In this case one can show that if the operator $A$ satisfies the condition of Theorem 34 and then (3.19) holds, we have that the distribution

$$-\sum_{|\alpha|\leqslant k}(-1)^{|\alpha|}\partial^\alpha a_\alpha$$

is a positive measure.

In the particular case of constant coefficients and smooth boundary, we have also

**Theorem 35 ([20])** *Suppose that* $1 < p < \infty$, *that* $\Omega \subset \mathbb{R}^n$ *is open, bounded and has* $C^\infty$*-boundary and that the real constant coefficients* $\{a_{ijk\ell}\}$ *fulfill*

$$a_{ijk\ell} = a_{jki\ell} = a_{j\ell ik}$$

*for all $i, j, k$ and $\ell$, and also fulfill the relation*

$$\sum_{1\leqslant i,j,k,\ell\leqslant n} a_{ijk\ell}\xi_{ij}\xi_{k\ell} \geqslant 0$$

*for all real symmetric $n\times n$-matrices $\xi = \{\xi_{ij}\}$. Then*

$$\int_\Omega (a_{ijk\ell}\partial_i\partial_j\partial_k\partial_\ell u)u^{p-1}\,dx \geqslant 0$$

*for all nonnegative functions $u \in W^{4,p}(\Omega)\cap W_0^{2,p}(\Omega)$ if and only if $\frac{3}{2} \leqslant p \leqslant 3$.*

**Corollary 22 ([20])** *Suppose that $\frac{3}{2} \leqslant p \leqslant 3$ and that $\Omega$ and the coefficients of the operator*

$$A = -a_{ijk\ell}\partial_{ijk\ell},$$

*with domain $W^{4,p}(\Omega)\cap W_0^{2,p}(\Omega)$, fulfill the hypotheses of Theorem 35. Then any differentiable solution $s$ of the Cauchy problem (3.17) with nonnegative initial value $s(0) \in D(A)$ fulfills*

$$\frac{d}{dt}\|s(t)\|_p\bigg|_{t=0^+} \leqslant 0.$$

An operator satisfying the conditions of Corollary 22 is $A = -\Delta^2$.

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