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ON A BOUNDARY VALUE PROBLEM FOR THE ELLIPTIC EQUATION IN A UNIT DISK

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Let $D = \{z||z| < 1\}$– be the unit disk in a complex plane with boundary $\Gamma = \partial D$. We consider in D the following elliptic equation

$$\sum_{k=0}^{N} A_k \frac{\partial^N u}{\partial x^k \partial y^{N-k}} = 0, \tag{1}$$

where A_k are the complex constants. The roots λ_k of the corresponding to (1) characteristic equation $\sum\limits_{k=0}^{N} A_k \lambda^{N-k} = 0$, satisfy the conditions

$$\lambda_k = \overline{\lambda}_{M+k}, \quad k = 1, \dots, M, \quad \lambda_k \neq \overline{\lambda}_j, \quad k, j = 2M+1, \dots, N. \tag{2}$$

Without loss of generality we suppose, that $\Im \lambda_k > 0$ when $k = 1, \dots, M$. We seek the solution u of the equation (1), which belongs to the class $C^N(D) \bigcap C^{(N-M-1,\alpha)}(D \bigcup \Gamma)$ and on the boundary Γ satisfies following boundary value conditions

$$\frac{\partial^k u}{\partial r^k} = f_k(x, y), \quad k = 0, \dots, M-1, \quad (x, y) \in \Gamma, \tag{3}$$

$$\Re \frac{\partial^k u}{\partial r^k} = f_k(x, y), \quad k = M, \dots, N-M-1, \quad (x, y) \in \Gamma. \tag{4}$$

Here $f_k \in C^{(N-M-k-1,\alpha)}(\Gamma)$ $(k = 0, \dots, N-M-1)$– are prescribed functions on Γ, $0 \leq 2M \leq N$. If $M = 0$ or $N = 2M$ then the conditions (3) or (4) respectively are missing.

The boundary conditions (3), (4) were considered in the case, when the number R_1 of the roots of equation (2) with positive imaginary part is not equal to the number R_2 of the roots of characteristic equation with negative imaginary part, i. e., when the equation (1) is improperly elliptic; in these works if $R_1 > R_2$, then $R_1 - R_2$ conditions are Riemann type conditions (4), and remaining $2R_2$ conditions are symmetric Dirichlet type conditions (3). The special case of the equation (1):

$$\frac{\partial^N u}{\partial z^M \partial \bar{z}^{N-M}}(x, y) = 0, \quad (x, y) \in G,$$

where G is a simply connected domain with a smooth boundary, was studied. In the paper it has been proved that inhomogeneous problem (1), (3), (4) has a solution, and the corresponding homogeneous problem has $(N-2M)^2$ linearly independent pure imaginary solutions. For this special case the characteristic equation has the roots $\lambda_1 = \ldots = \lambda_M = i$ and $\lambda_{M+1} = \ldots = \lambda_N = -i$, that is the equation (1) is improperly elliptic (if $N - 2M > 0$) and, from other side, these roots satisfy the conditions (2). Here we will show, that the problem (1), (3), (4) is well posed for arbitrary equation (1) with the roots, satisfying the conditions (2). We prove the following

Theorem 1. *The boundary value problem (1), (3), (4) is Nötherian. The inhomogeneous problem (1), (3), (4) has a solution for arbitrary boundary functions, and the corresponding homogeneous problem (when $f_k \equiv 0$) has $(N-2M)^2$ linearly independent solutions. The general solution of homogeneous problem is a pure imaginary polynomial of order $2(N-M)-2$.*