SHORT COMMUNICATIONS

DIRICHLET PROBLEM FOR THE MARGUERRE-VON KÁRMÁN EQUATIONS SYSTEM

Devdariani G., Janjgava R., Gulua B.

I. Vekua Institute of Applied Mathematics
of Iv. Javakhishvili Tbilisi State University

Abstract: In the present paper the nonlinear boundary value problem for the system of the Marguerre-von Karman equations is considered. Using the general theorem of Banach spaces, the existence of solutions has been proved.

Key words: Shallow shell, Marguerre-von Karman equations, existence theorem

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1. Introduction

The paper deals with the question of existence of solutions to the nonlinear boundary value problem of the Marguerre-von Karman equations. These equations describe the strong bending of the shallow shells. The Marguerre-von Karman equations are due to Marguerre [1] and von Karman & Tsien [2]. As shown by Ciarlet & Paumier [3], the method of formal asymptotic expansions, applied in the form of the displacement-stress approach, may be also used for justification of the Marguerre-von Karman equations.

The general theorem of Banach spaces [4] for the proof of existence of solutions has been used. This method has been used by Dubinski [4], [5] and Skripnik [6] for the different non-linear equations .

Let (e_i) denote the basis of the Euclidean space R^3, and let ω be a domain in the plane spanned on the vectors $\mathbf{e}_\alpha$. Assume that ω is bounded and connected and that its boundary γ is smooth enough. We donote by $Ox_1x_2x_3$ Cartesian coordinates and let

$$\partial_\alpha := \frac{\partial}{\partial x_\alpha}, \quad \alpha = 1, 2.$$

Let $\theta(x_1, x_2) : \overline{\omega} \to R$ be a function of class C^2 such that

$$\partial_\alpha \theta = 0 \quad \text{along} \ \ \gamma, \ \ \alpha = 1, 2.$$

$x_3 = h\theta(x_1, x_2), (x_1, x_2) \in \overline{\omega}$, is the equation of the middle surface of the shell, where h is a semi-thickness of the shell.

The system of equations under consideration for homogeneous isotropic shallow shells has the following form

$$\begin{aligned}&\frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\Delta^2\zeta-[\chi,\zeta+\theta]=p \quad \text{in} \quad \omega,\\&\frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\Delta^2\chi+[\zeta,\zeta+2\theta]=0 \quad \text{in} \quad \omega,\end{aligned} \tag{1}$$

where $h\zeta(x_1,x_2)$ is the deflection, $h^2\chi(x_1,x_2)$ is the Airy stress function, $h^4p(x_1,x_2)$ is function given in ω, $\lambda>0$ and $\mu>0$ are the Lame constants, $\Delta^2:=\partial_{1111}+2\partial_{1122}+\partial_{2222}$ is the two-dimensional biharmonic operator and

$$[\chi,\psi]:=\partial_{11}\chi\partial_{22}\psi+\partial_{22}\chi\partial_{11}\psi-2\partial_{12}\chi\partial_{12}\psi.$$

Dirichlet boundary conditions for equations (1) look like

$$\zeta=\chi=0,\ \ \partial_1\zeta=\partial_2\zeta=0,\ \ \partial_1\chi=\partial_2\chi=0,\ \ \text{on} \quad \gamma. \tag{2}$$

Let us reformulate the problem (1), (2) as the following equivalent problem

$$\begin{aligned}&\frac{8(\lambda+\mu)}{3k(\lambda+2\mu)}\Delta^2\zeta-[\chi^*,\zeta+\theta]=p^* \quad \text{in} \quad \omega,\\&\frac{k(\lambda+\mu)}{3\lambda+2\mu}\Delta^2\chi^*+[\zeta,\zeta+2\theta]=0 \quad \text{in} \quad \omega,\end{aligned} \tag{1*}$$

$$\zeta=\chi^*=0,\quad \partial_1\zeta=\partial_2\zeta=0,\ \ \partial_1\chi^*=\partial_2\chi^*=0 \quad \text{on} \quad \gamma \tag{2*}$$

where $\chi^*=\dfrac{\chi}{k\mu}$, $p^*=\dfrac{p}{k\mu}$, $k>0$ should be some real constant such that

$$min\left(\frac{8(\lambda+\mu)}{3k(\lambda+2\mu)},\frac{k(\lambda+\mu)}{3\lambda+2\mu)}\right)$$

be maximal. It takes place when

$$k=\sqrt{\frac{8(3\lambda+2\mu)}{3(\lambda+2\mu)}}.$$

Let us represent the problem $(1^*),(2^*)$ as the operational problem

$$A(u):=L(u)+B(u)+N(u)=f,$$

where $u=(\zeta(x_1,x_2),\chi^*(x_1,x_2))$ is the vector function,

$$L(u):=\frac{2\sqrt{2}(\lambda+\mu)}{\sqrt{3(\lambda+2\mu)(3\lambda+2\mu)}}(\Delta^2\zeta,\Delta^2\chi^*),$$

$$B(u):=(-[\chi^*,\theta],2[\zeta,\theta]),$$

$$N(u):=(-[\chi^*,\zeta],[\zeta,\zeta]),$$

$$f:=(p^*(x_1,x_2),0).$$

2. The Existence Theorem

In order to show the existence of solutions of the problem (1), (2) we remined the general theorem of the Banach spaces:

Theorem 1. *Let X is the separable and reflexive Banach space and let X^* is the dual space of that. Assume that the generally non-linear operator $A(u) : X \to X^*$ satisfies the following conditions:*

1. The condition of coerciveness. For any $u \in X$

$$\frac{< A(u), u >}{\|u\|_X} \to +\infty, \text{ when } \|u\|_X \to +\infty.$$

2. The condition of weakly compactness. If $u_n \rightharpoonup u$ weakly in X, then for any $v \in X$

$$\lim_{m\to\infty} < A(u_m), v >=< A(u), u >,$$

where u_m is the some subsequence of u_n.

Then for any $h \in X^*$ equation

$$A(u) = h$$

has at least one solution u in X.

Theorem 2. *Let $\theta(x_1, x_2) \in C^2(\overline{\omega})$, $\max_{(x_1,x_2)\in\overline{\omega}}(\theta(x_1, x_2) - æ) < \frac{2\sqrt{2}(\lambda+\mu)}{\sqrt{3(\lambda+2\mu)(3\lambda+2\mu)}}$, where $æ = \frac{1}{2}\left(\max_{(x_1,x_2)\in\overline{\omega}} \theta(x_1, x_2) + \min_{(x_1,x_2)\in\overline{\omega}} \theta(x_1, x_2)\right)$, $p^* \in W_2^{-2}(\omega)$, then the problem $(1^*), (2^*)$ has at least one generalized solution $\zeta \in \overset{0}{W}{}^2_2$, $\chi^* \in \overset{0}{W}{}^2_2$.*

Proof. Because of $dim\omega = 2$, $\theta \in C^2(\overline{\omega})$, and $\overset{0}{W}{}^2_2(\omega) \subset C(\overline{\omega})$ we have $B(u) \in L_2(\omega) \subset W_2^{-2}(\omega) = X^*$ and $N(u) \in L_1(\omega) \subset W_2^{-2}(\omega) = X^*$. It is evident, that $L(u) : X \to X^*$. Thus, $A(u) : X \to X^*$.

For any $u \in \overset{0}{W}{}^2_2(\omega)$ we have

$$< Lu, u >= \frac{2\sqrt{2}(\lambda + \mu)}{\sqrt{3(\lambda + 2\mu)(3\lambda + 2\mu)}}\|u\|_X. \quad (3)$$

For all $\chi^*, \zeta \in \overset{0}{W}{}^2_2(\omega)$ and $\theta \in C^2(\omega)$ we have $(dx := dx_1dx_2)$

$$\int_\omega [\chi^*, \theta]\zeta dx = \int_\omega [\zeta, \theta]\chi^* dx,$$

$$\int_\omega [\chi^*, \theta]\zeta dx = \int_\omega [\chi^*, \zeta]\theta dx, \quad (4)$$

$$\int_\omega [\chi^*, \zeta] dx = 0.$$

By means of the formulas (4) we show

$$|< B(u), u >| = \left| \int\limits_{\omega} [\chi^*, \zeta](\theta - æ) dx \right| =$$

$$= \left| \int\limits_{\omega} (\partial_{11}\chi^* \partial_{22}\zeta + \partial_{22}\chi^* \partial_{11}\zeta - 2\partial_{12}\chi^* \partial_{12}\zeta)(\theta - æ) dx \right| \leq$$

$$\leq \max_{(x_1,x_2)\in\overline{\omega}} (\theta(x_1, x_2) - æ) \int\limits_{\omega} \left\{ \frac{1}{2}[(\partial_{11}\chi^*)^2 + (\partial_{22}\zeta)^2 + (\partial_{22}\chi^*)^2 + (\partial_{11}\zeta)^2] + \right.$$

$$\left. +(\partial_{12}\chi^*)^2 + (\partial_{12}\zeta)^2 \right\} dx \leq \max_{(x_1,x_2)\in\overline{\omega}} (\theta(x_1, x_2) - æ) \|u\|_X^2,$$

Now, we have

$$< L(u), u > + < B(u), u > \geq c\|u\|_X^2, \tag{5}$$

where

$$c := \frac{2\sqrt{2}(\lambda + 2\mu)}{\sqrt{3(\lambda + 2\mu)(3\lambda + 2\mu)}} - \max_{(x_1,x_2)\in\overline{\omega}} (\theta(x_1, x_2) - æ).$$

In view of (5), $L(u) + B(u)$ is coercive. As $L(u)$ and $B(u)$ are the linear bounded operators, the condition of coercivennes is fulfilled for them.

Let us show that a non-linear operator $N(u)$ is orthogonal, i.e., for any $u \in \overset{0}{W}{}_2^2(\omega)$

$$< N(u), u >= 0.$$

Let $\zeta(x_1, x_2) \in D(\omega)$, $\chi^*(x_1, x_2) \in D(\omega)$, then

$$-\int\limits_{\omega} [\chi^*, \zeta]\zeta dx = \int\limits_{\omega} [\partial_{22}\chi^*(\partial_1\zeta)^2 - 2\partial_{12}\chi^*\partial_1\zeta\partial_2\zeta + \partial_{11}\chi^*(\partial_2\zeta)^2] dx, \tag{6}$$

$$\int\limits_{\omega} [\zeta, \zeta]\chi^* dx = -\int\limits_{\omega} [\partial_{22}\chi^*(\partial_1\zeta)^2 - 2\partial_{12}\chi^*\partial_1\zeta\partial_2\zeta + \partial_{11}\chi^*(\partial_2\zeta)^2] dx. \tag{7}$$

By (6) and (7), for any finite function $< N(u), u >= 0$. As $\overline{D(\omega)} = \overset{0}{W}{}_2^2(\omega)$ and $\overset{0}{W}{}_2^2(\omega) \subset C(\overline{\omega})$ $(dim\omega = 2)$, operator $N(u)$ is orthogonal for all $u \in \overset{0}{W}{}_2^2(\omega)$.

Since the operator $N(u)$ is orthogonal, the operator $A(u)$ is coercive.

Now, we show that the operator $N(u)$ is weakly compact.

Lemma 3. *For any* $\zeta(x_1, x_2) \in \overset{0}{W}{}_2^2(\omega)$ $\chi^* \in \overset{0}{W}{}_2^2(\omega)$ we have

$$\begin{aligned} \int\limits_{\omega} [\chi^*, \zeta] v dx = -\int\limits_{\omega} [&\partial_{22}\chi^*\partial_1\zeta\partial_1 v - \partial_{12}\chi^*(\partial_1\zeta\partial_2 v - \partial_2\zeta\partial_1 v) \\ &+\partial_{11}\chi^*\partial_2\zeta\partial_1 v] dx; \end{aligned} \tag{8}$$

$$\int\limits_{\omega}[\zeta,\zeta]v dx = -\int\limits_{\omega}[\partial_1\zeta\partial_{12}\zeta\partial_1 v + \partial_2\zeta\partial_{12}\zeta\partial_1 v - \partial_1\zeta\partial_2\zeta\partial_{12}v]dx. \qquad (9)$$

Lemma 4. *If $u_n(x) \rightharpoonup u(x)$ weakly in $L_2(\omega)$, $q_n(x) \to q(x)$ in $L_2(\omega)$, then for any bounded $v(x)$ function*

$$\int\limits_{\omega} u_n q_n v dx \to \int\limits_{\omega} uqv dx.$$

By virtue of (8), (9) and the lemma 4 we get the weakly compactness of $N(u)$.

Now the proof of the theorem 2 immediately follows from the theorem 1. ■

R e f e r e n c e s

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