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Preface

The present lecture notes contain results concerning elastic cusped Euler-Bernoulli beams and Kirchhoff-Love plates reported by the author at workshops and minisymposia organized by TICMI and mostly belonging to him.

In practice, such plates and beams are often encountered in spatial structures with partly fixed edges, e.g., stadium ceilings, aircraft wings, submarine wings etc., in machine-tool design, as in cutting-machines, planning-machines, in astronautics, turbines, and in many other areas of engineering. The problem mathematically leads to the question of setting and solving boundary value problems for even order equations and systems of elliptic type with the order degeneration in the static case and of initial boundary value problems for even order equations and systems of hyperbolic type with the order degeneration in the dynamical case.

To end this preamble, I would like to express my gratitude to Alois Kufner and Natalia Chinchaladze for valuable comments and also to Mary Bitsadze and Mary Sharikadze who contributed with efficiency to type this text on LaTex.

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Introduction

In 1955 I.Vekua [85]-[87] raised the problem of investigation of elastic cusped plates, i.e., such ones whose thickness on the part of the plate boundary or on the whole one vanishes. Such bodies considered as three-dimensional models occupy three-dimensional domains with, in general, non-Lipschitz boundaries. In practice, such plates and beams are often encountered in spatial structures with partly fixed edges, e.g., stadium ceilings, aircraft wings, submarine wings etc., in machine-tool design, as in cutting-machines, planning-machines, in astronautics, turbines, and in many other areas of engineering. The problem mathematically leads to the question of setting and solving boundary value problems for even order equations and systems of elliptic type with the order degeneration in the static case and of initial boundary value problems for even order equations and systems of hyperbolic type with the order degeneration in the dynamical case (for corresponding investigations see the survey [30], [53], and also I. Vekua's comments in [87, p.86]).

The first works concerning classical bending of cusped elastic plates were done by E. Makhover [61], [62] and S. Mikhlin [64]. Namely, in 1957, 1958 E. Makhover [61], [62], by using the results of S. Mikhlin [64], had considered such a cusped plate with the flexural rigidity $D(x_1, x_2)$ satisfying

$$D_1 x_2^{\varkappa_1} \leq D(x_1, x_2) \leq D_2 x_2^{\varkappa_1}, \quad D_1, D_2, \varkappa_1 = \text{const} > 0, \tag{1}$$

within the framework of classical bending theory. She particularly studied in which cases the deflection ($\varkappa_1 < 2$) or its normal derivative ($\varkappa_1 < 1$) on the cusped edge of the plate can be given. In 1971, A. Khvoles [59] represented the forth order Airy stress function operator as the product of two second order operators in the case when the plate thickness $2h$ was given by

$$2h = h_0 x_2^{\varkappa_2}, \quad h_0, \ \varkappa_2 = \text{const} > 0, \ \ x_2 \geq 0, \tag{2}$$

and investigated the question of the general representation of corresponding solutions.

Since 1972 the works of G. Jaiani [31]-[52] have also been devoted to these problems. By using more natural spaces than E. Makhover had, G. Jaiani in [43] has analyzed in which cases the cusped edge can be freed ($\varkappa_1 > 0$) or simply supported ($\varkappa_1 < 3$). Moreover, he established well-posedness and the correct formulation of all the admissible principal boundary value problems (BVPs). In [36], [37], [42] he also investigated the tension-compression problem of cusped plates, based on I. Vekua's model of shallow prismatic shells ($N = 0$). G. Jaiani's results can be summarized as follows:

Let n be the inward normal of the plate boundary (i.e., of the boundary of the plate projection). In the case of the tension-compression ($N = 0$) problem on the cusped edge, where

$$0 \leq \frac{\partial h}{\partial n} < +\infty \text{ (in the case (2) this means } \varkappa_2 \geq 1),$$

which will be called a sharp cusped edge, one can not prescribe the displacement vector; while on the cusped edge, where

$$\frac{\partial h}{\partial n} = +\infty \text{ (in the case (2) this means } \varkappa_2 < 1),$$

which will be called a blunt cusped edge, the displacement vector can be prescribed. In the case of the classical bending problem with a cusped edge, where

$$\frac{\partial h}{\partial n} = O(d^{\varkappa - 1}) \text{ as } d \to 0, \quad \varkappa = \text{const} > 0 \tag{3}$$

and where d is the distance between an interior reference point of the plate projection and the cusped edge, the edge can not be fixed if $\varkappa \geq 1/3$, but it can be fixed if $0 < \varkappa < 1/3$; it can not be simply supported if $\varkappa \geq 1$, and it can be simply supported if $0 < \varkappa < 1$ (here 1 should be replaced by $\frac{2}{3}$ when the bending moment prescribed on the cusped edge is not identically zero i.e., when we consider inhomogeneous boundary conditions); it can be free or arbitrarily loaded by a shear force and a bending moment if $\varkappa > 0$. Note that in the case (2), the condition (3) implies that $d_2 = x_2$ and $\varkappa = \varkappa_2 = \varkappa_1/3$. The above conditions are also reformulated as some integral conditions on the plate thickness.

For the specific cases of cusped cylindrical and conical shell bending, the above results remain valid as it has been shown by G. Tsiskarishvili and N. Khomasuridse [80]-[82]. These results also remain valid in the case of classical bending of orthotropic cusped plates (see [46]). However, for general cusped shells and also for general anisotropic cusped plates, corresponding analysis is yet to be done.

As it was already mention the problems involving cusped plates lead to the problem of correct mathematical formulations of BVPs for even order elliptic equations and systems whose orders degenerate on the boundary (see [42], [47], [48]).

Applying the functional-analytic method developed by G. Fichera in [24], [25] (see also [18], [19]), in [42] the particular case ($\lambda = \mu$) of Vekua's system for general cusped plates has been investigated.

I.Vekua's system in the $N = 1$ approximation in the case (2) is investigated in [17].

The classical bending of plates with the stiffness (1) in energetic and in weighted Sobolev spaces has been studied by G. Jaiani in [43], [46] (see also Sections 2.2, 2.4, and 2.5 of the present book). In the energetic space some restrictions on the lateral load has been relaxed by G. Devdariani in [16]. G. Tsiskarishvili [80] characterized completely the classical axial symmetric bending of specific circular cusped plates without or with a hole.

In the case (2), the basic BVPs have been explicitly solved in [38] and [48] with the help of singular solutions depending only on the polar angle.

If we consider the cylindrical bending of a plate, in particular of a cusped one, with rectangular projection $a \leq x_1 \leq b$, $0 \leq x_2 \leq l$, we actually get the corresponding results also for cusped beams (see [44], [38], [9]-[11], [49], [50]).

In 1980-1986 S. Uzunov [83] numerically solved the problem of bending of the cusped circular beam on an elastic foundation with constant compliance. The moment of inertia of the beam had the form

$$I(x_2) = \frac{\pi r^4}{4}, \quad r = c x_2^{\gamma}, \quad c, \gamma = \text{const} > 0, \quad \gamma < 1$$

(r is the cross-section radius). The cusped end was free and the non-cusped end was clamped.

In 1990-1995 the bending vibration of homogeneous Euler-Bernoulli cone beams and beams of continuously varying rectangular cross-sections, when one side of the cross-section is constant, while the other side is proportional to $x_2^{\varkappa}$, $\varkappa = \text{const} > 0$, where x_2 is the axial coordinate measured from the cusped end, were considered by S. Naguleswaran [65 - 68]. Firstly, the concrete cases of $\varkappa = 1, 1/2$, and finally, the general case of k were investigated. In these investigations the cusped end is always free; direct analytical solutions were constructed for the mode shape equation and the frequences were also tabulated.

In 1999-2001 two contact problems were considered by N. Shavlakadze [77], [78], namely, the contact problem for an unbounded elastic medium composed of two half-planes $x_1 > 0$ and $x_1 < 0$ having different elastic constants and strengthened on the semi-axis $x_2 > 0$ by an inclusion of variable thickness (cusped beam) with constant Young's modulus and Poisson's ratio. It was assumed that the plate is subjected to plane deformation, the flexural rigidity D had the form

$$D = D_0 x_2^{\varkappa}, \quad D_0, \ \varkappa = \text{const} > 0,$$

and the cusped end $x_2 = 0$ of the beam was free.

The second contact problem considered in [77], [78] was the problem of bending of an isotropic plate of constant thickness reinforced by a finite elastic rib (beam) with the flexural rigidity D of the form

$$D = (a^2 - x_2^2)^{n+1/2} P(x_2),$$

where $a = \text{const} \geq 0$, $n \geq 1$ was an integer and $P(x_2)$ was a polynomial which satisfied certain restrictions. It was assumed that the rib was not loaded.

In the fifties of the twentieth century, I.Vekua [85] introduced a new mathematical model for elastic prismatic shells (i.e., of plates of variable thickness) which was based on expansions of the three-dimensional displacement vector fields and the strain and stress tensors of linear elasticity into orthogonal Fourier-Legendre series with respect to the variable of plate thickness. By taking only the first $N + 1$ terms of the expansions, he introduced the so-called N-th approximation. Each of these approximations for $N = 0, 1, ...$ can be considered as an independent mathematical model of plates. In particular, the approximation for $N = 1$ corresponds to the classical Kirchhoff plate model. In the sixties of the XX century, I. Vekua [86] developed the analogous mathematical model for thin shallow shells. All his results concerning plates and shells are collected in his monograph [87]. Works of I. Babuška, D. Gordeziani, V. Guliaev, I. Khoma, A. Khvoles, T. Meunargia, C. Schwab, T. Vashakmadze, V. Zhgenti, M. Avalishvili, G. Avalishvili, and others (see [1], [2], [26], [28], [58], [59], [63], [76], [84], [90] and the references therein) are devoted to further analysis of I.Vekua's models (rigorous estimation of the modelling error, numerical solutions, etc.) and their generalizations (to non-shallow shells, to the anisotropic case, etc.).

In [53] variational hierarchical two-dimensional models for cusped elastic plates are constructed by G. Jaiani, S. Kharibegashvili, D. Natroshvili, and W. Wendland. With the help of variational methods, existence and uniqueness theorems for the corresponding two-dimensional boundary value problems are proved in appropriate weighted function spaces. By means of the solutions of these two-dimensional boundary value problems, a sequence of approximate solutions in the corresponding three-dimensional region is constructed. This sequence converges in the Sobolev space H^1 to the solution of the original three-dimensional boundary value problem. The systems of differential equations corresponding to the two-dimensional variational hierarchical models are explicitly given for a general system and for Legendre polynomials, in particular.

The direct and inverse problems connected with the interaction between different vector fields of different dimension have been recently given much attention and intensively investigated in the mathematical and engineering scientific literature. They arise in many physical and mechanical models describing the interaction of two different media where the whole process is characterised by a vector-function of dimension k in one medium and by a vector-function of dimension n in another one (e.g., fluid-structure interaction where a streamlined body is an elastic obstacle, scattering of acoustic and electromagnetic waves, interaction between an elastic body and seismic waves, etc.).

A lot of authors have considered and studied in detail the direct problems of interaction between an elastic isotropic body, which occupies a bounded region Ω and where a three-dimensional elastic vector field is to be defined, and some isotropic medium (fluid, say), which occupies the unbounded exterior region, the compliment of Ω with respect to the whole space, where a scalar field is to be defined. The time-harmonic dependent unknown vector and scalar fields are coupled by some kinematic and dynamical conditions on the boundary $\partial\Omega$, which lead to various types of non-classical interface problems of steady oscillations for a piecewise homogeneous isotropic medium. An exhaustive information in this direction concerning theoretical and numerical results can be found in [3]-[5], [20]-[23], [56], [27], [29], [69].

Some particular cases when the elastic body under consideration is an anisotropic one have been treated in [55].

Various authors dedicated their works to the solid-fluid (see e.g., [75], [88], [89], [73], [74], [6]-[8]) contact problems but interaction problems when the profile of an elastic part is cusped one on some part or on the whole boundary of its projection was not considered there. In [9-15] cylindrical bending of a plate with two cusped edges under the action of a fluid has been considered by N. Chinchaladze: the peculiarities of setting of boundary value problems of the classical bending theory caused by sharpening of plates are established; the well-posedness of these problems have been studied; admissible dynamical problems are also investigated; general and harmonic vibration of such plates are studied; the setting of boundary conditions at the plates edges depends on the geometry of sharpenings of plate edges, while the setting of initial conditions is independent of them; in some cases the solutions of these problems are represented explicitly either by integrals or by series; the transmission conditions of interaction problem between an above elastic cusped plate and a fluid are established; the bending of such plates under the action of incompressible ideal and viscous fluids has been considered, in particular, harmonic vibration is studied.

The present lecture notes contain results concerning elastic cusped Euler-Bernoulli beams and Kirchhoff-Love plates reported by the author at workshops and minisymposia organized by TICMI and mostly belonging to him.

This book is divided into two Chapters.

The first Chapter is devoted to elastic cusped beams. In section 1.1 the elastic cusped Euler-Bernoulli beam is introduced. Section 1.2 deals with the properties of the general solution of the degenerate Euler-Bernoulli equation. In Section 1.3 we solve all the admissible boundary value problems of cusped beam's bending. In section 1.4 we study dynamical problems, namely the existence of weak solutions to vibration problems.

The second Chapter is devoted to cusped elastic plates. In Section 2.1 the elastic cusped Kirchhoff-Love plate is introduced. Section 2.2 deals with the admissible bending problems in the energetic space. In Section 2.3 we prove a modification of the Lax-Milgram theorem. In Section 2.4 and 2.5 we study cusped plate's bending and bending vibration problems in the weighted Sobolev spaces.

Chapter 1

Cusped Euler-Bernoulli Beams

The aim of the present chapter is to consider an elastic cusped beam with a continuously varying cross-section of an arbitrary form.

Section 1.2 is devoted to the investigation of properties of solutions of degenerate Euler-Bernoulli equation (see Theorem 1.2.1).

Section 1.3 deals with the well-posedness and correct formulation of all admissible bending BVPs for cusped elastic beams. In contrast to the case of non-cusped beams, when the beam end can always be either clamped or freely supported, for cusped beams this is not the case. The admissibility of these boundary conditions (BCs) depends on the geometry of the beam end sharpening, which is expressed by the convergence-divergence of the integrals I_k^0, I_k^l, $k = 0, 1, 2, ...$ (see Theorem 1.3.1). For the indicated cases of the beam end sharpening some BCs completely disappear and are replaced by the boundedness of the deflection and its derivative. In particular, mechanically free ends are also free of mathematical BCs (see Remarks 1.3.3 and 1.3.4). The BVPs formulated in Theorem 1.3.1 are solved in the explicit form.

A bending vibration of the cusped beam is considered in Section 1.4 (see also [54]). The investigation is based on the Lax-Milgram theorem. It is established that BCs preserve their peculiarities from the static case, while the presence of cusped ends does not affect the setting of initial conditions.

1.1 Cusped Euler-Bernoulli beam

Let the barycenters of cross-sections lie on the axis x_2 of the Cartesian system of coordinates $Ox_1x_2x_3$. The dynamical bending equation of such a beam (i.e., Euler-Bernoulli beam) has the following form [52]

$$(D(x_2)w,_{22}),_{22} = f(x_2,t) - \rho\sigma(x_2)\frac{\partial^2 w}{\partial t^2}, \quad 0 < x_2 < l, \tag{1.1.1}$$

where $w(x_2,t)$ is a deflection of the beam, $f(x_2,t)$ is an intensity of the load,

$$D(x_2) := E(x_2)I(x_2), \tag{1.1.2}$$

$E(x_2)$ is Young's modulus, $I(x_2)$ is the moment of inertia with respect to the barycentric axis normal to the plane x_2x_3, $\rho(x_2)$ is a density, $\sigma(x_2)$ is the area of a transverse section lying in the plane x_1x_3, and index 2 after comma means differentiation with respect to x_2. Such a beam will be called a cusped one if $I(x_2)$ vanishes at least on one of the ends $x_2 = 0$, l of the beam (see Appendix, Figures 1-19).

Let us remark that if we consider a cylindrical bending of the cusped plate (see Chapter 2) with the flexural rigidity

$$D(x_2) := \frac{2E(x_2)h^3(x_2)}{3(1-\nu^2)}, \tag{1.1.3}$$

where ν is Poisson's ratio and $2h(x_2)$ is a thickness of the plate then the bending equation for the plate coincides with (1.1.1), where $\sigma(x_2)$ should be replaced by $2h$.

In the case of a beam vibration with a frequence $\omega = \text{const}$ (i.e., $w(x_2,t) = w(x_2)e^{i\omega t}$, $f(x_2,t) = f(x_2)e^{i\omega t}$), from (1.1.1) we obtain the following vibration equation

$$(D(x_2)w,_{22}),_{22} - \omega^2\rho(x_2)\sigma(x_2)w(x_2) = f(x_2), \quad 0 < x_2 < l. \tag{1.1.4}$$

1.2 Properties of the general solution of the Euler-Bernoulli equation

In the static case, the equation (1.1.1) becomes

$$(D(x_2)w,_{22}),_{22} = f(x_2). \tag{1.2.1}$$

But (1.2.1) coincides with the equation of cylindrical bending of the cusped plate with the flexural rigidity (1.1.3) and projection

$$\omega := \{x_1, x_2 : \ -\infty < x_1 < +\infty, \ 0 < x_2 < l\}$$

on the plane $x_3 = 0$.

The well-posedness of BVPs for such plates when the thickness can vanish only at points $(-\infty < x_1 < +\infty, x_2 = 0)$ was investigated in [46]. After reformulation of the corresponding results for (1.2.1) (see Chapter 2 bellow), where D is given by (1.1.3), the case $I(0) = 0$, $I(l) \neq 0$ will be completely studied. Below in an analogous way we consider the general case when both $I(0) = 0$ and $I(l) = 0$ are admissible. Obviously, the results will be applicable also for cylindrical bending of a plate (1.2.1), where D is given by (1.1.3), with the cusped edges, i.e., both $h(x_1, 0) = 0$ and $h(x_1, l) = 0$ for arbitrary x_1 will be admissible.

Now, let us consider (1.2.1), where D is given by (1.1.2), with $D(x_2) \in C([0,l]) \cap C^2(]0,l[)$ and recall that the bending moment and shearing force are (see also (2.1.6)-(2.1.9) below):

$$M_2 = -Dw_{,22}, \tag{1.2.2}$$

$$Q_2 = M_{2,2}. \tag{1.2.3}$$

At the ends of a beam, where $I(x_2)$ vanishes all quantities will be defined as limits from inside of $]0, l[$.

From (1.2.1)-(1.2.3) follows

$$Q_{2,2} = -f(x_2), \quad M_{2,22} = -f(x_2),$$

where a fixed $x_0 \in]0, l[$ and $C_1, C_2 = \text{const}$.

Hence,

$$Q_2 = -\int_{x_0}^{x_2} f(t)dt + C_1, \tag{1.2.4}$$

$$M_2 = -\int_{x_0}^{x_2} (x_2 - t)f(t)dt + C_1(x_2 - x_0) + C_2, \tag{1.2.5}$$

taking into account (1.2.2),

$$\begin{aligned} w_{,2} &= -\int_{x_0}^{x_2} M_2(\tau)D^{-1}(\tau)d\tau + C_3 \\ &= \int_{x_0}^{x_2} K_1(\tau)D^{-1}(\tau)d\tau + \int_{x_0}^{x_2} K_2(\tau)\tau D^{-1}(\tau)d\tau + C_3 \\ &= \int_{x_0}^{x_2} K(\tau)D^{-1}(\tau)d\tau + C_3, \end{aligned} \tag{1.2.6}$$

$$\begin{aligned} w &= -\int_{x_0}^{x_2} (x_2 - \tau)M_2(\tau)D^{-1}(\tau)d\tau + C_3(x_2 - x_0) + C_4 \\ &= \int_{x_0}^{x_2} (x_2 - \tau)K_1(\tau)D^{-1}(\tau)d\tau \\ &+ \int_{x_0}^{x_2} (x_2 - \tau)K_2(\tau)\tau D^{-1}(\tau)d\tau + C_3(x_2 - x_0) + C_4 \\ &= \int_{x_0}^{x_2} (x_2 - \tau)K(\tau)D^{-1}(\tau)d\tau + C_3(x_2 - x_0) + C_4, \end{aligned} \tag{1.2.7}$$

where

$$K(\tau) := K_1(\tau) + \tau K_2(\tau) \tag{1.2.8}$$

with

$$K_1(\tau) := C_1 x_0 - C_2 - \int_{x_0}^{\tau} f(t) t dt, \tag{1.2.9}$$

$$K_2(\tau) := -C_1 + \int_{x_0}^{\tau} f(t) dt. \tag{1.2.10}$$

Clearly,

$$K'(\tau) = K_2(\tau).$$

From (1.2.4), (1.2.5), (1.2.8)-(1.2.10) we conclude that

$$K_2(\tau) = -Q_2(\tau), \;\; K(\tau) = -M_2(\tau), \;\; K_1(\tau) = \tau Q_2(\tau) - M_2(\tau). \tag{1.2.11}$$

For f summable on $]0, l[$, i.e., $f \in L(]0, l[)$, obviously,

$$Q_2, \;\; M_2 \in C([0, l]); \quad w, \; w_{,2} \in C(]0, l[);$$

the behavior of

$$w_{,2} \quad \text{and} \quad w \quad \text{when} \;\; x_2 \to 0+, \, l-$$

depends, in view of (1.2.6), (1.2.7), on the convergence of the integrals

$$I_i^0 := \int_0^{\varepsilon} \tau^i D^{-1}(\tau) d\tau, \;\; I_i^l := \int_{l-\varepsilon}^{l} (l-\tau)^i D^{-1}(\tau) d\tau,$$
$$i = 0, 1, 2, \dots, \;\; l > \varepsilon = \text{ const} > 0.$$

Evidently, for any nonnegative integer i :

$$\text{if} \;\; I_i^{0(l)} < +\infty, \;\; \text{then} \;\; I_{i+1}^{0(l)} < +\infty, \quad i \geq 0,$$

and

$$\text{if} \;\; I_i^{0(l)} = +\infty, \;\; \text{then} \;\; I_{i-1}^{0(l)} = +\infty, \;\; i \geq 1.$$

Theorem 1.2.1 *Let $f \in L(]0, l[)$, $D \in C^2(]0, l[) \cap C([0, l])$, and $w \in C^4(]0, l[)$ be a solution of equation (1.2.1).*

Case I. *If I_0^0 $(I_0^l) < +\infty$, then*

$$w, \;\; w_{,2} \in C([0, l[) \; (C(]0, l])). \tag{1.2.12}$$

Case II. $I_0^0\ (I_0^l) = +\infty,\ I_1^0\ (I_1^l) < +\infty.$

$$\begin{array}{l} If\ either\ D \in C^2([0,l[)(C^2(]0,l])) \text{ or the value of} \\ \text{the first or second order derivative of } D \\ \text{tends to infinity as } x_2 \to 0+(l-) \text{ and } f \text{ is bounded} \\ \text{in some neighbourhood }]0,\varepsilon]([l-\varepsilon,l[) \text{ of } 0(l), \end{array} \tag{1.2.13}$$

then

$$w \in C([0,l[)\ (C(]0,l])). \tag{1.2.14}$$

If

$$K(0) = 0\ (K(l) = 0), \tag{1.2.15}$$

then

$$w_{,2} = O(1) \text{ as } x_2 \to 0+\ (l-) \tag{1.2.16}$$

(condition (1.2.15) is necessary and sufficient).

Case III. *If* $I_1^0(I_1^l) = +\infty$, $I_2^0(I_2^l) < +\infty$, *and either* $D \in C^3([0,l[)\ (C^3(]0,l]))$ *or the value of the first, second, or third order derivative of* D *tends to infinity as* $x_2 \to 0+(l-)$, *and* f *is bounded with its first derivative in some right (left) neighbourhood of the point* $0(l)$ *then*

$$w = O(1) \text{ as } x_2 \to 0+\ (l-), \tag{1.2.17}$$

if and only if (iff) (1.2.15) is fulfilled.

Case IV. *If* $I_2^0(I_2^l) = +\infty$ *and, moreover, for a fixed* $k \geq 2$

$$I_k^0(I_k^l) = +\infty, \quad I_{k+1}^0(I_{k+1}^l) < +\infty; \tag{1.2.18}$$

$$\begin{array}{r} f^{(j)}(0) = 0\ \ (f^{(j)}(l) = 0), \quad j = 0,1,...,k-2, \\ f^{k-1}(x_2) \quad \text{is continuous at } 0(l), \end{array} \tag{1.2.19}$$

then (1.2.17) *is valid iff*

$$K(0) = 0,\ \ K_2(0) = 0\ \ (K(l) = 0,\ \ K_2(l) = 0) \tag{1.2.20}$$

hold.

Case V. *If* $I_1^0(I_1^l) = +\infty$ *and either (1.2.18), (1.2.19) are fulfilled for* $k \geq 2$ *or (1.2.18) is fulfilled for* $k = 1$ *and* $f(x_2)$ *is continuous at* $0(l)$, *then (1.2.16) is valid iff (1.2.20) holds.*

In order to prove Theorem 1.2.1. beforehand we prove some lemmas

Lemma 1.2.2 *If*

$$I_0^0(I_0^l) = +\infty \tag{1.2.21}$$

and moreover, for a fixed integer $k \geq 0$

$$I_k^0(I_k^l) = +\infty,\ \ I_{k+1}^0(I_{k+1}^l) < +\infty; \tag{1.2.22}$$

$$f^{(j)}(0) = 0\ (f^{(j)}(l) = 0),\ \ j = 0, 1, ..., k-2\ (\textit{for the case}\ k \geq 2); \tag{1.2.23}$$

$$f^{(k-1)}(x_2)\ \ \textit{is continuous at}\ 0(l)\ (\textit{for the case}\ k \geq 1), \tag{1.2.24}$$

then

$$\begin{aligned}&\left|\int\limits_{x_2}^{x_0} K(\tau)D^{-1}(\tau)d\tau\right| \leq \int\limits_{x_2}^{x_0} |K(\tau)|D^{-1}(\tau)d\tau\ \ \left(\int\limits_{x_0}^{x_2} |K(\tau)|D^{-1}(\tau)d\tau\right)\\ &\leq\ \ const < +\infty\ \ \forall x_2 \in]0, x_0]\ \ (\forall\ \ x_2 \in [x_0, l[)\end{aligned} \tag{1.2.25}$$

iff (1.2.15) and (1.2.20) are fulfilled for $k = 0$, and $k \geq 1$, respectively.

Proof. Obviously, in the case $k = 0$

$$\begin{aligned}\left|\int\limits_{x_2}^{x_0} K(\tau)D^{-1}(\tau)d\tau\right| &\leq \int\limits_{x_2}^{x_0} \left|\frac{K(\tau)}{\tau}\right| \tau D^{-1}(\tau)d\tau\\ &\leq C\int\limits_{0}^{x_0} \tau D^{-1}(\tau)d\tau = \text{const} < +\infty \quad \forall\ x_2 \in]0, x_0],\end{aligned} \tag{1.2.26}$$

since, by virtue of $K(0) = 0$ and $K'(\tau) = K_2(\tau)$,

$$\lim_{\tau\to 0+} \frac{K(\tau)}{\tau} = K'(0) = K_2(0) < +\infty,$$

i.e.,

$$\left|\frac{K(\tau)}{\tau}\right| \leq C\ \ \forall\tau \in]0, x_0].$$

Analogously,

$$\begin{aligned}\left|\int\limits_{x_0}^{x_2} K(\tau)D^{-1}(\tau)d\tau\right| &\leq \int\limits_{x_0}^{x_2} \left|\frac{K(\tau)}{l-\tau}\right| (l-\tau)D^{-1}(\tau)d\tau\\ &\leq C\int\limits_{x_0}^{l} (l-\tau)D^{-1}(\tau)d\tau = \text{const} < +\infty \quad \forall x_2 \in [x_0, l[,\end{aligned} \tag{1.2.27}$$

since, using the substitution $l - \tau = \xi$,

$$\lim_{\tau\to l-} \frac{K(\tau)}{l-\tau} = \lim_{\xi\to 0+} \frac{K(l-\xi)}{\xi} = -K'(l) = -K_2(l) < +\infty,$$

i.e.,

$$\left|\frac{K(\tau)}{l-\tau}\right| \leq C\ \ \forall\tau \in [x_0, l[.$$

In the case $k \geq 1$, evidently,

$$\left| \int\limits_{x_2}^{x_0} K(\tau) D^{-1}(\tau) d\tau \right| \leq \int\limits_{x_2}^{x_0} \left| \frac{K(\tau)}{\tau^{k+1}} \right| \tau^{k+1} D^{-1}(\tau) d\tau$$

$$\leq C \int\limits_{0}^{x_0} \tau^{k+1} D^{-1}(\tau) d\tau = \text{const} < +\infty \quad \forall x_2 \in]0, x_0], \tag{1.2.28}$$

since, in view of, (1.2.20), (1.2.23), (1.2.24),

$$\begin{aligned} \lim_{\tau \to 0+} \frac{K(\tau)}{\tau^{k+1}} &= \lim_{\tau \to 0+} \frac{K'(\tau)}{(k+1)\tau^k} = \lim_{\tau \to 0+} \frac{K_2(\tau)}{(k+1)\tau^k} \\ &= \lim_{\tau \to 0+} \frac{f^{(k-1)}(\tau)}{(k+1)!} = \frac{1}{(k+1)!} f^{(k-1)}(0), \end{aligned}$$

i.e., $\left| \dfrac{K(\tau)}{\tau^{k+1}} \right| \leq C \quad \forall \tau \in]0, x_0]$.

Analogously,

$$\left| \int\limits_{x_0}^{x_2} K(\tau) D^{-1}(\tau) d\tau \right| \leq \int\limits_{x_0}^{x_2} |K(\tau)|\, D^{-1}(\tau) d\tau \leq \text{const} < +\infty \quad \forall x_2 \in [x_0, l[, \tag{1.2.29}$$

since, using the substitution $l - \tau = \xi$,

$$\begin{aligned} &\lim_{\tau \to l-} \frac{K(\tau)}{(l-\tau)^{k+1}} = \lim_{\xi \to 0+} \frac{K(l-\xi)}{\xi^{k+1}} \\ = &\lim_{\xi \to 0+} \frac{-K'(l-\xi)}{(k+1)\xi^k} = -\lim_{\xi \to 0+} \frac{K_2(l-\xi)}{(k+1)\xi^k} \\ = &\frac{(-1)^{k+1}}{(k+1)!} f^{(k-1)}(l), \end{aligned}$$

i.e., $\left| \dfrac{K(\tau)}{(l-\tau)^{k+1}} \right| \leq C \quad \forall \tau \in [x_0, l[$.

Let us consider the end $x_2 = 0$ and show that the condition (1.2.15) is also necessary for (1.2.25). In fact, if we assume that (1.2.25) takes place and at the same time, without loss of generality, suppose that $K(0) > 0$, then $K(\tau) > \widetilde{C} =$ const > 0 in some neighbourhood $[0, \varepsilon]$ of 0, and

$$+\infty > \text{ const} \geq \int\limits_{x_2}^{\varepsilon} K(\tau) D^{-1}(\tau) d\tau > \widetilde{C} \int\limits_{x_2}^{\varepsilon} D^{-1}(\tau) d\tau, \tag{1.2.30}$$

whence,

$$\int_{x_2}^{\varepsilon} D^{-1}(\tau)d\tau \leq \text{ const} < +\infty \quad \text{for } x_2 \in]0,\varepsilon].$$

But the last inequality would contradict (1.2.21). Thus, $K(0) = 0$.

Analogously, we can show the necessity of the conditions (1.2.20) for the case $k \geq 1$. The necessity of $K(0) = 0$ follows from the previous assertion. Now, let (1.2.25) be valid and let $K(0) = 0$ but $K_2(0) > 0$. Then, in view of (1.2.8), from $K(0) = 0$ we have $K_1(0) = 0$. By virtue of $K_1'(x_2) = -x_2 f(x_2)$, similarly to the proof of (1.2.28) we can show that

$$\left|\int_{x_2}^{x_0} K_1(\tau)D^{-1}(\tau)d\tau\right| \leq \text{ const} < +\infty \quad \forall x_2 \in]0,x_0], \text{ iff } K_1(0) = 0. \tag{1.2.31}$$

From (1.2.25) and (1.2.31), because of $\tau K_2(\tau) = K(\tau) - K_1(\tau)$, we immediately get

$$\left|\int_{x_2}^{x_0} \tau K_2(\tau)D^{-1}(\tau)d\tau\right| \leq \text{ const} < +\infty \quad \forall x_2 \in]0,x_0]. \tag{1.2.32}$$

But the necessary condition for (1.2.32) is the condition $K_2(0) = 0$. Indeed, if $K_2(0) > 0$, then similar to (1.2.30) we get

$$\left|\int_{x_2}^{\varepsilon} \tau D^{-1}(\tau)d\tau\right| \leq \text{ const} < +\infty \quad \forall x_2 \in]0,\varepsilon],$$

which contradicts $I_1^0 = +\infty$. Thus, $K_2(0) = 0$.

Let us now consider the end $x_2 = l$. The proof of necessity of the conditions (1.2.15) and (1.2.20) is similar to the case of the end $x_2 = 0$. In this case, when $k \geq 1$, we use the following identity

$$\begin{aligned}
&\int_{x_0}^{x_2} (l-\tau)K_2(\tau)D^{-1}(\tau)d\tau \\
&= \int_{x_0}^{x_2} [K_1(\tau) + lK_2(\tau)]D^{-1}(\tau)d\tau \\
&- \int_{x_0}^{x_2} K(\tau)D^{-1}(\tau)d\tau \quad \forall x_2 \in [x_0, l[.
\end{aligned} \tag{1.2.33}$$

Which is obvious in view of (1.2.8). Bearing in mind that

$$K_1(l) + lK_2(l) = K(l) = 0$$

and, hence,

$$\lim_{\tau\to l-}\frac{K_1(\tau)+lK_2(\tau)}{(l-\tau)^{k+1}}=\lim_{\tau\to l-}\frac{-f(\tau)\tau+lf(\tau)}{-(k+1)(l-\tau)^k}$$
$$=\lim_{\tau\to l-}\frac{-f(\tau)}{(k+1)(l-\tau)^{k-1}}=\lim_{\xi\to 0+}\frac{-f(l-\xi)}{(k+1)\xi^{k-1}},$$

in the right hand side of (1.2.33) we prove the boundedness as $x_2\to l-$ of the first integral like the proof of (1.2.29). Therefore, taking into account that we assumed the validity of (1.2.25), the left hand side is bounded for $x_2\in[l-\varepsilon,l[$, since such is the right hand side of (1.2.33). But the necessary condition for it is $K_2(l)=0$. □

Finally, let us note, that (1.2.25) implies

$$\int_{x_2}^{x_0}K(\tau)D^{-1}(\tau)d\tau\in C([0,x_0])\ \ (C([x_0,l]))\,.$$

Corollary 1.2.3 *Under assumptions of Lemma 1.2.2 we have*

$$\lim_{x_2\to 0+}x_2\int_{x_0}^{x_2}K(\tau)D^{-1}(\tau)d\tau=0, \tag{1.2.34}$$

$$\int_{x_0}^{x_2}K(\tau)\tau D^{-1}(\tau)d\tau\in C([0,x_0]), \tag{1.2.35}$$

$$\lim_{x_2\to l-}(x_2-l)\int_{x_0}^{x_2}K(\tau)D^{-1}(\tau)d\tau=0,$$

$$\int_{x_0}^{x_2}K(\tau)(l-\tau)D^{-1}(\tau)d\tau\in C([x_0,l]). \tag{1.2.36}$$

Lemma 1.2.4 *If $I_0^0=+\infty$, $I_1^0<+\infty$ ($I_0^l=+\infty$, $I_1^l<+\infty$), then*

$$\left|x_2\int_{x_2}^{x_0}K(\tau)D^{-1}(\tau)d\tau\right|\le\ const<+\infty\ \ \forall x_2\in]0,x_0] \tag{1.2.37}$$

$$\left(\left|(x_2-l)\int_{x_2}^{x_0}K(\tau)D^{-1}(\tau)d\tau\right|\le\ const<+\infty\ \ \forall x_2\in[x_0,l[\right)\,. \tag{1.2.38}$$

Proof. Evidently,

$$\left| x_2 \int\limits_{x_2}^{x_0} K(\tau) D^{-1}(\tau) d\tau \right| = \left| \int\limits_{x_2}^{x_0} K(\tau) \frac{x_2}{\tau} \tau D^{-1}(\tau) d\tau \right|$$
$$\leq C \int\limits_{0}^{x_0} \tau D^{-1}(\tau) d\tau = \text{ const} < +\infty \quad \forall x_2 \in]0, x_0],$$

because of

$$|K(\tau)| \leq C, \quad \tau \in [0, x_0]; \quad 0 < \frac{x_2}{\tau} \leq 1,$$

since $0 < x_2 \leq \tau \leq x_0$.

Taking into account that

$$0 < \frac{l - x_2}{l - \tau} \leq 1,$$

because of

$$0 < x_0 \leq \tau \leq x_2,$$

i.e.,

$$l > l - x_0 \geq l - \tau \geq l - x_2 > 0,$$

we analogously prove (1.2.38). □

Lemma 1.2.5 *If $I_0^0 = +\infty$, $I_1^0 < +\infty$ ($I_0^l = +\infty$, $I_1^l < +\infty$), and either $D \in C^2([0, l[)$ ($D \in C^2(]0, l])$) or the value of the first or second derivative of D tends to infinity as $x_2 \to 0+$ $(l-)$, and f is bounded in some neighbourhood $]0, \varepsilon]$ ($[l - \varepsilon, l[$) of $0(l)$, then*

$$\lim_{x_2 \to 0+} x_2 \int\limits_{x_2}^{x_0} K(\tau) D^{-1}(\tau) d\tau$$
$$= \begin{cases} 0 & if \;\; K(0) = 0; \;\; if \;\; K(0) \neq 0 \;\; and \;\; either \;\; D'(0) \neq 0 \\ & or \;\; D'(0) = \infty \;\; or \;\; D'(0) = 0 \;\; and \;\; D''(0) = \infty. \end{cases} \tag{1.2.39}$$

The case $D'(0) = 0$, $D''(0) = 0$, $K(0) \neq 0$ and the case $D'(0) = 0$, $D''(0) \neq 0$ cannot occur;

$$\Bigg(\lim_{x_2 \to l-} (x_2 - l) \int\limits_{x_0}^{x_2} K(\tau) D^{-1}(\tau) d\tau$$
$$= \begin{cases} 0 & if \;\; K(l) = 0; \;\; if \;\; K(l) \neq 0 \;\; and \;\; either \;\; D'(l) \neq 0 \\ & or \;\; D'(l) = \infty \;\; or \;\; D'(l) = 0 \;\; and \;\; D''(l) = \infty. \end{cases} \Bigg) \tag{1.2.40}$$

The case $D'(l)=0,\ D''(l)=0,\ K(l)\neq 0$ and the case $D'(l)=0,\ D''(l)\neq 0$ cannot occur;

Proof. Let us note that because of $I_0^0=+\infty$, evidently, $D(0)=0$. If $K(0)=0$, then according to Lemma 1.2.2 for $k=0$ we have (1.2.25), and, hence,

$$\lim_{x_2\to 0+} x_2 \int\limits_{x_2}^{x_0} K(\tau)D^{-1}(\tau)d\tau = 0.$$

Let, now, $K(0)\neq 0$. By virtue of

$$K'(x_2)=K_2(x_2), \tag{1.2.41}$$

we obtain

$$\lim_{x_2\to 0+} x_2 \int\limits_{x_2}^{x_0} K(\tau)D^{-1}(\tau)d\tau = \lim_{x_2\to 0+}\frac{x_2^2K(x_2)}{D(x_2)}$$

$$= \lim_{x_2\to 0+}\frac{2x_2K(x_2)+x_2^2K_2(x_2)}{D'(x_2)}$$

$$= \begin{cases} 0 \quad \text{if } D'(0)\neq 0 \quad \text{or } D'(0)=\infty; \\ \lim\limits_{x_2\to 0+}\dfrac{2K(x_2)+4x_2K_2(x_2)+x_2^2f(x_2)}{D''(x_2)} \quad \text{if } D'(0)=0. \end{cases} \tag{1.2.42}$$

Therefore, when $D'(0)=0$, we obtain

$$\lim_{x_2\to 0+} x_2 \int\limits_{x_2}^{x_0} K(\tau)D^{-1}(\tau)d\tau = \begin{cases} 0 \quad \text{if} \quad \mathrm{D}''(0)=\infty; \\ \dfrac{2K(0)}{D''(0)} \quad \text{if } \mathrm{D}''(0)\neq 0, \end{cases}$$

and

$$\lim_{x_2\to 0+} x_2 \int\limits_{x_2}^{x_0} K(\tau)D^{-1}(\tau)d\tau = \infty \quad \text{if} \quad \mathrm{D}''(0)=0,\ \mathrm{K}(0)\neq 0. \tag{1.2.43}$$

But $D''(0)$ cannot be equal to 0, when $K(0)\neq 0$, otherwise (1.2.37) and (1.2.43) will contradict each other. Hence, (1.2.43) is excluded. Also the case $D'(0)=0,\ D''(0)\neq 0$ cannot occur since, otherwise,

$$\lim_{\tau\to 0+}\tau^\gamma \tau D^{-1}(\tau) = \lim_{\tau\to 0+}\frac{(\gamma+1)\tau^\gamma}{D'(\tau)} = \lim_{\tau\to 0+}\frac{(\gamma+1)\gamma\tau^{\gamma-1}}{D''(\tau)} = \frac{2}{D''(0)} > 0 \ \ for \ \ \gamma=1.$$

Hence, $I_1^0=+\infty$. But the latter contradicts the assumption $I_1^0<+\infty$.

Similarly, we can prove (1.2.40). If $K(l)=0$, then according to to Lemma 1.2.2 for $k=0$ we have (1.2.25), and, hence,

$$\lim_{x_2\to l-}(x_2-l) \int\limits_{x_0}^{x_2} K(\tau)D^{-1}(\tau)d\tau = 0.$$

Let, now, $K(l) \neq 0$. Then, by virtue of (1.2.41), we obtain

$$\lim_{x_2\to l-}(x_2-l)\int\limits_{x_0}^{x_2} K(\tau)D^{-1}(\tau)d\tau = \lim_{x_2\to l-}\frac{-(x_2-l)^2K(x_2)}{D(x_2)}$$

$$= -\lim_{x_2\to l-}\frac{2(x_2-l)K(x_2)+(x_2-l)^2K_2(x_2)}{D'(x_2)}$$

$$= \begin{cases} 0 \quad \text{if } D'(l)\neq 0 \quad \text{or } D'(l)=+\infty; \\ -\lim\limits_{x_2\to l-}\dfrac{2K(x_2)+4(x_2-l)K_2(x_2)+(x_2-l)^2f(x_2)}{D''(x_2)} \quad \text{if } D'(l)=0. \end{cases} \tag{1.2.44}$$

Hence, when $D'(l)=0$, we have

$$\lim_{x_2\to l-}(x_2-l)\int\limits_{x_0}^{x_2} K(\tau)D^{-1}(\tau)d\tau = \begin{cases} 0 \quad \text{if} \quad D''(l)=\infty; \\ -\dfrac{2K(l)}{D''(l)} \quad \text{if } D''(l)\neq 0; \end{cases}$$

and

$$\lim_{x_2\to l-}(x_2-l)\int\limits_{x_0}^{x_2} K(\tau)D^{-1}(\tau)d\tau = \infty \text{ if } D''(l)=0,\ K(l)\neq 0. \tag{1.2.45}$$

But $D''(l)$ can not be equal to 0, when $K(l)\neq 0$, otherwise (1.2.38) and (1.2.45) will contradict each other. Hence, (1.2.45) is excluded. Because of $I_1^l < +\infty$, the case $D'(l) = 0$, $D''(l)\neq 0$ cannot occur as well. □

Lemma 1.2.6 *If $K(0)=0$ ($K(l)=0$), $I_1^0=+\infty$ and $I_2^0<+\infty$ ($I_1^l=+\infty,\ I_2^l<+\infty$), then (1.2.37) ((1.2.38)) is valid.*

Proof. Evidently, by virtue of $I_2^0 < +\infty$, we have

$$\begin{aligned} \Big|x_2\int\limits_{x_2}^{x_0} K(\tau)D^{-1}(\tau)d\tau\Big| &= \Big|\int\limits_{x_2}^{x_0}\frac{K(\tau)}{\tau}\,\frac{x_2}{\tau}\tau^2D^{-1}(\tau)d\tau\Big| \\ &\leq C\int\limits_{0}^{x_0}\tau^2D^{-1}(\tau)d\tau \\ &= const < +\infty \quad \forall x_2\in]0,x_0], \end{aligned}$$

because of

$$0<\frac{x_2}{\tau}\leq 1$$

(since $0 < x_2 \leq \tau \leq x_0$) and

$$\Big|\frac{K(\tau)}{\tau}\Big| < C \quad \forall \tau \in]0, x_0] \tag{1.2.46}$$

(since $\lim\limits_{\tau\to 0+} \frac{K(\tau)}{\tau} = K_2(0) < +\infty$).

Similarly, by virtue of $I_2^l < +\infty$, we have

$$\begin{aligned}
\Big|(x_2 - l)\int\limits_{x_2}^{x_0} K(\tau)D^{-1}(\tau)d\tau\Big| &= \Big|\int\limits_{x_2}^{x_0} \frac{K(\tau)}{l-\tau}\,\frac{x_2 - l}{l-\tau}(l-\tau)^2 D^{-1}(\tau)d\tau\Big| \\
&\leq C\int\limits_{x_0}^{l}(l-\tau)^2 D^{-1}(\tau)d\tau \\
&= const < +\infty \ \ \forall x_2 \in [x_0, l[,
\end{aligned}$$

because of

$$0 < \frac{l - x_2}{l - \tau} \leq 1$$

(since $x_0 \leq \tau \leq x_2 < l$, i.e., $0 < l - x_2 \leq l - \tau$) and

$$\Big|\frac{K(\tau)}{l-\tau}\Big| < C \quad \forall \tau \in [x_0, l[\tag{1.2.47}$$

(since $\lim\limits_{\tau\to l-} \frac{K(\tau)}{l-\tau} = -\lim\limits_{\tau\to l-} K'(\tau) = -\lim\limits_{\tau\to l-} K_2(\tau) = -K_2(l) < +\infty$). □

Lemma 1.2.7 *Let either $D \in C^3([0, l[)$ $(D \in C^3(]0, l]))$ or the value of the first, second, or third order derivative of D tends to infinity as $x_2 \to 0+$ $(l-)$. Let further f be bounded with its first derivative in a neighbourhood $]0, \varepsilon[$ ($]l - \varepsilon, l[$) of the point $x_2 = 0$ $(x_2 = l)$. If $I_1^0 = +\infty$ and $I_2^0 < +\infty$ $(I_1^l = +\infty$ and $I_2^l = +\infty)$, then*

1.

$$\lim_{x_2\to 0+} x_2 \int\limits_{x_2}^{x_0} K(\tau)D^{-1}(\tau)d\tau = \begin{cases} 0 & \textit{when } D'(0) \neq 0 \textit{ or } D'(0) = \infty \\ & \textit{or } D'(0) = 0 \textit{ and } D''(0) = \infty; \\ \dfrac{2K(0)}{D''(0)} & \textit{when } D'(0) = 0 \\ & \textit{and } D''(0) \neq 0; \\ \infty & \textit{when } D'(0) = 0 \textit{ and } D''(0) = 0 \end{cases} \tag{1.2.48}$$

$$\left(\lim_{x_2\to l-} (x_2 - l) \int\limits_{x_0}^{x_2} K(\tau)D^{-1}(\tau)d\tau = \begin{cases} 0 & \textit{when } D'(l) \neq 0 \textit{ or } D'(l) = \infty) \\ & \textit{or } D'(l) = 0 \textit{ and } D''(l) = \infty; \\ -\dfrac{2K(l)}{D''(l)} & \textit{when } D'(l) = 0 \\ & \textit{and } D''(l) \neq 0; \\ \infty & \textit{when } D'(l) = 0 \textit{ and } D''(l) = 0 \end{cases}\right) \tag{1.2.49}$$

if

$$K(0) \neq 0 \ \ (K(l) \neq 0);$$

2.

$$\lim_{x_2 \to 0+} x_2 \int_{x_2}^{x_0} K(\tau) D^{-1}(\tau) d\tau$$
$$= \begin{cases} 0 & \textit{when } K_2(0) = 0; \textit{ when } K_2(0) \neq 0 \textit{ and} \\ & \textit{either } D'(0) \neq 0 \textit{ or } D'(0) = \infty \\ & \textit{or } D'(0) = 0 \textit{ and } D''(0) = \infty \\ & \textit{or } D'(0) = 0 \textit{ and } D''(0) \neq 0 \\ & \textit{or } D'(0) = 0, \ D''(0) = 0, \textit{ and } D'''(0) = \infty \end{cases} \tag{1.2.50}$$

(the case $D'(0) = 0, \ D''(0) = 0, \ D'''(0) = 0, \ K_2(0) \neq 0$ and the case $D'(0) = 0, \ D''(0) = 0, \ D'''(0) \neq 0$ cannot occur)

$$\Bigg(\lim_{x_2 \to l-} (x_2 - l) \int_{x_0}^{x_2} K(\tau) D^{-1}(\tau) d\tau$$
$$= \begin{cases} 0 & \textit{when } K_2(l) = 0; \textit{ when } K_2(l) \neq 0 \textit{ and} \\ & \textit{either } D'(l) \neq 0 \textit{ or } D'(l) = \infty \\ & \textit{or } D'(l) = 0 \textit{ and } D''(l) = \infty \\ & \textit{or } D'(l) = 0 \textit{ and } D''(l) \neq 0 \\ & \textit{or } D'(l) = 0, \ D''(l) = 0, \ \textit{ and } D'''(l) = \infty \end{cases} \tag{1.2.51}$$

(the case $D'(l) = 0, \ D''(l) = 0, \ D'''(l) = 0, \ K_2(l) \neq 0$ and the case $D'(l) = 0, \ D''(l) = 0, \ D'''(l) \neq 0$ cannot occur)
if

$$K(0) = 0 \quad (K(l) = 0).$$

Proof. In both the cases the reasonings (1.2.42), (1.2.43) are valid since by their derivation it was not used that $I_1^0 \ (I_1^l) < +\infty$. Therefore, (1.2.48) easily follows from (1.2.42) if $K(0) \neq 0$. If $K(0) = 0$, when $D'(0) = 0$, from (1.2.42) we get

$$\lim_{x_2 \to 0+} x_2 \int_{x_2}^{x_0} K(\tau) D^{-1}(\tau) d\tau = \begin{cases} 0 \quad if \quad D''(0) \neq 0; \\ \lim\limits_{x_2 \to 0+} \dfrac{6K_2(x_2) + 6x_2 f(x_2) + x_2^2 f'(x_2)}{D'''(x_2)} \\ \qquad if \quad D''(0) = 0. \end{cases}$$

Hence, when $D'(0) = 0$, $D''(0) = 0$, we have

$$\lim_{x_2 \to 0+} x_2 \int_{x_2}^{x_0} K(\tau) D^{-1}(\tau) d\tau \ = \begin{cases} 0 \quad if \quad D'''(0) = \infty; \\ \dfrac{6K_2(0)}{D'''(0)} \quad if \quad D'''(0) \neq 0, \end{cases}$$

$$\lim_{x_2 \to 0+} x_2 \int_{x_2}^{x_0} K(\tau) D^{-1}(\tau) d\tau = \infty \ if \ D'''(0) = 0, \quad K_2(0) \neq 0. \tag{1.2.52}$$

But $D'''(0)$ and $K_2(0) \neq 0$ cannot take place at the same time, otherwise (1.2.52) and (1.2.37) (see Lemma 1.2.6 which has been proved under the assumptions $I_1^0 = +\infty$, $I_2^0 < +\infty$, without any requirment of differentiability of $D(x_2)$) will contradict each other. Thus, (1.2.52) is excluded. Also the case $D'(0) = 0$, $D''(0) = 0$, $D'''(0) \neq 0$ cannot occur since in this case $I_2^0 = +\infty$ which is in contradiction with our assumption $I_2^0 < +\infty$. Indeed,

$$\lim_{\tau \to 0+} \tau^\gamma \tau^2 D^{-1}(\tau) = \lim_{\tau \to 0+} \frac{(\gamma+2)\tau^{\gamma+1}}{D'(\tau)}$$

$$= \lim_{\tau \to 0+} \frac{(\gamma+2)(\gamma+1)\tau^\gamma}{D''(\tau)} = \lim_{\tau \to 0+} \frac{(\gamma+2)(\gamma+1)\gamma\tau^{\gamma-1}}{D'''(\tau)} = \frac{6}{D'''(0)} > 0 \;\; for \;\; \gamma = 1.$$

But it means that $I_1^0 = +\infty$. When $K_2(0) = 0$, then according to the Lemma 1.2.2 for $k = 1$, (1.2.25) holds iff (1.2.20) is valid. Therefore,

$$\lim_{x_2 \to 0+} x_2 \int_{x_2}^{x_0} K(\tau) D^{-1}(\tau) d\tau = 0 \;\; if \;\; K_2(0) = 0.$$

So, (1.2.50) is proved.

Similarly, in both the cases the reasonings (1.2.44) are valid. Therefore, (1.2.49) easily follows if $K(l) \neq 0$. If $K(l) = 0$, when $D'(l) = 0$, from (1.2.44) we get

$$\lim_{x_2 \to l-} (x_2 - l) \int_{x_0}^{x_2} K(\tau) D^{-1}(\tau) d\tau$$
$$= \begin{cases} 0 \quad if \quad D''(l) \neq 0; \\ -\lim\limits_{x_2 \to 0+} \dfrac{6K_2(x_2) + 6(x_2 - l) f(x_2) + (x_2 - l)^2 f'(x_2)}{D'''(x_2)} \quad if \quad D''(l) = 0. \end{cases}$$

Hence, when $D'(l) = 0$, $D''(l) = 0$, we have

$$\lim_{x_2 \to l-} (x_2 - l) \int_{x_0}^{x_2} K(\tau) D^{-1}(\tau) d\tau = \begin{cases} 0 \quad if \quad D'''(l) = \infty; \\ -\dfrac{6K_2(l)}{D'''(l)} \quad if \quad D'''(l) \neq 0, \end{cases}$$

$$\lim_{x_2 \to l-} (x_2 - l) \int_{x_2}^{x_0} K(\tau) D^{-1}(\tau) d\tau = \infty \; if \; D'''(l) = 0, \quad K_2(l) \neq 0. \tag{1.2.53}$$

But $D'''(l)$ and $K_2(l) \neq 0$ cannot take place at the same time, otherwise (1.2.53) and (1.2.38) (see Lemma 1.2.6 which has been proved under the assumptions $I_1^l = +\infty$, $I_2^l < +\infty$, without any requirement of differentiability of $D(x_2)$) will contradict each other. Thus, (1.2.53) is excluded. Because of $I_2^l < +\infty$, the case $D'(l) = 0$, $D''(l) = 0$,

$D'''(l) \neq 0$ cannot occur as well. When $K_2(l) = 0$, then according to the Lemma 1.2.2 for $k = 1$, (1.2.25) holds iff (1.2.20) is valid. Therefore,

$$\lim_{x_2 \to 0+} (x_2 - l) \int_{x_0}^{x_2} K(\tau) D^{-1}(\tau) d\tau = 0 \quad if \quad K_2(l) = 0.$$

So, (1.2.51) is proved. □

Lemma 1.2.8 *If $I_1^0 = +\infty$ and $I_2^0 < +\infty$ ($I_1^l = +\infty$ and $I_2^l < +\infty$), then*

$$\lim_{x_2 \to 0+} \int_{x_2}^{x_0} K(\tau) \tau D^{-1}(\tau) d\tau = \int_{0}^{x_0} K(\tau) \tau D^{-1}(\tau) d\tau < +\infty \tag{1.2.54}$$

$$\left(\lim_{x_2 \to l-} \int_{x_0}^{x_2} (l-\tau) K(\tau) D^{-1}(\tau) d\tau = \int_{x_0}^{l} (l-\tau) K(\tau) D^{-1}(\tau) d\tau < +\infty \right) \tag{1.2.55}$$

iff

$$K(0) = 0 \quad (K(l) = 0). \tag{1.2.56}$$

Proof. By virtue of (1.2.46), for every $\tau \in]0, x_0]$ we have

$$|K(\tau) \tau D^{-1}(\tau) d\tau| = \left| \frac{K(\tau)}{\tau} \right| |\tau^2 D^{-1}(\tau)| \leq C |\tau^2 D^{-1}(\tau)|. \tag{1.2.57}$$

But the right hand side of (1.2.57) is integrable on $]0, x_0[$, because of $I_2^0 < +\infty$. Therefore, the left hand side of (1.2.57) will be also integrable on $]0, x_0[$, and so, we arrive at (1.2.54). The necessity of (1.2.56) can be shown with the help of (1.2.57) in a usual way by contradiction (see e.g., (1.2.30)).

Similarly, by virtue of (1.2.47), for every $\tau \in [x_0, l[$ we have

$$|K(\tau)(l-\tau) D^{-1}(\tau) d\tau| = \left| \frac{K(\tau)}{l-\tau} \right| |(l-\tau)^2 D^{-1}(\tau)| \leq C |(l-\tau)^2 D^{-1}(\tau)|. \tag{1.2.58}$$

But the right hand side of (1.2.58) is integrable on $]x_0, l[$, because of $I_2^l < +\infty$. Therefore, the left hand side of (1.2.58) will be also integrable on $]x_0, l[$, and so, we arrive at (1.2.55). The necessity of (1.2.56) can be shown with the help of (1.2.58) in a usual way by contradiction (see e.g., (1.2.30)). □

Lemma 1.2.9 *If either $I_1^0 = +\infty$ and $I_2^0 < +\infty$ ($I_1^l = +\infty$ and $I_2^l < +\infty$), and (1.2.15) is violated or $I_k^0 = +\infty$ and $I_{k+1}^0 < +\infty$ ($I_k^l = +\infty$ and $I_{k+1}^l < +\infty$), $k \in \{2, 3, \ldots\}$, and (1.2.20) is violated, then*

$$\lim_{\substack{x_2 \to 0+ \\ (x_2 \to l-)}} \int_{x_2}^{x_0} (\tau - x_2)(M_2 w)(\tau) D^{-1}(\tau) d\tau = \infty.$$

Proof: Let first $(M_2w)(0) > 0$, then on the one hand, both (1.2.15) and (1.2.20) are violated for $k \geq 1$ and on the other hand, there exists an $\varepsilon = const > 0$ such that

$$(M_2w)(x_2) \geq \widetilde{C} = const > 0 \quad \forall x_2 \in [0, \varepsilon].$$

After substitution $\tau - x_2 = t$, we get

$$\begin{aligned} &\lim_{x_2 \to 0+} \int_{x_2}^{x_0} (\tau - x_2)(M_2w)(\tau) D^{-1}(\tau) d\tau \\ = &\lim_{x_2 \to 0+} \int_{0}^{x_0 - x_2} t(M_2w)(x_2 + t) D^{-1}(x_2 + t) dt \\ = &\lim_{x_2 \to 0+} \int_{0}^{\varepsilon/2} t(M_2w)(x_2 + t) D^{-1}(x_2 + t) dt \\ + &\int_{\varepsilon/2}^{x_0} t(M_2w)(t) D^{-1}(t) dt = +\infty, \end{aligned}$$

since

$$\begin{aligned} &\lim_{x_2 \to 0+} \int_{0}^{\varepsilon/2} t(M_2w)(x_2 + t) D^{-1}(x_2 + t) dt \\ \geq \; &\widetilde{C} \lim_{x_2 \to 0+} \int_{0}^{\varepsilon/2} t D^{-1}(x_2 + t) dt = +\infty, \quad x_2, t \in [0, \varepsilon/2[, \end{aligned}$$

because of $x_2 + t < \varepsilon$ and $I_1 = +\infty$.

Let, now, $(M_2w)(0) = 0$ but $(Q_2w)(0) > 0$, i.e., (1.2.20) is violated for $k \geq 2$ and there exists an $\varepsilon = const > 0$ such that $(Q_2w)(x_2) \geq \widetilde{C} > 0 \; \forall x_2 \in [0, \varepsilon]$. Similarly, in view of

$$x_2(Q_2w)(x_2) = (M_2w)(x_2) + K_1(x_2),$$

we obtain

$$\begin{aligned} &\lim_{x_2 \to 0+} \int_{x_2}^{x_0} (\tau - x_2)(M_2w)(\tau) D^{-1}(\tau) d\tau \\ = &\lim_{x_2 \to 0+} \int_{0}^{x_0 - x_2} t[(t + x_2)(Q_2w)(x_2 + t) - K_1(x_2 + t)] D^{-1}(x_2 + t) dt \end{aligned}$$

$$= \lim_{x_2\to 0+}\int_0^{x_0-x_2} t(t+x_2)(Q_2w)(x_2+t)D^{-1}(x_2+t)dt$$

$$- \lim_{x_2\to 0+}\int_0^{x_0-x_2} tK_1(x_2+t)D^{-1}(x_2+t)dt$$

$$= \lim_{x_2\to 0+}\int_0^{\varepsilon/2} t(t+x_2)(Q_2w)(x_2+t)D^{-1}(x_2+t)dt$$

$$+ \int_{\varepsilon/2}^{x_0} t^2(Q_2w)(t)D^{-1}(t)dt - \int_0^{x_0} tK_1(t)D^{-1}(t)dt = +\infty,$$

since

$$\lim_{x_2\to 0+}\int_0^{\varepsilon/2} t(t+x_2)(Q_2w)(x_2+t)D^{-1}(x_2+t)dt$$

$$\geq \widetilde{C}\lim_{x_2\to 0+}\int_0^{\varepsilon/2} t(t+x_2)D^{-1}(x_2+t)dt = +\infty, \quad x_2, t\in [0,\varepsilon/2[,$$

because of $I_2^0 = +\infty$ and

$$\left|\int_{\varepsilon/2}^{x_2} t^2(Q_2w)(t)D^{-1}(t)dt\right| < +\infty,$$

$$\left|\int_0^{x_2} tK_1(t)D^{-1}(t)dt\right| < +\infty$$

(the finiteness of the last term readily follows from (1.2.31)). Using substitution $\tau - x_2 = t - l$, the case $x_2 \to l-$ we consider analogously. Thus, Lemma 1.2.9 is completely proved. □

Proof of Theorem 1.2.1.

Case I is evident in view of (1.2.7), (1.2.6), and I_0^0 (I_0^l) $< +\infty$.

Case II. $I_0^0 = +\infty$, $I_1^0 < +\infty$ ($I_0^l = +\infty$, $I_1^l < +\infty$). Then, in view of Lemma 1.2.4, the estimate (1.2.37) ((1.2.38)) is valid. Taking into account the fact that the other term

$$-\int_{x_0}^{x_2} \tau K(\tau)D^{-1}(\tau)d\tau$$

in (1.2.7) is bounded on $]0, x_0]$ ($[x_0, l[$) because of $I_1^0 < +\infty$ ($I_1^l < +\infty$), we conclude that

$$w(x_2) = O(1) \quad \text{as} \ \ x_2 \to 0+(l-).$$

Moreover, if (1.2.13) is fulfilled then (1.2.14) is valid. Indeed, from Lemma 1.2.5 it follows that

$$x_2 \int\limits_{x_0}^{x_2} K(\tau)D^{-1}(\tau)d\tau \in C([0,l[) \ \ (C(]0,l])) \,.$$

Obviously, the other term

$$-\int\limits_{x_0}^{x_2} \tau K(\tau)D^{-1}(\tau)d\tau$$

of (1.2.7) is also continuous on $[0, l[$ ($]0, l]$) and hence (1.2.14) is proved.

If (1.2.15) is fulfilled, then in view of (1.2.6), (1.2.25), obviously, $w_{,2}$ is bounded on $]0, l[$ and, moreover, continuous on $[0, l[$ ($]0, l]$) (see a note after proof of Lemma 1.2.2). So, (1.2.16) is proved. The necessity of condition (1.2.15) for (1.2.16) readily follows from (1.2.6) and Lemma 1.2.2 (see (1.2.25), (1.2.15)).

Case III. $I_1^0 = +\infty$, $I_2^0 < +\infty$ ($I_1^l = +\infty$, $I_2^l < +\infty$). Then, according to Corollary 1.2.3 for the case $k = 1$ (see also Lemma 1.2.2), from (1.2.7) we get (1.2.17) and, moreover (1.2.14). Let us note that in order to consider $x_2 \to l-$ we represent (1.2.7) as follows

$$w = \int\limits_{x_0}^{x_2} [(x_2 - l) + (l - \tau)]K(\tau)D^{-1}(\tau)d\tau + C_3(x_2 - x_0) + C_4.$$

The necessity of condition (1.2.15) for (1.2.27) follows from Lemma 1.2.9.

Case IV. Proof immediately follows from Corollary 1.2.3 and Lemma 1.2.9 for the case $k \geq 2$.

Case V is evident in view of Lemma 1.2.2 (see (1.2.6), (1.2.25), (1.2.15), (1.2.20)). □

Remark 1.2.10 *In Theorem 1.2.1 the existence of k was assumed such that $I_k^0 < +\infty$. If $I_k^0 = +\infty \ \forall k$, and $K(\tau) := K_1(\tau) + \tau K_2(\tau)$ is analytic in a right (left) neighbourhood of $\tau = 0$ (l), then, obviously, w and $w_{,2}$ are unbounded when $x_2 \to 0+$ ($l-$). We prove this by contradiction. Indeed, e.g., consider (1.2.6):*

$$w_{,2}(x_2) = \int\limits_{x_0}^{x_2} K(\tau)D^{-1}(\tau)d\tau + C_3.$$

Let this derivative be bounded when $x_2 \to 0+$, and $K(0) = 0$; the last condition is necessary for the boundednes of this derivative. Since the analytic function $K(\tau) \not\equiv 0$, there exists k such that

$$K^{(j)}(0) = 0, \quad j = 0, 1, ..., k-1, \quad K^{(k)}(0) \neq 0.$$

Further,

$$w_{,2}(x_2) = \int\limits_{x_0}^{x_2} \frac{K(\tau)}{\tau^k}\tau^k D^{-1}(\tau)d\tau + C_3,$$

where

$$\lim_{\tau\to 0+} \frac{K(\tau)}{\tau^k} = \frac{K^{(k)}(0)}{k!} \neq 0.$$

Then, taking into account the boundedness of $w_{,2}$*, similarly to the proof of Lemma 1.2.2 (see (1.2.30)) we can show*

$$\left| \int\limits_{x_2}^{\varepsilon} \tau^k D^{-1}(\tau)d\tau \right| < +\infty \qquad \textit{for} \quad x_2 \in]0,\varepsilon],$$

which would be in contradiction with $I_k = +\infty\ \forall k$*. Thus,* $w_{,2}$ *is unbounded when* $x_2 \to 0+$*.*

Remark 1.2.11 *In the case of the cusped beam with only one cusped end* $x_2 = 0$ *Theorem 1.2.1 formulated in the slightly different form is proved in [54] (see also [46]). In our general case of two cusped ends it looks like*

Theorem 1.2.12 *Let* $f \in L(]0,l[)$*,* $D \in C^2(]0,l[) \cap C([0,l])$*, and* $w \in C^4(]0,l[)$ *be a solution of equation (1.2.1). Then:*

1) If $I_1^0(I_1^l) < +\infty$*, then* $w \in C([0,l[)(C(]0,l]))$*.*

[For $I_0^0(I_0^l) = +\infty$*, we additionally assume that either* $D \in C^2([0,l[)\ (C^2(]0,l]))$ *or the value of the first or second order derivative of* D *tends to infinity as* $x_2 \to 0+\ (l-)$ *and* f *is bounded in some neighbourhood* $]0,\varepsilon]([l-\varepsilon,l[)$ *of* $0(l)$*.]*

2) If $I_1^0(I_1^l) = +\infty$ *and* $I_2^0(I_2^l) < +\infty$*, then* $w \in C([0,l[)\ (C(]0,l]))$ *iff*

$$(M_2w)(0) = 0\ \ ((M_2w)(l) = 0). \tag{1.2.59}$$

$$\begin{aligned}&[\textit{We additionally assume that either}\\ &D \in C^3([0,l[) \textit{ or the value of the first, second, or third order derivative}\\ &\textit{of } D \textit{ tends to infinity as } x_2 \to 0+\ (l-). \textit{ Further, we suppose that}\\ &(\textit{see Remark } 1.2.13 \textit{ below}) f \textit{ is bounded with its first derivative}\\ &\textit{in some right } (\textit{left}) \textit{ neighbourhood of the point } 0\ (l).]\end{aligned} \tag{1.2.60}$$

If (1.2.59) is violated, then w *is unbounded as* $x_2 \to 0+\ (l-)$*.*

3) If $I_k^0(I_k^l) = +\infty$ *and* $I_{k+1}^0(I_{k+1}^l) < +\infty$ *for a fixed* $k \in \{2,3,\ldots\}$*, then* $w \in C([0,l[)\ (C(]0,l]))$ *iff*

$$(M_2w)(0) = 0 \ \ ((M_2w)(l) = 0) \ \ \textit{and} \ \ (Q_2w)(0) = 0 \ \ ((Q_2w)(l) = 0). \tag{1.2.61}$$

$$[\textit{We additionally assume that } f^{(j)}(0) = 0 \ (f^{(j)}(l) = 0), \ j = 0, 1, \dots, k-2, \\ \textit{and } f^{(k-1)} \textit{ is continuous at the point } 0(l).] \tag{1.2.62}$$

If (1.2.61) is violated, then w *is unbounded as* $x_2 \to 0+ \ (l-)$.

4) If $I_0^0 \ (I_0^l) < +\infty$, *then* $w_{,2} \in C([0,l[) \ (C(]0,l]))$.

5) If $I_0^0 \ (I_0^l) = +\infty$ *and* $I_1^0 \ (I_1^l) < +\infty$, *then* $w_{,2} \in C([0,l[) \ (C(]0,l]))$ *iff (1.2.59) holds.*

If (1.2.59) is violated, then $w_{,2}$ *is unbounded as* $x_2 \to 0+ \ (l-)$.

6) If $I_k^0 \ (I_k^l) = +\infty$ *and* $I_{k+1}^0(I_{k+1}^l) < +\infty$ *for a fixed* $k \in \{1, 2, \dots\}$, *then* $w_{,2} \in C([0,l[) \ C(]0,l])$ *iff (1.2.61) holds.*

[We additionally assume that $f^{(j)}(0) = 0$ *(*$f^{(j)}(l) = 0$*),* $j = 0, 1, \dots, k-2$ *(if* $k \geq 2$*),* $f^{(k-1)}(x_2)$ *is continuous at the point 0 (l) (if* $k \geq 1$*).]*

If (1.2.61) is violated, then $w_{,2}$ *is unbounded as* $x_2 \to 0+ \ (l-)$.

7) $(M_2w)(x_2) \in C([0,l])$.

8) $(Q_2w)(x_2) \in C([0,l])$.

Remark 1.2.13 *In Theorem 1.2.12, the restrictions on* f *are not substantial; we could take even* $f \equiv 0$ *on* $[0,l]$. *On the other hand, the above restrictions can be weakened without influence on the kernel of this statement which consists in the clarification of the question of boundedness/unboundedness of* $w, w_{,2}, M_2w$ *and* Q_2w *as* $x_2 \to 0+ \ (l-)$ *in dependence on the behaviour (i.e., the nature of vanishing) of* $D(x_2)$ *at the point* $x_2 = 0 \ (l)$.

Remark 1.2.14 *The unboundedness of* $w_{,2}$ *geometrically means that the axis of the beam is tangent to the axis* x_3 *which mechanically seems hard to realize, but acceptable in some sense. The unboundedness of the deflection is not acceptable from the point of view of mechanics but can be justified like the case of concentrated forces.*

Remark 1.2.15 *Substituting (1.2.5) in (1.2.7), we rewrite (1.2.7) in the following two forms:*

$$\begin{aligned} w(x_2) &= (C_1x_0 - C_2)\int_{x_0}^{x_2}(x_2-\tau)D^{-1}(\tau)d\tau - \int_{x_0}^{x_2}(x_2-\tau)\int_{x_0}^{\tau} f(t)tdtD^{-1}(\tau)d\tau \\ &- C_1\int_{x_0}^{x_2}(x_2-\tau)\tau D^{-1}(\tau)d\tau + \int_{x_0}^{x_2}(x_2-\tau)\int_{x_0}^{\tau} f(t)dt\tau D^{-1}(\tau)d\tau \\ &+ C_3(x_2-x_0) + C_4, \end{aligned} \tag{1.2.63}$$

$$\begin{aligned} w(x_2) &= [C_1(x_0-l)+C_1l-C_2]\int\limits_{x_0}^{x_2}[(x_2-l)+(l-\tau)]\,D^{-1}(\tau)d\tau \\ &- \int\limits_{x_0}^{x_2}(x_2-\tau)\int\limits_{x_0}^{\tau}f(t)tdtD^{-1}(\tau)d\tau - C_1\int\limits_{x_0}^{x_2}[(x_2-l)+(l-\tau)]\,\tau D^{-1}(\tau)d\tau \\ &+ \int\limits_{x_0}^{x_2}(x_2-\tau)\int\limits_{x_0}^{\tau}f(t)dt\tau D^{-1}(\tau)d\tau + C_3(x_2-x_0)+C_4. \end{aligned} \tag{1.2.64}$$

Introducing the notation

$$I_k(x_2,x_0) := \int\limits_{x_2}^{x_0} t^k D^{-1}(t)dt,\ \ I_k(x_2) := I_k(x_2,l),\ \ I_k := I_k(0), \tag{1.2.65}$$

$$x_2 \in]0,x_0],\ \ x_0 \in]0,l[,$$

$$I^k(x_0,x_2) := \int\limits_{x_0}^{x_2}(l-t)^k D^{-1}(t)dt,\ \ I^k(x_2) := I^k(0,x_2),\ \ I^k := I^k(l), \tag{1.2.66}$$

$$x_2 \in [x_0,l[,\ \ x_0 \in]0,l[,$$

from (1.2.63) and (1.2.64) we get

$$\begin{aligned} w(x_2) &= -(C_1x_0-C_2)\,[x_2I_0(x_2,x_0)-I_1(x_2,x_0)] + C_1\,[x_2I_1(x_2,x_0)-I_2(x_2,x_0)] \\ &+ C_3(x_2-x_0)+C_4 - \int\limits_{x_0}^{x_2}(x_2-\tau)\int\limits_{x_0}^{\tau}f(t)tdtD^{-1}(\tau)d\tau \\ &+ \int\limits_{x_0}^{x_2}(x_2-\tau)\int\limits_{x_0}^{\tau}f(t)dt\tau D^{-1}(\tau)d\tau \end{aligned} \tag{1.2.67}$$

and

$$\begin{aligned} w(x_2) &= [C_1(x_0-l)-C_2]\left[(x_2-l)I^0(x_0,x_2)+I^1(x_0,x_2)\right] \\ &+ C_1\left[(x_2-l)I^1(x_0,x_2)+I^2(x_0,x_2)\right]+C_3(x_2-x_0)+C_4 \\ &- \int\limits_{x_0}^{x_2}(x_2-\tau)\int\limits_{x_0}^{\tau}f(t)tdtD^{-1}(\tau)d\tau + \int\limits_{x_0}^{x_2}(x_2-\tau)\int\limits_{x_0}^{\tau}f(t)dt\tau D^{-1}(\tau)d\tau, \end{aligned} \tag{1.2.68}$$

respectively.

Using obvious relations

$$x_2I_{k,2}(x_2,x_0) = I_{k+1,2}(x_2,x_0),\ \ (l-x_2)I^k{}_{,2}(x_0,x_2) = I^{k+1}{}_{,2}(x_0,x_2),\ \ k=0,1,\ldots,.$$

after differentiation of (1.2.67) and (1.2.68), we obtain

$$w,_2(x_2) = -(C_1x_0 - C_2)I_0(x_2,x_0) + C_1I_1(x_2,x_0) + C_3 - \int_{x_0}^{x_2}\int_{x_0}^{\tau} f(t)tdtD^{-1}(\tau)d\tau + \int_{x_0}^{x_2}\int_{x_0}^{\tau} f(t)dt\tau D^{-1}(\tau)d\tau, \quad (1.2.69)$$

$$(M_2w)(x_2) = -D(x_2)w,_{22}(x_2) = -(C_1x_0 - C_2) + C_1x_2 + \int_{x_0}^{x_2} f(t)tdt - x_2\int_{x_0}^{x_2} f(t)dt, \quad (1.2.70)$$

$$(Q_2w)(x_2) = C_1 - \int_{x_0}^{x_2} f(t)dt, \quad (1.2.71)$$

and

$$w,_2(x_2) = [C_1(x_0 - l) - C_2]\, I^0(x_0,x_2) + C_1I^1(x_0,x_2) + C_3 - \int_{x_0}^{x_2}\int_{x_0}^{\tau} f(t)tdtD^{-1}(\tau)d\tau + \int_{x_0}^{x_2}\int_{x_0}^{\tau} f(t)dt\tau D^{-1}(\tau)d\tau, \quad (1.2.72)$$

$$(M_2w)(x_2) = -D(x_2)w,_{22}(x_2) = -[C_1(x_0 - l) - C_2] - C_1(l - x_2) + \int_{x_0}^{x_2} f(t)tdt - x_2\int_{x_0}^{x_2} f(t)dt, \quad (1.2.73)$$

$$(Q_2w)(x_2) = C_1 - \int_{x_0}^{x_2} f(t)dt, \quad (1.2.74)$$

respectively.

On the other hand, evidently, (1.2.71) and (1.2.74) coincide with (1.2.4); (1.2.70) and (1.2.73) coincide with (1.2.5); finally, from (1.2.6) we could obtain (1.2.69) and (1.2.72) analogously to (1.2.67) and (1.2.68).

If $f \equiv 0$, then from (1.2.67), (1.2.69)-(1.2.71) and (1.2.68), (1.2.72)-(1.2.74) we have

$$w(x_2) = -(C_1x_0 - C_2)\,[x_2I_0(x_2,x_0) - I_1(x_2,x_0)] + C_1\,[x_2I_1(x_2,x_0) - I_2(x_2,x_0)] + C_3(x_2 - x_0) + C_4, \quad (1.2.75)$$

$$w,_2(x_2) = -(C_1x_0 - C_2)I_0(x_2,x_0) + C_1I_1(x_2,x_0) + C_3, \quad (1.2.76)$$

$$(M_2w)(x_2) = -D(x_2)w,_{22}(x_2) = -(C_1x_0 - C_2) + C_1x_2, \quad (1.2.77)$$

$$(Q_2w)(x_2) = C_1, \quad (1.2.78)$$

and

$$\begin{aligned} w(x_2) &= [C_1(x_0-l)-C_2]\left[(x_2-l)I^0(x_0,x_2)+I^1(x_0,x_2)\right] \\ &+ C_1\left[(x_2-l)I^1(x_0,x_2)+I^2(x_0,x_2)\right]+C_3(x_2-x_0)+C_4, \end{aligned} \tag{1.2.79}$$

$$w_{,2}(x_2) = [C_1(x_0-l)-C_2]\,I^0(x_0,x_2)+C_1I^1(x_0,x_2)+C_3, \tag{1.2.80}$$

$$(M_2w)(x_2) = -D(x_2)w_{,22}(x_2) = -[C_1(x_0-l)-C_2]-C_1(l-x_2), \tag{1.2.81}$$

$$(Q_2w)(x_2) = C_1, \tag{1.2.82}$$

respectively.

Taking into account

$$M_2(0)=C_2-C_1x_0, \quad M_2(l)=C_2-C_1(x_0-l),$$

we rewrite (1.2.75)-(1.2.77) and (1.2.79)-(1.2.81) as follows

$$\begin{aligned} w(x_2) &= x_2\left[M_2(0)I_0(x_2,x_0)+C_1I_1(x_2,x_0)\right]-M_2(0)I_1(x_2,x_0) \\ &- C_1I_2(x_2,x_0)+C_3(x_2-x_0)+C_4, \end{aligned} \tag{1.2.83}$$

$$w_{,2}(x_2) = M_2(0)I_0(x_2,x_0)+C_1I_1(x_2,x_0)+C_3, \tag{1.2.84}$$

$$(M_2w)(x_2) = M_2(0)+C_1x_2, \tag{1.2.85}$$

and

$$\begin{aligned} w(x_2) &= (l-x_2)\left[M_2(l)I^0(x_0,x_2)-C_1I^1(x_0,x_2)\right]-M_2(l)I^1(x_0,x_2) \\ &+ C_1I^2(x_0,x_2)+C_3(x_2-x_0)+C_4, \end{aligned} \tag{1.2.86}$$

$$w_{,2}(x_2) = -M_2(l)I^0(x_0,x_2)+C_1I^1(x_0,x_2)+C_3, \tag{1.2.87}$$

$$(M_2w)(x_2) = M_2(l)-C_1(l-x_2), \tag{1.2.88}$$

where

$$C_1=(Q_2w)(x_2)=:Q_2=const. \tag{1.2.89}$$

Let us note, that

$$\int\limits_{x_2}^{x_0} K(\tau)\tau^kD^{-1}(\tau)d\tau \sim K(0)I_k(x_2,x_0) \ \ as \ \ x_2\to 0+, \ \ K(0)\neq 0;$$

$$\left(\int\limits_{x_2}^{x_0} K(\tau)(l-\tau)^kD^{-1}(\tau)d\tau \sim K(l)I^k(x_0,x_2) \ \ as \ \ x_2\to l-, \ \ K(l)\neq 0\right)$$

when

$$I_k(x_2,x_0)\to+\infty \;\; as \;\; x_2\to 0+$$
$$\left(I^k(x_0,x_2)\to+\infty \;\; as \;\; x_2\to l-\right),$$

since, e.g.,

$$\lim_{x_2\to 0+}\frac{\int\limits_{x_2}^{x_0}K(\tau)\tau^kD^{-1}(\tau)d\tau}{K(0)I_k(x_2,x_0)}=\lim_{x_2\to 0+}\frac{K(x_2)x_2^kD^{-1}(x_2)}{K(0)x_2^kD^{-1}(x_2)}=1.$$

At last we prove the following

Lemma 1.2.16 *Let either* $D\in C^{k+2}([0,l[)\ \left(C^{k+2}(]0,l])\right)$, $k=\{0,1,\ldots\}$, *or there exist such* j $(1\le j\le k+2)$ *that*

$$D\in C^{j-1}([0,l[),\;\; D^{j-1}(0)=0,\;\; and\;\; D^j(0)=\infty$$
$$\left(D\in C^{j-1}(]0,l]),\;\; D^{j-1}(l)=0,\;\; and\;\; D^j(l)=\infty\right).$$

If

$$I_k(0,x_0)=+\infty \;\; and \;\; I_{k+1}(0,x_0)<+\infty \;\; for\;\; a\;\; fixed\;\; k\in\{0,1,\ldots\},\;\; x_0\in]0,l[$$
$$\left(I^k(x_0,l)=+\infty \;\; and \;\; I^{k+1}(x_0,l)<+\infty \;\; for\;\; a\;\; fixed\;\; k\in\{0,1,\ldots\},\;\; x_0\in]0,l[\right),$$

then

$$x_2I_k(x_2,x_0)\le I_{k+1}(x_2,x_0)<I_{k+1}(0,x_0)<+\infty\;\;\forall x_2\in]0,x_0]$$
$$\left((l-x_2)I^k(x_0,x_2)\le I^{k+1}(x_0,x_2)<I^{k+1}(x_0,0)<+\infty\;\;\forall x_2\in[x_0,l[\right),$$

and moreover,

$$\lim_{x_2\to 0+}x_2I_k(x_2,x_0)$$
$$=\begin{cases}0 & if\; D^{(j)}(0)=0\;\;\forall j\in\{1,2,\ldots,k+1\}\; and\; D^{(k+2)}(0)=\infty;\\ & or\;\; if\; k=0\; and\;\; either\; D'(0)\neq 0\; or\; D'(0)=\infty;\\ & or\;\; if\; k\ge 1\; and\;\; D^{(j)}(0)=0\;\forall j\in\{1,2,\ldots,i\}\\ & for\;\; a\;\; fixed\; i\in\{1,2,\ldots,k\}\; and\;\; either\\ & D^{(i+1)}(0)\neq 0\;\; or\;\; D^{(i+1)}(0)=\infty\end{cases}$$

$$\left(\begin{array}{l}\lim\limits_{x_2\to l-}(l-x_2)I^k(x_0,x_2)\\ =\begin{cases}0 & if\; D^{(j)}(l)=0\;\;\forall j\in\{1,2,\ldots,k+1\}\; and\; D^{(k+2)}(l)=\infty;\\ & or\;\; if\; k=0\; and\;\; either\; D'(l)\neq 0\; or\; D'(l)=\infty;\\ & or\;\; if\; k\ge 1\; and\;\; D^{(j)}(l)=0\;\forall j\in\{1,2,\ldots,i\}\\ & for\;\; a\;\; fixed\; i\in\{1,2,\ldots,k\}\; and\;\; either\\ & D^{(i+1)}(l)\neq 0\;\; or\;\; D^{(i+1)}(l)=\infty\end{cases}\end{array}\right).$$

The cases

$$D^{(j)}(0)=0\;\;\forall j\in\{1,2,\ldots,k+2\}$$

and

$$D^{(j)}(0)=0\ \ \forall j\in\{1,2,\ldots,k+1\},\ \ D^{(k+2)}(0)\neq 0$$
$$\big(D^{(j)}(l)=0\ \ \forall j\in\{1,2,\ldots,k+2\}$$

and

$$D^{(j)}(l)=0\ \ \forall j\in\{1,2,\ldots,k+1\},\ \ D^{(k+2)}(l)\neq 0\big)$$

cannot hold.

Proof is evident from the following relations:

$$x_2 I_k(x_2,x_0)=\int\limits_{x_2}^{x_0}\frac{x_2}{t}t^{k+1}D^{-1}(t)dt\leq\int\limits_{x_2}^{x_0}t^{k+1}D^{-1}(t)dt=I_{k+1}(x_2,x_0)<I_{k+1}(0,x_0)<+\infty$$
$$\forall x_2\in]0,x_0],$$

$$(l-x_2)I^k(x_0,x_2)=\int\limits_{x_0}^{x_2}\frac{l-x_2}{l-t}(l-t)^{k+1}D^{-1}(t)dt\leq\int\limits_{x_0}^{x_2}(l-t)^{k+1}D^{-1}(t)dt=I^{k+1}(x_0,x_2)$$
$$<I^{k+1}(x_0,l)<+\infty\ \ \forall x_2\in[x_0,l[,$$

$$\lim_{x_2\to 0+}x_2 I_k(x_2,x_0)=\lim_{x_2\to 0+}\frac{I_k(x_2,x_0)}{x_2^{-1}}=\lim_{x_2\to 0+}\frac{x_2^k D^{-1}(x_2)}{x_2^{-2}}=\lim_{x_2\to 0+}\frac{x_2^{k+2}}{D(x_2)},$$
$$\lim_{x_2\to l-}(l-x_2)I^k(x_0,x_2)=\lim_{x_2\to l-}\frac{I^k(x_0,x_2)}{(l-x_2)^{-1}}=\lim_{x_2\to l-}\frac{(l-x_2)^k D^{-1}(x_2)}{(l-x_2)^{-2}}=\lim_{x_2\to l-}\frac{(l-x_2)^{k+2}}{D(x_2)}.$$

Obviously,

$$D^{(j)}(0)=0\ \ \forall j\in\{1,2,\ldots,k+2\}$$
$$\big(D^{(j)}(l)=0\ \ \forall j\in\{1,2,\ldots,k+2\}\big)$$

cannot occur, otherwise the third (fourth) from above relations would be in contradiction with the first (second) one.

Also the case

$$D^{(j)}(0)=0\ \ \forall j\in\{1,2,\ldots,k+1\},\ \ D^{(k+2)}(0)\neq 0$$

cannot occur since, otherwise,

$$\lim_{\tau\to 0+}\tau^{\gamma}\tau^{k+1}D^{-1}(\tau)=\lim_{\tau\to 0+}\frac{\tau^{\gamma+k+1}}{D(\tau)}=\lim_{\tau\to 0+}\frac{(\gamma+k+1)\tau^{\gamma+k}}{D'(\tau)}$$
$$=\lim_{\tau\to 0+}\frac{(\gamma+k+1)(\gamma+k)\tau^{\gamma+k-1}}{D''(\tau)}=\lim_{\tau\to 0+}\frac{(\gamma+k+1)(\gamma+k)(\gamma+k-1)\cdots\gamma\tau^{\gamma-1}}{D^{(k+2)}(\tau)}$$
$$=\frac{(k+2)!}{D^{(k+2)}(0)}>0\ \ for\ \ \gamma=1,$$

i.e., $I_{k+1}(0, x_0) = +\infty$ which contradicts the assumption $I_{k+1}(0, x_0) < +\infty$.

The case

$$D^{(j)}(l) = 0 \;\; \forall j \in \{1, 2, \dots, k+1\}, \;\; D^{(k+2)}(l) \neq 0$$

can be considered analogously. □

For $k = 0$ and $k = 1$ Lemma 1.2.16 immediately follows from Lemma 1.2.5 and Lemma 1.2.7, respectively, when $f \equiv 0$ and we take $K(\tau) \equiv 1$, i.e., $C_1 = 0$, $C_2 = -1$, and $K(\tau) = \tau$, i.e., $C_1 = -1$, $C_2 = -x_0$ ($K(\tau) = l - \tau$, i.e., $C_1 = 1$, $C_2 = l - x_0$), respectively.

1.3 Solution of boundary value problems

From Theorem 1.2.1 we conclude that:

On the cusped edge $x_2 = 0$ (correspondingly, $x_2 = l$) we can admit the following classical BCs:

$$\begin{aligned} &w = w_0 \text{ (correspondingly, } w_l), \\ &w_{,2} = w'_0 \text{ (correspondingly, } w'_l) \end{aligned} \tag{1.3.1}$$

iff I_0^0 (correspondingly, I_0^l) $< +\infty$;

$$w_{,2} = w'_0 \; (w'_l), \;\; Q_2 = Q_0(Q_l) \text{ iff } I_0^0 \; (I_0^l) < +\infty; \tag{1.3.2}$$

$$w = w_0 \; (w_l), \;\; M_2 = M_0(M_l) \neq 0 \text{ iff } I_1^0 \; (I_1^l) < +\infty; \tag{1.3.3}$$

$$M_2 = M_0(M_l), \;\; Q_2 = Q_0(Q_l) \text{ if } I_0^0 \; (I_0^l) \leq +\infty, \tag{1.3.4}$$

and the following non-classical (in the sense of the bending theory) conditions (replacing BCs):

$$w = w_0 \; (w_l), \;\; w_{,2} = O(1) \text{ when } x_2 \to 0+ \;\; (x_2 \to l-) \tag{1.3.5}$$

if

$$I_0^0 \; (I_0^l) = +\infty, \quad I_1^0 \; (I_1^l) < +\infty;$$

$$w = O\;(1), \;\; w_{,2} = O(1) \text{ when } x_2 \to 0+ \;\; (x_2 \to l-) \tag{1.3.6}$$

if

$$I_1^0 \; (I_1^l) = +\infty,$$

where $w_0, w_l, w'_0, w'_l, M_0, M_l, Q_0, Q_l$ are given constants, O is a Landau symbol ($O(1)$ means boundedness).

Theorem 1.3.1 *Let the conditions of Theorem 1.2.1 be fulfilled. Then the following BVPs are well-posed in the sense of Hadamard:*

1. (1.2.1), $(1.3.1)_0$ $(1.3.1)_l$, $w \in C^4(\,]0, l[\,) \cap C^1([0, l])$;
2. (1.2.1), $(1.3.2)_0$ $(1.3.1)_l$, $w \in C^4(\,]0, l[\,) \cap C^1([0, l])$;
3. (1.2.1), $(1.3.3)_0$ $(1.3.1)_l$, $w \in C^4(\,]0, l[\,) \cap C^1(\,]0, l]) \cap C([0, l])$;
4. (1.2.1), $(1.3.4)_0$ $(1.3.1)_l$, $w \in C^4(\,]0, l[\,) \cap C^1(]0, l])$;

5. (1.2.1), $(1.3.1)_0$ $(1.3.2)_l$, $w \in C^4(\,]0,l[\,) \cap C^1([0,l])$;
6. (1.2.1), $(1.3.3)_0$ $(1.3.2)_l$, $w \in C^4(\,]0,l[\,) \cap C^1(\,]0,l]) \cap C([0,l])$;
7. (1.2.1), $(1.3.1)_0$ $(1.3.3)_l$, $w \in C^4(\,]0,l[\,) \cap C^1([0,l[\,) \cap C([0,l])$;
8. (1.2.1), $(1.3.2)_0$ $(1.3.3)_l$, $w \in C^4(\,]0,l[\,) \cap C^1([0,l[) \cap C([0,l])$;
9. (1.2.1), $(1.3.3)_0$ $(1.3.3)_l$, $w \in C^4(\,]0,l[\,) \cap C([0,l])$;
10. (1.2.1), $(1.3.1)_0$ $(1.3.4)_l$, $w \in C^4(\,]0,l[\,) \cap C^1([0,l[\,)$.

Remark 1.3.2 *Indices* 0 *and* l *at (1.3.1)-(1.3.5) mean the corresponding formulas for the points* 0 *and* l, *respectively.*

Remark 1.3.3 *Actually, conditions (1.3.6) and the second of (1.3.5) are not BCs. They are the conditions on* w *in a neighbourhood of the boundary point. That is why we say that in these cases BCs disappear at the cusped end of the beam (see Remark 1.3.4 bellow).*

Proof of Theorem 1.3.1. Using Theorem 1.2.1 or Theorem 1.2.12, Corollary 1.2.3, and Lemmas 1.2.5, 1.2.7, 1.2.8, we solve all the BVPs 1-10 in the explicit form. The uniqueness of solutions is guaranted by their construction from the general representation (1.2.7) of the solution w of the Euler-Bernoulli equation (1.2.1) in the class $C^4_J(]0,l[)$ (see Section 1.4 below). The continuous dependence of the solution w and of $w_{,2}$ [in the case of BVPs 3, 4, 6, 9 (7-10) with the weights

$$[I_k(x_2,x_0)]^{-1} := \left[\int_{x_2}^{x_0} t^k D^{-1}(t)dt\right]^{-1}, \quad x_2 \in]0,x_0], \quad x_0 \in]0,l[$$

$$\left(\left[I^k(x_0,x_2)\right]^{-1} := \left[\int_{x_0}^{x_2} (l-t)^k D^{-1}(t)dt\right]^{-1}, \quad x_2 \in [x_0,l[, \quad x_0 \in]0,l[\right)$$

by $k=1$ and $k=0$, respectively] on the boundary data easily follows from the explicit representations of the solutions of BVPs. Let us recall

$$I_k := I_k(0), \quad I_k(x_2) := I_k(x_2,l),$$
$$I^k := I^k(l), \quad I^k(x_2) := I^k(0,x_2).$$

SOLUTION of BVP 1. Since $I_0^0, I_0^l < +\infty$, obviously, we can take $x_0 = l$. Then, in view of (1.2.6), (1.2.7), from $(1.3.1)_l$ we have

$$C_4 = w_l, \quad C_3 = w'_l.$$

For determination of constants C_1, C_2, from $(1.3.1)_0$ we have the following algebraic system

$$C_1 \int_0^l \tau(\tau - l)D^{-1}(\tau)d\tau + C_2 \int_0^l \tau D^{-1}(\tau)d\tau$$

$$= \int_0^l \tau D^{-1}(\tau) \int_l^\tau f(t)(\tau - t)dtd\tau - lw_l' + w_l - w_0,$$

$$-C_1 \int_0^l (\tau - l)D^{-1}(\tau)d\tau - C_2 \int_0^l D^{-1}(\tau)d\tau$$

$$= - \int_0^l D^{-1}(\tau) \int_l^\tau f(t)(\tau - t)dtd\tau + w_l' - w_0',$$

which is solvable as its determinant satisfies

$$\Delta := \left[\int_0^l \tau D^{-1}(\tau)d\tau \right]^2 - \int_0^l \tau^2 D^{-1}(\tau)d\tau \cdot \int_0^l D^{-1}(\tau)d\tau < 0.$$

The last assertion follows from the Hölder inequality which is strong since $\tau D^{-\frac{1}{2}}(\tau)$ and $D^{-\frac{1}{2}}(\tau)$ are positive on $]0, l[$, and $\tau^2 D^{-1}(\tau)$ and $D^{-1}(\tau)$ differ from each other by a non-constant factor τ^2.

SOLUTION of BVP 9. From (1.2.5), taking into account the second conditions from $(1.3.3)_0$, $(1.3.3)_l$, we obtain

$$\begin{aligned} \int_{x_0}^0 tf(t)dt - C_1 x_0 + C_2 &= M_0, \\ - \int_{x_0}^l (l - t)f(t)dt + C_1(l - x_0) + C_2 &= M_l. \end{aligned}$$

Solving this system, we get

$$C_1 = \int_{x_0}^l f(t)dt + l^{-1} \left[\int_l^0 tf(t)dt + M_l - M_0 \right], \tag{1.3.7}$$

$$\begin{aligned} C_2 &= M_0 - \int_{x_0}^0 tf(t)dt + x_0 \int_{x_0}^l f(t)dt + \\ &+ \frac{x_0}{l} \left[\int_l^0 tf(t)dt + M_l - M_0 \right]. \end{aligned} \tag{1.3.8}$$

Hence, in view of (1.2.5), (1.2.11), we have

$$K(x_2) = -M_2(x_2) = \int\limits_{x_0}^{x_2} (x_2 - t) f(t) dt + \int\limits_{x_0}^{0} t f(t) dt - M_0 \\ -x_2 \int\limits_{x_0}^{l} f(t) dt - \frac{x_2}{l} \left[\int\limits_{l}^{0} t f(t) dt + M_l - M_0 \right]. \tag{1.3.9}$$

Further, from (1.2.7), by virtue of the first conditions from $(1.3.3)_0$, $(1.3.3)_l$ and Lemma 1.2.5, we obtain

$$-\int\limits_{x_0}^{0} \tau K(\tau) D^{-1}(\tau) d\tau - C_3 x_0 + C_4 = w_0,$$

$$\int\limits_{x_0}^{l} (l-\tau) K(\tau) D^{-1}(\tau) d\tau + C_3 (l - x_0) + C_4 = w_l.$$

Solving this system, we get

$$C_3 = -l^{-1} \left[w_0 - w_l + \int\limits_{x_0}^{0} \tau K(\tau) D^{-1}(\tau) d\tau \right. \\ \left. + \int\limits_{x_0}^{l} (l-\tau) K(\tau) D^{-1}(\tau) d\tau \right], \tag{1.3.10}$$

$$C_4 = -l^{-1} \left[-x_0 w_l - (l - x_0) w_0 \right. \\ \left. + x_0 \int\limits_{x_0}^{l} (l-\tau) K(\tau) D^{-1}(\tau) d\tau - (l - x_0) \int\limits_{x_0}^{0} \tau K(\tau) D^{-1}(\tau) d\tau \right]. \tag{1.3.11}$$

Thus, the solution has the form (1.2.7) with $K(\tau)$, C_3, C_4 given by (1.3.9)-(1.3.11), respectively.

If $I_0^0 = +\infty$ and $I_1^0 < +\infty$ ($I_0^l = +\infty$ and $I_1^l < +\infty$), then $w_{,2}$ is bounded as $x_2 \to 0+$ ($x_2 \to l-$) iff $M_0 = 0$ ($M_l = 0$) (see Theorem 1.2.1, the second part of Case II or Theorem 1.2.12, Case 5)).

Let us note that if either $I_1^l < +\infty$, $I_1^0 = +\infty$ but $I_2^0 < +\infty$ and $M_2(0) = M_0 = 0$ or $I_1^0 < +\infty$, $I_1^l = +\infty$ but $I_2^l < +\infty$ and $M_2(l) = M_l = 0$ or $I_1^0 = +\infty$, $I_1^l = +\infty$ but $I_2^0 < +\infty$, $I_2^l < +\infty$ and $M_2(0) = M_0 = 0$, $M_2(l) = M_l = 0$, then BVP 9 (call your attention to the change of the restrictions on I_1^0, I_1^l) will be uniquely solvable. The proof immediately follows from Lemmas 1.2.7 and 1.2.8. In these three cases the expressions for C_i, $i = 1, 2, 3, 4$, coincide with (1.3.7), (1.3.8), (1.3.10), and (1.3.11).

Let us note that in the above three cases $w,_2$ is unbounded as $x_2 \to 0+$ ($x_2 \to l-$) unless $(Q_2 w)(0) = 0$ ($(Q_2 w)(l) = 0$). But nevertheless

$$(M_2 w \cdot w,_2)\,|_{x_2=0} = 0 \quad ((M_2 w \cdot w,_2)\,|_{x_2=l} = 0)\,.$$

Indeed, if $Q_2(0) \neq 0$, then taking into account (1.2.6), (1.2.3), we have

$$\lim_{x_2\to 0+}(M_2 w)(x_2)\cdot w,_2(x_2) = \lim_{x_2\to 0+}\frac{w,_2}{M_2^{-1}} = \lim_{x_2\to 0+}\frac{M_2\cdot D^{-1}}{M_2^{-2}\cdot Q_2}$$

$$= \lim_{x_2\to 0+}\frac{M_2^3}{D\cdot Q_2} = \lim_{x_2\to 0+}\frac{3M_2^2\cdot Q_2}{D'\cdot Q_2 - D\cdot f}.$$

Hence,

$$\lim_{x_2\to 0+}(M_2 w)(x_2)\cdot w,_2(x_2) = 0 \;\; when \;\; D'(0)\neq 0 \;\; or \;\; D'(0)=\infty.$$

The same holds when either $D'(0) \neq 0$, $D''(0) \neq 0$ or $D'(0) = 0$, $D''(0) = 0$, $D'''(0) = \infty$, what follows from

$$\lim_{x_2\to 0+}(M_2 w)(x_2)\cdot w,_2(x_2) = \lim_{x_2\to 0+}\frac{6M_2\cdot Q_2 - 3M_2^2\cdot f}{D''\cdot Q_2 - 2D'\cdot f - D\cdot f'}$$

and

$$\lim_{x_2\to 0+}(M_2 w)(x_2)\cdot w,_2(x_2) = \lim_{x_2\to 0+}\frac{6Q_2^2 - 12M_2\cdot f - 3M_2^2\cdot f'}{D'''\cdot Q_2 - 3D''\cdot f - 3D'\cdot f' - D\cdot f''},$$

respectively.

As it was already shown (see Lemma 1.2.7, when $K(0) = 0$) the case $D'(0) = 0$, $D''(0) = 0$, $D'''(0) = 0$ (because of $(Q_2 w)(0) \neq 0$), and the case $D'(0) = 0$, $D''(0) = 0$, $D'''(0) \neq 0$ (because of $I_2^0 < +\infty$) cannot occur.

The case $x_2 \to l-$ can be considered analogously.

In these three cases BVP 9 is not well-posed since the arbitrarily small change of BCs $M_2(0) = 0$, $M_2(l) = 0$ implies the unsolvability of the BVP under consideration.

This important note with the other cases, when the similar situation can arise, we summarize as follows:

when $M_0\,(M_l) = 0$, BVPs 3 (7) and 6 (8) and when $M_0 = 0$, $M_l = 0$, BVP 9 are uniquely solvable if $I_2^0\,(I_2^l) < +\infty$ and $I_2^0 < +\infty$, $I_2^l < +\infty$, respectively; but they are correct in the sense of Hadamard only if $I_1^0\,(I_1^l) < +\infty$, and $I_1^0 < +\infty$, $I_1^l < +\infty$, respectively.

SOLUTION of BVP 10. Since $I_0^0 < +\infty$, without loss of generality, we assume $x_0 = 0$. From $(1.3.1)_0$, $(1.3.4)_l$ we get

$$C_4 = w_0, \quad C_3 = w_0', \quad C_2 = M_l - lQ_l - \int_0^l tf(t)dt, \;\; C_1 = Q_l + \int_0^l f(t)dt.$$

Thus,

$$w(x_2) = \int_0^{x_2} (x_2 - \tau) \left[(l-\tau)Q_l - M_l + \int_\tau^l tf(t)dt \right. \\ \left. -\tau \int_\tau^l f(t)dt \right] D^{-1}(\tau)d\tau + w_0' x_2 + w_0, \tag{1.3.12}$$

$$w_{,2}(x_2) = \int_0^{x_2} \left[(l-\tau)Q_l - M_l + \int_\tau^l tf(t)dt - \tau \int_\tau^l f(t)dt \right] D^{-1}(\tau)d\tau + w_0'.$$

Representing $(x_2 - \tau)$ in (1.3.12) as $(x_2 - l) + (l - \tau)$, it is not difficult to see (see (1.2.66)) that

$$|w(x_2)| \le (l - x_2) \left[\widetilde{C}_1 I^0(x_2) + |Q_l| I^1(x_2) \right]$$

$$+\widetilde{C}_1 I^1(x_2) + |Q_l| I^2(x_2) + |w_0'| x_2 + |w_0| \quad for \;\; all \;\; x_2 \in [0, l[,$$

where

$$\widetilde{C}_1 := |M_l| + \int_0^l t|f(t)|dt + l \int_0^l |f(t)|dt.$$

Therefore,

$$|w(x_2)| \le 2\widetilde{C}_1 I^1(x_2) + |Q_l| \left[(l - x_2) I^1(x_2) + I^2(x_2) \right] \\ +|w_0'| x_2 + |w_0| \quad for \;\; all \quad x_2 \in [0, l[, \quad \text{if} \quad I_1^l < +\infty, \tag{1.3.13}$$

and

$$|\left[I^1(x_2) \right]^{-1} w(x_2)| \le 2\widetilde{C}_1 + |Q_l| \left[(l - x_2) + \widetilde{C}_2 \right] \\ +\widetilde{C}_3 (|w_0'| x_2 + |w_0|) \quad \text{for} \quad x_2 \in [0, l[\;\; \text{if} \quad I_1^l = +\infty, \tag{1.3.14}$$

since

$$(l - x_2) I^0(x_2) \le I^1(x_2)$$

(because of

$$I^1(x_2) - (l - x_2) I^0(x_2) = \int_0^{x_2} (x_2 - t) D^{-1}(t) dt \ge 0);$$

$$I^2(x_2) \le \widetilde{C}_2 I^1(x_2), \quad \widetilde{C}_2 = \text{const} > 0, \quad \forall x_2 \in [0, l[$$

(because of

$$\lim_{x_2\to l-}\frac{I^2(x_2)}{I^1(x_2)}=\lim_{x_2\to l-}\frac{(l-x_2)^2D^{-1}(x_2)}{(l-x_2)D^{-1}(x_2)}=\lim_{x_2\to l-}(l-x_2)=0$$

if $I_1^l=+\infty$);

$$[I^1(x_2)]^{-1}\le \widetilde{C}_3=\text{const}>0\quad \forall x_2\in[0,l[.$$

The continuous dependence in the class of continuous on $[0,l]$ functions of the solution $w(x_2)$ and of $[I^1(x_2)]^{-1}w(x_2)$ for $I_1^l<+\infty$ and $I_1^l=+\infty$, respectively, on the boundary data and on the right hand side f immediately follows from the estimates (1.3.13) and (1.3.14), correspondingly. Let us note that for $I_1^l=+\infty$, the solution $w(x_2)$ for a fixed $x_2\in[0,l[$ continuously dependence on the boundary data and the right hand side f. Similar conclusions can be made with respect to $w_{,2}$, which follow from the following evident estimates:

$$|w_{,2}(x_2)|\le \widetilde{C}_1I^0(x_2)+|Q_l|I^1(x_2)+|w_0'|\quad for\ \ x_2\in[0,l[\ \ if\ \ I_0^l<+\infty,$$

$$\left|\left[I^0(x_2)\right]^{-1}w_{,2}(x_2)\right|\le \widetilde{C}_1+|Q_l|\widetilde{C}_4+\widetilde{C}_5|w_0'|\quad for\ \ x_2\in[0,l[\ \ if\ \ I_0^l=+\infty,$$

since

$$I^1(x_2)\le\widetilde{C}_4I^0(x_2)\ \ and\ \ \left[I^0(x_2)\right]^{-1}\le\widetilde{C}_5\ \ for\ \ all\ \ x_2\in[0,l[,\ \ \widetilde{C}_4,\ \widetilde{C}_5=\text{const}>0.$$

The other BVPs 2-8 can be solved in an analogous way. For the sake of simplicity, we take $f\equiv 0$.

Using (1.2.75)-(1.2.89) along with the Lemma 1.2.16, we get the following solutions to BVPs 1-8 when $f\equiv 0$.

A unique solution of BVP 1 has the form

$$\begin{aligned}
w(x_2)&=[x_2I_1(x_2)-I_2(x_2)]C_1-[x_2I_0(x_2)-I_1(x_2)](C_1l-C_2)+w_l'(x_2-l)+w_l,\\
w_{,2}(x_2)&=C_1I_1(x_2)-(C_1l-C_2)I_0(x_2)+w_l',\\
(M_2w)(x_2)&=C_1(x_2-l)+C_2,\ (Q_2w)(x_2)=C_1,
\end{aligned}$$

where

$$\begin{aligned}
C_1&=\Delta^{-1}[-I_1(w_0'-w_l')-I_0(w_0-w_l+w_l'l],\\
C_2&=\Delta^{-1}[(I_1-lI_0)(w_0-w_l+w_l'l)+(I_2-lI_1)(w_0'-w_l')].
\end{aligned}$$

A unique solution of BVP 2 has the form

$$\begin{aligned}
w(x_2)&=[x_2I_1(x_2)-I_2(x_2)]Q_0-(Q_0l-C_2)[x_2I_0(x_2)-I_1(x_2)]+w_l'(x_2-l)+w_l,\\
w_{,2}(x_2)&=Q_0I_1(x_2)-(Q_0l-C_2)I_0(x_2)+w_l',\\
(M_2w)(x_2)&=Q_0(x_2-l)+C_2,\ (Q_2w)(x_2)=Q_0,
\end{aligned}$$

where

$$C_2 = \frac{w_0' - w_l' - Q_0(I_1 - lI_0)}{I_0}.$$

A unique solution of BVP 3 has the form

$$\begin{aligned} w(x_2) &= x_2[C_1 I_1(x_2) + M_0 I_0(x_2)] - C_1 I_2(x_2) - M_0 I_1(x_2) + w_l'(x_2 - l) + w_l, \\ w_{,2}(x_2) &= C_1 I_1(x_2) + M_0 I_0(x_2) + w_l', \\ (M_2 w)(x_2) &= C_1 x_2 + M_0, \ (Q_2 w)(x_2) = C_1, \end{aligned}$$

where

$$C_1 = \frac{-M_0 I_1 - w_l' l + w_l - w_0}{I_2}.$$

As we see from this solution of the BVP 3 the function w is bounded, but $w_{,2}$ is bounded as $x_2 \to 0+$ if and only if $M_0 = 0$ for $I_0(0) = I_0 = +\infty$. Therefore, the solution of the BVP 3 under the additional restriction of boundedness of $w_{,2}$ exists if and only if the condition $M_0 = 0$ when $I_0 = +\infty$ holds. Here, it was important that $I_1 < +\infty$ [see (1.3.3)]. If, now, $I_1 = +\infty$ but $I_2 < +\infty$, then for $M_0 = 0$, a unique solution of BVP 3 has the form

$$\begin{aligned} w(x_2) &= C_1[x_2 I_1(x_2) - I_2(x_2)] + w_l'(x_2 - l) + w_l, \\ w_{,2}(x_2) &= C_1 I_1(x_2) + w_l', \\ (M_2 w)(x_2) &= C_1 x_2, \ (Q_2 w)(x_2) = C_1, \end{aligned}$$

where

$$C_1 = \frac{-w_l' l + w_l - w_0}{I_2}.$$

Obviously, $I_2 > 0$, because of $t^2 D^{-1}(t) > 0 \ \forall t \in]0, l]$. Let us note that in the last case $w_{,2}$ is unbounded unless $C_1 = 0$. But nevertheless, according to Lemma 1.2.16, $(M_2 w \cdot w_{,2})|_{x_2=0} = 0$.

If $I_1 = +\infty$ but $I_2 < +\infty$ and $M_0 \neq 0$, then the BVP 3 is ill-posed in the sense of nonsolvability.

A unique solution of BVP 4 has the form

$$\begin{aligned} w(x_2) &= x_2[Q_0 I_1(x_2) + M_0 I_0(x_2)] - Q_0 I_2(x_2) - M_0 I_1(x_2) + w_l'(x_2 - l) + w_l, \\ w_{,2}(x_2) &= Q_0 I_1(x_2) + M_0 I_0(x_2) + w_l', \\ (M_2 w)(x_2) &= x_2 Q_0 + M_0, \ (Q_2 w)(x_2) = Q_0. \end{aligned}$$

As we see from this solution of BVP 4, both the functions w and $w_{,2}$ are bounded as $x_2 \to 0+$ if and only if $M_0 = 0$ for $I_0(0) = I_0 = +\infty$ and $Q_0 = 0$ for $I_1(0) = I_1 = +\infty$ (in the general case, i.e., when $f \not\equiv 0$, this assertion follows from Theorem 1.2.12). Therefore, the solution of the problem BVP 4 under the additional restriction of boundedness of the solution and of its derivative exists if and only if the above conditions hold.

It is not difficult to see that

$$|w(x_2)| \leq |Q_0|(x_2 I_1 + I_2) + 2|M_0|I_1 + |w_l'|l + |w_l| \ for\ I_1 < \infty$$

and

$$|I_1^{-1}(x_2)w(x_2)| \leq |Q_0|(x_2 + \widetilde{C}) + 2M_0 + C^*(|w_l'|l + |w_l|) \ for\ I_1 = +\infty,$$

since

$$\begin{array}{rcl} x_2 I_0(x_2) & \leq & I_1(x_2) \quad \forall\ x_2 \in]0, l], \\ I_1^{-1}(x_2) & < & C^* = const > 0 \quad \forall\ x_2 \in]0, l], \end{array}$$

and

$$I_2(x_2) \leq \widetilde{C} I_1(x_2),\ \widetilde{C} = const > 0,\ \forall\ x_2 \in]0, l],$$

because of

$$\lim_{x_2 \to 0+} \frac{I_2(x_2)}{I_1(x_2)} = \lim_{x_2 \to 0+} \frac{I_2'(x_2)}{I_1'(x_2)} = \lim_{x_2 \to 0+} x_2 = 0 \ if\ I_1 = +\infty.$$

The continuous dependence of $w(x_2)$ and $I_1^{-1}(x_2)w(x_2)$ for $I_1 < +\infty$ and $I_1 = +\infty$, respectively, on the boundary data immediately follows from the above estimates for the solution $w(x_2)$. Similar conclusion can be made with respect to $w_{,2}(x_2)$ and $I_0^{-1}(x_2)w(x_2)$ for $I_0 < +\infty$ and $I_1 = +\infty$, respectively.

A unique solution of BVP 5 has the form

$$\begin{array}{l} w(x_2) = (I_0)^{-1}(w_0' - w_l' - Q_l I_1)\,[x_2 I_0(x_2) - I_1(x_2) + I_1] \\ +x_2 Q_l I_1(x_2) - Q_l I_2(x_2) + w_l' x_2 + w_0 + Q_l I_2, \\ w_{,2}(x_2) = (I_0)^{-1}(w_0' - w_l' - Q_l I_1) I_0(x_2) + Q_l I_1(x_2) + w_l', \\ (M_2 w)(x_2) = (I_0)^{-1}(w_0' - w_l' - Q_l I_1) + Q_l x_2, \quad (Q_2 w)(x_2) = Q_l. \end{array}$$

A unique solution of BVP 6 has the form

$$\begin{array}{l} w(x_2) = x_2\,[M_0 I_0(x_2) + Q_l I_1(x_2)] - M_0 I_1(x_2) - Q_l I_2(x_2) \\ +x_2 w_l' + M_0 I_1 + Q_l I_2 + w_0, \\ w_{,2}(x_2) = M_0 I_0(x_2) + Q_l I_1(x_2) + w_l', \\ (M_2 w)(x_2) = Q_l x_2 + M_0, \quad (Q_2 w)(x_2) = Q_l. \end{array}$$

If $I_0^0 = +\infty$ and $I_1^0 < +\infty$, then $w_{,2}$ is bounded as $x_2 \to 0+$ iff $M_0 = 0$.

If $M_0 = 0$, then there exists a unique solution of the same BVP even when $I_1^0 = +\infty$ and $I_2^0 < +\infty$, which has the following form

$$\begin{array}{l} w(x_2) = Q_l\,[x_2 I_1(x_2) - I_2(x_2)] + w_l' x_2 + Q_l I_2 + w_0, \\ w_{,2}(x_2) = Q_l I_1(x_2) + w_l', \\ (M_2 w)(x_2) = Q_l x_2, \quad (Q_2 w)(x_2) = Q_l. \end{array}$$

If $Q_l \neq 0$, $w_{,2}$ is unbounded as $x_2 \to 0+$ but nevertheless, according to Lemma 1.2.16, $(M_2 w \cdot w_{,2})|_{x_2=0} = 0$.

A unique solution of BVP 7 has the form

$$\begin{aligned}
&w(x_2) = M_l \left[(l-x_2)I^0(x_2) - I^1(x_2)\right] + C_1 \left[I^2(x_2) - (l-x_2)I^1(x_2)\right] \\
&+w_0' x_2 + w_0, \\
&w_{,2}(x_2) = -M_l I^0(x_2) + C_1 I^1(x_2) + w_0', \\
&(M_2 w)(x_2) = M_l + C_1(l-x_2), \;\; (Q_2 w)(x_2) = C_1,
\end{aligned}$$

where

$$C_1 := \frac{w_l - w_0 - w_0' l + M_l I^1}{I^2}.$$

If $I_0^l = +\infty$ and $I_1^l < +\infty$, then $w_{,2}$ is bounded as $x_2 \to l-$ iff $M_l = 0$.

If now $I_1^l = +\infty$ and $I_2^l < +\infty$, then for $M_l = 0$ a unique solution of BVP 7 has the form

$$\begin{aligned}
&w(x_2) = C_1 \left[I^2(x_2) - (l-x_2)I^1(x_2)\right] + w_0' x_2 + w_0, \\
&w_{,2}(x_2) = C_1 I^1(x_2) + w_0', \\
&(M_2 w)(x_2) = -C_1(l-x_2), \;\; (Q_2 w)(x_2) = C_1,
\end{aligned}$$

where

$$C_1 := \frac{w_l - w_0 - w_0' l}{I^2}.$$

Obviously, $I^2 > 0$, because of $(l-t)^2 D^{-1}(t) > 0 \;\forall t \in]0, l[$. Let us note that in the last case $w_{,2}$ is unbounded unless $C_1 = 0$. But nevertheless, according to Lemma 1.2.16,

$$(M_2 w \cdot w_{,2})|_{x_2=l} = 0.$$

A unique solution of BVP 8 has the form

$$\begin{aligned}
&w(x_2) = M_l \left[(l-x_2)I^0(x_2) - I^1(x_2)\right] + Q_0 \left[I^2(x_2) - (l-x_2)I^1(x_2)\right] \\
&+w_0'(x_2 - l) + w_l - M_l I^1 - Q_0 I^2, \\
&w_{,2}(x_2) = -M_l I^0(x_2) + Q_0 I^1(x_2) + w_0', \\
&(M_2 w)(x_2) = M_l - Q_0(l-x_2), \;\; (Q_2 w)(x_2) = Q_0.
\end{aligned}$$

If $I_0^l = +\infty$ and $I_1^l < +\infty$, then $w_{,2}$ is bounded as $x_2 \to l-$ iff $M_l = 0$.

If now $I_1^l = +\infty$ and $I_2^l < +\infty$, then for $M_l = 0$ a unique solution of BVP 8 has the form

$$\begin{aligned}
&w(x_2) = Q_0 \left[I^2(x_2) - (l-x_2)I^1(x_2) - I^2\right] + w_0'(x_2 - l) + w_l, \\
&w_{,2}(x_2) = Q_0 I^1(x_2) + w_0', \\
&(M_2 w)(x_2) = -Q_0(l-x_2), \;\; (Q_2 w)(x_2) = Q_0.
\end{aligned}$$

If $Q_0 \neq 0$, then $w_{,2}$ is unbounded as $x_2 \to l-$ but nevertheless, according to Lemma 1.2.16, $(M_2 w \cdot w_{,2})|_{x_2=l_-} = 0$. □

Remark 1.3.4 *According to (1.2.11)*

$$K_2(0) = -Q_2(0),\ \ K(0) = -M_2(0),\ \ K_2(l) = -Q_2(l),\ \ K(l) = -M_2(l)$$

and conditions (1.2.15) and (1.2.20) can be rewritten in the form

$$M_2(0) = 0\ \ (M_2(l) = 0)$$

and

$$M_2(0) = 0,\ Q_2(0) = 0\ \ (M_2(l) = 0,\ Q_2(l) = 0),$$

respectively. Now, by virtue of Theorem 1.2.1 (see (1.2.15), (1.2.20), (1.2.16), (1.2.17)), the following assertions become evident:

1) if $I_0^0\,(I_0^l) = +\infty$, $I_1^0\,(I_1^l) < +\infty$, then conditions

$$w_{,2} = O(1),\ \ x_2 \to 0+\ \ (x_2 \to l-) \tag{1.3.15}$$

can be replaced by BCs

$$M_2(0) = 0\ \ (M_2(l) = 0) \tag{1.3.16}$$

and vice versa, i.e., (1.3.15) and (1.3.16) are equivalent conditions.

2) if $I_1^0\,(I_1^l) = +\infty$, then conditions (1.3.6) can be replaced by BCs

$$M_2(0) = 0,\ \ Q_2(0) = 0\ \ (M_2(l) = 0,\ \ Q_2(l) = 0)$$

and vice versa, i.e., the last conditions and (1.3.6) are equivalent conditions.

Remark 1.3.5 *Let $D(0) = 0$, $D(l) > 0$. Homogeneous BVP 1 (see Theorem 1.3.1) corresponds to the three-dimensional problem when the lateral surfaces are loaded by surface forces, the edge $x_2 = l$ is fixed and the edge $x_2 = 0$ is glued to the absolutely rigid tangent plane. In the case of homogeneous BVP 3 the above mentioned plane is rigid parallel to the axis x_3. BVP 4 corresponds to the three-dimensional problem when along the edge $x_2 = 0$ the concentrated along the above edge force and moment are applied which are equal to Q_0 and M_0, respectively.*

For forces and moments concentrated along the line (in particular, at a point of a cusped edge) see [38], [51].

Remark 1.3.6 *By setting of BVPs we have to take into account peculiarities of classical bending that by the arbitrary load f, the shearing force Q_2 (see (1.2.4)) can be given only on one edge; from $Q_2(0)$ (or $Q_2(l)$), $M_2(0)$, $M_2(l)$ (see (1.2.5)) only two can participate in BCs on the both edges together (these peculiarities are not caused by cusps they arise even in the case of bending of a beam of a constant cross-section). If we choose f correspondingly (see (1.2.4), (1.2.5)), we can avoid these peculiarities but restriction on choice of f would be artificial (in the mathematical sense but natural in the physical sense). Nevertheless, problems posed in this way can also make practical sense. Obviously, solutions to all these problems can be constructed in explicit forms. Some of them are unique, some are defined either up to a rigid translation along the axis x_3 or an infinitesimal rigid rotation at the axis x_1 or a general rigid motion (combination of above mentioned). We omit the exact formulation of these artificial BVPs. But for the sake of illustration, at the end of this section we set and solve a typical one.*

Remark 1.3.7 *From Theorem 1.3.1. and Remark 1.3.4 we arrive at the following conclusions. In the case of BVPs 3 and 6 the derivative of solution* $w_{,2}$ *is bounded if either* $I_0^0 < +\infty$ *or* $I_0^0 = +\infty$ *and* $M_0 = 0$. *In the case of BVP 4: the solution* w *is bounded if either* $I_1^0 < +\infty$ *or* $I_1^0 = +\infty$ *and* $\exists k \geq 2$ *such that* $I_k^0 < +\infty$ *and* $M_0 = 0$ *(for* $k \geq 2$*),* $Q_0 = 0$ *(for* $k \geq 3$*); the derivative of solution* $w_{,2}$ *is bounded if either* $I_0^0 < +\infty$ *or* $I_0^0 = +\infty$ *and* $\exists k \geq 1$ *such that* $I_k^0 < +\infty$ *and* $M_0 = 0$ *(for* $k \geq 1$*),* $Q_0 = 0$ *(for* $k \geq 2$*). In the case of BVPs 7 and 8 the derivative of solution* $w_{,2}$ *is bounded if either* $I_0^l < +\infty$ *or* $I_0^l = +\infty$ *and* $M_l = 0$. *In the BVP 9 the derivative of solution* $w_{,2}$ *is bounded if either* $I_0^0 < +\infty$, $I_0^l < +\infty$ *or* $I_0^0 = +\infty$ *with* $M_0 = 0$ *and* $I_0^l = +\infty$ *with* $M_l = 0$. *In the case of BVP 10: the solution* w *is bounded if either* $I_1^l < +\infty$ *or* $I_1^l = +\infty$ *and* $\exists k \geq 2$ *such that* $I_k^l < +\infty$ *and* $M_l = 0$ *(for* $k \geq 2$*),* $Q_l = 0$ *(for* $k \geq 3$*); the derivative of solution* $w_{,2}$ *is bounded if either* $I_0^l < +\infty$ *or* $I_0^l = +\infty$ *and* $\exists k \geq 1$ *such that* $I_k^l < +\infty$ *and* $M_l = 0$ *(for* $k \geq 1$*),* $Q_l = 0$ *(for* $k \geq 2$*).*

Remark 1.3.8 *If* $I_1^0 = +\infty$, *BVP 4 with homogeneous boundary data* $M_0 = 0$, $Q_0 = 0$ *is equivalent to BVP*

$$(1.2.1),\ (1.3.6)_0,\ (1.3.1)_l, \quad w \in C^4(]0,l[) \cap C^1(]0,l]).$$

If $I_1^l = +\infty$, *BVP 10 with homogeneous boundary data* $M_l = 0$, $Q_l = 0$ *is equivalent to BVP*

$$(1.2.1),\ (1.3.1)_0,\ (1.3.6)_l, \quad w \in C^4(]0,l[) \cap C^1([0,l[).$$

If $I_0^0 = +\infty$, $I_1^0 < +\infty$, *BVP 4 with a homogeneous boundary datum* $M_0 = 0$, *is equivalent to BVP*

$$\begin{aligned}(1.2.1),\ w_{,2} = O(1) \text{ as } x_2 \to 0+,\ Q_2(0) = Q_0,\ (1.3.1)_l,\\ w \in C^4(]0,l[) \cap C^1(]0,l]).\end{aligned}$$

If $I_0^l = +\infty$, $I_1^l < +\infty$, *BVP 10 with a homogeneous boundary datum* $M_l = 0$ *is equivalent to BVP*

$$\begin{aligned}(1.2.1),\ (1.3.1)_0,\ w_{,2} = O(1) \text{ as } x_2 \to l-,\ Q_2(l) = Q_l,\\ w \in C^4(]0,l[) \cap C^1([0,l[).\end{aligned}$$

If $I_0^0 = +\infty$, $I_1^0 < +\infty$, *BVP 3 with a homogeneous boundary datum* $M_0 = 0$, *is equivalent to BVP*

$$(1.2.1),\ (1.3.5)_0,\ (1.3.1)_l, \quad w \in C^4(]0,l[) \cap C^1(]0,l]) \cap C([0,l]).$$

If $I_0^l = +\infty$, $I_1^l < +\infty$, *BVP 7 with a homogeneous boundary datum* $M_l = 0$ *is equivalent to BVP*

$$(1.2.1),\ (1.3.1)_0,\ (1.3.5)_l, \quad w \in C^4(]0,l[) \cap C^1([0,l[) \cap C([0,l]).$$

If $I_0^0 = +\infty$, $I_0^l = +\infty$, $I_1^0 < +\infty$, $I_1^l < +\infty$, *BVP 9 with homogeneous boundary data* $M_0 = 0$ *and* $M_l = 0$ *is equivalent to BVP*

$$(1.2.1),\ (1.3.5)_0,\ \ (1.3.5)_l, \quad w \in C^4(]0,l[) \cap C([0,l]).$$

Let us now consider an example mentioned in Remark 1.3.6. Let moments and shearing forces be applied at the both ends of the beam, i.e.,

$$M_2(0) = M_0, \tag{1.3.17}$$
$$Q_2(0) = Q_0, \tag{1.3.18}$$
$$M_2(l) = M_l, \tag{1.3.19}$$
$$Q_2(l) = Q_l. \tag{1.3.20}$$

In order to determine constants $C_1,\ C_2$ from (1.2.5), (1.3.17), (1.3.19) we get the following system

$$\begin{aligned} \int_{x_0}^{0} tf(t)dt - C_1 x_0 + C_2 &= M_0, \\ -\int_{x_0}^{l} (l-t)f(t)dt + C_1(l-x_0) + C_2 &= M_l, \end{aligned}$$

whence,

$$C_1 = \frac{1}{l}\left[M_l - M_0 + l\int_{x_0}^{l} f(t)dt - \int_{0}^{l} tf(t)dt\right], \tag{1.3.21}$$

$$\begin{aligned} C_2 &= \frac{1}{l}\left[lM_0 + x_0(M_l - M_0) - l\int_{x_0}^{0} tf(t)dt + \right. \\ &+ \left. lx_0\int_{x_0}^{l} f(t)dt - x_0\int_{0}^{l} tf(t)dt\right]. \end{aligned} \tag{1.3.22}$$

In view of (1.2.5), (1.2.4), (1.3.21), (1.3.22) we have

$$\begin{aligned} M_2(x_2) &= -\int_{x_0}^{x_2} (x_2 - t)f(t)dt + (x_2 - x_0)\frac{1}{l}\left[M_l - M_0 + l\int_{x_0}^{l} f(t)dt\right. \\ &- \left.\int_{0}^{l} tf(t)dt\right] + \frac{1}{l}\left[lM_0 + x_0(M_l - M_0) - l\int_{x_0}^{0} tf(t)dt\right. \\ &+ \left. lx_0\int_{x_0}^{l} f(t)dt - x_0\int_{0}^{l} tf(t)dt\right] \end{aligned}$$

$$
\begin{aligned}
&= x_2 \int_{x_2}^{l} f(t)dt + \int_{0}^{x_2} tf(t)dt \\
&+ \frac{x_2}{l}\left[M_l - M_0 - \int_0^l tf(t)dt\right] + M_0,
\end{aligned}
$$

$$
Q_2(x_2) = \int_{x_2}^{l} f(t)dt + \frac{1}{l}\left[M_l - M_0 - \int_0^l tf(t)dt\right]. \tag{1.3.23}
$$

Now, we must find conditions on $f(t)$ which guarantee satisfaction of BCs (1.3.18), (1.3.20). To this end we substitute (1.3.23) in (1.3.18), (1.3.20):

$$
l\int_0^l f(t)dt - \int_0^l tf(t)dt + M_l - M_0 = lQ_0, \tag{1.3.24}
$$

$$
-\int_0^l tf(t)dt + M_l - M_0 = lQ_l. \tag{1.3.25}
$$

The difference of (1.3.24) and (1.3.25) gives

$$
\int_0^l f(t)dt = Q_0 - Q_l. \tag{1.3.26}
$$

(1.3.26) with either (1.3.24) or (1.3.25) yields the conditions we were looking for. These conditions are natural in the physical sense since they express the fact that the resultant vector and resultant moment of the applied forces should be equal to zero.

Let us observe that C_3, C_4 in (1.2.6), (1.2.7) remain arbitrary. This means that we found the solution up to a rigid translation along the axis x_3 and an infinitesimal rigid rotation at the axis x_1, which are expressed by arbitrary C_4 and C_3, respectively.

In particular, let the both ends be free:

$$
M_2(0) = Q_2(0) = M_2(l) = Q_2(l) = 0.
$$

Then the conditions (1.3.26),(1.3.24) and their equivalent conditions (1.3.26),(1.3.25) become

$$
\int_0^l f(t)dt = 0, \quad \int_0^l tf(t)dt = 0.
$$

This means that the lateral load and its moment are self-balanced.

Remark 1.3.9 *It is easy to see that the assertions of Sections 1.2 and 1.3 are also true if at the ends of the beam either* $\sigma(x_2) > 0$ *and Young's modulus* $E(x_2) = 0$ *or both vanish. In particular, this means that the peculiarities of the cusped beams will be preserved if we consider a beam of uniform cross-section with an appropriately chosen variable Young's modulus which vanishes at the ends. Moreover, if we consider a cusped beam, e.g., with* $D = EI = const$*, then the effect of geometry of the cusped beam on setting of BCs will be cancelled because of appropriately chosen nonhomogeneity of medium and all the BCs can be set without any restrictions on the sharpening geometry. The same is true for cusped plates as well, what readily follows from the expression of the flexural rigidity (1.1.3).*

1.4 Vibration problem

Let $I(0) \geq 0$, $I(l) > 0$ and $C^4_J(]0,l[)$ be a class of functions belonging to $C^4(]0,l[)$ with the properties 1-8 stated in Theorem 1.2.12. Let further $w, v \in C^4_J(]0,l[)$ and either conditions (1.2.59) or (1.2.61) be fulfilled in the corresponding cases. Then after multiplying both sides of equation (1.1.4) by v and integrating twice by parts, we get:

$$\int\limits_0^l \left(Dw_{,22}v_{,22} - \omega^2\rho\sigma wv\right) dx_2 = \int\limits_0^l fv dx_2 + (Q_2 w \cdot v)\left.\right|_0^l - (M_2 w \cdot v_{,2})\left.\right|_0^l. \qquad (1.4.1)$$

It is clear that, by virtue of Theorem 1.2.12, when $I_0 = +\infty$ and $I_1 < +\infty$, under condition (1.2.59) the last term at 0 will be missing in (1.4.1) with $v = w$, while when $I_1 = +\infty$ and $\exists k \in \{2, 3, \ldots\}$ such that $I_k < +\infty$, under conditions (1.2.61) the last and penultimate one at 0 will be lacking as well. It should be noted here that as it follows from a note to Solution of BVP 9, (1.4.1) remains true when $I_1^0 = +\infty$ ($I_1^l = +\infty$), $I_2^0 = +\infty$ ($I_2^l = +\infty$) even if $(Q_2 w)(0) \neq 0$ ($(Q_2 w)(l) \neq 0$), i.e., $w_{,2}$ is unbounded as $x_2 \to 0+$ ($x_2 \to l-$), provided $(M_2 w)(0) = 0$ ($(M_2 w)(l) = 0$), since in this case

$$\lim_{x_2 \to 0+} (M_2 w)(x_2) \cdot w_{,2}(x_2) = 0$$

$$\left(\lim_{x_2 \to l-} (M_2 w)(x_2) \cdot w_{,2}(x_2) = 0\right).$$

The relation (1.4.1) connects (in some sense) classical and weak solutions, and it is crucial in view of the definition of the latter in the sense of expression of unstable BCs (see Remark 1.4.5 below). Therefore, considering weak solutions of the vibration problem, by setting of BCs we will not be able to avoid the restrictions (1.2.61) and (1.2.59) in the corresponding cases. The more so, as BCs at cusped ends, on the other hand, should be chosen in such a way that the terms at endpoints in (1.4.1) disappear, provided the above-mentioned BCs are homogeneous [compare with the case of cusped plates, where contour integrals should disappear (see proof of (2.2.11) below)].

Problem 1.4.1 *Let us consider the vibration equation (1.1.4) with the following inhomogeneous BCs:*

- *at the non-cusped end* $x_2 = l$ *of the beam conditions* $(1.3.1)_l$,
- *at the other end* $x_2 = 0$ *which is a cusped one if* $D(0) = 0$ *either conditions* $(1.3.1)_0$, *or* $(1.3.2)_0$, *or* $(1.3.3)_0$ *provided* $I_2 < \infty$, *or* $(1.3.4)_0$ *with*

$$M_0 = 0 \quad if \quad I_0 = \infty \tag{1.4.2}$$

and

$$Q_0 = 0 \quad if \quad I_1 = \infty. \tag{1.4.3}$$

Remark 1.4.2 *Problem 1.4.1 is the common formulation of the following four BVPs:*
(i) (1.1.4), $(1.3.1)_l$, $(1.3.1)_0$;
(ii) (1.1.4), $(1.3.1)_l$, $(1.3.2)_0$;
(iii) (1.1.4), $(1.3.1)_l$, $(1.3.3)_0$ *provided* $I_2 < +\infty$;
(iv) (1.1.4), $(1.3.1)_l$, $(1.3.4)_0$.

Such a formulation is convenient since it makes possible to investigate all the four BVPs at the same time.

Definition 1.4.3 *Let*

$$W^{2,2}(]0, l[; \rho_0, \rho_2) \tag{1.4.4}$$

be the set of all measurable functions $w = w(x_2)$ *defined on* $]0, l[$ *which have on* $]0, l[$ *generalized derivatives* $\partial^\alpha_{x_2} w$, $\alpha \in \{0, 1, 2\}$ $(\partial^0_{x_2} w \equiv w)$ *such that*

$$\begin{aligned} &w \in L^2(]0, l[; \rho_0), \quad i.e., \quad \int\limits_0^l |w(x_2)|^2 \rho_0(x_2) dx_2 < +\infty, \\ &\partial^1_{x_2} w \in L^1_{loc}(]0, l[), \\ &\partial^2_{x_2} w \in L^2(]0, l[; \rho_2), \quad i.e., \quad \int\limits_0^l |\partial^2_{x_2} w(x_2)|^2 \rho_2(x_2) dx_2 < +\infty. \end{aligned} \tag{1.4.5}$$

Here ρ_0, ρ_1 *are weight functions, i.e., functions measurable and positive a.e. in* $]0, l[$.

The condition

$$\rho_0^{-1}(x_2),\ \rho_2^{-1}(x_2) \in L^1_{loc}(]0, l[)$$

guarantees [60] that $W^{2,2}(]0, l[; \rho_0, \rho_2)$ is a Banach space and even a Hilbert space under the norm

$$\|w\|^2_{W^{2,2}(]0,l[;\rho_0,\rho_2)} := \int\limits_0^l [w^2 \rho_0 + (\partial^2_{x_2} w)^2 \rho_2] dx_2 \tag{1.4.6}$$

and with the appropriate scalar product.

We recall that in what follows, we will use the notation $w_{,2}$ and $w_{,22}$ for $\partial^1_{x_2} w$ and $\partial^2_{x_2} w$, respectively.

First, we will consider the special case $\rho_0 \equiv 1$, $\rho_2(x_2) = D(x_2)$ with $D(x_2) > 0$ for $x_2 \in]0,l[$, $D(0) \geq 0$. In this case, we will denote the space $W^{2,2}(]0,l[;\rho_0,\rho_1)$ as

$$W^{2,2}(]0,l[,D). \tag{1.4.7}$$

Obviously, the last space is a Hilbert space if $\frac{1}{D(x_2)} \in L^1_{\text{loc}}(]0,l[)$ which holds, e.g., if $D \in C([0,l])$.

Now we can constitute subspaces $V_{\gamma_1,\gamma_2}$of $W^{2,2}(]0,l[,D)$, $\gamma_1,\gamma_2 \in \{0,1\}$, as follows:

(i) In the case of the BVP (1.1.4), $(1.3.1)_l$, $(1.3.1)_0$ we define

$$\begin{aligned} V_{0,0} \;:=\; & \{v \in W^{2,2}(]0,l[,D) : v(0)=0,\ v_{,2}(0)=0 \ \textit{and} \\ & v(l)=0,\ v_{,2}(l)=0 \ \textit{in the sense of traces}\}. \end{aligned} \tag{1.4.8}$$

(ii) In the case of the BVP (1.1.4), $(1.3.1)_l$, $(1.3.2)_0$ we define

$$\begin{aligned} V_{0,1} \;:=\; & \{v \in W^{2,2}(]0,l[,D) : v_{,2}(0)=0 \ \textit{and} \\ & v(l)=0,\ v_{,2}(l)=0 \ \textit{in the sense of traces}\}. \end{aligned} \tag{1.4.9}$$

(iii) In the case of the BVP (1.1.4), $(1.3.1)_l$, $(1.3.3)_0$ provided $I_2 < \infty$ we define

$$\begin{aligned} V_{1,0} \;:=\; & \{v \in W^{2,2}(]0,l[,D) : v(0)=0 \ \textit{and}\ v(l)=0,\ v_{,2}(l)=0 \\ & \textit{in the sense of traces}\}. \end{aligned} \tag{1.4.10}$$

(iv) In the case of the BVP (1.1.4), $(1.3.1)_l$, $(1.3.4)_0$ we define

$$V_{1,1} := \{v \in W^{2,2}(]0,l[,D) : v(l)=0,\ v_{,2}(l)=0 \ \textit{in the sense of traces}\}. \tag{1.4.11}$$

Notice that these spaces are defined in terms of traces. If these traces exist, it is not difficult to show that all spaces V_{γ_1,γ_2} are complete (see, e.g., proof of completness of $W^2_2(\Omega,\widetilde{D})$ in proof of Theorems 2.4.6 and 2.4.7 below). Now, the traces at the point $x_2 = l$ always exist since

$$W^{2,2}(]\varepsilon,l[,D) \subset W^{2,2}(]\varepsilon,l[) \quad \textit{for} \quad 0<\varepsilon<l \tag{1.4.12}$$

(where the second space is the "classical" Sobolev space), and, moreover, by virtue of embending theorems, $v \in C^1([\varepsilon,l])$.

In order to clarify the question of the existence of the traces at the point $x_2 = 0$, we make the function $D(x_2)$ subject to the following unilateral condition:

$$D(x_2) \geq D_\varkappa x_2^\varkappa \quad \forall x_2 \in]0,l[, \tag{1.4.13}$$

$D_\varkappa = const > 0$, $\varkappa = const \geq 0$ [1]. In other words

$$0 < D_\varkappa := \inf_{]0,l[} \frac{D(x_2)}{x_2^\varkappa}. \tag{1.4.14}$$

[1]By $\varkappa$ we denote the minimal among possible exponents $\delta \geq 1$ for which $D(x_2) \geq const\, x_2^\delta$ holds. For $\varkappa < 1$ it is not necessary to find the minimal possible exponent since in this case we have the same result concerning traces for all $\varkappa < 1$. Let us note that $D(x_2) = D_0[\ln(\widetilde{l}/x_2)]^{-1}$, $\widetilde{l} > l$, satisfies (1.4.13) for any $\varkappa > 0$. Condition (1.4.13) is obviously important in the neighbourhood of $x_2 = 0$.

It follows from (1.4.5) for $\rho_2(x_2) = D(x_2)$ and (1.4.13) that

$$\int_0^l x_2^{\varkappa}[w_{,22}(x_2)]^2 dx_2 < +\infty, \tag{1.4.15}$$

and, hence, under condition (1.4.13),

$$W^{2,2}(]0,l[,D) \subset W^{2,2}(]0,l[,x_2^{\varkappa}). \tag{1.4.16}$$

The last space is a special case of (1.4.4) with $\rho_0 \equiv 1$, $\rho_2(x_2) = x_2^{\varkappa}$. The obvious inequality

$$x_2(l - x_2) \leq l x_2 \quad for \quad x_2 \in [0,l]$$

together with (1.4.15) and (1.4.16) implies

$$W^{2,2}(]0,l[,D) \subset W^{2,2}(]0,l[,x_2^{\varkappa}) \subset W^{2,2}(]0,l[,x_2^{\varkappa}(l-x_2)^{\varkappa}). \tag{1.4.17}$$

The last space is a special case of (1.4.4) with $\rho_0 \equiv 1$, $\rho_2(x_2) = x_2^{\varkappa}(\rho - x_2)^{\varkappa}$. But any function

$$w \in W^{2,2}(]0,l[,x_2^{\varkappa}(l-x_2)^{\varkappa})$$

has a trace at the point $x_2 = 0$ if

$$\varkappa \in [0,3[\tag{1.4.18}$$

while its derivative $w_{,2}$ has a trace at $x_2 = 0$ if

$$\varkappa \in [0,1[. \tag{1.4.19}$$

More precisely, after a suitable change of the values of w at a set of measure zero, these functions became continuous on $[0,l]$, i.e., $w \in C([0,l])$ for (1.4.18) and $w_{,2} \in C([0,l])$ for (1.4.19) (see, e.g., [70]).

Let us note that if there exists such γ that

$$\lim_{x_2 \to 0+} x_2^{\gamma} D(w_{,22})^2 = \text{const} > 0$$

and along with (1.4.13) inequality (1.4.25) (see below) takes place, then from

$$w \in W^{2,2}(]0,l[,D) \subset W^{2,2}(]0,l[,x_2^{\varkappa})$$

there follows

$$(M_2 w)(0) = 0 \;\; for \;\; \varkappa \geq 1 \; (I_0 = +\infty).$$

Indeed, since

$$D(w_{,22})^2 \in L_1(]0,l[),$$

we have

$$D(w_{,22})^2 = O(x_2^{-\gamma}), \;\; x_2 \to 0+, \;\; \gamma < 1.$$

Otherwise, i.e., if $\gamma \geq 1$, the penultimate relation will be violated according to the convergence criterion in the limit form for improper integrals.

Consequently,

$$D^{\frac{1}{2}} w_{,22} = O(x_2^{-\frac{\gamma}{2}}), \;\; x_2 \to 0+, \;\; \gamma < 1,$$

and according to (1.4.25),

$$|M_2 w| = |-Dw_{,22}| = D^{\frac{1}{2}} O(x_2^{-\frac{\gamma}{2}}) \leq (D^{\varkappa})^{\frac{1}{2}} x_2^{\frac{\varkappa}{2}} O(x_2^{-\frac{\gamma}{2}}) = O(x_2^{\frac{\varkappa-\gamma}{2}}), \;\; x_2 \to 0+, \;\; \gamma < 1.$$

Whence,

$$(M_2 w)(0) = 0 \;\; when \;\; \varkappa > \gamma < 1, \;\; i.e., \;\; \varkappa \geq \; 1 \; (I_0 = +\infty).$$

This is one more argument for the condition (1.4.2).

Remark 1.4.4 *The obvious inequality*

$$x_2^4 \leq l^{4-\varkappa} x_2^{\varkappa} \quad for \quad x_2 \in [0, l] \quad and \quad \varkappa \leq 4 \tag{1.4.20}$$

implies

$$W^{2,2}(]0, l[, D) \subset W^{2,2}(]0, l[, x_2^{\varkappa}) \subset W^{2,2}(]0, l[, x_2^4) \tag{1.4.21}$$

for $\varkappa \leq 4$.

Inequality (1.4.13) can be rewritten as

$$\frac{1}{D(x_2)} \leq D_{\varkappa}^{-1} x_2^{-\varkappa}, \quad x_2 \in]0, l]. \tag{1.4.22}$$

Whence, we immediately conclude that (1.4.18) and (1.4.19) imply

$$I_2 < \infty \tag{1.4.23}$$

and

$$I_0 < \infty^2, \tag{1.4.24}$$

respectively.

Thus, the traces $v(0)$ and $v_{,2}(0)$ mentioned in (1.4.8)–(1.4.10) exist by (1.4.18) (i.e., by (1.4.23)) and by (1.4.19) (i.e., by (1.4.24)), respectively, provided (1.4.13) holds.

If instead of (1.4.13) the following inequality takes place

$$D(x_2) \leq D^{\varkappa} x_2^{\varkappa} \quad for \quad x_2 \in]0, l[, \tag{1.4.25}$$

$D^{\varkappa} = const > 0$, $\varkappa = const \geq 0$, then

$$W^{2,2}(]0, l[, x_2^{\varkappa}) \subset W^{2,2}(]0, l[, D).$$

[2] If $I_0 = +\infty$, then from (1.4.22) it follows that $\varkappa$ cannot be less than 1 (otherwise, i.e., if $\varkappa < 1$, we have (1.4.24) and come to the contradiction). Thus, the conditions (1.4.19) and (1.4.24) are equivalent in this sense. Clearly, we have analogous equivalence of the conditions (1.4.18) and (1.4.23).

In this case

$$(D^{\varkappa})^{-1}\frac{1}{x_2^{\varkappa}} \le \frac{1}{D(x_2)}, \quad x_2 \in]0,l[,$$

and (1.4.23) and (1.4.24) imply (1.4.18) and (1.4.19), respectively.

Finally, if both (1.4.13) and (1.4.25) are fulfilled, then (1.4.18) and (1.4.19) are equivalent to (1.4.23) and (1.4.24), respectively, and

$$W^{2,2}(]0,l[), D) = W^{2,2}(]0,l[, x_2^{\varkappa})$$

in the sense of equivalent norms.

Remark 1.4.5 *According to the customary terminology, the BCs*

$$w(0) = w_0 \qquad if \quad I_2 < \infty \tag{1.4.26}$$

and

$$w_{,2}(0) = w_0' \qquad if \quad I_0 < \infty \tag{1.4.27}$$

with prescribed constants w_0 and w_0' are the stable (principal) BCs for the operator J_ω since they are fulfilled by functions from both sets $C_J^4(]0,l[)$ and $W^{2,2}(]0,l[,D)$. Against it, the BCs

$$(M_2 w)(0) = M_0 \text{ and } (Q_2 w)(0) = Q_0 \tag{1.4.28}$$

with prescribed constants M_0, Q_0 are the unstable (natural) conditions since they are fulfilled by functions from $C_J^4(]0,l[)$ but not by functions from $W^{2,2}(]0,l[,D)$ due to the fact that traces at $x_2 = 0$ of the second and third order derivatives of functions from the latter class do not exist, in general.

In what follows, let $u \in W^{2,2}(]0,l[,D)$ and $f \in L^2(]0,l[)$ be given. Taking into account (1.4.1), we introduce the following definitions:

Definition 1.4.6 *The function $w \in W^{2,2}(]0,l[,D)$ will be called a weak solution of the BVP (1.1.4), (1.3.1)$_l$, (1.3.1)$_0$ in the space $W^{2,2}(]0,l[,D)$ if*

$$w - u \in V_{0,0} \tag{1.4.29}$$

and

$$J_\omega(w,v) := \int_0^l B_\omega(w,v)dx_2 = \int_0^l f v dx_2, \tag{1.4.30}$$

where

$$B_\omega(w,v) := Dw_{,22}v_{,22} - \omega^2\rho\sigma w v, \tag{1.4.31}$$

holds for every $v \in V_{0,0}$.

Definition 1.4.7 *The function* $w \in W^{2,2}(]0,l[,D)$ *will be called a weak solution of the BVP (1.1.4), (1.3.1)*$_l$*, (1.3.2)*$_0$ *in the space* $W^{2,2}(]0,l[,D)$ *if*

$$w - u \in V_{0,1} \tag{1.4.32}$$

and

$$J_\omega(w,v) = \int_0^l f v dx_2 + Q_0 v(0) \tag{1.4.33}$$

holds for every $v \in V_{0,1}$.

Definition 1.4.8 *The function* $w \in W^{2,2}(]0,l[,D)$ *will be called a weak solution of the BVP (1.1.4), (1.3.1)*$_l$*, (1.3.3)*$_0$ *provided* $I_2 < +\infty$ *in the space* $W^{2,2}(]0,l[,D)$ *if*

$$w - u \in V_{1,0} \tag{1.4.34}$$

and

$$J_\omega(w,v) = \int_0^l f v dx_2 - \begin{cases} M_0 v_{,2}(0) & \text{if } I_0 < +\infty, & (1.4.35) \\ 0 & \text{if } I_0 = +\infty & (1.4.36) \end{cases}$$

holds for every $v \in V_{1,0}$.

Definition 1.4.9 *The function* $w \in W^{2,2}(]0,l[,D)$ *will be called a weak solution of the BVP (1.1.4), (1.3.1)*$_l$*, (1.3.4)*$_0$ *in the space* $W^{2,2}(]0,l[,D)$ *if*

$$w - u \in V_{1,1} \tag{1.4.37}$$

and

$$J_\omega(w,v) = \int_0^l f v dx_2 + \begin{cases} Q_0 v(0) - M_0 v_{,2}(0) & \text{if } I_0 < +\infty, & (1.4.38) \\ Q_0 v(0) & \text{if } I_0 = +\infty \text{ and } I_1 < +\infty, & (1.4.39) \\ 0 & \text{if } I_1 = +\infty & (1.4.40) \end{cases}$$

holds for every $v \in V_{1,1}$.

Remark 1.4.10 *The conditions (1.4.29), (1.4.32), (1.4.34), (1.4.37) express the fact that the BCs (1.3.1)*$_l$*, (1.3.1)*$_0$*, the first BC of (1.3.2)*$_0$*, the first BC of (1.3.3)*$_0$ *provided* $I_2 < +\infty$ *are fulfilled. The BCs (1.3.4)*$_0$*, the second BC of (1.3.2)*$_0$*, the second BC of (1.3.3)*$_0$ *provided* $I_2 < +\infty$ *can be found directly in the identities (1.4.33), (1.4.35), (1.4.36), (1.4.38)–(1.4.40). These identities are derived from the identity (1.4.1). As we see from (1.4.36), (1.4.39), and (1.4.40), these last mentioned BCs cannot be specified in these identities if* $M_0 \neq 0$ *for* $I_0 = +\infty$ *and* $Q_0 \neq 0$ *for* $I_1 = +\infty$ *since for* $I_0 = +\infty$ *and* $I_1 = +\infty$*, the traces of* $v_{,2}$ *and* v*, respectively, at the point* $x_2 = 0$ *do not exist, in general. Hence, the restrictions (1.4.2) and (1.4.3) are natural in this sense, too (see also Remark 1.2.14).*

Remark 1.4.11 *In view of (1.4.1) with $\omega = 0$, the classical solutions of the static BVPs 1–4 from Theorem 1.3.1 constructed in Section 1.3 satisfy (1.4.30), (1.4.33), (1.4.35), (1.4.36), (1.4.38)–(1.4.40) by $\omega = 0$ in the corresponding cases. Obviously they satisfy also (1.4.29), (1.4.32), (1.4.34), (1.4.37).*

Besides the space (1.4.7) let us consider the space (1.4.4) with

$$\rho_0(x_2) = x_2^{\varkappa-4}, \ \rho_2(x) = x_2^{\varkappa}. \tag{1.4.41}$$

We will denote this space by

$$\widetilde{W}^{2,2}(]0,l[,x_2^{\varkappa}); \tag{1.4.42}$$

it is equipped with the norm

$$\|w\|_{\widetilde{W}^{2,2}(]0,l[,x_2^{\varkappa})} := \Big(\int\limits_0^l [x_2^{\varkappa-4}w^2(x_2) + x_2^{\varkappa}w_{,22}^2(x_2)]dx_2\Big)^{1/2}. \tag{1.4.43}$$

The space (1.4.42) is a Hilbert space with the appropriate scalar product, since $x_2^{4-\varkappa}$, $x_2^{-\varkappa} \in L^1_{loc}(]0,l[)$. We can easily see that

$$\begin{aligned}
\widetilde{W}^{2,2}(]0,l[,x_2^{\varkappa}) &\subset W^{2,2}(]0,l[),x_2^{\varkappa}) \quad if \quad \varkappa < 4, & (1.4.44)\\
\widetilde{W}^{2,2}(]0,l[,x_2^{4}) &= W^{2,2}(]0,l[,x_2^{4}), & (1.4.45)\\
\widetilde{W}^{2,2}(]0,l[,x_2^{\varkappa}) &\supset W^{2,2}(]0,l[,x_2^{\varkappa}) \quad if \quad \varkappa > 4. & (1.4.46)
\end{aligned}$$

Let us consider the space

$$V_\varepsilon(x_2^{\varkappa}) := \{v \in \widetilde{W}^{2,2}(]\varepsilon,l[,x_2^{\varkappa}), v(l) = 0, \ v_{,2}(l) = 0\}. \tag{1.4.47}$$

The traces $v(l)$ and $v_{,2}(l)$ are well-defined since for $\varepsilon \in]0,l[$

$$\widetilde{W}^{2,2}(]\varepsilon,l[,x_2^{\varkappa}) \subset W^{2,2}(]\varepsilon,l[), \tag{1.4.48}$$

and hence, v and $v_{,2}$ are absolutely continuous on $[\varepsilon,l]$. Thus,

$$v_{,2}\,, \ v \in AC_R(\varepsilon,l) \tag{1.4.49}$$

(see [72], p.5, Definition 1.2) and in view of the first boundary condition in (1.4.47), if $\varkappa > 1$, the following Hardy inequality holds (see [72], p. 69)

$$\int\limits_\varepsilon^l x_2^{\varkappa-2}v^2dx_2 \le \frac{4}{(\varkappa-1)^2}\int\limits_\varepsilon^l x_2^{\varkappa}(v_{,2})^2dx_2, \ \varkappa > 1. \tag{1.4.50}$$

Therefore, taking into account the second boundary condition in (1.4.47), we can write

$$\int\limits_\varepsilon^l x_2^{\varkappa-2}(v_{,2})^2dx_2 \le \frac{4}{(\varkappa-1)^2}\int\limits_\varepsilon^l x_2^{\varkappa}(v_{,22})^2dx_2, \ \varkappa > 1. \tag{1.4.51}$$

Replacing in (1.4.50) $\varkappa$ by $\varkappa - 2$, we obtain

$$\int_{\varepsilon}^{l} x_2^{\varkappa-4} v^2 dx_2 \leq \frac{4}{(\varkappa-3)^2} \int_{\varepsilon}^{l} x_2^{\varkappa-2} (v_{,2})^2 dx_2, \ \varkappa > 3. \tag{1.4.52}$$

Combining (1.4.51) and (1.4.52), we have

$$\int_{\varepsilon}^{l} x_2^{\varkappa-4} v^2 dx_2 \leq \frac{16}{(\varkappa-1)^2(\varkappa-3)^2} \int_{\varepsilon}^{l} x_2^{\varkappa} (v_{,22})^2 dx_2, \ \varkappa > 3. \tag{1.4.53}$$

Now, considering the limit procedure as $\varepsilon \to 0+$, since the limits of the integrals in (1.4.53) exist for $v \in \widetilde{W}^{2,2}(]0,l[, x_2^{\varkappa})$, we immediately get the following

Lemma 1.4.12 *If $v \in V_0(x_2^{\varkappa})$, then*

$$\int_{0}^{l} x_2^{\varkappa-4} v^2(x_2) dx_2 \leq \frac{16}{(\varkappa-1)^2(\varkappa-3)^2} \int_{0}^{l} x_2^{\varkappa} [v_{,22}(x_2)]^2 dx_2, \ \varkappa > 3. \tag{1.4.54}$$

Corollary 1.4.13 *If $v \in V_0(x_2^4)$, from* (1.4.54) *we obtain*

$$\int_{0}^{l} v^2 dx_2 \leq \frac{16}{9} \int_{0}^{l} x_2^4 (v_{,22})^2 dx_2. \tag{1.4.55}$$

Remark 1.4.14 *Obviously, all the spaces V_{γ_1,γ_2}, $\gamma_1, \gamma_2 \in \{0,1\}$, constructed by (1.4.8)–(1.4.11) are contained in $V_0(x_2^4)$ if $\varkappa \leq 4$ (see (1.4.47), (1.4.45), and the relations (1.4.21) of Remark 1.4.4).*

First we consider the case

$$0 \leq \varkappa < 4,$$

i.e.,

$$I_3 < +\infty.$$

Theorem 1.4.15 *If $0 \leq \varkappa < 4$ (i.e., $I_3 < +\infty$) and*

$$\omega^2 < \frac{9 D_{\varkappa} l^{\varkappa-4}}{16 \max\limits_{[0,l]} \rho\sigma}, \tag{1.4.56}$$

then the BVPs

1. $(1.1.4), (1.3.1)_l, (1.3.1)_0;$
2. $(1.1.4), (1.3.1)_l, (1.3.2)_0;$
3. $(1.1.4), (1.3.1)_l, (1.3.3)_0$ *provided* $I_2 < +\infty;$
4. $(1.1.4), (1.3.1)_l, (1.3.4)_0$

have unique solutions. These solutions are such that

$$\|w\|_{W^{2,2}(]0,l[,D)} \leq C[\|f\|_{L_2(]0,l[)} + \|u\|_{W^{2,2}(]0,l[,D)} + \gamma_1|M_0| + \gamma_2|Q_0|],$$

where the constant C is independent of f, u, M_0, Q_0, and

$$\begin{gathered}\gamma_1 = 0,\ \gamma_2 = 0 \ for\ the\ first\ problem,\\ \gamma_1 = 0,\ \gamma_2 = 1 \ for\ the\ second\ problem,\\ \gamma_1 = 1,\ \gamma_2 = 0 \ for\ the\ third\ problem,\\ \gamma_1 = 1,\ \gamma_2 = 1 \ for\ the\ fourth\ problem.\end{gathered}$$

Proof. It is easy to see that

$$|J_\omega(w,v)| \leq (1+T)\|w\|_{W^{2,2}(]0,l[,D)}\|v\|_{W^{2,2}(]0,l[,D)}, \tag{1.4.57}$$

where

$$T := \omega^2 \max_{[0,l]} \rho(x_2)\sigma(x_2), \tag{1.4.58}$$

and the functional

$$F_\omega v := \int_0^l f(x_2)v(x_2)dx_2 - J_\omega(u,v) + \gamma_2 v(0)Q_0 - \gamma_1 v_{,2}(0)M_0,\ v \in V_{\gamma_1,\gamma_2}$$

(see (1.4.8)–(1.4.11) and (1.4.30), (1.4.33), (1.4.35), (1.4.36), (1.4.38)–(1.4.40)) is bounded in V_{γ_1,γ_2}:

$$|F_\omega v| \leq [\|f\|_{L_2(]0,l[)} + (1+T)\|u\|_{W^{2,2}(]0,l[,D)} + C_0(\gamma_2|Q_0| + \gamma_1|M_0|)]\|v\|_{V_{\gamma_1,\gamma_2}}, \tag{1.4.59}$$

where we have used the theorem of traces (the constant C_0 is from this theorem) and

$$\|v\|_{V_{\gamma_1,\gamma_2}} := \|v\|_{W^{2,2}(]0,l[,D)}. \tag{1.4.60}$$

Now, taking into account (1.4.7), (1.4.31), (1.4.58), Remark 1.4.14, Corollary 1.4.13, (1.4.20), (1.4.13), and introducing the notation

$$T_0 := \frac{16 l^{4-\varkappa}}{9 D_\varkappa}(1+T), \tag{1.4.61}$$

we have

$$\|v\|^2_{V_{\gamma_1,\gamma_2}} := \int_0^l [v^2 + D(v_{,22})^2]dx_2 = \int_0^l v^2 dx_2 + J_\omega(v,v) + \omega^2 \int_0^l \rho\sigma v^2 dx_2$$

$$
\begin{aligned}
&\leq (1+T)\int\limits_0^l v^2dx_2 + J_\omega(v,v) \leq \frac{16}{9}(1+T)\int\limits_0^l x_2^4(v_{,22})^2dx_2 + J_\omega(v,v)\\
&\leq \frac{16l^{4-\varkappa}}{9D_\varkappa}(1+T)\int\limits_0^l D_\varkappa x_2^\varkappa(v_{,22})^2dx_2 + J_\omega(v,v)\\
&\leq T_0\int\limits_0^l D(v_{,22})^2dx_2 + J_\omega(v,v)\\
&= J_\omega(v,v) + T_0\Big[J_\omega(v,v) + \omega^2\int\limits_0^l \rho\sigma v^2dx_2\Big]\\
&\leq J_\omega(v,v) + T_0\Big[J_\omega(v,v) + T\int\limits_0^l v^2dx\Big]\\
&\leq J_\omega(v,v) + T_0\Big[J_\omega(v,v) + \frac{16l^{4-\varkappa}T}{9D_\varkappa}\int\limits_0^l D(v_{,22})^2dx_2\Big]\\
&= J_\omega(v,v) + T_0\Big\{J_\omega(v,v) + \frac{16l^{4-\varkappa}T}{9D_\varkappa}\Big[J_\omega(v,v) + \omega^2\int\limits_0^l \rho\sigma v^2dx_2\Big]\Big\}\\
&\leq J_\omega(v,v) + T_0\{J_\omega(v,v) + \frac{16l^{4-\varkappa}T}{9D_\varkappa}\Big[J_\omega(v,v) + T\int\limits_0^l v^2dx_2\Big]\Big\}\\
&\leq J_\omega(v,v) + T_0\{J_\omega(v,v) + \frac{16l^{4-\varkappa}T}{9D_\varkappa}\Big[J_\omega(v,v) + \frac{16l^{4-\varkappa}T}{9D_\varkappa}\int\limits_0^l D(v_{,22})^2dx_2\Big]\Big\}\\
&= J_\omega(v,v) + T_0\Big\{J_\omega(v,v) + \frac{16l^{4-\varkappa}T}{9D_\varkappa}\Big[J_\omega(v,v) + \frac{16l^{4-\varkappa}T}{9D_\varkappa}(J_\omega(v,v)\\
&+ \omega^2\int\limits_0^l \rho\sigma v^2dx_2\Big)\Big]\Big\}\\
&= J_\omega(v,v) + T_0\Big\{J_\omega(v,v)\Big[1 + \frac{16l^{4-\varkappa}T}{9D_\varkappa} + \Big(\frac{16l^{4-\varkappa}T}{9D_\varkappa}\Big)^2\Big]\\
&+ \Big(\frac{16l^{4-\varkappa}}{9D_\varkappa}\Big)^2\omega^2\int\limits_0^l \rho\sigma v^2dx_2\Big\}\\
&\quad (repeating\ \ the\ \ same\ \ (n-2)-times\ \ more)
\end{aligned}
$$

$$\leq \quad J_\omega(v,v) + T_0\Big[J_\omega(v,v)\frac{1-(\frac{16l^{4-\varkappa}T}{9D_\varkappa})^{n+1}}{1-\frac{16l^{4-\varkappa}T}{9D_\varkappa}} + \Big(\frac{16l^{4-\varkappa}T}{9D_\varkappa}\Big)^n\omega^2\int\limits_0^l \rho\sigma v^2 dx_2\Big).$$

Now, tending n to infinity and taking into account that, in view of (1.4.56) and (1.4.58),

$$\frac{16l^{4-\varkappa}T}{9D_\varkappa} < 1,$$

we obtain

$$\|v\|^2_{V_{\gamma_1,\gamma_2}} \leq J_\omega(v,v) + \frac{T_0}{1-\frac{16l^{4-\varkappa}T}{9D_\varkappa}} J_\omega(v,v),$$

i.e., in view of (1.4.61),

$$J_\omega(v,v) \geq \frac{9D_\varkappa - 16l^{4-\varkappa}T}{9D_\varkappa + 16l^{4-\varkappa}}\|v\|^2_{V_{\gamma_1,\gamma_2}}. \tag{1.4.62}$$

Thus, by virtue of (1.4.57), (1.4.62), and (1.4.59), according to the Lax-Milgram theorem (see (2.4.36) below) there exists a unique $z \in V_{\gamma_1,\gamma_2}$ such that

$$J_\omega(z,v) = F_\omega v := \int\limits_0^l fv dx_2 - J_\omega(u,v) + \gamma_2 v(0)Q_0 - \gamma_1 v_{,2}(0)M_0 \qquad \forall\ v \in V_{\gamma_1,\gamma_2},$$

whence,

$$J_\omega(w,v) = \int\limits_0^l fv dx_2 + \gamma_2 v(0)Q_0 - \gamma_1 v_{,2}(0)M_0 \qquad \forall\ v \in V_{\gamma_1,\gamma_2}, \tag{1.4.63}$$

where

$$w := u + z \in W^{2,2}(]0,l[),D). \tag{1.4.64}$$

So,

$$w - u = z \in V_{\gamma_1,\gamma_2},$$

and (1.4.63) means that (1.4.30), (1.4.33), (1.4.35), (1.4.36), (1.4.38)–(1.4.40) hold in the corresponding cases.

Besides, according to the Lax-Milgram theorem (see (2.4.37) below)

$$\|z\|_{V_{\gamma_1,\gamma_2}} \leq \frac{9D_\varkappa + 16l^{4-\varkappa}}{9D_\varkappa - 16l^{4-\varkappa}T}\|F_\omega\|_{V^*_{\gamma_1,\gamma_2}} \tag{1.4.65}$$

where $V^*_{\gamma_1,\gamma_2}$ is dual to V_{γ_1,γ_2}. From (1.4.59) it follows that

$$\|F_\omega\|_{V^*_{\gamma_1,\gamma_2}} \leq \|f\|_{L_2(]0,l[)} + (1+T)\|u\|_{W^{2,2}(]0,l[,D)} + C_0(\gamma_2|Q_0| + \gamma_1|M_0|). \tag{1.4.66}$$

By virtue of (1.4.64)–(1.4.66), we have

$$\begin{aligned}\|w\|_{W^{2,2}(]0,l[,D)} &\leq \|u\|_{W^{2,2}(]0,l[,D)} + \|z\|_{V_{\gamma_1,\gamma_2}} \leq \|u\|_{W^{2,2}(]0,l[,D)} \\ &+ \frac{9D_\varkappa + 16l^{4-\varkappa}}{9D_\varkappa - 16l^{4-\varkappa}T}[\|f\|_{L^2(]0,l[)} + (1+T)\|u\|_{W^{2,2}(]0,l[,D)} + C_0(\gamma_2|Q_0| + \gamma_1|M_0|)] \\ &\leq C[\|f\|_{L_2(]0,l[)} + \|u\|_{W^{2,2}(]0,l[,D)} + \gamma_2|Q_0| + \gamma_1|M_0|],\end{aligned}$$

where

$$C := \max\Big\{1 + \frac{9D_\varkappa + 16l^{4-\varkappa}}{9D_\varkappa - 16l^{4-\varkappa}T}(1+T),\ \frac{9D_\varkappa + 16l^{4-\varkappa}}{9D_\varkappa - 16l^{4-\varkappa}T}C_0\Big\}.$$

□

Now, let us consider the case

$$\varkappa \geq 4,$$

i.e.,

$$I_k = +\infty \ and \ I_{k+1} < +\infty \ for \ a \ fixed \ k \in \{3,4,\ldots\}.$$

Instead of the space $W^{2,2}(]0,l[,D)$ (see (1.4.7)) with the norm (1.4.6) we look for a solution in a wider space

$$\widetilde{W}^{2,2}(]0,l[,D) \tag{1.4.67}$$

with the norm

$$\|w\|^2_{\widetilde{W}^{2,2}(]0,l[,D)} := \int_0^l [x_2^{\varkappa-4}w^2 + D(w_{,22})^2]dx_2. \tag{1.4.68}$$

More precisely,

$$\widetilde{W}^{2,2}(]0,l[,D) \supset W^{2,2}(]0,l[,D) \ for \ \varkappa > 4 \tag{1.4.69}$$

and

$$\widetilde{W}^{2,2}(]0,l[,D) = W^{2,2}(]0,l[,D) \ for \ \varkappa = 4. \tag{1.4.70}$$

In the case under consideration, as it follows from the previous arguments (see Problem 1.4.1 and compare $(1.3.4)_0$ with $(1.3.1)_0$–$(1.3.3)_0$), only the BVP (1.1.4), $(1.3.1)_l$, $(1.3.4)_0$ is admissible.

Let

$$V := \{v \in \widetilde{W}^{2,2}(]0,l[,D) : v(l) = 0,\ v_{,2}(l) = 0\}. \tag{1.4.71}$$

In view of (1.4.13),

$$\widetilde{W}^{2,2}(]0,l[,x_2^\varkappa) \supset \widetilde{W}^{2,2}(]0,l[,D) \ for \ \varkappa \geq 4.$$

Therefore, Lemma 1.4.12 is also valid for $v \in V$.

Definition 1.4.16 *Let $u \in \widetilde{W}^{2,2}(]0,l[,D)$ be given and $x_2^{\frac{4-\varkappa}{2}} f \in L_2(]0,l[)$. A function $w \in \widetilde{W}^{2,2}(]0,l[,D)$ will be called a weak solution of the BVP (1.1.4), $(1.3.1)_l$, $(1.3.4)_0$ in the space $\widetilde{W}^{2,2}(]0,l[,D)$ if*

$$w - u \in V$$

with V defined by (1.4.71), and if (1.4.40) holds for every $v \in V$.

Theorem 1.4.17 *Let $\rho(x_2)\sigma(x_2)x_2^{4-\varkappa} \in C([0,l])$. If $\varkappa \geq 4$ (i.e., $I_k = +\infty$ and $I_{k+1} < +\infty$ for a fixed $k \in \{3,4,\ldots\}$) and*

$$\omega^2 < \frac{(\varkappa-1)^2(\varkappa-3)^2 D_\varkappa}{16 \max\limits_{[0,l]} \rho(x_2)\sigma(x_2)x_2^{4-\varkappa}}, \tag{1.4.72}$$

then the BVP (1.1.4), $(1.3.1)_l$, $(1.3.4)_0$ with $M_0 = 0$, $Q_0 = 0$ has a unique weak solution in $\widetilde{W}^{2,2}(]0,l[,D)$ such that

$$\|w\|_{\widetilde{W}^{2,2}(]0,l[,D)} \leq C[\|x_2^{\frac{4-\varkappa}{2}} f(x_2)\|_{L_2(]0,l[)} + \|u\|_{\widetilde{W}^{2,2}(]0,l[,D)}],$$

where the constant C is independent of f and u.

Proof. Let

$$T_* := \omega^2 \max_{[0,l]} \rho\sigma x_2^{4-\varkappa},$$

$$T_\varkappa := \frac{16(1+T_*)}{(\varkappa-1)^2(\varkappa-3)^2 D_\varkappa}.$$

Using Lemma 1.4.12 and relations

$$\begin{aligned} \omega^2 \int_0^l \rho\sigma|wv|dx_2 &= \omega^2 \int_0^l (\rho\sigma x_2^{4-\varkappa})(x_2^{\frac{\varkappa-4}{2}}|w|)(x_2^{\frac{\varkappa-4}{2}}|v|)dx_2 \\ &\leq T_* \left(\int_0^l x_2^{\varkappa-4} w^2 dx_2\right)^{1/2} \left(\int_0^l x_2^{\varkappa-4} v^2 dx_2\right)^{1/2}, \\ \int_0^l \rho\sigma v^2 dx_2 &= \int_0^l (\rho\sigma x_2^{4-\varkappa})(x_2^{\varkappa-4} v^2)dx_2, \end{aligned}$$

similarly to the proof of Theorem 1.4.15 (compare also with the proof of Theorem 2.5.12 below), we get

$$|J_\omega(w,v)| \leq (1+T_*)\|w\|_{\widetilde{W}^{2,2}(]0,l[,D)} \cdot \|v\|_{\widetilde{W}^{2,2}(]0,l[,D)},$$

where J_ω is defined by (1.4.30), (1.4.31),

$$|F_\omega v| \leq [\|x_2^{4-\varkappa} f\|_{L_2(]0,l[)} + (1+T_*)\|u\|_{\widetilde{W}^{2,2}(]0,l[,D)}]\|v\|_V,$$

where

$$F_\omega v := \int_0^l f(x_2)v(x_2)dx_2 - J_\omega(u,v), \ \ v \in V,$$

and

$$J_\omega(v,v) \geq \frac{(\varkappa-1)^2(\varkappa-3)^2 D_\varkappa - 16T_*}{(\varkappa-1)^2(\varkappa-3)^2 D_\varkappa + 16}\|v\|_V^2 \ \ \forall v \in V.$$

Thus, all the conditions of the Lax-Milgram theorem are fulfilled and it is not difficult to finish the proof. □

Remark 1.4.18 *The restriction*

$$\rho(x_2)\sigma(x_2)x_2^{4-\varkappa} \in C([0,l])$$

is not heavy because of $\sigma(0)=0$. *For instance, if we consider a beam with a rectangular cross-section, with the unit width and the thickness*

$$2h = h_0 x_2^{\varkappa/3}, \ h_0 = const > 0, \tag{1.4.73}$$

then $\sigma(x_2) = h_0 x_2^{\frac{\varkappa}{3}}$ *and for* $4 \leq \varkappa \leq 6$

$$\rho(x_2)\sigma(x_2)x_2^{4-\varkappa} = \rho(x_2)h_0 x_2^{4-\frac{2\varkappa}{3}} \in C([0,l]).$$

Remark 1.4.19 *In the case (1.4.73)* $D(x_2)$ *has the form*

$$D(x_2) = D_* x_2^{\varkappa}, \ D_* = const > 0,$$

provided $E = const$, $\nu = const$. *If we additionally suppose that*

$$\rho(x_2) = \rho_* x_2^{\frac{2\varkappa}{3}-4}, \ \rho_* = const > 0,$$

then

$$\rho(x_2)\sigma(x_2)x_2^{4-\varkappa} = \rho_* h_0 = const.$$

Hence, from (1.4.72) we have

$$\omega^2 < \frac{(\varkappa-1)^2(\varkappa-3)^2 D_*}{16\rho_* h_0}.$$

Whence, the greater is $\varkappa$ *the greater is the lower bound of eigenvalues of the operator* J_ω. *If now* $\varkappa$ *tends to* $+\infty$, *then the above bound tends to* $+\infty$ *as well.*

Remark 1.4.20 *Let* $l=1$. *In the case of the homogeneous BCs for the BVP (1.1.4),* $(1.3.1)_l$, $(1.3.1)_0$, *from the results of [57] (see theorem* 1.6_1, *and Lemma* 1.5_1*) there follows the following sufficient condition of the unique solvability on the vibration frequence*

$$\omega^2 < \min\left\{ \frac{3}{\int\limits_0^{\tau_0}(\tau_0-\tau)^3 D^{-1}(\tau)d\tau}, \ \frac{3}{\int\limits_{\tau_0}^{1}(\tau-\tau_0)^3 D^{-1}(\tau)d\tau} \right\}$$

for a fixed $\tau_0 \in]0,l[$. *Here we do not precise the other restrictions.*

Now, we consider the general case, i.e., we refuse (1.4.13). So,

$$D(x_2) \in C([0,l]),\ D(x_2) > 0 \quad \forall\, x_2 \in]0,l],\ D(0) \geq 0. \tag{1.4.74}$$

Under these assumptions, obviously,

$$\int_{x_2}^{l} D^{-1}(\tau)d\tau < +\infty \quad for\ \ every \quad x_2 \in]0,l]. \tag{1.4.75}$$

Let further

$$P(x_2) \ := \ D^{-1}(x_2)\left[\int_{x_2}^{l} D^{-1}(\tau)d\tau\right]^{-2}, \quad x_2 \in]0,l[, \tag{1.4.76}$$

$$Q(x_2) \ := \ D(x_2)\left[\int_{x_2}^{l} D^{-1}(\tau)d\tau\right]^{2}\left\{\int_{x_2}^{l} D(t)\left[\int_{t}^{l} D^{-1}(\tau)d\tau\right]^{2}dt\right\}^{-2}, \tag{1.4.77}$$

$$x_2 \in]0,l[.$$

Evidently,

$$P(x_2), Q(x_2) \in C(]0,l[), \tag{1.4.78}$$

and

$$P(x_2) > 0,\ Q(x_2) > 0 \quad \forall\, x_2 \in]0,l[. \tag{1.4.79}$$

Definition 1.4.21 *Let*

$$\overset{*}{W}{}^{2,2}(]0,l[,D) \tag{1.4.80}$$

be a special case of (1.4.4) with

$$\rho_0 = Q(x_2),\ \rho_2 = D(x_2).$$

Since

$$Q^{-1}(x_2),\ D^{-1}(x_2) \in L^1_{loc}(]0,l[),$$

the space (1.4.80) is a Hilbert space.

Now, we consider Problem 1.4.1, where w_0, w_l and w_0', w_l' are the traces of a certain given function $u \in \overset{*}{W}{}^{2,2}(]0,l[,D)$ and of its derivative, respectively.

Let

$$\begin{aligned} \overset{*}{V} \ := \ & \Big\{ v \in \overset{*}{W}{}^{2,2}(]0,l[,D) : v(l) = 0,\ v_{,2}(l) = 0, \\ & and\ \ additionally \\ & v(0) = 0,\ v_{,2}(0) = 0\ in\ \ the\ \ case\ \ of\ \ BCs\ \ (1.3.1)_0, \\ & v_{,2}(0) = 0\ in\ \ the\ \ case\ \ of\ \ BCs\ \ (1.3.2)_0, \\ & v(0) = 0\ in\ \ the\ \ case\ \ of\ \ BCs\ (1.3.3)_0\ \ provided\ I_2 < +\infty \\ & (in\ \ the\ \ sense\ \ of\ \ traces)\Big\}. \end{aligned} \tag{1.4.81}$$

Remark 1.4.22 *In (1.4.81) the existence of traces in the indicated cases is assumed. But if we additionally suppose that* $\int\limits_0^{x_2} D^{-1}(t)dt < +\infty$ *for* $x_2 \in]0, l]$ *(so, with (1.4.74) it implies* $0 < \int\limits_0^{l} D^{-1}(t)dt < +\infty$*) and consider the space*

$$\overset{*}{W}{}^{2,2}(]0, l-\varepsilon[, D) \supset \overset{*}{W}{}^{2,2}(]0, l[, D),\ \varepsilon = const > 0,$$

then, in view of (1.4.77),

$$Q(x_2) = D(x_2) \cdot \widetilde{D}(x_2) \quad \forall\ x_2 \in [0, l-\varepsilon]$$

with the positive continuous $\widetilde{D}(x_2)$ *on* $[0, l-\varepsilon]$*. If, now, we assume (1.4.13), we will have*

$$D(x_2) \geq D_\varkappa x_2^\varkappa \quad and \quad Q(x_2) \geq \widetilde{D} \cdot D_\varkappa x_2^\varkappa \quad \forall\ x_2 \in]0, l-\varepsilon[,$$

where

$$\widetilde{D} := \min_{[0,l-\varepsilon]} \widetilde{D}(x_2).$$

Hence,

$$D(x_2) \geq D_* x_2^\varkappa, \quad Q(x_2) \geq D_* x_2^\varkappa$$

with

$$D_* := \min\{D_\varkappa,\ \widetilde{D} D_\varkappa\}.$$

Therefore,

$$u \in \overset{*}{W}{}^{2,2}(]0, l-\varepsilon[, D)$$

implies

$$u \in {}_2W^{2,2}(]0, l-\varepsilon[, x_2^\varkappa),$$

where

$${}_2W^{2,2}(]0, l-\varepsilon[, x_2^\varkappa) := \left\{ u : \|u\|_{{}_2W^{2,2}(]0,l-\varepsilon[,x_2^\varkappa)} := \int\limits_0^{l-\varepsilon} [x_2^\varkappa u^2 + x_2^\varkappa (u_{,22})^2] dx_2 < +\infty \right\}.$$

So,

$$\overset{*}{W}{}^{2,2}(]0, l-\varepsilon[, D) \subset {}_2W^{2,2}(]0, l-\varepsilon[, x_2^\varkappa).$$

But on the one hand,

$${}_2W^{2,2}(]0, l-\varepsilon[, x_2^\varkappa) \subset {}_2W^{2,2}(]0, l-\varepsilon[, x_2^\varkappa (l-x_2)^\varkappa),$$

because of

$$x_2^\varkappa (l-x_2)^\varkappa \leq l^\varkappa x_2^\varkappa \quad \forall\ x_2 \in [0, l].$$

On the other hand (see [71], Theorem 1.1.4),

$${}_2W^{2,2}(]0, l-\varepsilon[, x_2^\varkappa (l-x_2)^\varkappa) = W^{2,2}(]0, l-\varepsilon[, x_2^\varkappa (l-x_2)^\varkappa) \quad \forall\ \varkappa \in]-1, 4].$$

Thus, if (1.4.13) holds, then the traces of u *at* $x_2 = 0$ *in the mentioned in (1.4.81) cases exist (see [71], Theorem 1.1.2 or [70]).*

Obviously, from

$$v \in \overset{*}{V}$$

there follows

$$v \in \overset{*}{V}_\varepsilon,$$

where

$$\overset{*}{V}_\varepsilon := \{v \in \overset{*}{W}{}^{2,2}(]\varepsilon, l[, D) : v(l) = 0,\ v_{,2}(l) = 0\} \tag{1.4.82}$$

with arbitrarily small $\varepsilon > 0$.

On $[\varepsilon, l]$:

1.

$$D(x_2) \geq \min_{[\varepsilon,l]} D(x_2) =: D^\varepsilon > 0 \ and \ \frac{D(x_2)}{D^\varepsilon} \geq 1, \tag{1.4.83}$$

because of

(i) $D(x_2) \in C([\varepsilon, l])$;

(ii) $D(x_2) > 0 \ \forall \ x_2 \in [\varepsilon, l]$.

2.

$$P(x_2) = D^{-1}(x_2) \left[\int_{x_2}^{l} D^{-1}(\tau) d\tau \right]^{-2} \geq \min_{[\varepsilon,l]} P(x_2) =: P^\varepsilon > 0 \ and \ \frac{P(x_2)}{p^\varepsilon} \geq 1, \tag{1.4.84}$$

because of

i) $P(x_2) \in C([\varepsilon, l[)$;

ii) $P(x_2) > 0 \quad \forall \ x_2 \in [\varepsilon, l[$;

iii) $\lim\limits_{x_2 \to l-} P(x_2) = \lim\limits_{x_2 \to l-} D^{-1}(x_2) \left[\int_{x_2}^{l} D^{-1}(\tau) d\tau \right]^{-2} = +\infty$, since $0 < D^{-1}(l) < +\infty$.

3.

$$Q(x_2) := D(x_2) \left[\int_{x_2}^{l} D^{-1}(\tau) d\tau \right]^2 \left\{ \int_{x_2}^{l} D(t) \left[\int_{t}^{l} D^{-1}(\tau) d\tau \right]^2 dt \right\}^{-2} \geq$$

$$\geq \min_{[\varepsilon,l]} Q(x_2) =: Q^\varepsilon > 0 \ and \ \frac{Q(x_2)}{Q^\varepsilon} \geq 1, \tag{1.4.85}$$

because of

i) $Q(x_2) \in C([\varepsilon, l[)$;

ii) $Q(x_2) > 0 \quad \forall \ x_2 \in [\varepsilon, l[$;

iii)

$$\lim_{x_2 \to l-} Q(x_2) = D(l) \lim_{x_2 \to l-} \frac{\left[\int_{x_2}^{l} D^{-1}(\tau) dt \right]^2}{\left\{ \int_{x_2}^{l} D(t) \left[\int_{t}^{l} D^{-1}(\tau) d\tau \right]^2 dt \right\}^2}$$

$$= D(l) \lim_{x_2 \to l-} \frac{2\int\limits_{x_2}^{l} D^{-1}(\tau)d\tau \cdot [-D^{-1}(x_2)]}{2\int\limits_{x_2}^{l} D(t)\left[\int\limits_{t}^{l} D^{-1}(\tau)d\tau\right]^2 dt \left\{-D(x_2)\left[\int\limits_{x_2}^{l} D^{-1}(\tau)d\tau\right]^2\right\}} = +\infty$$

since $0 < D^{-1}(l) < +\infty$, $0 < D(l) < +\infty$.

Evidently,

$$u \in \overset{*}{W}{}^{2,2}(]0,l[, D)$$

implies

$$u \in \overset{*}{W}{}^{2,2}(]\varepsilon,l[, D). \tag{1.4.86}$$

But from (1.4.83), (1.4.85) we have

$$|u|^2 \le |u|^2 \frac{Q(x_2)}{Q^\varepsilon}, \quad |u_{,22}|^2 \le |u_{,22}|^2 \frac{D(x_2)}{D^\varepsilon} \quad \forall\, x_2 \in [\varepsilon, l[.$$

Hence, in view of (1.4.86), we get

$$u \in W^{2,2}(]\varepsilon,l[).$$

All the more, for

$$v \in \overset{*}{W}{}^{2,2}(]0,l[, D)$$

with

$$v(l) = 0, \quad v_{,2}(l) = 0,$$

we have

$$v \in W^{2,2}(]\varepsilon,l[) \tag{1.4.87}$$

with

$$v(l) = 0, \quad v_{,2}(l) = 0$$

in the usual sense, since by virtue of (1.4.87) v and its derivative are absolutely continuous on $[\varepsilon, l]$ (more precisely, after maybe necessary change on the set of the measure 0). Thus,

$$v \text{ and } v_{,2} \in AC_R(\varepsilon, l)$$

(see [72], p. 5, Definition 1.2) and the following Hardy type inequalities hold (see [72], p. 66, Theorem 6.4):

$$\int\limits_{\varepsilon}^{l} Q(x_2)v^2(x_2)dx_2 \le 4\int\limits_{\varepsilon}^{l} P(x_2)[v_{,2}(x_2)]^2 dx_2, \tag{1.4.88}$$

$$\int\limits_{\varepsilon}^{l} P(x_2)[v_{,2}(x_2)]^2 dx_2 \le 4\int\limits_{\varepsilon}^{l} D(x_2)[v_{,22}(x_2)]^2 dx_2, \tag{1.4.89}$$

whence,

$$\int_{\varepsilon}^{l} Q(x_2)v^2(x_2)dx_2 \leq 16 \int_{\varepsilon}^{l} D(x_2)[v_{,22}(x_2)]^2 dx_2. \qquad (1.4.90)$$

Considering limit procedure as $\varepsilon \to 0+$, since all the limit integrals exist because of $v \in \overset{*}{W}{}^{2,2}(]0,l[,D)$, we immediately get the following

Lemma 1.4.23 *If $v \in \overset{*}{W}{}^{2,2}(]0,l[,D)$ and $v(l)=0,\ v_{,2}(l)=0$, then*

$$\int_{0}^{l} Q(x_2)v^2(x_2)dx_2 \leq 16 \int_{0}^{l} D(x_2)[v_{,22}(x_2)]^2 dx_2.$$

Definition 1.4.24 *Let $Q^{-\frac{1}{2}}(x_2)f(x_2) \in L_2(]0,l[)$. A function $w \in \overset{*}{W}{}^{2,2}(]0,l[,D)$ will be called a weak solution of the Problem 1.4.1 in the space $\overset{*}{W}{}^{2,2}(]0,l[,D)$ if it satisfies the following conditions*

$$w - u \in \overset{*}{V}$$

and

$$J_\omega(w,v) := \int_0^l B_\omega(w,v)dx_2 = \int_0^l vf dx_2 + \gamma_2 v(0)Q_0 - \gamma_1 v_{,2}(0)M_0 \quad \forall\ v \in \overset{*}{V},$$

where

$$\begin{aligned}
&B_\omega(w,v) := Dw_{,22}v_{,22} - \omega^2\rho(x_2)\sigma(x_2)wv,\\
&\gamma_1 = 0,\ \gamma_2 = 0 \textit{ in the case of } BCs\ (1.3.1)_0,\\
&\gamma_1 = 0,\ \gamma_2 = 1 \textit{ in the case of } BCs\ (1.3.2)_0,\\
&\gamma_1 = 1,\ \gamma_2 = 0 \textit{ in the case of } BCs(1.3.3)_0 \textit{ provided } I_2 < \infty,\\
&\gamma_1 = 1,\ \gamma_2 = 1 \textit{ in the case of } BCs(1.3.4)_0.
\end{aligned}$$

Theorem 1.4.25 *Let $Q^{-1}(x_2)\rho(x_2)\sigma(x_2) \in C([0,l])$ and*

$$\omega^2 < \frac{1}{16\max\limits_{[0,l]}(\rho\sigma Q^{-1})}. \qquad (1.4.91)$$

Then there exists a unique weak solution of Problem 1.4.1 (more precisely of the four BVPs stated there) in $\overset{*}{W}{}^{2,2}(]0,l[,D)$. This solution is such that

$$\begin{aligned}
\|w\|_{\overset{*}{W}{}^{2,2}(]0,l[,D)} &\leq C[\|Q^{-\frac{1}{2}}f\|_{L_2(]0,l[)} +\\
&+ \|u\|_{\overset{*}{W}{}^{2,2}(]0,l[,D)} + \gamma_1|M_0| + \gamma_2|Q_0|],
\end{aligned}$$

where the constant C is independent of f, u, M_0, Q_0.

Proof. It is easy to see that

$$\begin{aligned}
|J_\omega(w,v)| &= \left| \int_0^l D^{\frac{1}{2}} w_{,22} D^{\frac{1}{2}} v_{,22} dx_2 - \omega^2 \int_0^l \rho(x_2)\sigma(x_2) Q^{-1}(x_2) Q^{\frac{1}{2}} w \cdot Q^{\frac{1}{2}} v dx_2 \right| \\
&\leq \left[\int_0^l D(w_{,22})^2 dx_2 \right]^{1/2} \left[\int_0^l D(v_{,22})^2 dx_2 \right]^{1/2} + \\
&+ \overset{*}{T} \left[\int_0^l Q w^2 dx_2 \right]^{1/2} \left[\int_0^l Q v^2 dx_2 \right]^{1/2} \leq \\
&\leq (1+\overset{*}{T}) \|v\|_{\overset{*}{W}^{2,2}(]0,l[,D)} \|v\|_{\overset{*}{W}^{2,2}(]0,l[,D)},
\end{aligned}$$

where

$$\overset{*}{T} := \omega^2 \max_{[0,l]} (\rho\sigma Q^{-1}). \tag{1.4.92}$$

Hence, the functional

$$F_\omega v := \int_0^l v(x_2) f(x_2) dx_2 - J_\omega(u,v) + \gamma_2 v(0) Q_0 - \gamma_1 v_{,2}(0) M_0, \ v \in \overset{*}{V}$$

is bounded in $\overset{*}{V}$:

$$|F_\omega v| \leq [\|Q^{-1} f\|_{L_2(\Omega)} + (1+\overset{*}{T}) \|u\|_{\overset{*}{W}^{2,2}(]0,l[,D)} + C_0(\gamma_2 |Q_0| + \gamma_1 |M_0|)] \|v\|_{\overset{*}{V}},$$

where C_0 is the constant from the trace theorem. In order to use the Lax-Milgram theorem it remains to show the $\overset{*}{V}$-ellipticity of $J_\omega(w,v)$. Indeed, using Lemma 1.4.23 and introducing the notation

$$T_0 := 16(1+\overset{*}{T}), \tag{1.4.93}$$

we have

$$\begin{aligned}
\|v\|^2_{\overset{*}{V}} &:= \int_0^l Q(x_2) v^2 dx_2 + \int_0^l D(x_2)(v_{,22})^2 dx_2 \\
&= \int_0^l Q(x_2) v^2 dx_2 + J_\omega(v,v) + \omega^2 \int_0^l \rho\sigma Q^{-1} Q v^2 dx_2 \\
&\leq (1+\overset{*}{T}) \int_0^l Q(x_2) v^2 dx_2 + J_\omega(v,v)
\end{aligned}$$

$$
\begin{aligned}
&\leq && 16(1+\overset{*}{T})\int_0^l D(x_2)(v_{,22})^2dx_2 + J_\omega(v,v)\\
&= && T_0\left[J_\omega(v,v)+\omega^2\int_0^l \rho\sigma Q^{-1}Qv^2dx_2\right]+J_\omega(v,v)\\
&\leq && J_\omega(v,v)+T_0\left[J_\omega(v,v)+\overset{*}{T}\int_0^l Qv^2dx_2\right]\\
&\leq && J_\omega(v,v)+T_0\left[J_\omega(v,v)+16\overset{*}{T}\int_0^l D(v_{,22})^2dx_2\right]\\
&= && J_\omega(v,v)+T_0\left[J_\omega(v,v)+16\overset{*}{T}J_\omega(v,v)+16\overset{*}{T}\omega^2\int_0^l \rho\sigma Q^{-1}Qv^2dx_2\right]\\
&\leq && J_\omega(v,v)+T_0\left[J_\omega(v,v)+16\overset{*}{T}J_\omega(v,v)+(16\overset{*}{T})^2\int_0^l D(v_{,22})^2dx_2\right]\\
&= && J_\omega(v,v)+T_0\Big[J_\omega(v,v)+16\overset{*}{T}J_\omega(v,v)+(16\overset{*}{T})^2J_\omega(v,v)\\
&+ && (16\overset{*}{T})^2\overset{*}{T}\int_0^l Qv^2dx_2\Big] \leq J_\omega(v,v)\\
&+ && T_0\left\{J_\omega(v,v)\left[1+16\overset{*}{T}+(16\overset{*}{T})^2+(16\overset{*}{T})^3\int_0^l D(v_{,22})^2dx_2\right]\right\}\\
& && (repeating\ \ the\ \ same\ (n-2)-times\ \ more)\\
&\leq && J_\omega(v,v)+T_0\left[J_\omega(v,v)\frac{1-(16\overset{*}{T})^{n+1}}{1-16\overset{*}{T}}+(16\overset{*}{T})^{n+1}\int_0^l D(v_{,22})^2dx_2\right].
\end{aligned}
$$

Now, tending n to infinity and taking into account that

$$16\overset{*}{T}<1,$$

because of (1.4.91), (1.4.92), we obtain

$$\|v\|^2_{\overset{*}{V}} \leq J_\omega(v,v)+\frac{T_0}{1-16\overset{*}{T}}J_\omega(v,v).$$

Whence,

$$J_\omega(v,v)\geq \frac{1-16\overset{*}{T}}{1-16\overset{*}{T}+T_0}\|v\|^2_{\overset{*}{V}}=\frac{1-16\overset{*}{T}}{17}\|v\|^2_{\overset{*}{V}},$$

since, in view of (1.4.93),

$$1 - 16\overset{*}{T} + T_0 = 17.$$

Now, we can use the Lax-Milgram theorem and complete the proof similarly to the proof of Theorem 1.4.15. □

Chapter 2

Cusped Kirchhoff-Love Plates

2.1 Cusped orthotropic plate

Let $Ox_1x_2x_3$ be the Cartesian coordinate system, and let Ω be a domain in the plane Ox_1x_2 with a piecewise smooth boundary. The body bounded from above by the surface $x_3 = h(x_1, x_2) \geq 0,\ (x_1, x_2) \in \Omega$, from below by the surface $x_3 = -h(x_1, x_2),\ (x_1, x_2) \in \Omega$, from the side by a cylindrical surface parallel to the x_3-axis, will be called a symmetric cusped plate. The points $P \in \partial\Omega$, at which s.c. plate thickness $2h(x_1, x_2) = 0$, will be called plate cusps. If $h \in C^1(\Omega)$, obviously,

$$0 \leq L := \lim_{Q \to P} \frac{\partial 2h(Q)}{\partial n} \leq +\infty, \quad Q \in \Omega, \quad P \in \partial\Omega,$$

provided the finite or infinite limit L exists; if P is an angular point of the boundary $\partial\Omega$, then under inward to $\partial\Omega$ normal n we mean bisectrix of an angle between unilateral tangents to $\partial\Omega$ at P. Ω will be called the projection of the plate. $\partial\Omega$ will be called the plate boundary. In appendix on Figures 11-19 the possible normal sections (profiles) of an asymmetric, in general, plate at the point P in its neighbourhood are represented (see also Figures 21-23 there).

Let us, now, consider an orthotropic cusped plate.

The equation of classical bending theory of orthotropic plates has the following form (see [79])

$$\begin{aligned} Jw &:= (D_1 w,_{11}),_{11} + (D_2 w,_{22}),_{22} + (D_3 w,_{22}),_{11} \\ &+ (D_3 w,_{11}),_{22} + 4(D_4 w,_{12}),_{12} \\ &= f(x_1, x_2) \ \text{ in } \ \Omega \subset R^2, \end{aligned} \tag{2.1.1}$$

where w is a deflection; f is a lateral load; $D_i \in C^2(\Omega)$, $i = 1, 2, 3, 4$, and

$$D_i := \frac{2\,E_i\,h^3}{3}, \quad i = 1, 2, 3, \quad D_4 := \frac{2\,G\,h^3}{3};$$

$$D_\alpha - D_3 > 0, \quad \alpha = 1, 2 \quad \text{ if } \ h > 0 \tag{2.1.2}$$

(for all known orthotropic plates these last conditions are fulfilled (see [79])); E_i, $i = 1, 2, 3$, and G are elastic constants for the orthotropic case; indices after comma mean differentiation with respect to corresponding variables.

In particular, if the plate is isotropic,

$$E_\alpha = \frac{E}{1-\nu^2}, \quad \alpha = 1,2, \quad E_3 = \frac{\nu E}{1-\nu^2}, \quad G = \frac{E}{2(1+\nu)},$$

where E is Young's modulus and ν is Poisson's ratio.

Let $\partial\Omega$ be the piecewise smooth boundary of the domain Ω with a part Γ_1 lying on the axis Ox_1 and a part Γ_2 lying in the upper half-plane $x_2 > 0$ ($\partial\Omega \equiv \overline{\Gamma}_1 \cup \overline{\Gamma}_2$).

Let further the thickness $2h > 0$ $in\, \Omega \cup \Gamma_2$, and $2h \geq 0$ on Γ_1. Therefore,

$$D_i(x_1, x_2) > 0 \quad \text{in} \quad \Omega \cup \Gamma_2, \quad D_i(x_1, x_2) \geq 0 \quad \text{on} \quad \Gamma_1, \quad i = 1,2,3,4. \tag{2.1.3}$$

In particular case let

$$D_{1i} x_2^{\varkappa} \leq D_i(x_1, x_2) \leq D_{2i} x_2^{\varkappa}, \quad i = 1,2,3,4, \quad \text{in } \Omega, \tag{2.1.4}$$

where

$$D_{\alpha i} = const > 0, \quad \alpha = 1,2, \quad i = 1,2,3,4, \quad \varkappa = \text{ const} \geq 0,$$

i.e.,

$$D_i(x_1, x_2) = \widetilde{D}_i(x_1, x_2) x_2^{\varkappa}, \quad D_{1i} \leq \widetilde{D}_i(x_1, x_2) \leq D_{2i},$$

$$D_{1\alpha} > D_{23}, \quad \alpha = 1,2, \tag{2.1.5}$$

(otherwise there would exist such points of Ω where (2.1.2) will be violated). In the case under consideration, (2.1.1) is an elliptic equation, in general, with order degeneration on Γ_1.

We recall (see [79]) that

$$\begin{aligned} M_\alpha &= -(D_\alpha w,_{\alpha\underline{\alpha}} + D_3 w,_{\beta\underline{\beta}}), \quad \alpha \neq \beta, \quad \alpha, \beta = 1,2, & (2.1.6)\\ M_{12} &= -M_{21} = 2D_4 w,_{12}, & (2.1.7)\\ Q_\alpha &= M_{\alpha,\underline{\alpha}} + M_{21,\beta}, \quad \alpha \neq \beta, \quad \alpha,\beta = 1,2, & (2.1.8)\\ Q_\alpha^* &= Q_\alpha + M_{21,\beta}, \quad \alpha \neq \beta, \quad \alpha,\beta = 1,2, & (2.1.9) \end{aligned}$$

where M_α are bending moments, $M_{\alpha\beta}$, $\alpha \neq \beta$, are twisting moments, Q_α are shearing forces and Q_α^* are generalized shearing forces (bar under repeated indices means that we do not sum with respect to these indices).

At points of the plate boundary, where the thickness vanishes, all quantities will be defined as limits from inside of Ω.

2.2 Bending in the energetic space

Let $D_i \in C^2(\Omega \cup \Gamma_2),\ i = 1,2,3,4$. Let us consider the operator J (acting in $L_2(\Omega)$) on D_J :

1. $w \in C^4(\Omega \cup \Gamma_2)$;
$Jw \in L_2(\Omega)$;

$$w \begin{cases} \in C(\overline{\Omega}) & \text{when } \ I_{1i}|_{\Gamma_1} < +\infty \ \text{ in case (2.1.3)} \ \ (0 \le \varkappa < 2 \ \text{ in case } \ (2.1.4)) \\ & \textit{provided} \ \ (M_2 w)|_{\Gamma_1} \neq 0 \ \ \textit{or when} \ \ I_{2i}|_{\Gamma_1} < +\infty \ \ \textit{in case} \ \ (2.1.3) \\ & (0 \le \varkappa < 3 \ \text{ in case } \ (2.1.4)) \ \ \textit{provided} \ \ (M_2 w)|_{\Gamma_1} = 0 \\ = O(1), & x_2 \to 0+, \quad \text{when } I_{1i}|_{\Gamma_1} = +\infty \ \ (\varkappa \ge 2) \ \ \textit{provided} \ \ (M_2 w)|_{\Gamma_1} \neq 0 \\ & \textit{or} \text{ when } I_{2i}|_{\Gamma_1} = +\infty \ \ (\varkappa \ge 3) \ \ \textit{provided} \ \ (M_2 w)|_{\Gamma_1} = 0; \end{cases} \tag{2.2.1}$$

$$w_{,\alpha} \begin{cases} \in C(\overline{\Omega}) & \text{when } \ I_{0i}|_{\Gamma_1} < +\infty \ \ (0 \le \varkappa < 1), \\ = \mathrm{O}(1), & x_2 \to 0+, \quad \text{when } \ I_{0i}|_{\Gamma_1} = +\infty \ \ (1 \le \varkappa < +\infty), \ \ \alpha = 1,2; \end{cases} \tag{2.2.2}$$

$$I_{ki} \equiv I_{ki}(x_1) := \int\limits_0^{l(x_1)} x_2^k D_i^{-1}(x_1, x_2) dx_2, \ i = 1,2,3,4, \ k = 0,1,...,$$
$$(x_1, 0) \in \Gamma_1, \ (x_1, l(x_1)) \in \Omega,$$

$$(D_2 - D_3)^{\frac{1}{2}} w_{,22} \in L_2(\Omega) \tag{2.2.3}$$

(this restriction can be avoided when we consider only solutions with a finite energy); the bending moment, and the generalized shearing force

$$(M_2 w) := -(D_2 w,_{22} + D_3 w,_{11}) \in C(\overline{\Omega}), \tag{2.2.4}$$

$$(Q_2^* w) := -[(D_2 w,_{22} + D_3 w,_{11}),_2 + 4(D_4 w,_{12}),_1] \in C(\overline{\Omega}); \tag{2.2.5}$$

2.

$$w|_{\Gamma_2} = 0, \quad \left.\frac{\partial w}{\partial n}\right|_{\Gamma_2} = 0 \tag{2.2.6}$$

where n is the inward normal;

3. On Γ_1 one of the following pairs of BCs is fulfilled:

$$w = 0, \ w,_2 = 0 \ \text{ if } \ I_{0i} < +\infty, \ i = 1,2,3,4, \ (0 \le \varkappa < 1); \tag{2.2.7}$$
$$w,_2 = 0, \ (Q_2^* w) = 0 \ \text{ if } \ I_{0i} < +\infty, \ i = 1,2,3,4, \ (0 \le \varkappa < 1); \tag{2.2.8}$$
$$w = 0, \ (M_2 w) = 0 \ \text{ if } \ I_{2i} < +\infty, \ i = 1,2,3,4, \ (0 \le \varkappa < 3); \tag{2.2.9}$$
$$(M_2 w) = 0, \ (Q_2^* w) = 0 \ \text{ if } \ I_{0i} \le +\infty, \ i = 1,2,3,4, \ (0 \le \varkappa < +\infty). \tag{2.2.10}$$

Obviously, $D_J \subset L_2(\Omega)$ and is dense in $L_2(\Omega)$ since D_J contains the set of finite functions $C_0^\infty(\Omega)$ which is dense in $L_2(\Omega)$.

Remark 2.2.1 *How it follows from the case of cylindrical bending (see Section 1 of [46], [38], p. 96 and also Section 1.2 of the present book), the BCs (2.2.7)-(2.2.9) cannot be posed (in the sense of solvability and uniqueness) for other values of $\varkappa$ except indicated ones, or in the general case (2.1.3) if $I_{0i}|_{\Gamma_1} = +\infty$, and $I_{2i}|_{\Gamma_1} = +\infty$, correspondingly.*

Statement 2.2.2 *The operator J is linear, symmetric, and positive on the lineal D_J, and*

$$(Jw, v) := \int\limits_\Omega vJwd\Omega = \int\limits_\Omega [D_1 v,_{11} w,_{11} + D_2 v,_{22} w,_{22} + D_3(v,_{11} w,_{22}$$
$$+ v,_{22} w,_{11}) + 4D_4 v,_{12} w,_{12}]d\Omega =: \int\limits_\Omega B(v,w)d\Omega \quad \forall v, w \in D_J. \qquad (2.2.11)$$

In particular, if $v = w$,

$$\begin{aligned}(Jw, w) &:= \int\limits_\Omega [D_1(w,_{11})^2 + D_2(w,_{22})^2 + 2D_3\ w,_{11} w,_{22} \\ &+ 4D_4(w,_{12})^2]d\Omega = \int\limits_\Omega [D_3(w,_{11} + w,_{22})^2 + (D_1 - D_3)(w,_{11})^2 \\ &+ 4D_4(w,_{12})^2 + (D_2 - D_3)(w,_{22})^2]d\Omega. \qquad (2.2.12)\end{aligned}$$

Proof. It is obvious that J is a linear operator on the lineal D_J (the latter about D_J easily follows from the linearity of operators J, M_2, and Q_2^* on $C^4(\Omega \cup \Gamma_2)$). Since $D_J \subset L_2(\Omega)$ and $Jw \in L_2(\Omega)$, we can consider the following scalar product in $L_2(\Omega)$

$$(Jw, v) := \int\limits_\Omega vJwd\Omega = \lim_{\delta \to 0} \int\limits_{\Omega_\delta} vJwd\Omega_\delta \ \ \forall v, w \in D_J,$$

where

$$\Omega_\delta := \{(x_1, x_2) \in \Omega \ : \ x_2 > \delta = \text{ const} > 0\}.$$

After integration by parts twice and using formulas (d), (c) on page 87 (page 105 of Russian edition) of [79] we have

$$(Jw, v) := \lim_{\delta \to 0} \left[\int\limits_{\partial\Omega_\delta} \left(v(Q_n w) - \frac{\partial v}{\partial n}(M_n w) + \frac{\partial v}{\partial s}(M_{ns} w) \right) ds + \int\limits_{\Omega_\delta} B(v,w)d\Omega_\delta \right],$$

where ds is the arc element, $(Q_n w)$ is the shearing force, $(M_n w)$ is the bending moment, $(M_{ns} w)$ is the twisting moment, which act on the plate cross-section with the normal n.

But

$$\int\limits_{\partial\Omega_\delta} \frac{\partial v}{\partial s}(M_{ns}w)ds = \int\limits_{\partial\Omega_\delta} \frac{\partial v(M_{ns}w)}{\partial s}ds - \int\limits_{\partial\Omega_\delta} v\frac{\partial(M_{ns}w)}{\partial s}ds = -\int\limits_{\partial\Omega_\delta} v\frac{\partial(M_{ns}w)}{\partial s}ds$$

as $v,\ (M_{ns}w) \in C(\overline{\Omega}_\delta)$. Hence,

$$(Jw, v) := \lim_{\delta\to 0} \int\limits_{\partial\Omega_\delta} \left(v(Q_n^* w) - \frac{\partial v}{\partial n}(M_n w)\right) ds + \lim_{\delta\to 0} \int\limits_{\Omega_\delta} B(v, w) d\Omega_\delta, \qquad (2.2.13)$$

where

$$(Q_n^* w) := (Q_n w) - \frac{\partial(M_{ns}w)}{\partial s}.$$

In view of (2.2.6),

$$\int\limits_{\partial\Omega_\delta} \left(v(Q_n^* w) - \frac{\partial v}{\partial n}(M_n w)\right) ds = \int\limits_{\Gamma_1^\delta} \left(v(Q_2^* w) - v_{,2}\,(M_2 w)\right) ds,$$

where

$$\Gamma_1^\delta := \{(x_1, x_2) \in \Omega \ :\ x_2 = \delta = \text{ const} > 0\}.$$

By virtue of (2.2.1), (2.2.2), (2.2.4), (2.2.5), (2.2.7)-(2.2.10), $\forall \varepsilon = const > 0\ \exists \delta(\varepsilon) = const > 0$ such that

$$|vQ_2^* - v_{,2}\, M_2| \le |v||Q_2^*| + |v_{,2}\,||M_2| < \varepsilon, \quad \text{when } \ 0 < x_2 < \delta,$$

i.e., taking into account (2.2.6),

$$\left|\int\limits_{\partial\Omega_\delta} \left(v\,(Q_n^* w) - \frac{\partial v}{\partial n}\,(M_n w)\right) ds\right| = \left|\int\limits_{\Gamma_1^\delta} \left(v\,(Q_2^* w) - v_{,2}\,(M_2 w)\right) ds\right| < \varepsilon|\Gamma_1^\delta| < \varepsilon|\partial\Omega_\delta| \le \varepsilon|\partial\Omega|$$

($|\partial\Omega|$ is the length of the curve $\partial\Omega$). So that

$$\lim_{\delta\to 0} \int\limits_{\partial\Omega_\delta} \left(v\,(Q_n^* w) - \frac{\partial v}{\partial n}\,(M_n w)\right) ds = \lim_{\delta\to 0} \int\limits_{\Gamma_1^\delta} \left(v\,(Q_2^* w) - v_{,2}\,(M_2 w)\right) ds = 0.$$

Therefore, because of existence of the integral on the left-hand side of (2.2.13), limit of the second addend on the right hand side of (2.2.13), also exists, and (2.2.11) is valid. (2.2.12) is obvious.

From (2.2.11) there follows

$$(Jw, v) = (Jv, w) = (w, Jv), \ \ \forall v, w \in D_J.$$

Hence, the operator J is symmetric.

From (2.2.12), taking into account (2.1.2), we have

$$(Jw, w) \geq 0.$$

But

$$(Jw, w) = 0, \quad w \in D_J,$$

iff

$$w_{,11} = 0, \quad w_{,22} = 0, \quad w_{,12} = 0 \text{ in } \Omega,$$

i.e.,

$$w = k_1 x_1 + k_2 x_2 + k_3, \quad k_i = \text{ const}, \; i = 1, 2, 3, \quad \text{in } \Omega.$$

The latter, by virtue of (2.2.6), should be zero on Γ_2 and, because of its linearity, $k_i = 0$, $i = 1, 2, 3$. Therefore, $w \equiv 0$ on $\overline{\Omega}$. □

Introduce

$$D_0 := \inf_{\Omega} \frac{D_2 - D_3}{x_2^4} \geq 0.$$

Statement 2.2.3 *The operator J is positive definite if only $D_0 > 0$ ($0 \leq \varkappa \leq 4$).*

Proof. Let $D_0 = 0$, and consider the particular case (2.1.4) with $\widetilde{D_i}(x_1, x_2) \in C^1(\overline{\Omega})$. Then

$$D_0 = \inf_{\Omega} (\widetilde{D}_2 - \widetilde{D}_3) x_2^{\varkappa - 4} = 0 \quad \text{if} \quad \text{only} \quad \varkappa > 4.$$

Now, we show that J is not positive definite. Indeed, let the rectangle

$$\Pi_0 := \{(x_1, x_2) : \; a < x_1 < b, \; 0 < x_2 < \delta\}$$

be cut out from Ω. Let (see [64])

$$w_\delta(x_1, x_2) := \begin{cases} (\delta - x_2)^5 sin^5 \dfrac{\pi(x_1 - a)}{b - a} & \text{when } (x_1, x_2) \in \Pi_0; \\ \\ 0 & \text{when } (x_1, x_2) \in \Omega \backslash \Pi_0. \end{cases}$$

Obviously, $w_\delta \in D_J$, and because of $\varkappa > 4$, (2.2.10) should be and, in fact, is fulfilled by w_δ. It is easy to see that

$$0 \leq \frac{(Jw_\delta, w_\delta)}{\|w_\delta\|^2_{L_2(\Omega)}} \leq \overset{*}{C} \delta^{\varkappa - 4}, \quad \overset{*}{C} = \text{ const} > 0,$$

since

$$\|w_\delta\|^2_{L_2(\Omega)} = \frac{1}{11} \, \frac{b - a}{\pi} \frac{1}{2^{10}} \binom{10}{5} \pi \delta^{11},$$

and, in view of (2.1.4),

$$0 \leq (Jw_\delta, w_\delta) \leq \int\limits_{\Pi_0} \max_{i\in\{1,2,3,4,\}} \{D_{2i}\} x_2^{\varkappa}[(w_{\delta,11})^2 + (w_{\delta,22})^2 +$$

$$+2w_{\delta,11}w_{\delta,22} + 4(w_{\delta,12})^2]dx_1dx_2 \leq \overset{**}{C}(\delta^{\varkappa+11} + \delta^{\varkappa+7} + \delta^{\varkappa+9}),$$

$$\overset{**}{C} = \text{const} > 0.$$

Hence, J is not positive definite on D_J.

Now, let us return to the general case (2.1.3). Further $D_0 > 0$ $(0 \leq \varkappa \leq 4)$, and prove that J is positive definite.

From (2.2.12), taking into account (2.2.3), (2.1.2) and (2.1.3), we obtain

$$(Jw, w) \geq \int\limits_{\Omega} (D_2 - D_3)(w_{,22})^2 d\Omega$$

$$\geq D_0 \int\limits_{\Omega} x_2^4 (w_{,22})^2 d\Omega = D_0 \int\limits_{\Pi} x_2^4 (w_{,22})^2 dx_1 dx_2,$$

where

$$\Pi := \{(x_1, x_2) : \ a < x_1 < b, \ 0 < x_2 < 1\}, \tag{2.2.14}$$

and without loss of generality, it is supposed that the domain Ω lies inside of the rectangle Π, and a definition of the function w is completed assuming w equal to zero outside of Ω. Then w will be continuous in Π with its first derivatives, and its derivatives of second order, in general, will have discontinuity of the first kind on the arc Γ_2. Further

$$\begin{aligned}(Jw, w) \ &\geq \ D_0 \int\limits_a^b \int\limits_0^1 x_2^4 (w_{,22})^2 dx_1 dx_2 \geq \frac{9D_0}{16} \int\limits_a^b \int\limits_0^1 w^2 dx_1 dx_2 \\ &= \ \gamma \int\limits_{\Pi} w^2 dx_1 dx_2 = \gamma \int\limits_{\Omega} w^2 dw = \gamma \|w\|^2_{L_2(\Omega)},\end{aligned}$$

where

$$\gamma := \frac{9}{16} D_0.$$

In the previous reasonings we have used the following

Lemma 2.2.4 *Let $w(x_1, .)$ be a real function of x_2 for fixed x_1 satisfying the following conditions:*

1) w and $w_{,2}$ are absolutely continuous on $[\delta, 1]$ $\forall \delta \in]0, 1[$;

2) $w, \ w_{,2} = O(1)$ when $x_2 \to 0+$;

3) $x_2^2 w,_{22} \in L_2(]0,1[)$;

4) $w(x_1,1) = w,_2(x_1,1) = 0$.

Then

$$\int_0^1 x_2^4 (w,_{22})^2 dx_2 \geq \frac{9}{16} \int_0^1 w^2 dx_2, \tag{2.2.15}$$

$$\int_0^1 x_2^4 (w,_{22})^2 dx_2 \geq \frac{9}{4} \int_0^1 x_2^2 (w,_2)^2 dx_2. \tag{2.2.16}$$

Proof is similar to the one used in [64] for the case $\delta = 0$ but we have to consider all integrals from δ to 1 and then let δ tend to zero (see also Corollary 1.4.13 and (1.4.52) for $\varkappa = 4$). □

Let H_J be the energetic space (see, e.g., [64]) corresponding to the operator J defined on D_J and acting in $L_2(\Omega)$.

Theorem 2.2.5 *Let $f \in L_2(\Omega)$. If $D_0 > 0$ $(0 \leq \varkappa \leq 4)$, there exists a unique generalized solution of (2.1.1) in the energetic space H_J. If $D_0 = 0$ $(\varkappa > 4)$, and $f(x_1,x_2) = 0$ in $\Omega \backslash \Omega_\delta$ then there exists a unique generalized solution with a finite energy.*

Proof. First we prove that the solution with the finite energy exists for $D_0 \geq 0$ $(\varkappa \geq 0)$ if $f = 0$ in $\Omega \backslash \Omega_\delta$ (the last restriction of f can be weakened [16]). Let $w \in D_J$. Then

$$w|_{\Gamma_2} = 0, \quad \frac{\partial w}{\partial n}|_{\Gamma_2} = 0,$$

and there exist continuous on Γ_2 from inside derivatives $w,_{\alpha\beta}$. We put the domain Ω inside of the rectangle (2.2.14) and complete a definition of the function w assuming it to be equal to zero outside of Ω.

We have

$$\begin{aligned} |(w,f)|^2 &= \left| \int_\Omega w f dx_1 dx_2 \right|^2 = \left| \int_{\Omega_\delta} w f dx_1 dx_2 \right|^2 \\ &\leq \int_{\Omega_\delta} f^2 dx_1 dx_2 \int_{\Omega_\delta} w^2 dx_1 dx_2 = C \int_{\Pi_\delta} w^2 dx_1 dx_2 \quad \forall w \in D_J, \end{aligned} \tag{2.2.17}$$

where

$$C := \int_\Omega f^2 dx_1 dx_2 \geq 0,$$

$$\Pi_\delta := \{(x_1,x_2) \in \Pi \ : \ x_2 > \delta = const > 0\}.$$

Obviously, when $x_2 > 0$

$$\int_a^{x_1} w_{,1}\, dx_1 = w(x_1, x_2) - w(a, x_2) = w(x_1, x_2) \qquad (2.2.18)$$

since $w(a, x_2) = 0$ as $(a, x_2) \in \Pi \backslash \Omega$. According to Cauchy - Bunyakovskii inequality, from (2.2.18) we have

$$w^2 \leq \int_a^{x_1} 1^2 dx_1 \int_a^{x_1} (w_{,1})^2 dx_1 \leq (b-a) \int_a^{b} (w_{,1})^2 dx_1.$$

Integrating both sides in limits $a \leq x_1 \leq b,\ \delta \leq x_2 \leq 1$,

$$\int_{\Pi_\delta} w^2 dx_1 dx_2 \leq (b-a)^2 \int_{\Pi_\delta} (w_{,1})^2 dx_1 dx_2$$

$$\leq (b-a)^4 \int_{\Pi_\delta} (w_{,11})^2 dx_1 dx_2 = (b-a)^4 \int_{\Omega_\delta} (w_{,11})^2 dx_1 dx_2 \qquad (2.2.19)$$

(in the second inequality the first inequality is applied to $w_{,1}$)

$$= \quad (b-a)^4 \int_{\Omega_\delta} \frac{(D_1 - D_3)(w_{,11})^2}{D_1 - D_3} dx_1 dx_2$$

$$\leq \quad \frac{(b-a)^4}{D_\delta} \int_{\Omega_\delta} (D_1 - D_3)(w_{,11})^2 dx_1 dx_2$$

$$\leq \quad \frac{(b-a)^4}{D_\delta} \int_{\Omega_\delta} [(D_1 - D_3)(w_{,11})^2$$

$$+ \quad D_3(w_{,11} + w_{,22})^2 + 4D_4(w_{,12})^2 + (D_2 - D_3)(w_{,22})^2] dx_1 dx_2$$

$$\leq \quad \frac{(b-a)^4}{D_\delta}(Jw, w) = \frac{(b-a)^4}{D_\delta} \|w\|^2_{H_J},$$

$$D_\delta := \min_{\Omega_\delta}(D_1 - D_3).$$

From (2.2.17) and (2.2.19) there follows

$$|(w, f)|^2 \leq \frac{C(b-a)^4}{D_\delta} \|w\|^2_{H_J},$$

i.e., (w, f) can be considered as a linear bounded functional with respect to the energetic norm. But then, according to the well-known theory [64], there exists a solution with a finite energy.

In case $D_0 > 0$ $(0 \leq \varkappa \leq 4)$, moreover, according to the general theory [64], there exists a generalized solution since J is positive definite (see Statement 2.2.3). $\square$

Remark 2.2.6 *In the particular case (2.1.4):*

$$I_{0i}(x_1) = \int_0^l \widetilde{D}_i^{-1}(x_1,\tau)\tau^{-\varkappa}d\tau \leq D_{1i}^{-1}\frac{\tau^{1-\varkappa}}{1-\varkappa}\Big|_0^l = D_{1i}^{-1}\frac{l^{1-\varkappa}}{1-\varkappa} < +\infty \quad \text{if } \varkappa < 1,$$

and, when $\varkappa \geq 1$,

$$I_{0i}(x_1) \geq D_{2i}^{-1} \lim_{\varepsilon\to 0} \int_\varepsilon^l \tau^{-\varkappa}d\tau = +\infty.$$

Similarly,

$$I_{1i}(x_1)\begin{cases} < +\infty & \text{if} \quad \varkappa < 2, \\ = +\infty & \text{if} \quad \varkappa \geq 2, \end{cases} \qquad I_{2i}(x_1)\begin{cases} < +\infty & \text{if} \quad \varkappa < 3, \\ = +\infty & \text{if} \quad \varkappa \geq 3. \end{cases}$$

2.3 On a modification of the Lax-Milgram theorem

This section deals with the following modification of the Lax-Milgram theorem:

Theorem 2.3.1 *Let V be a real Hilbert space, and let $J(u,v)$ be a bilinear form defined on $V \times V$. Let there exist a constant $k > 0$ such that*

$$|J(u,v)| \leq K\|u\|_V\|v\|_V \quad \forall u,v \in V, \tag{2.3.1}$$

and let

$$J(v,v) = 0 \ \Rightarrow \ v = \theta \ \text{ in } V \tag{2.3.2}$$

(θ is the zero element of V). Then for any bounded linear functional F defined on V there exists a unique functional $F_{z_0} \in V^*$ (V^* is the space conjugate to V) such that

$$Fv = F_{z_0}v := \lim_{k\to\infty} J(z_k,v) \quad \forall v \in V, \tag{2.3.3}$$

where

$$z_k := C^{-1}t_k \tag{2.3.4}$$

for any sequence $t_k \in C(V) \subset V$ converging to t_0 uniquely defined by F in view of Riesz theorem. C^{-1} is the inverse operator of the bounded linear operator C:

$$t = Cz \tag{2.3.5}$$

defined in the space V by the relation

$$J(z,v) = (v,t) \quad \forall v \in V \tag{2.3.6}$$

and fixed $z \in V$.

Proof. In view of Riesz theorem, it is possible to express every bounded linear functional F in V in the following form

$$Fv = (v, t_0) \ \ \forall v \in V, \tag{2.3.7}$$

where the element $t_0 \in V$ is uniquely determined by the functional F and $\|t_0\|_V = \|F\|_{V^*}$.

If $z \in V$ is fixed, then the bilinear form $J(z, v)$ represents, obviously, a linear functional in V. This functional is bounded since by (2.3.1)

$$|J(z,v)| \leq \widetilde{k}\|v\|_V, \ \ \widetilde{k} = K\|z\|_V = const > 0. \tag{2.3.8}$$

Then according to the above Riesz theorem, there exists a unique $t \in V$ such that (2.3.6) holds, and also, by virtue of (2.3.6), (2.3.8),

$$\|t\|_V \leq \widetilde{k} = K\|z\|_V. \tag{2.3.9}$$

By the relation (2.3.6) to every $z \in V$ a unique $t \in V$ is assigned. This defines by (2.3.5) an operator C in V. C is, obviously, a linear one, and, in view of (2.3.9), also bounded. The range $L \equiv C(V)$ of this operator C is a certain linear set in V. More precisely, L is a metric space whose elements are the elements of that linear set L with the metric of the space V.

We will prove that the mapping (2.3.5) from V onto L is one-to-one, i.e., $L \sim V$. To this end it is sufficient to prove that to the zero-element of L there corresponds the zero-element of V. Thus, let $\theta = Cz$, i.e., by virtue of (2.3.6),

$$J(z, v) = (v, \theta) = 0 \ \ \forall v \in V. \tag{2.3.10}$$

In particular, for $v = z$, (2.3.10) yields

$$J(z, z) = 0.$$

But then, according to (2.3.2), $z = \theta$. Hence, $\exists C^{-1}$:

$$z = C^{-1}t. \tag{2.3.11}$$

Let $\{t_k\}$ be a fundamental sequence in L, and, thus, also in V. Since V is complete, $\exists t_0 \in V$ such that

$$\lim_{k\to\infty} t_k = t_0, \ \ \text{in } V. \tag{2.3.12}$$

Therefore, complete $\overline{L}$ is a subspace of V.

Now, we will prove that $\overline{L} \equiv V$. The proof will be performed by contradiction. Let $\overline{L} \neq V$. Then there exists an element $w \neq \theta$ in V orthogonal to the subspace $\overline{L}$, so that

$$(w, t) = 0 \tag{2.3.13}$$

holds $\forall t \in \overline{L}$. Since $w \in V$, in view of (2.3.6), a unique $t_* \in L \subset \overline{L}$ exists such that

$$J(w, v) = (v, t_*) \ \ \forall v \in V.$$

In particular, for $v = w$, we have

$$J(w, w) = (w, t_*) = 0$$

because of (2.3.13). Therefore, by virtue of (2.3.2), $w = \theta$ in V, which is in contradiction with assumption $w \neq \theta$. Hence, $\overline{L} \equiv V$.

For any bounded linear functional F in V we have (2.3.7), where $t_0 \in V \equiv \overline{L}$ is uniquely determined by F. For the above $t_0 \in \overline{L}$ there exists a sequence $t_k \in L$ which is convergent to t_0 in V. According to (2.3.11) $\forall t_k \in L \ \exists z_k \in V$ such that

$$J(z_k, v) = (v, t_k) \ \ \forall v \in V. \tag{2.3.14}$$

Functionals $J(z_k, v)$ and (v, t_k) are bounded linear functionals from V^* for fixed k. Now, tending $k \to +\infty$ in (2.3.14), since, in view of (2.3.12), there exists a limit (which is equal to (v, t_0) because of continuity of a scalar product) in the right-hand side, the limit of the left-hand side will also exist, and

$$\lim_{k\to\infty} J(z_k, v) = (v, t_0) \ \ \forall v \in V. \tag{2.3.15}$$

Then, by virtue of an immediate corollary of the Banach-Steinhaus theorem, linear form

$$F_{z_0} : \ v \to \lim_{k\to\infty} J(z_k, v) \tag{2.3.16}$$

is a bounded linear functional on V, which does not depend on the choice of $\{z_k\}$, i.e., of $\{t_k\}$ since for any sequence $t_k \to t_0$ in V, on the right hand side of (2.3.15) we have the same limit (v, t_0).Thus, from (2.3.7), (2.3.15) and (2.3.16) we get (2.3.3). ☐

Remark 2.3.2 *If the sequence $\{z_k\}$, $z_k \in V$, corresponding to $\{t_k\}$, ($t_k \in L$ is from (2.3.12)) is fundamental in V, then because of completness of V $\exists z_0 \in V$ such that*

$$\lim_{k\to\infty} z_k = z_0 \quad \textit{in } V.$$

Therefore, taking into account (2.3.1), we have

$$F_{z_0} v := \lim_{k\to\infty} J(z_k, v) = J(z_0, v)$$

(this is justification of notation F_{z_0}), and from (2.3.3) it follows that there exists a unique $z_0 \in V$ such that

$$Fv = J(z_0, v) \ \ \forall v \in V$$

which coincides with the assertion of the Lax-Milgram theorem (see Section 2.4 below). Therefore, $F_{z_0} \in V^*$ can be identified with $z_0 \in V$. If the sequence $\{z_k\}$ is not fundamental in V (let us note that the numerical sequence $\{J(z_k, v)\}$ is fundamental for fixed $v \in V$), then $F_{z_0} \in V^*$ will be identified with the ideal element z_0 which does not belong to V. Let us denote by V_i the set of the ideal elements z, and by $\widetilde{V} := V \cup V_i$. Let us remind that when $\{z_k\}$ is fundamental, the ideal element $z_0 \in V$.

Under the product λz_0, $\lambda \in R$, $z_0 \in \widetilde{V}$, we understand the ζ_0 identified with the functional

$$F_{\zeta_0} v := \lim_{k\to\infty} J(\lambda z_k, v) = \lim_{k\to\infty} \lambda J(z_k, v) =: \lambda F_{z_0} v.$$

Under the sum $z_0' + z_0''$ of $z_0', z_0'' \in \widetilde{V}$ we understand ζ_0, identified with the functional

$$F_{\zeta_0} v := \lim_{k\to\infty} J(z_0' + z_0'', v) = \lim_{k\to\infty} J(z_k', v) + \lim_{k\to\infty} J(z_k'', v) =: F_{z_0'} v + F_{z_0''} v,$$

where

$$z_k' := C^{-1} t_k', \quad z_k'' := C^{-1} t_k'',$$

$$\lim_{k\to\infty} t_k' = t_0', \quad \lim_{k\to\infty} t_k'' = t_0'' \text{ in } V,$$

t_0' and t_0'' are uniquely defined, in view of Riesz theorem, by bounded linear functionals $F' := (v, t_0')$ and $F'' := (v, t_0'')$, corespondingly. Obviously $\widetilde{V}$ is a linear vector space.

Now, introducing in $\widetilde{V}$ the norm as

$$\|z_0\|_{\widetilde{V}} := \|F_{z_0}\|_{V^*}, \tag{2.3.17}$$

$\widetilde{V}$ will be Banach, and moreover Hilbert space since such is V^*. Indeed,

$$\begin{aligned} \|z_0' + z_0''\|^2_{\widetilde{V}} + \|z_0' - z_0''\|^2_{\widetilde{V}} &:= \|F_{z_0'} + F_{z_0''}\|^2_{V^*} + \|F_{z_0'} - F_{z_0''}\|^2_{V^*} \\ &= 2(\|F_{z_0'}\|^2_{V^*} + \|F_{z_0''}\|^2_{V^*}) \\ &=: 2(\|z_0'\|^2_{\widetilde{V}} + \|z_0''\|^2_{\widetilde{V}}). \end{aligned}$$

Therefore, the scalar product can be defined as

$$(z_0', z_0'')_{\widetilde{V}} := 4^{-1}(\|z_0' + z_0''\|^2_{\widetilde{V}} - \|z_0' - z_0''\|^2_{\widetilde{V}}).$$

The completness of $\widetilde{V}$ is obvious from (2.3.17).

Remark 2.3.3 *If $C^{-1}$is a bounded operator then from (2.3.11), (2.3.12) it follows that $\{z_k\}$ is a fundamental sequence.*

Remark 2.3.4 *If J is coercive, i.e.,*

$$|J(v, v)| \geq c\|v\|^2_V, \quad c = const > 0, \quad \forall v \in V,$$

then C^{-1} is a bounded operator.

Remark 2.3.5 *If (2.3.2) is fulfilled, then either $J(v, v) \geq 0$ $\forall v \in V$ or $J(v, v) \leq 0$ $\forall v \in V$.*

Proof (belongs to S.S. Kharibegashvili). Let us take arbitrary fixed $v_0 \in V$, $v_0 \neq \theta$, from $J(v_0, v_0) \neq 0$ we have either

$$J(v_0, v_0) > 0 \tag{2.3.18}$$

or

$$J(v_0, v_0) < 0 \tag{2.3.19}$$

Let us, now, show that if (2.3.18) is fulfilled, then $\forall v \in V$, $v \neq \theta$ we get $J(v, v) > 0$, while if (2.3.19) is fulfilled, then $J(v, v) < 0$.

Let first $v \in V$, $v \neq \theta$ be not linearly dependent on v_0 then for $\forall t \in]-\infty, +\infty[$, we have

$$0 \neq J(v_0 + tv, v_0 + tv) = J(v_0, v_0) + [J(v_0, v) + J(v, v_0)]t + J(v, v)t^2, \tag{2.3.20}$$

since $v_0 + tv \neq \theta \quad \forall t \in]-\infty, +\infty[$. Therefore, according to the well-known property of the quadratic trinomial

$$J(v_0, v_0) \cdot J(v, v) > \frac{1}{4}[J(v_0, v_0) + J(v, v)]^2 \geq 0. \tag{2.3.21}$$

But if (2.3.18) is fulfilled, then in view of (2.3.21), obviously $J(v, v) > 0$, for arbitrary $v \in V\backslash\{\theta\}$ which is lineary independent of v_0; if (2.3.19) is fulfilled, then from (2.3.21), we get, similarly, $J(v, v) < 0$ for arbitrary $v \in V\backslash\{\theta\}$ which is lineary independent of v_0.

Let, now, $v \in V$, $v \neq \theta$, and be lineary dependent on v_0, i.e., $\exists t_0 \in]-\infty, +\infty[$, such that $v_0 + t_0 v = \theta$. Obviously, such t_0 is unique, i.e., the equation $v_0 + tv = \theta$ with respect to t has a unique solution $t = t_0$. On the other hand, from

$$J(v_0 + tv, v_0 + tv) = 0 \;\Leftrightarrow\; v_0 + tv = \theta$$

it follows that the trinomial (2.3.20) has a unique zero $t = t_0$. This is equivalent with the assertion that the discriminant of the trinomial (2.3.20) is equal to zero:

$$J(v_0, v_0)J(v, v) = \frac{1}{4}[J(v_0, v_0) + J(v, v)]^2 > 0 \tag{2.3.22}$$

(the last inequality is strong since in the left-hand side of the equality $J(v_0, v_0) \neq 0$, $J(v, v) \neq 0$). Finaly, from (2.3.22) follows $J(v, v) > 0$ and $J(v, v) < 0$ when correspondingly (2.3.18) and (2.3.19) are fulfilled. Thus, the remark is proved. □

2.4 Bending in the weighted Sobolev space

Let us consider for the equation (2.1.1) the following inhomogenuoes BCs:
on Γ_2

$$w = g_{12}, \quad \frac{\partial w}{\partial n} = g_{22}, \tag{2.4.1}$$

and on Γ_1 either

$$w = g_{11}, \quad w,_2 = g_{21} \quad \text{if} \quad I_{0i} < +\infty \;\; (0 < \varkappa < 1), \tag{2.4.2}$$

or

$$w,_2 = g_{21}, \quad Q_2^* = h_2 \quad \text{if} \quad I_{0i} < +\infty \;\; (0 < \varkappa < 1), \tag{2.4.3}$$

or

$$w = g_{11},$$

$$(M_2 w) = h_1 \begin{cases} \not\equiv 0 & \text{when } I_{0i} < +\infty \;\; (0 \leq \varkappa < 1), \\ \equiv 0 & \text{when } \; I_{0i} = +\infty \;\; (1 \leq \varkappa < 2) \end{cases} \tag{2.4.4}$$

if $I_{2i} < +\infty \;\; (0 \leq \varkappa < 3)$,

or

$$(M_2 w) = h_1 \begin{cases} \not\equiv 0 \text{ when } I_{0i} < +\infty \;\; (0 \leq \varkappa < 1), \\ \equiv 0 \text{ when } I_{0i} = +\infty \;\; (1 \leq \varkappa < +\infty), \end{cases}$$
$$(Q_2^* w) = h_2 \begin{cases} \not\equiv 0 \text{ when } I_{1i} < +\infty \; (0 \leq \varkappa < 2), \\ \equiv 0 \text{ when } I_{1i} = +\infty \; (2 \leq \varkappa < +\infty) \end{cases} \tag{2.4.5}$$

if $I_{0i} \leq +\infty \; (0 \leq \varkappa < +\infty)$.

Let

$$g_{\alpha\beta}, \; h_\alpha \in L_2(\Gamma_1), \;\; \alpha, \beta = 1, 2, \tag{2.4.6}$$

and g_{11}, g_{21}, g_{12}, g_{22} be traces of a certain given function $u \in W_2^2(\Omega, \widetilde{D})$ (see below (2.4.7), (2.4.10)).

Remark 2.4.1 *Conditions $h_\alpha = 0, \; \alpha = 1, 2,$ in (2.4.4), (2.4.5) are necessary conditions (see Section 1.2) of boundedness of deflection w and $w,_2$ correspondingly when $I_{1i}|_{\Gamma_1} = +\infty$ $(2 \leq \varkappa < +\infty)$, and $I_{0i}|_{\Gamma_1} = +\infty$ $(1 \leq \varkappa < +\infty)$. The demand of boundedness of w and $w,_2$ is natural in the mechanical point of view since we do not consider the case of concentrated shearing forces and moments, when w and $w,_2$ should be, in general, unbounded (see also Remark 1.2.14).*

Remark 2.4.2 *In the particular case (2.1.4), let*

$$g_{12} \in W_2^{\frac{3}{2}}(\Gamma_2), \;\; g_{22} \in W_2^{\frac{1}{2}}(\Gamma_2), \;\; g_{11} \in W_2^{\frac{3-\varkappa}{2}}(\Gamma_1),$$
$$g_{21} \in W_2^{\frac{1-\varkappa}{2}}(\Gamma_1), \quad h_1, \, h_2 \in L_2(\Gamma_1),$$

and g_{11}, g_{21}, g_{12}, g_{22} be traces of a certain given function $u \in W_2^2(\Omega, \widetilde{D})$ (see below (2.4.15), and Remark 2.4.8) and its derivative of the first order (if $\partial\Omega$ is of the class C^3, they exist, on Γ_2 always, and on Γ_1 when $0 < \varkappa < 3$ and $0 < \varkappa < 1$ respectively (see [70]).

Let us note that if there exist such γ_α, $\alpha = 1, 2$, that

$$\lim_{x_2 \to 0+} x_2^{\gamma_1} D_3(w,_{11})^2 = C_1(x_1) > 0,$$

$$\lim_{x_2 \to 0+} x_2^{\gamma_2} D(w,_{22})^2 = C_2(x_1) > 0,$$

$$C_\alpha(x_1) < const \quad \forall (x_1, 0) \in \Gamma_1, \quad \alpha = 1, 2.$$

Then from

$$w \in W_2^2(\Omega, D) \;\; or \;\; W_2^2(\Omega, \widetilde{D})$$

there follows

$$(M_2 w)(x_1, 0) = 0 \;\; for \;\; \varkappa \geq 1 \; (I_{0i} = +\infty, \; i = 2, 3).$$

Indeed, since

$$D_2(w,_{22})^2, \; D_3(w,_{11})^2 \; \in L_1(\Omega),$$

we have

$$D_2(w,_{22})^2 = O(x_2^{-\gamma_2}), \quad D_3(w,_{11})^2 = O(x_2^{-\gamma_1}), \quad x_2 \to 0+, \quad \gamma_\alpha < 1, \quad \alpha = 1, 2.$$

Consequently,

$$D_2^{\frac{1}{2}} w,_{22} = O(x_2^{-\frac{\gamma_2}{2}}), \quad D_3^{\frac{1}{2}} w,_{11} = O(x_2^{-\frac{\gamma_1}{2}}), \quad x_2 \to 0+, \quad \gamma_\alpha < 1,, \quad \alpha = 1, 2,$$

and, according to (2.1.4), (2.1.6),

$$|(M_2 w)| = | - D_2 w,_{22} - D_3 w,_{11} | \leq \left(D_2^{\frac{1}{2}} + D_3^{\frac{1}{2}} \right) O(x_2^{-\frac{\gamma}{2}}) = O(x_2^{\frac{\varkappa - \gamma}{2}}), \quad x_2 \to 0+,$$
$$\gamma := \max\{\gamma_1, \gamma_2\} < 1.$$

Whence,

$$(M_2 w)(x_1, 0) = 0 \;\; when \; (x_1, 0) \in \Gamma_1, \; \varkappa > \gamma < 1,$$

$$i.e., \;\; \varkappa \geq \; 1 \; (I_{0i} = +\infty, \; i = 2, 3).$$

This is one more argument for the condition (2.4.5).

Let further (compare with Definition 2.5.2 below)

$$W_2^2(\Omega, \, D) \tag{2.4.7}$$

be the set of all measurable functions $u = u(x_1, x_2)$ defined on Ω which have on Ω generalized derivatives $D_{x_1, x_2}^{(\alpha_1, \alpha_2)} u$ for $\alpha_1 + \alpha_2 \leq 2$, $\alpha_1, \alpha_2 \in \{0, 1, 2\}$ such that

$$\int_\Omega |D_{x_1, x_2}^{(\alpha_1, \alpha_2)} u|^2 \rho_{\alpha_1, \alpha_2}(x_1, x_2) d\Omega < +\infty \tag{2.4.8}$$

for $\rho_{0,0} := 1$, $\rho_{2,0} := D_1(x_1, x_2)$, $\rho_{1,1} := D_4(x_1, x_2)$, $\rho_{0,2} := D_2(x_1, x_2)$. D_i, $i = 1, 2, 3, 4$, are bounded measurable on Ω functions satisfying (2.1.2), (2.1.3). Therefore, since $D_\alpha \geq D_3$, $\alpha = 1, 2$, in $\bar{\Omega}$,

$$\int_\Omega D_3(u,_{\alpha\underline{\alpha}})^2 d\Omega \leq \int_\Omega D_\alpha(u,_{\alpha\underline{\alpha}})^2 d\Omega < +\infty, \quad \alpha = 1, 2, \tag{2.4.9}$$

$$\int_\Omega D_3(u,_{11} + u,_{22})^2 d\Omega \leq \int_\Omega D_1(u,_{11})^2 d\Omega + 2\int_\Omega D_1^{\frac{1}{2}} u,_{11} \cdot D_2^{\frac{1}{2}} u,_{22}\, d\Omega$$

$$+ \int_\Omega D_2(u,_{22})^2 d\Omega \leq \left\{ \left[\int_\Omega D_1(u,_{11})^2 d\Omega \right]^{\frac{1}{2}} + \left[\int_\Omega D_2(u,_{22})^2 d\Omega \right]^{\frac{1}{2}} \right\}^2 < +\infty.$$

Let

$$D := \{\rho_{0,0},\ \rho_{2,0}, \rho_{1,1}, \rho_{0,2}\},$$

and

$$\widetilde{D} := D \cup \{\rho_{0,1} := x_2^2\}.$$

Then, in view of (2.4.7), (2.4.8), the sense of the notation $W_2^2(\Omega, \widetilde{D})$ is clear. Obviously,

$$W_2^2(\Omega, \widetilde{D}) \subset W_2^2(\Omega, D). \tag{2.4.10}$$

From (2.1.3), it is clear that if $D_i \in C(\overline{\Omega})$, then

$$\rho_{\alpha_1,\alpha_2}^{-1} \in L_1^{loc}(\Omega).$$

Hence, according to [60] $W_2^2(\Omega, D)$ and $W_2^2(\Omega, \widetilde{D})$, by virtue of (2.4.8), (2.4.9), will be Banach spaces under the norms

$$\begin{aligned} ||u||^2_{W_2^2(\Omega,D)} &:= \int_\Omega [u^2 + D_3(u,_{11} + u,_{22})^2 \\ +(D_1 - D_3)(u,_{11})^2 &+ 4D_4(u,_{12})^2 + (D_2 - D_3)(u,_{22})^2] d\Omega, \end{aligned} \tag{2.4.11}$$

$$||u||^2_{W_2^2(\Omega,\widetilde{D})} := ||u||^2_{W_2^2(\Omega,D)} + \int_\Omega x_2^2 (u,_2)^2 d\Omega, \tag{2.4.12}$$

respectively, and moreover, Hilbert spaces under the scalar products

$$\begin{aligned} (u, v)_{W_2^2(\Omega,D)} &:= \int_\Omega [uv + D_3(u,_{11} + u,_{22})(v,_{11} + v,_{22}) + (D_1 - D_3)u,_{11} v,_{11} \\ &+ 4D_4 u,_{12} v,_{12} + (D_2 - D_3)u,_{22} v,_{22}] d\Omega, \end{aligned}$$

$$(u, v)_{W_2^2(\Omega,\widetilde{D})} := (u, v)_{W_2^2(\Omega,D)} + \int_\Omega x_2^2 u,_2 v,_2\, d\Omega,$$

respectively. Let further $f \in L_2(\Omega)$, and

$$V := \overset{0}{W}{}_2^2(\Omega, \widetilde{D}) = \overline{C_0^\infty(\Omega)} \quad \text{in the norm of} \quad W_2^2(\Omega, \widetilde{D}). \tag{2.4.13}$$

Since $\rho_{\alpha_1,\alpha_2} \in L_1^{loc}(\Omega)$ we have $C_0^\infty(\Omega) \subset W_2^2(\Omega, \widetilde{D})$, and (2.4.13) has the meaning. In particular case (2.1.4), we can take as V also

$$V := \{v \in W_2^2(\Omega, \widetilde{D}) : v|_{\Gamma_2} = 0, \quad \frac{\partial v}{\partial n}\Big|_{\Gamma_2} = 0, \quad \text{and either}$$
$$v|_{\Gamma_1} = 0, \ \ v,_2|_{\Gamma_1} = 0 \quad \text{by BCs (2.4.2) or} \quad v,_2|_{\Gamma_1} = 0 \quad \text{by BCs (2.4.3)} \tag{2.4.14}$$
$$\text{or} \quad v|_{\Gamma_1} = 0 \quad \text{by BCs (2.4.4) in the sense of traces}\}.$$

In case (2.1.4) we could introduce weights and norm as follows:

$$\rho_{0,0} := 1, \ \ \rho_{2,0} \equiv \rho_{1,1} \equiv \rho_{0,2} := x_2^{\varkappa},$$

$$||u||^2_{W_2^2(\Omega, x_2^{\varkappa})} := \int_\Omega \{u^2 + x_2^{\varkappa}[(u,_{11} + u,_{22})^2 + (u,_{11})^2 + (u,_{12})^2 + (u,_{22})^2]\} d\Omega. \tag{2.4.15}$$

It is obvious, in view of (2.1.4), that the latter norm and (2.4.11) are equivalent in $W_2^2(\Omega, D)$. But we prefer (2.4.11) since the above resonings are valid for the more general case (2.1.3).

Definition 2.4.3 *A function $w \in W_2^2(\Omega, \widetilde{D})$ will be called a weak solution of the BVP (2.1.1), (2.1.3), (2.4.1)-(2.4.5) in the space $W_2^2(\Omega, \widetilde{D})$ if it satisfies the following conditions:*

$$w - u \in V, \tag{2.4.16}$$

and $\forall v \in V$

$$J(w, v) := \int_\Omega B(w, v) d\Omega = \int_\Omega v f d\Omega, \tag{2.4.17}$$

where (defined in (2.2.11))

$$\begin{aligned} B(v, w) &:= D_3(w,_{11} + w,_{22})(v,_{11} + v,_{22}) + \\ &+ (D_1 - D_3) w,_{11} v,_{11} + 4D_4 w,_{12} v,_{12} + (D_2 - D_3) w,_{22} v,_{22}, \end{aligned} \tag{2.4.18}$$

or corespondingly, for the particular case (2.1.4),

$$J(w, v) := \int_\Omega B(w, v) d\Omega = \int_\Omega v f d\Omega + \gamma_2 \int_{\Gamma_1} h_2 v dx_1 - \gamma_1 \int_{\Gamma_1} h_1 v,_2 dx_1, \tag{2.4.19}$$

where $\gamma_1 = \gamma_2 = 0$ *by BCs (2.4.2);* $\gamma_1 = 0, \ \gamma_2 = 1$ *by BCs (2.4.3);* $\gamma_1 = 1, \ \gamma_2 = 0$ *by BCs (2.4.4); and by BCs (2.4.5)* $\gamma_1 = 1$ *when* $0 \le \varkappa < 1$, *and* $\gamma_1 = 0$ *when* $1 \le \varkappa < +\infty$; $\gamma_2 = 1$ *when* $0 \le \varkappa < 2$, *and* $\gamma_2 = 0$ *when* $2 \le \varkappa < +\infty$.

Remark 2.4.4 *The BCs (2.4.1), (2.4.2), the first ones of (2.4.3), (2.4.4) and (2.4.5), the second ones of (2.4.3), (2.4.4) are specified in (2.4.16) and (2.4.17) ((2.4.19)), correspondingly.*

Remark 2.4.5 *Obviously, if the solution of the above problem exists in the classical sense then (2.4.16), (2.4.17) ((2.4.19)) will be fulfilled.*

Theorem 2.4.6 *In case (2.1.3), if $D_0 > 0$, under other above conditions there exists the a unique weak solution in $W_2^2(\Omega, \widetilde{D})$ of the BVP (2.1.1), (2.1.3), (2.4.1)-(2.4.5). This solution is such that*

$$||w||_{W_2^2(\Omega,D)} \leq \widetilde{C}\left[||f||_{L_2(\Omega)} + ||u||_{W_2^2(\Omega,D)}\right], \tag{2.4.20}$$

where constant $\widetilde{C}$ is independent of f and u.

Theorem 2.4.7 *In case (2.1.4), if $0 < \varkappa \leq 4$, under other above conditions there exists a unique weak solution in $W_2^2(\Omega, \widetilde{D})$ of the BVP (2.1.1), (2.1.4), (2.4.1)-(2.4.5). This solution is such that*

$$||w||_{W_2^2(\Omega,D)} \leq \widetilde{C}\left[||f||_{L_2(\Omega)} + ||u||_{W_2^2(\Omega,D)} + \gamma_1||h_1||_{L_2(\Gamma_1)} + \gamma_2||h_2||_{L_2(\Gamma_1)}\right], \tag{2.4.21}$$

where constant $\widetilde{C}$ is independent of $f, u, h_1,$ and h_2.

Proof of the Theorems 2.4.6 and 2.4.7. First of all, let us prove that V is a subspace of $W_2^2(\Omega, \widetilde{D})$. In case (2.1.3) it is obvious. In case (2.1.4) to this end we have to show its completeness. Because of linearity of the trace operators and operators in (2.4.1)-(2.4.4), obviously, V is a lineal. Since $u \in W_2^2(\Omega, \widetilde{D})$ has the traces [70]

$$\begin{aligned} u\big|_{\Gamma_1} &\in W_2^{\frac{3-\varkappa}{2}}(\Gamma_1) \ \text{ for } \ 0 \leq \varkappa < 3, \\ u\big|_{\Gamma_2} &\in W_2^{\frac{3}{2}}(\Gamma_2) \quad \text{for } \ 0 \leq \varkappa < +\infty, \\ u,_2\big|_{\Gamma_1} &\in W_2^{\frac{1-\varkappa}{2}}(\Gamma_1) \ \text{ for } \ 0 \leq \varkappa < 1, \\ \frac{\partial u}{\partial n}\big|_{\Gamma_2} &\in W_2^{1/2}(\Gamma_2) \ \text{ for } \ 0 \leq \varkappa < +\infty, \end{aligned}$$

then $\exists C_1 =$const> 0 such that

$$||u||_{W_2^{\frac{3-\varkappa}{2}}(\Gamma_1)} \leq C_1||u||_{W_2^2(\Omega,D)} \quad \text{for } \ 0 \leq \varkappa < 3, \tag{2.4.22}$$

$$||u||_{W_2^{\frac{3}{2}}(\Gamma_2)} \leq C_1||u||_{W_2^2(\Omega,D)} \quad \text{for } \ 0 \leq \varkappa < +\infty, \tag{2.4.23}$$

$$||u,_2||_{W_2^{\frac{1-\varkappa}{2}}(\Gamma_1)} \leq C_1||u||_{W_2^2(\Omega,D)} \quad \text{for } \ 0 \leq \varkappa < 1, \tag{2.4.24}$$

$$\left|\left|\frac{\partial u}{\partial n}\right|\right|_{W_2^{\frac{1}{2}}((\Gamma_2)} \leq C_1||u||_{W_2^2(\Omega,D)} \quad \text{for } \ 0 \leq \varkappa < +\infty. \tag{2.4.25}$$

Let $v_m \in V$ be a fundamental sequence. It will be also a fundamental sequence in $W_2^2(\Omega, \widetilde{D})$. But the latter is a complete set, i.e., $\exists v \in W_2^2(\Omega, \widetilde{D})$ such that

$$||v_m - v||_{W_2^2(\Omega,D)} \underset{m \to +\infty}{\longrightarrow} 0.$$

Then, by virtue of (2.4.22)-(2.4.25), respectively,

$$||v_m - v||_{W_2^{\frac{3-\varkappa}{2}}(\Gamma_1)} \leq C_1 ||v_m - v||_{W_2^2(\Omega,D)} \text{ for } 0 \leq \varkappa < 3,$$

$$||v_m - v||_{W_2^{\frac{3}{2}}(\Gamma_2)} \leq C_1 ||v_m - v||_{W_2^2(\Omega,D)} \text{ for } 0 \leq \varkappa < +\infty,$$

$$||v_{m,2} - v_{,2}||_{W_2^{\frac{1-\varkappa}{2}}(\Gamma_1)} \leq C_1 ||v_m - v||_{W_2^2(\Omega,D)} \text{ for } 0 \leq \varkappa < 1,$$

$$\left|\left|\frac{\partial v_m}{\partial n} - \frac{\partial v}{\partial n}\right|\right|_{W_2^{\frac{1}{2}}(\Gamma_2)} \leq C_1 ||v_m - v||_{W_2^2(\Omega,D)} \text{ for } 0 \leq \varkappa < +\infty.$$

Therefore,

$$||v_m - v||_{W_2^{\frac{3-\varkappa}{2}}(\Gamma_1)} \underset{m \to +\infty}{\longrightarrow} 0 \quad \text{for } 0 \leq \varkappa < 3, \tag{2.4.26}$$

$$||v_m - v||_{W_2^{\frac{3}{2}}(\Gamma_2)} \underset{m \to +\infty}{\longrightarrow} 0 \quad \text{for } 0 \leq \varkappa < +\infty, \tag{2.4.27}$$

$$||v_{m,2} - v_{,2}||_{W_2^{\frac{1-\varkappa}{2}}(\Gamma_1)} \underset{m \to +\infty}{\longrightarrow} 0 \quad \text{for } 0 \leq \varkappa < 1, \tag{2.4.28}$$

$$\left|\left|\frac{\partial v_m}{\partial n} - \frac{\partial v}{\partial n}\right|\right|_{W_2^{\frac{1}{2}}(\Gamma_2)} \underset{m \to +\infty}{\longrightarrow} 0 \quad \text{for } 0 \leq \varkappa < +\infty. \tag{2.4.29}$$

But since

$$v_m|_{\Gamma_1} = 0 \quad \text{for } 0 \leq \varkappa < 3, \tag{2.4.30}$$

$$v_m|_{\Gamma_2} = 0 \quad \text{for } 0 \leq \varkappa < +\infty, \tag{2.4.31}$$

$$v_{m,2}|_{\Gamma_1} = 0 \quad \text{for } 0 \leq \varkappa < 1, \tag{2.4.32}$$

$$\left.\frac{\partial v_m}{\partial n}\right|_{\Gamma_2} = 0 \quad \text{for } 0 \leq \varkappa < +\infty, \tag{2.4.33}$$

from (2.4.30) follows

$$||v_m||_{W_2^{\frac{3-\varkappa}{2}}(\Gamma_1)} = 0.$$

Then, taking into account (2.4.26),

$$0 \leq ||v||_{W_2^{\frac{3-\varkappa}{2}}(\Gamma_1)} = \left|\, ||v_m||_{W_2^{\frac{3-\varkappa}{2}}(\Gamma_1)} - ||v||_{W_2^{\frac{3-\varkappa}{2}}(\Gamma_1)} \right| \leq ||v_m - v||_{W_2^{\frac{3-\varkappa}{2}}(\Gamma_1)} \underset{m \to +\infty}{\longrightarrow} 0$$

i.e., almost everywhere (a.e.)

$$v|_{\Gamma_1} = 0 \quad \text{for } 0 \leq \varkappa < 3.$$

Similarly, in view of (2.4.27)-(2.4.29), (2.4.31)-(2.4.33), we have a.e.

$$v|_{\Gamma_2} = 0 \quad \text{for} \ \ 0 \leq \varkappa < +\infty,$$
$$v,_2|_{\Gamma_1} = 0 \quad \text{for} \ \ 0 \leq \varkappa < 1,$$
$$\frac{\partial v}{\partial n}\Big|_{\Gamma_2} = 0 \ \ \text{for} \ \ 0 \leq \varkappa < +\infty.$$

Thus V is complete, i.e., it is a Hilbert space and a subspace of $W_2^2(\Omega, \widetilde{D})$.

Further, the proof of Theorems 2.4.6 and 2.4.7 will be realized by means of

The Lax-Milgram theorem. *Let V be a real Hilbert space and let $J(w,v)$ be a bilinear form defined on $V \times V$. Let this form be continuous, i.e., let there exist a constant $K > 0$ such that*

$$|J(w,v)| \leq K\|w\|_V \|v\|_V \tag{2.4.34}$$

holds $\forall w, v \in V$ and V-elliptic, i.e., let there exist a constant $\alpha > 0$ such that

$$J(w,w) \geq \alpha \|w\|_V^2 \tag{2.4.35}$$

holds $\forall w \in V$. Further let F be a bounded linear functional from V^* dual of V. Then there exists one and only one element $z \in V$ such that

$$J(z,v) =< F, v >\equiv Fv \quad \forall v \in V \tag{2.4.36}$$

and

$$\|z\|_V \leq \alpha^{-1}\|F\|_{V^*}. \tag{2.4.37}$$

Obviously, for the bilinear form (2.4.17), in view of (2.4.18),

$$|J(w,v)| \leq \int_\Omega (D_1 - D_3)^{\frac{1}{2}}|w,_{11}| \cdot (D_1 - D_3)^{\frac{1}{2}}|v,_{11}|d\Omega$$
$$+ \int_\Omega (D_2 - D_3)^{\frac{1}{2}}|w,_{22}| \cdot (D_2 - D_3)^{\frac{1}{2}}|v,_{22}|d\Omega$$
$$+ \int_\Omega D_3^{\frac{1}{2}}|w,_{11} + w,_{22}| \cdot D_3^{\frac{1}{2}}|v,_{22} + v,_{11}|d\Omega$$
$$+4 \int_\Omega D_4^{\frac{1}{2}}|w,_{12}| \cdot D_4^{\frac{1}{2}}|v,_{12}|d\Omega$$
$$\leq \left[\int_\Omega (D_1 - D_3)(w,_{11})^2 d\Omega\right]^{\frac{1}{2}} \left[\int_\Omega (D_1 - D_3)(v,_{11})^2 d\Omega\right]^{\frac{1}{2}}$$
$$+ \left[\int_\Omega (D_2 - D_3)(w,_{22})^2 d\Omega\right]^{\frac{1}{2}} \cdot \left[\int_\Omega (D_2 - D_3)(v,_{22})^2 d\Omega\right]^{\frac{1}{2}}$$

$$+\left[\int_\Omega D_3(w,_{11}+w,_{22})^2 d\Omega\right]^{\frac{1}{2}}\cdot\left[\int_\Omega D_3(v,_{11}+v,_{22})^2 d\Omega\right]^{\frac{1}{2}}$$

$$+4\left[\int_\Omega D_4(w,_{12})^2 d\Omega\right]^{\frac{1}{2}}\cdot\left[\int_\Omega D_4(w,_{12})^2 d\Omega\right]^{\frac{1}{2}}\le 7||w||_{W_2^2(\Omega,D)}||v||_{W_2^2(\Omega,D)} \quad (2.4.38)$$

and, in particular (see Remark 2.4.8 below),

$$|J(w,v)|\le 7||w||_V||v||_V \quad \forall w,v\in V. \quad (2.4.39)$$

Hence, (2.4.34) is fulfilled.

Taking into account that $(D_2-D_3)(v,_{22})^2\in L_1(\Omega)$, because of

$$x_2^4\le\frac{D_2-D_3}{D_0}, \quad (2.4.40)$$

obviously,

$$x_2^4(v,_{22})^2\in L_1(\Omega).$$

Without loss of generality, we can suppose that Ω lies in Π (see (2.2.14)), and let $v\in V$ and $v\equiv 0$ in $R_+^2\backslash\Omega$. Then for fixed x_1

$$v(x_1,\cdot)\in W_2^2(]0,1[,\ x_2^4),\ ||v||^2_{W_2^2(]0,1[,x_2^4)}:=\int_0^1[v^2+x_2^4(v,_{22})^2]dx_2,$$

$$v(x_1,1)=0,\quad v,_2(x_1,1)=0,$$

and if we suppose that $(D_2-D_3)^{\frac{1}{2}}(v,_2)^2\in L_1(\Omega)$, i.e., $x_2^2(v,_2)^2\in L_1(\Omega)$ since $x_2^2\le\frac{(D_2-D_3)^{\frac{1}{2}}}{D_0^{\frac{1}{2}}}$ because of (2.4.40), it is easy to show (see below Lemma 2.4.9) that the inequalities (2.2.15), (2.2.16) are valid for such functions $v\in W_2^2(\Omega,\widetilde{D}),\ \Omega\subset\Pi$.

Remark 2.4.8 *In viev of (2.2.16), (2.4.40), when $D_0>0$ the norms (2.4.12), and (2.4.11) are equivalent in $W_2^2(\Omega,\widetilde{D}),\ \Omega\subset\Pi$. Consequently, (2.4.38) holds also for $W_2^2(\Omega,\widetilde{D})$.*

Lemma 2.4.9 *If $v\in W_2^2(]0,1[,x_2^4)$, $x_2^2(v,_2)^2\in L_1(]0,1[)$ and*

$$v(x_1,1)=0,\quad v,_2(x_1,1)=0,$$

then (2.2.15), (2.2.16) are valid, i.e.,

$$\int_0^1 x_2^4(v,_{22})^2dx_2\ge\frac{9}{16}\int_0^1 v^2dx_2,$$

$$\int_0^1 x_2^4 (v,_{22})^2 dx_2 \geq \frac{9}{4} \int_0^1 x_2^2 (v,_2)^2 dx_2.$$

Proof. In the case under consideration $v(x_1, \cdot) \in W_2^2(]c, 1[)$ for $\forall [c, 1] \subset]0, 1]$. Therefore, $v(x_{1,} \cdot)$ and $v,_2 (x_{1,} \cdot)$ are absolutely continuous on $[\varepsilon, 1]$ for arbitrarily small $\varepsilon =$const>0. Now, we have to repeat proof of Lemma 2.2.4 considering all integrals in limits $\varepsilon \leq x_2 \leq 1$, and then tend ε to $0+$, taking into account that from square summability of $v(x_{1,} \cdot)$ and $x_2 v,_2 (x_{1,} \cdot)$ there follow

$$\lim_{x_2 \to 0+} x_2 v^2(x_1, x_2) = 0 \ \ and \ \ \lim_{x_2 \to 0+} x_2^3 [v,_2 (x_1, x_2)]^2 = 0,$$

respectively. Otherwise if we assume $\lim\limits_{x_2 \to 0+} x_2 v^2(x_1, x_2) = c_0(x_1) > 0$, $\lim\limits_{x_2 \to 0+} x_2^3 [v,_2 (x_1, x_2)]^2 = c_1(x_1) > 0$ then in some right neighbourhood of point $(x_1, 0)$

$$v^2(x_1, x_2) > \frac{c_0(x_1)}{2x_2}, \ \ x_2^2 [v,_2 (x_1, x_2)]^2 > \frac{c_1(x_1)}{2x_2}.$$

But this is a contradiction since on the left-hand sides we have integrable functions while on the right-hand sides we have nonintegrable functions. □

Let us note that, on the other hand (see Section 1.4),

$$W_2^2(]0, 1[, \ x_2^4) \equiv W^{2,2}(]0, 1[, \ x_2^4) \equiv \widetilde{W}^{2,2}(]0, 1[, \ x_2^4)$$

and Lemma 2.4.9 immediately follows from (1.4.52) by $\varkappa = 4$, $l = 1$ if we tend $\varepsilon \to 0+$ and from Corollary 1.4.13 by $l = 1$.

In view of (2.2.15), as $0 < D_0 \leq \frac{D_2 - D_3}{x_2^4}$ (see (2.4.40)), for $v \in W_2^2(\Omega, \widetilde{D})$, we have

$$\int_\Omega v^2(x_1, x_2) d\Omega = \int_\Pi v^2(x_1, x_2) dx_1, dx_2 = \int_a^b dx_1 \int_0^1 v^2 dx_2 \leq$$

$$\leq \frac{16}{9} \int_a^b dx_1 \int_0^1 x_2^4 (v,_{22})^2 dx_2 \leq \frac{16}{9D_0} \int_a^b dx_1 \int_0^1 D_0 x_2^{\varkappa} (v,_{22})^2 dx_2 \leq$$

$$\leq \frac{16}{9D_0} \int_\Omega (D_2 - D_3)(v,_{22})^2 d\Omega.$$

Similarly, by virtue of (2.2.16),

$$\int_\Omega x_2^2 (v,_2)^2 d\Omega \leq \frac{4}{9D_0} \int_\Omega (D_2 - D_3)(v,_{22})^2 d\Omega.$$

Hence, according to (2.4.14), (2.4.12), (2.4.11), (2.2.12),

$$||v||_V^2 := \int\limits_{\Omega} [v^2 + x_2^2(v_{,2})^2 + D_3(v,_{11} + v,_{22})^2 + (D_1 - D_3)(v_{,11})^2 +$$

$$+4D_4(v,_{12})^2 + (D_2 - D_3)(v_{,22})^2] d\Omega \le \tag{2.4.41}$$

$$\le \frac{20}{9D_0} \int\limits_{\Omega} (D_2 - D_3)(v_{,22})^2 d\Omega + J(v,v) \le \overset{*}{C} J(v,v),$$

where

$$\overset{*}{C} := 1 + \frac{20}{9D_0}.$$

(2.4.41) means V-ellipticity of the bilinear form J. Thus, (2.4.35) is also fulfilled.

Now, let us consider the following functional

$$Fv := (v, f) - J(u, v) + \gamma_2 \int\limits_{\Gamma_1} v h_2 dx_1 - \gamma_1 \int\limits_{\Gamma_1} v,_2 h_1 dx_1, \quad v \in V \tag{2.4.42}$$

(For case (2.1.3) we have to take $\gamma_1 = \gamma_2 = 0$).

Further,

$$|(v, f)| \le ||v||_{L_2(\Omega)} ||f||_{L_2(\Omega)} \le ||v||_V ||f||_{L_2(\Omega)}, \tag{2.4.43}$$

and, since in case (2.1.4) traces belonging to $W_2^{\frac{3-\varkappa}{2}}(\Gamma_1)$, $0 \le \varkappa < 3$; $W_2^{\frac{1-\varkappa}{2}}(\Gamma_1)$, $0 \le \varkappa < 1$; are also traces belonging to $L_2(\Gamma_1)$,

$$\Big| \int\limits_{\Gamma_1} v\, h_2 dx_1 \Big| \le ||v||_{L_2(\Gamma_1)} ||h_2||_{L_2(\Gamma_1)} \le C ||v||_V ||h_2||_{L_2(\Gamma_1)}, \tag{2.4.44}$$

$$C = const > 0, \quad 0 \le \varkappa < 3,$$

$$\Big| \int\limits_{\Gamma_1} v_{,2}\; h_1 dx_1 \Big| \le ||v_{,2}||_{L_2(\Gamma_1)} ||h_1||_{L_2(\Gamma_1)} \le C ||v||_V ||h_1||_{L_2(\Gamma_1)}, \tag{2.4.45}$$

$$0 \le \varkappa < 1.$$

After substitution of (2.4.43), (2.4.38), (2.4.44), (2.4.45) in (2.4.42), we obtain

$$|Fv| \le [||f||_{L_2(\Omega)} + 7||u||_{W_2^2(\Omega, D)} + C(\gamma_2 ||h_2||_{L_2(\Gamma_1)} + \gamma_1 ||h_1||_{L_2(\Gamma_1)})] ||v||_V. \tag{2.4.46}$$

Let us note that by demonstration of boundedness of the functional F defined by (2.4.42), we did not use that $D_0 > 0$ $(0 \le \varkappa \le 4)$, i.e., the assertion is true for $D_0 \ge 0$ $(0 \le \varkappa < +\infty)$. Therefore, the linear functional (2.4.42) is bounded in V. So, in view of (2.4.39), (2.4.41), (2.4.46), according to the Lax-Milgram theorem there exists a unique $z \in V$ such that, by virtue of (2.4.36), we have

$$J(z,v) = Fv := (v,f) - J(u,v) + \gamma_2 \int\limits_{\Gamma_1} vh_2 dx_1 - \gamma_1 \int\limits_{\Gamma_1} v,_2 h_1 dx_1 \ \forall v \in V,$$

i.e.,

$$J(w,v) = (v,f) + \gamma_2 \int\limits_{\Gamma_1} vh_2 dx_1 - \gamma_1 \int\limits_{\Gamma_1} v,_2 h_1 dx_1 \ \ \forall v \in V, \tag{2.4.47}$$

where

$$w := u + z \in W_2^2(\Omega, \widetilde{D}). \tag{2.4.48}$$

So,

$$w - u = z \in V,$$

and (2.4.16) is fulfilled. (2.4.47) coincides with (2.4.19) (in case (2.1.3) with (2.4.17)). Thus, the existence of a unique weak solution $w \in W_2^2(\Omega, \widetilde{D})$ of the BVP (2.1.1), (2.1.4) [or (2.1.3)], (2.4.1)-(2.4.5) has been proved.

From (2.4.46) it follows that

$$||F||_{V^*} \le ||f||_{L_2(\Omega)} + 7||u||_{W_2^2(\Omega,D)} + C(\gamma_2 ||h_2||_{L_2(\Gamma_1)} + \gamma_1 ||h_1||_{L_2(\Gamma_1)}). \tag{2.4.49}$$

By virtue of (2.4.48), (2.4.37), (2.4.49),

$$\begin{aligned} &||w||_{W_2^2(\Omega,D)} \le ||u||_{W_2^2(\Omega,D)} + ||z||_V \le ||u||_{W_2^2(\Omega,D)} \\ &+\alpha^{-1}[||f||_{L_2(\Omega)} + 7||u||_{W_2^2(\Omega,D)} + C(\gamma_2||h_2||_{L_2(\Gamma_1)} + \gamma_1||h_1||_{L_2(\Gamma_1)})] \\ &\le \widetilde{C}[||f||_{L_2(\Omega)} + ||u||_{W_2^2(\Omega,D)} + \gamma_1||h_1||_{L_2(\Gamma_1)} + \gamma_2||h_2||_{L_2(\Gamma_1)}], \end{aligned}$$

where

$$\widetilde{C} := max\{7\alpha^{-1} + 1, \ \ \alpha^{-1}C\},$$

i.e., (2.4.20), and (2.4.21) are valid in cases (2.1.3) and (2.1.4), respectively. □

Remark 2.4.10 *Instead of V defined by (2.4.14), we could consider the space*

$$\overset{0}{W}{}_2^2(\Omega, \widetilde{D}).$$

Then taking into account that (2.2.15) is, obviously, valid for $v \in C_0^\infty(\,]0.1[\,)$, the condition (2.4.41) will be fulfilled for $v \in C_0^\infty(\Omega)$ and, hence, for $v \in \overset{0}{W}{}_2^2(\Omega, D)$. The condition (2.4.39) will be also realized on $\overset{0}{W}{}_2^2(\Omega, \widetilde{D})$ which is a subspace of $W_2^2(\Omega, \widetilde{D})$. (2.4.46) (where $\gamma_1 = \gamma_2 = 0$) will be also true for $v \in \overset{0}{W}{}_2^2(\Omega, \widetilde{D})$. Therefore, Theorem 2.4.7 will be valid if in the Definition 2.4.3 the space V is replaced by the space $\overset{0}{W}{}_2^2(\Omega, \widetilde{D}) \subset V$, and (2.4.19) is replaced by (2.4.17).

Let, now, $D_0 = 0$ $(\varkappa > 4)$.

In this case only the BVP (2.1.1), (2.1.3), (2.4.1), (2.4.5), can be correctly posed. Let

$$V \equiv \overset{0}{W}{}_2^2(\Omega, D) := \overline{C_0^\infty(\Omega)}$$

with the norm of $W_2^2(\Omega, D)$.

Definition 2.4.11 *Let $u \in W_2^2(\Omega, D)$ be given, and*

$$Fv := (v, f) - J(u, v), \quad v \in V, \tag{2.4.50}$$

where J is defined by (2.4.17). $z_0 + u$, where $z_0 \in \widetilde{V}$ is identified with $F_{z_0} \in V^*$ (see a modification of the Lax-Milgram theorem in Section 2.3), will be called the ideal solution of the BVP (2.1.1), (2.1.3), (2.4.1), (2.4.5) if it satisfies the following condition:

$$F_{z_0}v := \lim_{k\to\infty} J(z_k, v) = \int\limits_\Omega f v d\Omega - J(u, v) \ \ \forall v \in V \equiv \overset{0}{W}{}_2^2(\Omega, D). \tag{2.4.51}$$

Theorem 2.4.12 *There exists a unique ideal solution of the BVP (2.1.1), (2.1.3), (2.4.1), (2.4.5).*

Proof. Obviously,

$$|Fv| \le ||v||_{L_2(\Omega)}||f||_{L_2(\Omega)} + 7||u||_{W_2^2(\Omega,D)} \cdot ||v||_V$$

$$\le ||f||_{L_2(\Omega)}||v||_V + 7||u||_{W_2^2(\Omega,D)} \cdot ||v||_V,$$

since (2.4.38) is all the more fulfilled for $v \in V \equiv \overset{0}{W}{}_2^2(\Omega, D) \subset W_2^2(\Omega, D)$. Hence, F defined by (2.4.50) is a bounded linear functional on V. In view of (2.4.39), which is all the more valid for $V \equiv \overset{0}{W}{}_2^2(\Omega, D)$, (2.3.1) holds.

From $v \in V$ and

$$J(v, v) = 0$$

follows

$$v = k_1x_1 + k_2x_2 + k_3, \quad k_i = \text{const}, \quad i = 1, 2, 3, \quad \text{a.e. in} \quad \Omega,$$

since from (2.4.17), (2.4.18) we have

$$J(v, v) = \int\limits_\Omega \Big[D_1(v_{,11})^2 + D_2(v_{,22})^2 + 2D_3v_{,11} \cdot v_{,22} + 4D_4(v_{,12})^2\Big] d\Omega$$

$$= \int\limits_\Omega \Big[D_3(v_{,11} + v_{,22})^2 + (D_1 - D_3)(v_{,11})^2$$

$$+ (D_2 - D_3)(v_{,22})^2 + 4D_4(v_{,12})^2\Big] d\Omega = 0,$$

and, hence, a.e. in Ω

$$v_{,11}=0, \quad v_{,22}=0 \quad v_{,12}=0.$$

On the other hand, it is obvious that

$$u \in W_2^2(\Omega, D) \Rightarrow u \in W_2^2(\Omega_\delta, D) \equiv W_2^2(\Omega_\delta).$$

Hence, u and $\dfrac{\partial u}{\partial n}$ have traces on $\Gamma_2 \cap \bar{\Omega}_\delta$ $\forall \delta > 0$, and since $v \in V \equiv \overset{0}{W}{}_2^2(\Omega, D)$, similarly, in the sense of traces,

$$v\Big|_{\Gamma_2 \cap \bar{\Omega}_\delta} = \frac{\partial v}{\partial n}\Big|_{\Gamma_2 \cap \bar{\Omega}_\delta} = 0.$$

Therefore, $v = 0$ a.e. in Ω, i.e., $v \equiv \theta$ in V. Hence, (2.3.2) is fulfilled. □

Thus, we can apply the modified Lax-Milgram theorem, which asserts the existence of a unique ideal element z_0 such that (2.4.51) is fulfilled. □

Remark 2.4.13 *If, in particular, $z_0 \in V$ then $z_0 + u \in W_2^2(\Omega, D)$, and on Γ_2 the traces of $z_0 + u$, $\dfrac{\partial z_0 + u}{\partial n}$ and u, $\dfrac{\partial u}{\partial n}$ coincide.*

2.5 Vibration problem

The *vibration equation of isotropic Kirchhoff-Love plates* has the following form (see, e.g., [79]):

$$\begin{aligned} J_\omega w \;:=\; & (Dw_{,11})_{,11} + (Dw_{,22})_{,22} + \nu(Dw_{,22})_{,11} \\ +\; & \nu(Dw_{,11})_{,22} + 2(1-\nu)(Dw_{,12})_{,12} \\ -\; & \omega^2 2h(x_1,x_2)\rho(x_1,x_2)w = f(x_1,x_2) \text{ in } \Omega \subset \mathbb{R}^2, \end{aligned} \tag{2.5.1}$$

where $w = w(x_1, x_2)$ is the deflection, $\omega = const$ is the vibration frequence, f is the intensity of the lateral load, Ω is a bounded plane domain with Lipschitz boundary $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$ with Γ_1 lying on the x_1 axis and Γ_2 lying in the upper half plane $\{x_2 > 0\}$ (see Appendix, Fig. 20), $D \in C^2(\Omega) \cap C(\overline{\Omega})$ is the flexural rigidity of the plate,

$$D := \frac{2Eh^3}{3(1-\nu^2)}, \tag{2.5.2}$$

$2h(x_1, x_2)$ is the thickness of the plate, $E(x_1, x_2)$ is the Young's modulus, ν is the Poisson's ratio, $0 < \nu < 1$, $\rho(x_1, x_2) \in C(\overline{\Omega})$ is the density and indices after comma mean again differentiation with respect to the corresponding variables.

Throughout this section we assume once and for all that

$$D(x_1, x_2) > 0 \text{ on } \Omega \cup \Gamma_2, \; D(x_1, 0) \geq 0 \quad for \quad (x_1, 0) \in \Gamma_1. \tag{2.5.3}$$

If

$$2h(x_1, x_2)|_{\Gamma_1} = 0 \tag{2.5.4}$$

(i.e., $2h(x_1, 0) = 0$ for $(x_1, 0) \in \Gamma_1$), as it was already said, the plate is called a *cusped* one (see Appendix, Fig. 21–23, and Fig. 11–19 with all the possible profiles of the plates in the neighbourhood of the cusped edge). In this case

$$D(x_1, x_2)|_{\Gamma_1} = 0. \tag{2.5.5}$$

But (2.5.5) may also appear if $2h|_{\Gamma_1} > 0$ but $E|_{\Gamma_1} = 0$ (or if both quantities vanish). In all these cases, the plate will be called a cusped one, although it can be even of constant thickness but with properties of cusped plate caused by vanishing of the inhomogeneous Young's modulus E on Γ_1.

We recall that for the *bending moments* $M_\alpha w$, $\alpha = 1, 2$, the *twisting moments* $M_{12}w$, $M_{21}w$, the *shearing forces* $Q_\alpha w$, $\alpha = 1, 2$, and the *generalized shearing forces* $Q^*_\alpha w$, $\alpha = 1, 2$, we have the following expressions:

$$\begin{aligned}
M_\alpha w &= -D(w_{,\alpha\alpha} + \nu w_{,\beta\beta}),\ \alpha, \beta = 1, 2;\ \alpha \neq \beta, && (2.5.6)\\
M_{12}w &= -M_{21}w = 2(1-\nu)Dw_{,12}, && (2.5.7)\\
Q_\alpha w &= (M_\alpha w)_{,\alpha} + (M_{21}w)_{,\beta},\ \alpha = 1, 2;\ \alpha \neq \beta, && (2.5.8)\\
Q^*_\alpha w &= Q_\alpha w + (M_{21}w)_{,\beta},\ \alpha = 1, 2;\ \alpha \neq \beta. && (2.5.9)
\end{aligned}$$

At points of the boundary $\partial\Omega$ where D vanishes, all the above quantities will be defined as limits from the inside of Ω.

Problem 2.5.1 *Let us consider for equation (2.5.1) the following inhomogeneous BCs:*

– *on* Γ_2

$$w = g_1, \quad \frac{\partial w}{\partial n} = g_2, \tag{2.5.10}$$

– *on* Γ_1

$$\begin{aligned}
&\textit{either } w = w_0(x_1),\ w_{,2} = w_0^1(x_1) \quad \textit{iff } I_{02} < +\infty, && (2.5.11)\\
&\textit{or } w_{,2} = w_0^1(x_1),\ Q^*_2 = Q^0_2(x_1) \quad \textit{iff } I_{02} < +\infty, && (2.5.12)\\
&\textit{or } w = w_0(x_1),\ M_2 = M^0_2(x_1) \begin{cases} \not\equiv 0 \textit{ when } I_{02} < +\infty, \\ \equiv 0 \textit{ when } I_{02} = +\infty, \end{cases} \\
&\qquad \textit{iff } I_{22} < +\infty, && (2.5.13)
\end{aligned}$$

or

$$M_2 = M^0_2(x_1) \begin{cases} \not\equiv 0 \textit{ when } I_{02} < +\infty, \\ \equiv 0 \textit{ when } I_{02} = +\infty, \end{cases} \quad Q^*_2 = Q^0_2(x_1) \begin{cases} \not\equiv 0 \textit{ when } I_{12} < +\infty, \\ \equiv 0 \textit{ when } I_{12} = +\infty, \end{cases} \tag{2.5.14}$$

where g_1, g_2 and w_0, w_0^1, Q_2^0, M_2^0 are prescribed functions on Γ_2 and Γ_1, respectively,

$$I_{k2} \equiv I_{k2}(x_1) := \int_0^{l(x_1)} \tau^k D^{-1}(x_1,\tau)d\tau,\ k \in \{0,1,\ldots\},\ (x_1,0) \in \Gamma_2, \tag{2.5.15}$$

where $(x_1, l(x_1)) \in \Omega$ for $(x_1, 0) \in \Gamma_1$.

Let us now introduce some function spaces.

Definition 2.5.2 *Let*

$$W^{2,2}(\Omega, p) \text{ and } \widetilde{W}^{2,2}(\Omega, p) \tag{2.5.16}$$

be the sets of all measurable functions $w(x_1, x_2)$ defined on Ω which have on Ω locally summable generalized derivatives $\partial_{x_1,x_2}^{(\alpha_1,\alpha_2)} w$ for $\alpha_1 + \alpha_2 \le 2$, $\alpha_1, \alpha_2 \in \{0,1,2\}$, such that

$$\int_\Omega \rho_{\alpha_1,\alpha_2}(x_1,x_2)|\partial_{x_1,x_2}^{(\alpha_1,\alpha_2)} w|^2 d\Omega < +\infty,\ \partial_{x_1,x_2}^{(0,0)} w = w, \tag{2.5.17}$$

for

$$\rho_{0,0} := 1,\ \rho_{2,0} = \rho_{1,1} = \rho_{0,2} := p(x_1,x_2)$$

and

$$\rho_{0,0} := x_2^{\varkappa-4},\ \rho_{2,0} = \rho_{1,1} = \rho_{0,2} := p(x_1,x_2),$$

respectively, with a bounded measurable on Ω function $p(x_1, x_2)$.

Let us further consider the following sets for different cases of the function $p(x_1, x_2)$:

$$W^{2,2}(\Omega, D) \text{ and } \widetilde{W}^{2,2}(\Omega, D) \tag{2.5.18}$$

with $p(x_1, x_2) = D(x_1, x_2)$ satisfying (2.5.3), and

$$V^{2,2}(\Omega, x_2^\varkappa) := W^{2,2}(\Omega, x_2^\varkappa) \quad (p(x_1,x_2) = x_2^\varkappa), \tag{2.5.19}$$
$$\widetilde{V}^{2,2}(\Omega, x_2^\varkappa) := \widetilde{W}^{2,2}(\Omega, x_2^\varkappa) \quad (p(x_1,x_2) = x_2^\varkappa), \tag{2.5.20}$$

and

$$V^{2,2}(\Omega, d^\varkappa) := W^{2,2}(\Omega, d^\varkappa) \quad (p(x_1,x_2) = d(x_1,x_2)), \tag{2.5.21}$$

where

$$d(x_1,x_2) := dist\{(x_1,x_2) \in \Omega,\ \partial\Omega\}.$$

Further, let us introduce the following norms;

$$\|w\|^2_{W^{2,2}(\Omega,D)} := \int_\Omega [w^2 + \nu D(w_{,11} + w_{,22})^2 + (1-\nu)D(w_{,11})^2 + 2(1-\nu)D(w_{,12})^2 + (1-\nu)D(w_{,22})^2] d\Omega, \tag{2.5.22}$$

$$\|w\|^2_{\widetilde{W}^{2,2}}(\Omega, D) := \int_\Omega [x_2^{\varkappa-4} w^2 + \nu D(w_{,11} + w_{,22})^2 + (1-\nu)D(w_{,11})^2 + 2(1-\nu)D(w_{,12})^2 + (1-\nu)D(w_{,22})^2] d\Omega, \tag{2.5.23}$$

$$\|w\|^2_{\widetilde{V}^{2,2}(\Omega,x_2^\varkappa)} := \int_\Omega \{x_2^{\varkappa-4} w^2 + x_2^\varkappa[(w_{,21})^2 + (w_{,12})^2 + (w_{,22})^2]\} d\Omega, \tag{2.5.24}$$

$$\|w\|^2_{V^{2,2}(\Omega,x_2^\varkappa)} := \int_\Omega \{w^2 + x_2^\varkappa[(w_{,11})^2 + (w_{,12})^2 + (w_{,22})^2]\} d\Omega \tag{2.5.25}$$

$$\|w\|^2_{V^{2,2}(\Omega,d^\varkappa)} := \int_\Omega \{w^2 + d^\varkappa[(w_{,11})^2 + (w_{,12})^2 + (w_{,22})^2]\} d\Omega. \tag{2.5.26}$$

From (2.5.3) it is clear that in our cases if $D \in C(\overline{\Omega})$, then

$$\rho^{-1}_{\alpha_1,\alpha_2} \in L_1^{loc}(\Omega).$$

Therefore, according to [60], the spaces (2.5.18)–(2.5.21) with the norms (2.5.22)–(2.5.26), respectively, will be Banach spaces, and moreover, Hilbert spaces under the appropriate scalar products.

Lemma 2.5.3

$$V^{2,2}(\Omega, x_2^\varkappa) \subset V^{2,2}(\Omega, d^\varkappa(x_1,x_2)) \quad \forall\ \varkappa \geq 0. \tag{2.5.27}$$

Proof follows from the obvious inequality

$$d(x_1,x_2) \leq x_2 \quad for \quad (x_1,x_2) \in \Omega \tag{2.5.28}$$

(if $d(x_1,x_2)$ is a regularized distance, then in the inequality (2.5.28) arises a constant factor). □

Further, without loss of generality, we suppose that the domain Ω lies inside of the rectangle:

$$\Pi := \{(x_1,x_2) \in R^2 : a < x_1 < b,\ 0 < x_2 < l\}, \tag{2.5.29}$$

with a constant $l > \max\limits_{(x_1,x_2)\in\overline{\Omega}} \{x_2\}$.

Lemma 2.5.4

$$V^{2,2}(\Omega, x_2^{\varkappa}) \subset V^{2,2}(\Omega, x_2^4) \ for\ 0 \leq \varkappa < 4. \tag{2.5.30}$$
$$V^{2,2}(\Omega, x_2^4) = \widetilde{V}^{2,2}(\Omega, x_2^4). \tag{2.5.31}$$

Proof of (2.5.30) follows from $l^{4-\varkappa} x_2^{\varkappa} \geq x_2^4$ for $0 \leq x_2 < l$, (2.5.31) is evident. □

Let

$$\Omega_\delta := \{(x_1, x_2) \in \Omega : x_2 > \delta,\ \delta = const\ > 0\}.$$

Evidently,

$$\widetilde{V}^{2,2}(\Omega, x_2^{\varkappa}) \subset \widetilde{V}^{2,2}(\Omega_\delta, x_2^{\varkappa}) \subset W^{2,2}(\Omega_\delta), \tag{2.5.32}$$

where $W^{2,2}(\Omega_\delta) \equiv W_2^2(\Omega_\delta)$ is the usual (i.e., non-weighted) Sobolev space. Hence, there exist the traces

$$w|_{\Gamma_2} \in W^{\frac{3}{2},2}(\Gamma_2),\ \left.\frac{\partial w}{\partial n}\right|_{\Gamma_2} \in W^{\frac{1}{2},2}(\Gamma_2) \quad \forall\ v \in \widetilde{V}^{2,2}(\Omega, x_2^{\varkappa}).$$

Lemma 2.5.5 *If* $v \in \widetilde{V}^{2,2}(\Omega, x_2^{\varkappa})$ *and*

$$v|_{\Gamma_2} = 0,\ \left.\frac{\partial v}{\partial n}\right|_{\Gamma_2} = 0, \tag{2.5.33}$$

then

$$\int\limits_{\Omega} x_2^{\varkappa-4} v^2 d\Omega \leq \frac{16}{(\varkappa-1)^2(\varkappa-3)^2} \int\limits_{\Omega} x_2^{\varkappa} (v_{,22})^2 d\Omega,\ \ \varkappa > 3. \tag{2.5.34}$$

Proof. Let us complete the definition of the function v in $\Pi \setminus \Omega$, assuming v equal to zero there. Then evidently,

$$v \in \widetilde{V}^{2,2}(\Omega, x_2^{\varkappa})$$

implies

$$\int\limits_{\Pi} [x_2^{\varkappa-4} v^2 + x_2^{\varkappa}(v_{,22})^2] dx_1 dx_2 < +\infty,$$

i.e.,

$$v(x_1, \cdot) \in \widetilde{W}^{2,2}(]0, l[, x_2^{\varkappa})$$

(see (1.4.42)) and

$$v(x_1, l) = 0,\ v_{,2}(x_1, l) = 0$$

for almost every $x_1 \in]a, b[$. We can, now, apply Lemma 1.4.12, i.e.,

$$\int\limits_0^l x_2^{\varkappa-4} v^2(x_1, x_2) dx_2 \leq \frac{16}{(\varkappa-1)^2(\varkappa-3)^2} \int\limits_0^l x_2^{\varkappa}[v_{,22}(x_1, x_2)]^2 dx_2,\ \ \varkappa > 3, \tag{2.5.35}$$

for almost every $x_1 \in]a,b[$. Integrating by x_1 both the sides of (2.5.35) over $]a,b[$, we get

$$\int\limits_{\Omega} x_2^{\varkappa-4} v^2 d\Omega = \int\limits_{\Pi} x_2^{\varkappa-4} v^2 dx_1 dx_2 \leq \frac{16}{(\varkappa-1)^2(\varkappa-3)^2} \int\limits_{\Pi} x_2^{\varkappa}(v_{,22})^2 dx_1 dx_2 =$$
$$= \frac{16}{(\varkappa-1)^2(\varkappa-3)^2} \int\limits_{\Omega} x_2^{\varkappa}(v_{,22}) d\Omega \ for \ \varkappa > 3.$$

□

Corollary 2.5.6 *If $v \in \widetilde{V}^{2,2}(\Omega, x_2^4) = V^{2,2}(\Omega, x_2^4)$ and (2.5.33) is fulfilled, then*

$$\int\limits_{\Omega} v^2 d\Omega \leq \frac{16}{9} \int\limits_{\Omega} x_2^4 (v_{,22})^2 d\Omega.$$

Let

$$D(x_1,x_2) \geq D_{\varkappa} x_4^{\varkappa} \quad \forall\ (x_1,x_2) \in \Omega,\ i.e.,\ 0 < D_{\varkappa} := \inf_{\Omega} \frac{D(x_1,x_2)}{x_2^{\varkappa}}. \qquad (2.5.36)$$

If $\varkappa \geq 1$, then by $\varkappa$ we denote the minimal among all the exponents $\delta \geq \varkappa \geq 1$ for which (2.5.36) holds. It means that if we have the inequality (2.5.36) for $\varkappa \geq 1$, we have to check whether there exists or no the less exponent for which (2.5.36) is valid. If yes, then we have to continue this procedure untill we arrive at the minimal one.

If (2.5.36) holds for $\varkappa < 1$, then we need no additional revision since for all the $\varkappa < 1$ we have the same result concerning the traces.

The condition (2.5.36) is essential in a right neighbourhood of Γ_1. Then it can be easily extended for the whole domain Ω.

Let us note, that when $\omega = 0$, Problem 2.5.1 was considered in Section 2.4 (see also [46]) under the different from the condition (2.5.36) condition

$$0 \leq D_0 := \inf_{\Omega} \frac{D(x_1,x_2)}{x_2^4},$$

which does not make it possible to discuss the traces of solutions when the thickness is of the non-power type. Besides, in Section 2.4 the spaces, when $D_0 = 0$ (i.e., in the particular case of the power type thickness, when $\varkappa \geq 4$), are not transparent (in the sense of the so called ideal elements) even in the case of the power type thickness.

Lemma 2.5.7 *If (2.5.36) takes place, then*

$$W^{2,2}(\Omega, D) \subset V^{2,2}(\Omega, x_2^{\varkappa}) \subset V^{2,2}(\Omega, d^{\varkappa}(x_1,x_2)) \quad \forall\ \varkappa \geq 0, \qquad (2.5.37)$$

and

$$W^{2,2}(\Omega, D) \subset \widetilde{W}^{2,2}(\Omega, D) \subset \widetilde{V}^{2,2}(\Omega, x_2^{\varkappa}) \ for \ \varkappa \geq 4. \qquad (2.5.38)$$

Proof of (2.5.37) follows from (2.5.36), (2.5.27). Formula (2.5.38) follows from (2.5.36), since

$$\int_\Omega w^2 d\Omega < +\infty$$

implies

$$\int_\Omega x_2^{\varkappa-4} w^2 d\Omega < +\infty$$

for $\varkappa \geq 4$. □

Lemma 2.5.8 *If $w \in W^{2,2}(\Omega, D)$ and (2.5.36) is valid, then there exist traces*

$$w|_{\Gamma_1} \in B_2^{\frac{3-\varkappa}{2}}(\Gamma_1) \subset L_2(\Gamma_1) \quad if \quad 0 \leq \varkappa < 3 \ (i.e., \ I_{22}|_{\Gamma_1} < +\infty), \tag{2.5.39}$$

$$w_{,2}|_{\Gamma_1} \in B_2^{\frac{1-\varkappa}{2}}(\Gamma_1) \subset L_2(\Gamma_1) \quad if \quad 0 \leq \varkappa < 1 \ (i.e., \ I_{02}|_{\Gamma_1} < +\infty), \tag{2.5.40}$$

where $B_2^{\frac{3-\varkappa}{2}}(\Gamma_1)$ and $B_2^{\frac{1-\varkappa}{2}}(\Gamma_1)$ are Besov spaces.

Proof. Since (2.5.36) is valid, according to Lemma 2.5.7 (see (2.5.37)), $w \in W^{2,2}(\Omega, D)$ implies

$$w \in V^{2,2}(\Omega, d^{\varkappa}(x_1, x_2)).$$

But functions from this space (see [71], Theorem 1.1.2) have properties (2.5.39) and (2.5.40) if $\partial\Omega \in C^{1+\varepsilon}$ and $\partial\Omega \in C^{2+\varepsilon}$ (which means that the boundary is locally described by functions whose first and second derivatives, satisfy the Hölder condition with a Hölder exponent $\varepsilon \in]0, 1[$, respectively). Since in our case Γ_1 is a part of a straight line, these local conditions are fulfilled all the more. □

Now, we constitute the spaces V and $\widetilde{V}$ from the spaces $W^{2,2}(\Omega, D)$ and $\widetilde{W}^{2,2}(\Omega, D)$, correspondingly, as follows:

$$\begin{aligned} V \ := \ \Big\{ & v \in W^{2,2}(\Omega, D) : v|_{\Gamma_2} = 0, \ \frac{\partial v}{\partial n}\Big|_{\Gamma_2} = 0, \\ & \textit{and additionally} \\ & \textit{either } v|_{\Gamma_1} = 0, \ v_{,2}|_{\Gamma_1} = 0 \ (\textit{if we consider BCs } (2.5.11)) \\ & \textit{or } v_{,2}|_{\Gamma_1} = 0 \ (\textit{if we consider BCs } (2.5.12)) \\ & \textit{or } v|_{\Gamma_1} = 0 \ (\textit{if we consider BCs } (2.5.13)) \\ & \textit{in the sense of traces} \Big\} \end{aligned} \tag{2.5.41}$$

and

$$\widetilde{V} := \Big\{ v \in \widetilde{W}^{2,2}(\Omega, D) : v|_{\Gamma_2} = 0, \ \frac{\partial v}{\partial n}\Big|_{\Gamma_2} = 0 \textit{ in the sense of traces} \Big\}. \tag{2.5.42}$$

Using the trace theorem, it is not difficult to prove the completeness of V and $\widetilde{V}$.

In view of Lemma 2.5.8, we can suppose that the functions g_1, g_2, w_0, w_0^1 from Problem 2.5.1 are traces of a prescribed function

$$u \in W^{2,2}(\Omega, D). \tag{2.5.43}$$

Let further $Q_2^0, M_2^0 \in L_2(\Gamma_1)$.

Definition 2.5.9 *Let $f \in L_2(\Omega)$ and $\varkappa < 4$ (i.e., $I_{32}|_{\Gamma_1} < +\infty$). A function $w \in W^{2,2}(\Omega, D)$ will be called a weak solution of Problem 2.5.1, when $I_{32}|_{\Gamma_1} < +\infty$, in the space $W^{2,2}(\Omega, D)$ if it satisfies the following conditions:*

$$w - u \in V \tag{2.5.44}$$

and

$$\begin{aligned} J_\omega(w, v) := \int\limits_\Omega B_\omega(w, v) d\Omega &= \int\limits_\Omega f v d\Omega + \gamma_2 \int\limits_{\Gamma_1} Q_2^0 v dx_1 \\ &- \gamma_1 \int\limits_{\Gamma_1} M_2^0 v_{,2} dx_1 \quad \forall v \in V, \end{aligned} \tag{2.5.45}$$

where

$$\begin{aligned} &\gamma_1 = 0,\ \gamma_2 = 0 \ for\ \ the\ \ BCs\ \ (2.5.11), \\ &\gamma_1 = 0,\ \gamma_2 = 1 \ for\ \ the\ \ BCs\ \ (2.5.12), \\ &\gamma_1 = 1,\ \gamma_2 = 0 \ for\ \ the\ \ BCs\ \ (2.5.13), \\ &\gamma_1 = 1,\ \gamma_2 = 1 \ for\ \ the\ \ BCs\ \ (2.5.14), \end{aligned}$$

and

$$\begin{aligned} B_\omega(w, v) \ &:= \ \nu D(w_{,11} + w_{,22})(v_{,11} + v_{,22}) + (1-\nu) D w_{,11} v_{,11} \\ &+ \ 2(1-\nu) D w_{,12} v_{,12} + (1-\nu) D w_{,22} v_{,22} \\ &- \ \omega^2 2h\rho w v. \end{aligned} \tag{2.5.46}$$

Definition 2.5.10 *Let g_1 and g_2 be traces of a prescribed $u \in \widetilde{W}^{2,2}(\Omega, D)$, $x_2^{\frac{4-\varkappa}{2}} f \in L_2(\Omega)$, and $\varkappa \geq 4$ (i.e., $I_{k2}|_{\Gamma_1} = +\infty$ for a fixed $k \geq 3$). A function $w \in \widetilde{W}^{2,2}(\Omega, D)$ will be called a weak solution of the problem (2.5.1), (2.5.10), (2.5.14) (i.e., of the last BVP of Problem 2.5.1 when $I_{k2}|_{\Gamma_1} = +\infty$ for a fixed $k \geq 3$) in the space $\widetilde{W}^{2,2}(\Omega, D)$ if it satisfies the following conditions:*

$$w - u \in \widetilde{V} \tag{2.5.47}$$

and

$$J_\omega(w, v) := \int\limits_\Omega B_\omega(w, v) d\Omega = \int\limits_\Omega f v d\Omega \quad \forall\ v \in \widetilde{V} \tag{2.5.48}$$

where $B_\omega(w, v)$ is defined by (2.5.46).

Theorem 2.5.11 *Let*

$$\omega^2 < \frac{9(1-\nu)D_{\varkappa} l^{\varkappa-4}}{16 \max\limits_{\overline{\Omega}} 2h\rho}. \tag{2.5.49}$$

There exists a unique weak solution of Problem 2.5.1, when $I_{32}|_{\Gamma_1} < +\infty$ *(more precisely, of each of all the four BVPs stated in Problem 2.5.1). This solution is such that*

$$\|w\|_{W^{2,2}(\Omega,D)} \le C[\|f\|_{L_2(\Omega)} + \|u\|_{W^{2,2}(\Omega,D)} + \gamma_1 \|M_2^0\|_{L_2(\Gamma_1)} + \gamma_2 \|Q_2^0\|_{L_2(\Gamma_1)}], \tag{2.5.50}$$

where the constant C *is independent of* f, u, M_2^0*, and* Q_2^0*.*

Theorem 2.5.12 *Let* $2h\rho x_2^{4-\varkappa} \in C(\overline{\Omega})$ *and*

$$\omega^2 < \frac{(\varkappa-1)^2(\varkappa-3)^2(1-\nu)D_{\varkappa}}{16 \max\limits_{\overline{\Omega}} 2h\rho x_2^{4-\varkappa}}. \tag{2.5.51}$$

There exists a unique weak solution of Problem 2.5.1, when $I_{k2}|_{\Gamma_1} = +\infty$ *for a fixed* $k \ge 3$*. This solution is such that*

$$\|w\|_{\widetilde{W}^{2,2}(\Omega,D)} \le C[\|x_2^{\frac{4-\varkappa}{2}} f\|_{L_2(\Omega)} + \|u\|_{\widetilde{W}^{2,2}(\Omega,D)}], \tag{2.5.52}$$

where the constant C *is independent of* f *and* u*.*

Proof of Theorem 2.5.11 is similar to that of Theorem 1.4.15 and is based on the Lax-Milgram theorem. It is easy to show the following three inequalities (see (2.5.56), (2.5.59), (2.5.61) below which imply the proof).

In view of (2.5.45), (2.5.46), (2.5.22), we have

$$\begin{aligned}
|J_\omega(w,v)| \le & \int_\Omega (\nu D)^{\frac12}|w_{,11}+w_{,22}| \cdot (\nu D)^{\frac12}|v_{,11}+v_{,22}| d\Omega \\
+ & \int_\Omega [(1-\nu)D]^{\frac12}|w_{,11}| \cdot [(1-\nu)D]^{\frac12}|v_{,11}| d\Omega \\
+ & \int_\Omega [2(1-\nu)D]^{\frac12}|w_{,12}| \cdot [2(1-\nu)D]^{\frac12}|v_{,12}| d\Omega \\
+ & \int_\Omega [(1-\nu)D]^{\frac12}|w_{,22}| \cdot [(1-\nu)D]^{\frac12}|v_{,22}| d\Omega + T\int_\Omega |w||v| d\Omega \\
\le & \left[\int_\Omega \nu D(w_{,11}+w_{,22})^2 d\Omega\right]^{\frac12} \left[\int_\Omega \nu D(v_{,11}+v_{,22})^2 d\Omega\right]^{\frac12}
\end{aligned}$$

$$
\begin{aligned}
&+ \left[\int_\Omega (1-\nu)D(w_{,11})^2 d\Omega\right]^{\frac{1}{2}} \left[\int_\Omega (1-\nu)D(v_{,11})^2 d\Omega\right]^{\frac{1}{2}} \\
&+ \left[\int_\Omega 2(1-\nu)D(w_{,12})^2 d\Omega\right]^{\frac{1}{2}} \left[\int_\Omega 2(1-\nu)D(w_{,12})^2 d\Omega\right]^{\frac{1}{2}} \\
&+ \left[\int_\Omega (1-\nu)D(w_{,22})^2 d\Omega\right]^{\frac{1}{2}} \left[\int_\Omega (1-\nu)D(v_{,22})^2 d\Omega\right]^{\frac{1}{2}} \\
&+ T\left[\int_\Omega w^2 d\Omega\right]^{\frac{1}{2}} \left[\int_\Omega v^2 d\Omega\right]^{\frac{1}{2}} \\
&\leq (4+T)\|w\|_{W^{2,2}(\Omega,D)}\|v\|_{W^{2,2}(\Omega,D)}, \qquad (2.5.53)
\end{aligned}
$$

where

$$T := 2\omega^2 \max_{\overline{\Omega}}[h(x_1,x_2)\rho(x_1,x_2)]. \qquad (2.5.54)$$

In particular,

$$|J_\omega(w,v)| \leq (4+T)\|w\|_{W^{2,2}(\Omega,D)}\|v\|_V \quad \forall\ w \in W^{2,2}(\Omega,D)\ and\ \forall\ v \in V \qquad (2.5.55)$$

and

$$|J_\omega(w,v)| \leq (4+T)\|w\|_V\|v\|_V \quad \forall\ w,v \in V. \qquad (2.5.56)$$

Taking into account (2.5.55), and

$$
\begin{aligned}
\Big|\int_{\Gamma_1} vQ_2^0 dx_1\Big| &\leq \|v\|_{L_2(\Gamma_1)}\|Q_2^0\|_{L_2(\Gamma_1)} \leq C_0\|v\|_V\|Q_2^0\|_{L_2(\Gamma_1)}, \qquad (2.5.57)\\
\Big|\int_{\Gamma_1} v_{,2}M_2^0 dx_1\Big| &\leq \|v_{,2}\|_{L_2(\Gamma_1)}\|M_2^0\|_{L_2(\Gamma_1)} \qquad (2.5.58)\\
&\leq C_0\|v\|_V\|M_2^0\|_{L_2(\Gamma_1)}
\end{aligned}
$$

with the positive constant C_0 from the trace theorem, it is not difficult to see, that the functional

$$F_\omega v := \int_\Omega fv d\Omega - J_\omega(u,v) + \gamma_2\int_{\Gamma_1} Q_2^0 v dx_1 - \gamma_1\int_{\Gamma_1} M_2^0 v_{,2} dx_1,\ v \in V,$$

is bounded in V:

$$|F_\omega v| \leq \{\|f\|_{L_2(\Omega)} + (4+T)\|u\|_{W^{2,2}(\Omega,D)} + C_0[\|Q_2^0\|_{L_2(\Gamma_1)} + \|M_2^0\|_{L_2(\Gamma_1)}]\}\|v\|_V. \qquad (2.5.59)$$

Let

$$T_0 := \frac{16l^{4-\varkappa}(1+T)}{9(1-\nu)D_\varkappa}, \quad T_1 := \frac{16l^{4-\varkappa}T}{9(1-\nu)D_\varkappa}. \tag{2.5.60}$$

Analogously to (1.4.62), in view of Corollary 2.5.6, we get

$$\begin{aligned}
\|v\|_V^2 &= \int_\Omega \{v^2 + D[\nu(v_{,11}+v_{,22})^2 + (1-\nu)(v_{,11})^2 + 2(1-\nu)(v_{,12})^2 \\
&+ (1-\nu)(v_{,22})^2]\}d\Omega = \int_\Omega v^2 d\Omega + J_\omega(v,v) + 2\omega^2 \int_\Omega h\rho v^2 d\Omega \\
&\le J_\omega(v,v) + (1+T)\int_\Omega v^2 d\Omega \le J_\omega(v,v) + T_0 \int_\Omega (1-\nu)D_\varkappa x_2^\varkappa (v_{,22})d\Omega \\
&\le J_\omega(v,v) + T_0 \int_\Omega (1-\nu)D(v_{,22})^2 d\Omega \\
&\le J_\omega(v,v) + T_0 \int_\Omega D[(1-\nu)(v_{,22})^2 + \nu(v_{,11}+v_{,22})^2 + (1-\nu)(v_{,11})^2 \\
&+ 2(1-\nu)(v_{,12})^2]d\Omega = J_\omega(v,v) + T_0\Big[J_\omega(v,v) + 2\omega^2 \int_\Omega h\rho v^2 d\Omega\Big] \\
&\le J_\omega(v,v) + T_0\Big[J_\omega(v,v) + T\int_\Omega v^2 d\Omega\Big] \\
&\le J_\omega(v,v) + T_0\Big[J_\omega(v,v) + T_1\int_\Omega (1-\nu)D(v_{,22})^2 d\Omega\Big] \\
&\le J_\omega(v,v) + T_0\Big\{J_\omega(v,v) + T_1\Big[J_\omega(v,v) + T_1\int_\Omega (1-\nu)D(v_{,22})^2 d\Omega\Big]\Big\} \\
&\le J_\omega(v,v) + T_0\Big\{J_\omega(v,v) + T_1 J_\omega(v,v) + (T_1)^2\Big[J_\omega(v,v) \\
&+ T_1\int_\Omega (1-\nu)D(v_{,22})^2 d\Omega\Big]\Big\} = J_\omega(v,v) \\
&+ T_0\Big\{J_\omega(v,v)[1+T_1+(T_1)^2] + (T_1)^3\int_\Omega (1-\nu)D(v_{,22})^2 d\Omega\Big\} \\
&\quad (\textit{repeating the same } (n-2)-\textit{times more}) \\
&\le J_\omega(v,v) + T_0\Big[J_\omega(v,v)\frac{1-(T_1)^{n+1}}{1-T_1} + (T_1)^{n+1}\int_\Omega (1-\nu)D(v_{,22})^2 d\Omega\Big].
\end{aligned}$$

Now, tending n to infinity and taking into account that, by virtue of (2.5.60), (2.5.54), and (2.5.49),

$$T_1 < 1,$$

we obtain

$$\|v\|_V^2 \le J_\omega(v,v) + \frac{T_0}{1-T_1} J_\omega(v,v) = \frac{1-T_1+T_0}{1-T_1} J_\omega(v,v),$$

whence, in view of (2.5.60),

$$J_\omega(v,v) \ge \frac{9(1-\nu)D_\varkappa - 16 l^{4-\varkappa} T}{9(1-\nu)D_\varkappa + 16 l^{4-\varkappa}} \|v\|_V^2. \tag{2.5.61}$$

□

Proof of Theorem 2.5.12 is similar to that of Theorem 1.4.17 and Theorem 2.5.11. Obviously,

$$\begin{aligned}
|J_\omega(w,v)| &\le \int_\Omega (\nu D)^{\frac12} |w_{,11} + w_{,22}| \cdot (\nu D)^{\frac12} |v_{,11} + v_{,22}| d\Omega \\
&+ \int_\Omega [(1-\nu)D]^{\frac12} |w_{,11}| \cdot [(1-\nu)D]^{\frac12} |v_{,11}| d\Omega \\
&+ \int_\Omega [2(1-\nu)D]^{\frac12} |w_{,12}| \cdot [2(1-\nu)D]^{\frac12} |v_{,12}| d\Omega \\
&+ \int_\Omega [(1-\nu)D]^{\frac12} |w_{,22}| \cdot [(1-\nu)D]^{\frac12} |v_{,22}| d\Omega \\
&+ 2\omega^2 \int_\Omega h\rho x_2^{4-\varkappa} x_2^{\frac{\varkappa-4}{2}} |w| x_2^{\frac{\varkappa-4}{2}} |v| d\Omega \\
&\le \left[\int_\Omega \nu D (w_{,11} + w_{,22})^2 d\Omega\right]^{\frac12} \left[\int_\Omega \nu D (v_{,11} + v_{,22})^2 d\Omega\right]^{\frac12} \\
&+ \left[\int_\Omega (1-\nu) D (w_{,11})^2 d\Omega\right]^{\frac12} \left[\int_\Omega (1-\nu) D (v_{,11})^2 d\Omega\right]^{\frac12} \\
&+ \left[\int_\Omega 2(1-\nu) D (w_{,12})^2 d\Omega\right]^{\frac12} \left[\int_\Omega 2(1-\nu) D (w_{,12})^2 d\Omega\right]^{\frac12} \\
&+ \left[\int_\Omega (1-\nu) D (w_{,22})^2 d\Omega\right]^{\frac12} \left[\int_\Omega (1-\nu) D (v_{,22})^2 d\Omega\right]^{\frac12} \\
&+ T_* \left[\int_\Omega x_2^{\varkappa-4} w^2 d\Omega\right]^{\frac12} \left[\int_\Omega x_2^{\varkappa-4} v^2 d\Omega\right]^{\frac12} \\
&\le (4+T_*) \|w\|_{\widetilde{W}^{2,2}(\Omega,D)} \|v\|_{\widetilde{W}^{2,2}(\Omega,D)},
\end{aligned}$$

where

$$T_* := 2\omega^2 \max_{\bar{\Omega}} [h\rho x_2^{4-\varkappa}]. \tag{2.5.62}$$

Further, for

$$F_\omega v := \int_\Omega f v d\Omega - J_\omega(w, v), \quad v \in \widetilde{V},$$

we have

$$\begin{aligned} |F_\omega v| &\leq \{\|x_2^{\frac{4-\varkappa}{2}} f\|_{L_2(\Omega)} \|x_2^{\frac{\varkappa-4}{2}} v\|_{L_2(\Omega)} + (4+T_*)\|w\|_{\widetilde{W}^{2,2}(\Omega,D)} \|v\|_{\widetilde{W}^{2,2}(\Omega,D)}\} \\ &\leq \{\|x_2^{\frac{4-\varkappa}{2}} f\|_{L_2(\Omega)} + (4+T_*)\|w\|_{\widetilde{W}^{2,2}(\Omega,D)}\} \|v\|_{\widetilde{V}}, \end{aligned}$$

since $\|x_2^{\frac{\varkappa-4}{2}} v\|_{L_2(\Omega)} \leq \|v\|_{\widetilde{V}}$ and $\|v\|_{\widetilde{W}^{2,2}(\Omega,D)} = \|v\|_{\widetilde{V}}$.

Let

$$T_0^* := \frac{16(1+T_*)}{(\varkappa-1)^2(\varkappa-3)^2 D_\varkappa (1-\nu)}, \qquad T_1^* := \frac{16 T_*}{(\varkappa-1)^2(\varkappa-3)^2 D_\varkappa (1-\nu)}. \tag{2.5.63}$$

Evidently, taking into account the second imbedding of (2.5.38), Lemma 2.5.5, (2.5.36), and (2.5.46),

$$\begin{aligned} \|v\|_{\widetilde{V}}^2 &:= \int_\Omega \{x_2^{\varkappa-4} v^2 + D[\nu(v_{,11}+v_{,22})^2 + (1-\nu)(v_{,11})^2 \\ &+ 2(1-\nu)(v_{,12})^2 + (1-\nu)(v_{,22})^2]\} d\Omega \\ &= \int_\Omega x_2^{\varkappa-4} v^2 d\Omega + J_\omega(v,v) + 2\omega^2 \int_\Omega h\rho x_2^{4-\varkappa} x_2^{\varkappa-4} v^2 d\Omega \\ &\leq J_\omega(v,v) + (1+T_*) \int_\Omega x_2^{\varkappa-4} v^2 d\Omega \\ &\leq J_\omega(v,v) + T_0^* \int_\Omega (1-\nu) D_\varkappa x_2^\varkappa (v_{,22})^2 d\Omega \\ &\leq J_w(v,v) + T_0^* \int_\Omega (1-\nu) D (v_{,22})^2 d\Omega \\ &\leq J_\omega(v,v) + T_0^* \int_\Omega D[\nu(v_{,11}+v_{,22})^2 + (1-\nu)(v_{,11})^2 \\ &+ 2(1-\nu)(v_{,12})^2 + (1-\nu)(v_{,22})^2] d\Omega \\ &= J_\omega(v,v) + T_0^*[J_w(v,v) + 2\omega^2 \int_\Omega h\rho x_2^{4-\varkappa} x_2^{\varkappa-4} v^2 d\Omega] \end{aligned}$$

$$
\begin{aligned}
&\leq J_\omega(v,v) + T_0^*[J_\omega(v,v) + T_* \int\limits_\Omega x_2^{\varkappa-4} v^2 d\Omega] \\
&\leq J_\omega(v,v) + T_0^*[J_\omega(v,v) + T_1^* \int\limits_\Omega (1-\nu) D (v_{,22})^2 d\Omega \\
&\leq J_\omega(v,v) + T_0^*\{J_\omega(v,v) + T_1^*[J_\omega(v,v) + T_1^* \int\limits_\Omega (1-\nu) D (v_{,22})^2 d\Omega]\} \\
&\leq J_\omega(v,v) + T_0^*\{J_\omega(v,v) + T_1^* J_\omega(v,v) + (T_1^*)^2 \\
&\times [J_\omega(v,v) + T_1^* \int\limits_\Omega (1-\nu) D (v_{,22})^2 d\Omega]\} \\
&= J_\omega(v,v) + T_0^*\{J_\omega(v,v)[1 + T_1^* + (T_1^*)^2] + (T_1^*)^3 \int\limits_\Omega (1-\nu) D (v_{,22})^2 d\Omega\} \\
&\quad (repeating\ the\ same\ (n-2)-times\ more) \\
&= J_\omega(v,v) + T_0^*[J_\omega(v,v) \frac{1-(T_1^*)^{n+1}}{1-T_1^*} + (T_1^*)^{n+1} \int\limits_\Omega (1-\nu) D (v_{,22})^2 d\Omega].
\end{aligned}
$$

Now, tending n to infinity and taking into account that, by virtue of (2.5.63), (2.5.62), (2.5.51), obviously,

$$T_1^* < 1,$$

we obtain

$$\|v\|^2_{\widetilde{V}} \leq J_\omega(v,v) + \frac{T_0^*}{1-T_1^*} J_\omega(v,v) = \frac{1-T_1^*+T_0^*}{1-T_1^*} J_\omega(v,v).$$

But, in view of (2.5.63),

$$
\begin{aligned}
\frac{1-T_1^*+T_0^*}{1-T_1^*} &= \frac{(\varkappa-1)^2(\varkappa-3)^2 D_\varkappa (1-\nu) - 16T_* + 16(1+T_*)}{(\varkappa-1)^2(\varkappa-3)^2 D_\varkappa (1-\nu) - 16T_*} \\
&= \frac{(\varkappa-1)^2(\varkappa-3)^2 (1-\nu) D_\varkappa + 16}{(\varkappa-1)^2(\varkappa-3)^2 (1-\nu) D_\varkappa - 16T_*}.
\end{aligned}
$$

Thus,

$$J_\omega(v,v) \geq \frac{(\varkappa-1)^2(\varkappa-3)^2 (1-\nu) D_\varkappa - 16T_*}{(\varkappa-1)^2(\varkappa-3)^2 (1-\nu) D_\varkappa + 16} \|v\|^2_{\widetilde{V}} \quad \forall v \in \widetilde{V}.$$

□

Now, let us consider general case of the thickness.

Definition 2.5.13 *Let*

$$\overset{*}{W}{}^{2,2}(\Omega, D) \tag{2.5.64}$$

be the set of all measurable functions w defined on Ω which have on Ω locally summable generalized derivatives up to the order 2 such that (2.5.17) is valid for

$$\rho_{0,0} := Q(x_1, x_2), \quad \rho_{2,0} = \rho_{1,1} = \rho_{0,2} := D(x_1, x_2),$$

where

$$Q(x_1,x_2) := D(x_1,x_2)\left[\int_{x_2}^{l} D^{-1}(x_1,\tau)d\tau\right]^2 \left\{\int_{x_2}^{l} D(x_1,t)\left[\int_{t}^{l} D^{-1}(x_1,\tau)d\tau\right]^2 dt\right\}^{-2} \tag{2.5.65}$$

with $D \in C(\overline{\Omega})$ and

$$\int_{x_2}^{l(x_1)} D^{-1}(x_1,\tau)d\tau < +\infty \quad for \quad (x_1,0) \in \Gamma_1. \tag{2.5.66}$$

We recall that $(x_1, l(x_1)) \in \Omega$ for $(x_1, 0) \in \Gamma_1$.

Let us introduce the following norm:

$$\begin{aligned} \|w\|^2_{\overset{*}{W}{}^{2,2}(\Omega,D)} := & \int_\Omega \{Qw^2 + D[\nu(w_{,11} + w_{,22})^2 + (1-\nu)(w_{,11})^2 + \\ & +2(1-\nu)(w_{,12})^2 + (1-\nu)(w_{,22})^2]\}d\Omega. \end{aligned} \tag{2.5.67}$$

Since

$$Q^{-1}, D^{-1} \in L_1^{loc}(\Omega),$$

the space (2.5.64) with the norm (2.5.67) is a Banach space, and moreover, Hilbert space with the scalar product

$$\begin{aligned} (w,v)_{\overset{*}{W}{}^{2,2}(\Omega,D)} := & \int_\Omega \{Qwv + D[\nu(w_{,11} + w_{,22})(v_{,11} + v_{,22}) + (1-\nu)w_{,11}v_{,11} \\ & +2(1-\nu)w_{,12}v_{,12} + (1-\nu)w_{,22}v_{,22}]\}d\Omega. \end{aligned} \tag{2.5.68}$$

Lemma 2.5.14 *If*

$$v \in \overset{*}{W}{}^{2,2}(\Omega, D) \tag{2.5.69}$$

and

$$v|_{\Gamma_2} = 0, \quad \frac{\partial v}{\partial n}\Big|_{\Gamma_2} = 0, \tag{2.5.70}$$

in the sense of traces, then

$$\int_\Omega Q(x_1,x_2)v^2(x_1,x_2)d\Omega \le 16 \int_\Omega D(x_1,x_2)[v_{,22}(x_1,x_2)]^2 d\Omega. \tag{2.5.71}$$

Proof. Without loss of generality, we suppose that the domain Ω lies inside of the rectangle Π from (2.5.29) and complete the definition of the function v in $\Pi \setminus \Omega$, assuming v equal to zero there.

Evidently, (2.5.69) implies

$$\int_{\Pi} [Qv^2 + D(v_{,22})^2] d\Omega < +\infty,$$

i.e., for almost every fixed x_1, we have

$$v(x_1, \cdot) \in \overset{*}{W}{}^{2,2}(]0, l[, D)$$

(see (1.4.80)) and

$$v(x_1, l) = 0, \quad v_{,2}(x_1, l) = 0.$$

We recall that $l > \max\limits_{(x_1,x_2)\in\bar{\Omega}} \{x_2\}$. Now, we can apply Lemma 1.4.23, i.e.,

$$\int_0^l Q(x_1, x_2) v^2(x_1, x_2) dx_2 \leq 16 \int_0^l D(x_1, x_2)[v_{,22}(x_1, x_2)]^2 d\Omega \tag{2.5.72}$$

for almost every $x_1 \in]a, b[$. Integrating both the sides of (2.5.72) over $]a, b[$, we get (2.5.71).
□

Let

$$\begin{aligned}
\overset{*}{V} \;:=\; & \Big\{ v \in \overset{*}{W}{}^{2,2}(\Omega, D) : v|_{\Gamma_2} = 0,\ \frac{\partial v}{\partial u}\Big|_{\Gamma_2} = 0, \\
& \textit{and additionally} \\
& \textit{either } v|_{\Gamma_1} = 0,\ v_{,2}|_{\Gamma_1} = 0 \ (\textit{if we consider BCs } (2.5.11) \\
& \textit{or } v_{,2}|_{\Gamma_1} = 0 \ (\textit{if we consider BCs } (2.5.12)) \\
& \textit{or } v|_{\Gamma_1} = 0 \ (\textit{if we consider BCs } (2.5.13)) \\
& \textit{in the sense of traces} \Big\}.
\end{aligned} \tag{2.5.73}$$

Definition 2.5.15 *Let $Q^{-\frac{1}{2}} f \in L_2(\Omega)$ and g_1, g_2, w_0, w_0^1 be traces of a prescribed function $u \in \overset{*}{W}{}^{2,2}(\Omega, D)$ and its first derivatives. Let further M_2^0, $Q_2^0 \in L_2(\Gamma_1)$ be also prescribed. A function $w \in \overset{*}{W}{}^{2,2}(\Omega, D)$ will be called a weak solution of Problem 2.5.1 in the space $\overset{*}{W}{}^{2,2}(\Omega, D)$ if it satisfies the following conditions:*

$$w - u \in \overset{*}{V}$$

and (2.5.45) is valid for all $v \in \overset{*}{V}$.

Theorem 2.5.16 *Let* $2h\rho Q^{-1} \in C(\overline{\Omega})$ *and*

$$\omega^2 < \frac{1-\nu}{16 \max\limits_{\overline{\Omega}} 2h\rho Q^{-1}}.$$

Then there exists a unique weak solution w *of Problem 2.5.1, which satisfies*

$$\begin{aligned} \|w\|_{\overset{*}{W}^{2,2}(\Omega,D)} &\le C[\|Q^{-\frac{1}{2}} f\|_{L_2(\Omega)} + \|u\|_{\overset{*}{W}^{2,2}(\Omega,D)} \\ &\quad + \gamma_1 \|M_2^0\|_{L_2(\Gamma_1)} + \gamma_2 \|Q_2^0\|_{L^2(\Gamma_1)}] \end{aligned}$$

with a constant C *independent of* f, u, M_2^0 *and* Q_2^0.

Proof is similar to the proof of Theorem 1.4.25 and Theorem 2.5.12 (taking into account Lemma 2.5.14). □

Remark 2.5.17 *Evidently, the analogous to Section 2.5 investigation can be carried out for the cusped orthotropic plate considered in Sections 2.1, 2.2, 2.4.*

Appendix

Figures	$1-6$	*display examples of beams with one cusped end.*
Figures	$7-10$	*display beams with both cusped ends.*
Figures	$11-19$	*display longitudinal sections of cusped beams and profiles of cusped plates. The arrows denote tangents to the beam longitudinal sections and plate profiles boundaries at the cusped end (edge).*
Figures	20	*displays the projection of the plate into the* x_1x_2-*plane.*
Figures	$21-23$	*display examples of cusped plates.*

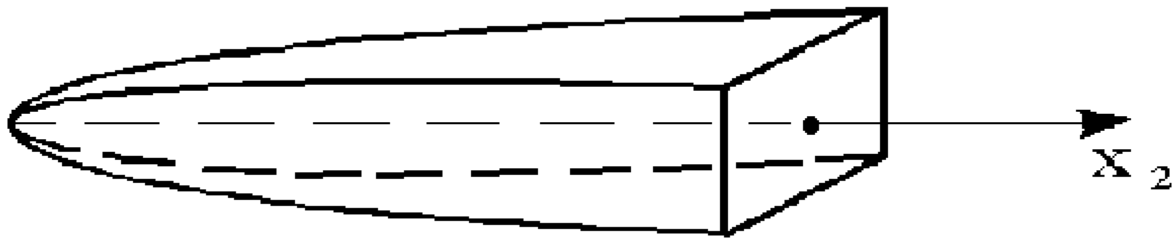

Fig. 1

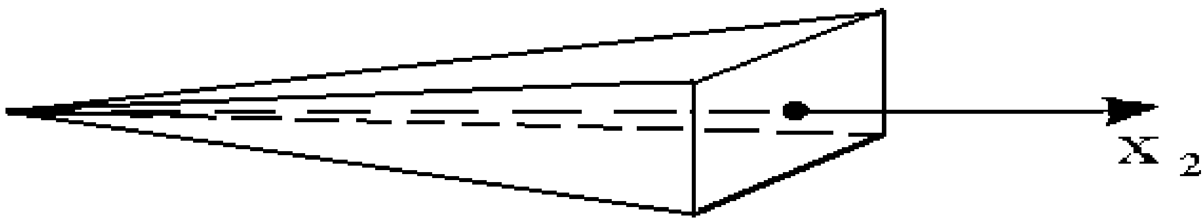

Fig. 2

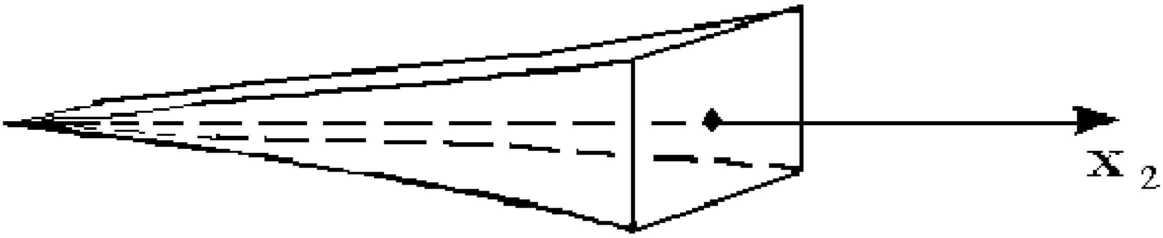

Fig. 3

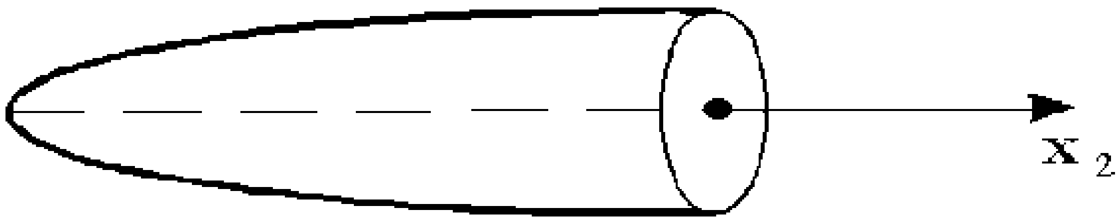

Fig. 4

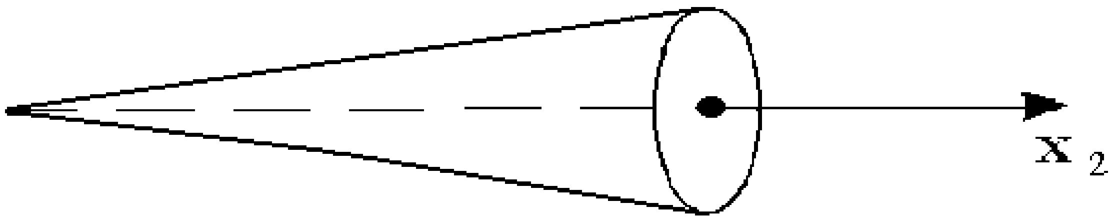

Fig. 5

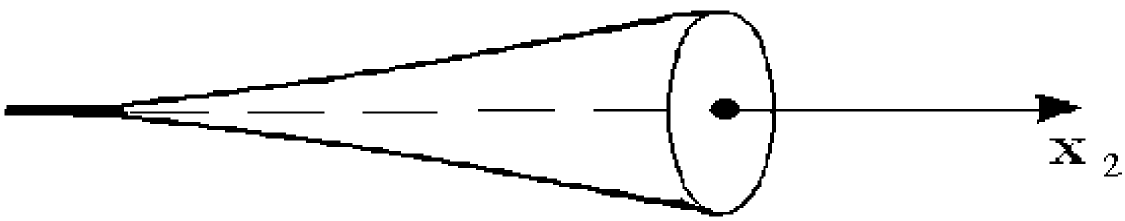

Fig. 6

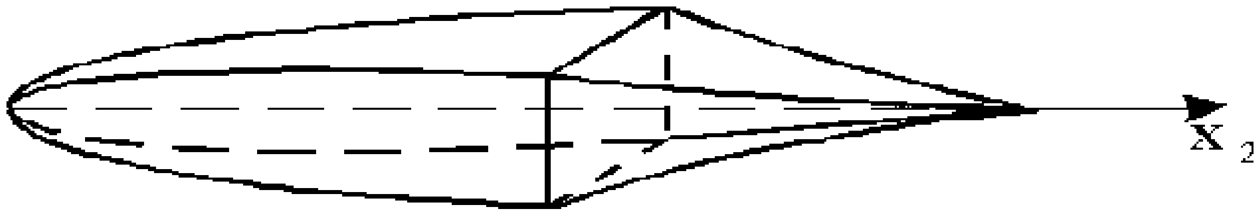

Fig. 7

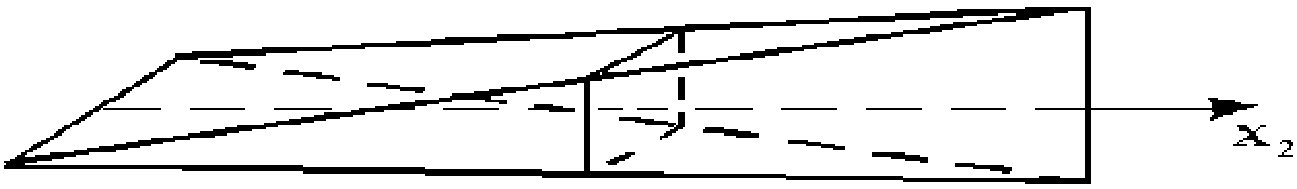

Fig. 8

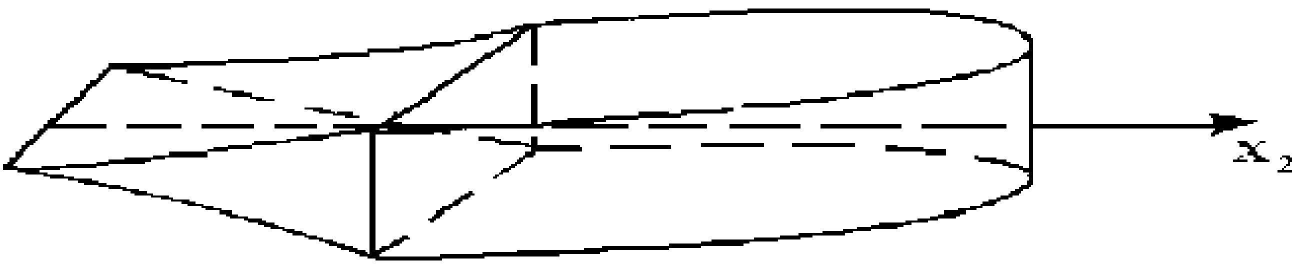

Fig. 9

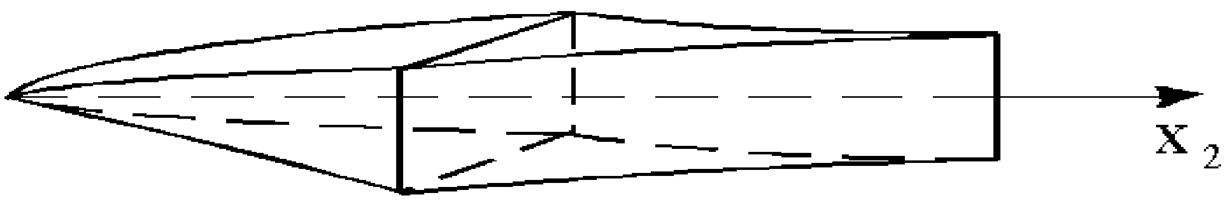

Fig. 10

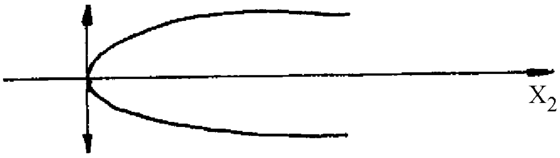

Fig. 11

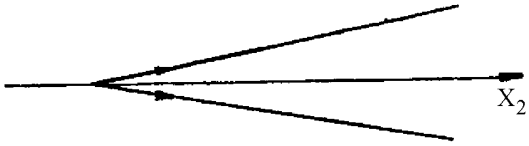

Fig. 12

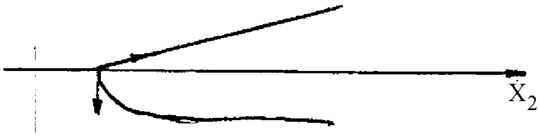

Fig. 13

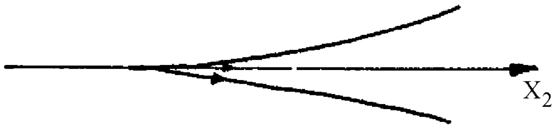

Fig. 14

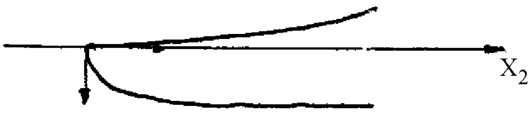

Fig. 15

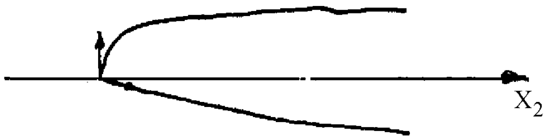

Fig. 16

Fig. 17

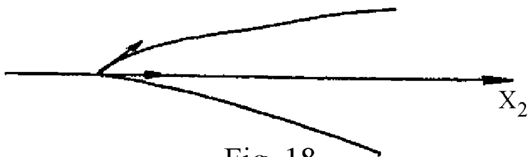

Fig. 18

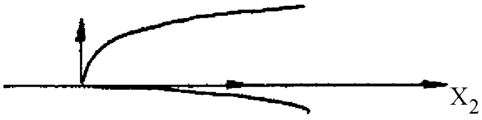

Fig. 19

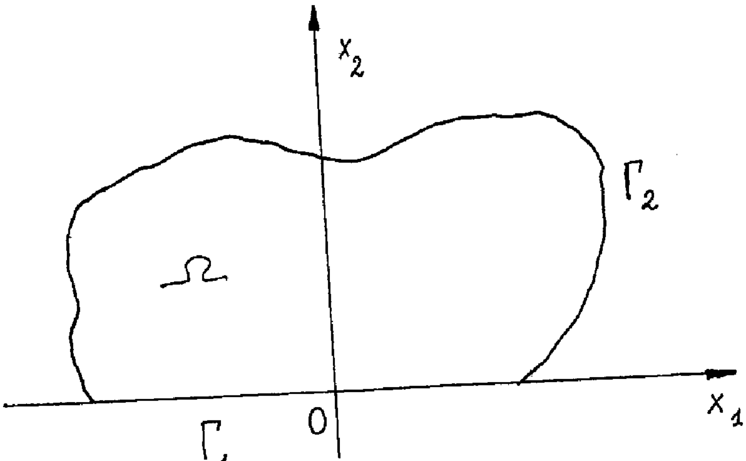

Fig. 20

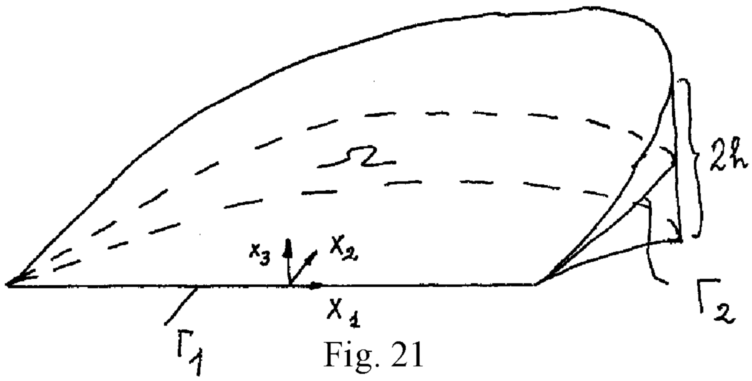

Fig. 21

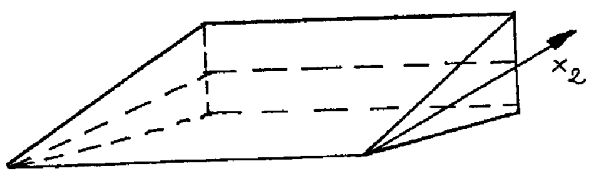

Fig. 22

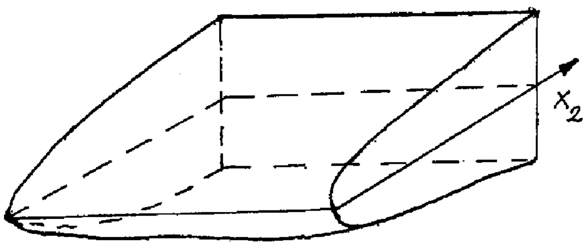

Fig. 23

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