

BITSADZE-SAMARSKY NONLOCAL BOUNDARY VALUE PROBLEM  
FOR A MIXED TYPE EQUATION WITH A PARABOLIC  
DEGENERATION IN THE DOMAIN

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*Abstract:* In the present work Bitsadze-Samarsky nonlocal problem for second order mixed type equation is investigated. The solution of the problem is represented by Fourier-Bessel series. The necessary condition for solvability of the problem is given.

*Key words:* Bitsadze-Samarsky nonlocal boundary value problem, Mixed type equation, degeneration

*MSC 2000:*

In the joint work of A. Bitsadze and A. Samarsky [1] for Laplace equation in a rectangular domain  $S$  ( $-l \leq x < l$ ,  $0 < y < 1$ ), it is studied a nonclassical, also known as a nonlocal boundary value problem, when the s.c. boundary conditions of the problem establish certain relationship between the values of unknown solution for the certain set of points in the domain and the values of the same function on the certain part of a boundary.

In the above indicated work one class of nonlocal problems for harmonic functions is stated and investigated. This problem is known as Bitsadze-Samarsky problem.

Below we consider analogous nonlocal boundary problem for the second order mixed type equation

$$u_{xx} + y u_{yy} + b u_y = 0, \quad b > 1, \quad b \neq [b] \quad (1)$$

in a rectangular domain  $D$  ( $0 < x < l$ ,  $-q < y < p$ ), where  $l, p, q > 0$  are given numbers. This equation is elliptic in the subdomain  $D^+ = D \cap (y > 0)$ , hyperbolic in the subdomain  $D^- = D \cap (y < 0)$  and parabolically degenerating on the segment  $0 \leq x \leq l$  of the axis  $y = 0$ . The segment of the type degeneration represents an envelope curve for both the sets of characteristic curves of equation (1). Therefore, this line represents a singular characteristic. In this case we say that we have a characteristic degeneration. Under these conditions the first boundary value problem for equation (1) in domain  $\bar{D}$  does not have a regular solution (first of all since the first boundary value problem for mixed and hyperbolic equations is posed incorrectly, and the most important, when  $b > 1$  the solution of equation (1) together with the derivative with respect to the variable  $y$  tend to infinity on the segment of parabolic degeneration, and the asymptotic form is defined by the numerical value of the coefficient  $b$ ). For these cases A. Bitsadze offered the conditions of weighted boundedness for the solution and its first order derivatives on the

line of equation degeneration [1]. Taking this fact into account in papers [6-12] it is studied the first boundary value problem for equation (1) on the straight line of degeneration with the conditions of weight compatibility of the solution and its derivative with respect to the variable  $y$ :

$$\begin{aligned} \lim_{y \rightarrow 0} y^{b-1} u^+(x, y) &= \lim_{y \rightarrow 0} (-y)^{b-1} u^-(x, y), \\ \lim_{y \rightarrow 0} y^b \frac{\partial u^+(x, y)}{\partial y} &= \lim_{y \rightarrow 0} (-y)^b \frac{\partial u^-(x, y)}{\partial y}, \end{aligned} \tag{2}$$

where  $u^+(x, y)$ ,  $u^-(x, y)$  are the solutions of equation (1) in subdomains  $D^+$ ,  $D^-$ , respectively. In the paper [11] for equation (1), in domain  $D$ , it is considered the first boundary problem with boundary conditions (2) and the following sufficient conditions for solvability are received:

$$\frac{\sqrt{q}}{l} \neq \frac{j_{b-1,m}}{2\pi n}, \quad m, n = 1, 2, \dots, \tag{3}$$

where  $j_{b-1,m}$  are the roots of the Bessel first kind functions  $J_{b-1}(z)$ . In the present paper we study the following Bitsadze-Samarsky type nonlocal problem for equation (1): *Find in domain  $D$  solution  $u(x, y)$  of (1), which satisfies the following boundary conditions*

$$u(0, y) = u(b, y) = 0, \tag{4}$$

$$u^+(x, p) = \varphi(x), \tag{5}$$

*nonlocal condition*

$$u^-(x, -q) + u^-(x, -\rho) = f(x) \quad 0 < \rho < q \tag{6}$$

*and the conditions of compatibility (2).*

The functions  $\varphi(x)$  and  $f(x)$  prolonged odd on segment  $[-l, 0]$  and with the period  $2l$  on whole straight line we denote by  $\vartheta(x)$  and  $F(x)$ , respectively. Let  $\vartheta(x)$  and  $F(x)$  be continuous up the order  $k$ ; the function  $\varphi(x)$  satisfy the conditions  $\varphi_{(0)}^{(k)} = \varphi_{(l)}^{(k)}$  ( $k = 0, 2, 4$ ), while for the function  $f(x)$ ,  $f_{(0)}^{(k)} = f_{(l)}^{(k)}$  ( $k = 0, 2$ ) is assumed. As it is well-known for Fourier series expansion of such periodic functions the coefficients converge to zero with orders  $O(n^{-4})$  and  $O(n^{-2})$ , respectively [4]. Note that for  $\rho = q$  the nonlocal problem coincides with the first boundary value problem. As it is known [11] for equation (1) the Dirichlet's problem in domain  $D$  with homogeneous boundary conditions on the sides of the rectangle and on the line of the type degeneration with conditions (2) can be represented by Fourier-Bessel series, which has the form

$$u^+(x, y) = (2\sqrt{y})^{1-b} \sum_{n=0}^{\infty} \left[ a_n I_{1-b} \left( \frac{2\pi n}{l} \sqrt{y} \right) + b_n I_{b-1} \left( \frac{2\pi n}{l} \sqrt{y} \right) \right] \sin \frac{\pi n}{l} x \tag{7}$$

in domain  $D^+$ , and the form

$$u^-(x, y) = (2\sqrt{-y})^{1-b} \sum_{n=0}^{\infty} \left[ k_n J_{1-b} \left( \frac{2\pi n}{l} \sqrt{-y} \right) + d_n J_{b-1} \left( \frac{2\pi n}{l} \sqrt{-y} \right) \right] \sin \frac{\pi n}{l} x \quad (8)$$

in domain  $D^-$ , where  $J_{1-b}$ ,  $J_{b-1}$ ,  $I_{1-b}$ ,  $I_{b-1}$  are the first and second kind Bessel functions respectively, while  $a_n$ ,  $b_n$ ,  $k_n$ ,  $d_n$  are arbitrary constants of integration [6]. Satisfying conditions (2), (5), (6) by series (7) and (8), we receive the following non-homogeneous systems of linear algebraic equations:

$$\frac{\left(\frac{2\pi n}{l}\right)^{1-b}}{\Gamma(2-b)} a_n - \frac{\left(\frac{2\pi n}{l}\right)^{1-b}}{\Gamma(2-b)} k_n = 0, \quad \frac{2^{1-b} \left(\frac{\pi n}{l}\right)^{-b}}{\Gamma(1-b)} a_n + \frac{2^{1-b} \left(\frac{\pi n}{l}\right)^{-b}}{\Gamma(1-b)} k_n = 0, \quad (9)$$

$$(2\sqrt{p})^{1-b} I_{1-b} \left( \frac{2\pi n}{l} \sqrt{p} \right) a_n + (2\sqrt{p})^{1-b} I_{b-1} \left( \frac{2\pi n}{l} \sqrt{p} \right) b_n = \varphi_n,$$

$$(2\sqrt{q})^{1-b} J_{1-b} \left( \frac{2\pi n}{l} \sqrt{q} \right) k_n + (2\sqrt{q})^{1-b} J_{b-1} \left( \frac{2\pi n}{l} \sqrt{q} \right) d_n$$

$$+ (2\sqrt{\rho})^{1-b} J_{1-b} \left( \frac{2\pi n}{l} \sqrt{\rho} \right) k_n + (2\sqrt{\rho})^{1-b} J_{b-1} \left( \frac{2\pi n}{l} \sqrt{\rho} \right) d_n = f_n,$$

$$n = 1, 2, \dots,$$

where  $\varphi_n$ ,  $f_n$  are Fourier coefficients of the functions  $\vartheta(x)$  and  $F(x)$ , while  $\Gamma$  is the Euler function.

From the system (9) we receive  $a_n = k_n = 0$ ,

$$b_n = \frac{\varphi_n}{(2\sqrt{p})^{1-b} I_{b-1} \left( \frac{2\pi n}{l} \sqrt{p} \right)},$$

$$d_n = \frac{f_n}{(2\sqrt{q})^{1-b} J_{b-1} \left( \frac{2\pi n}{l} \sqrt{q} \right) + (2\sqrt{\rho})^{1-b} J_{b-1} \left( \frac{2\pi n}{l} \sqrt{\rho} \right)},$$

where we see that for determination of the constants  $d_n$  it is necessary that

$$-J_{b-1} \left( \frac{2\pi n}{l} \sqrt{\rho} \right) \neq \sqrt{\frac{q}{\rho}}^{1-b} J_{b-1} \left( \frac{2\pi n}{l} \sqrt{q} \right), \quad n = 1, 2, \dots \quad (10)$$

Therefore, the functions (7) and (8) take the form

$$u^+(x, y) = (2\sqrt{y})^{1-b} \sum_{n=0}^{\infty} \varphi_n \frac{I_{b-1} \left( \frac{2\pi n}{l} \sqrt{y} \right)}{(2\sqrt{p})^{1-b} I_{b-1} \left( \frac{2\pi n}{l} \sqrt{p} \right)} \sin \frac{\pi n}{l} x, \quad (11)$$

$$u^-(x, y) = (2\sqrt{-y})^{1-b} \times \sum_{n=0}^{\infty} f_n \frac{J_{b-1} \left( \frac{2\pi n}{l} \sqrt{-y} \right)}{(2\sqrt{\rho})^{1-b} J_{b-1} \left( \frac{2\pi n}{l} \sqrt{\rho} \right) + (2\sqrt{q})^{1-b} J_{b-1} \left( \frac{2\pi n}{l} \sqrt{q} \right)} \sin \frac{\pi n}{l} x, \quad (12)$$

respectively. Let us rewrite functions (11) and (12) as follows

$$u^+(x, y) = \sum_{n=0}^{\infty} \varphi_n \Phi_n(y) \sin \frac{\pi n}{l} x, \tag{13}$$

$$u^-(x, y) = \sum_{n=0}^{\infty} f_n \Psi_n(y) \sin \frac{\pi n}{l} x, \tag{14}$$

where we introduce notations

$$\Phi_n(y) = \frac{(2\sqrt{y})^{1-b} I_{b-1} \left( \frac{2\pi n}{l} \sqrt{y} \right)}{(2\sqrt{p})^{1-b} I_{b-1} \left( \frac{2\pi n}{l} \sqrt{p} \right)}, \tag{15}$$

$$\Psi_n(y) = \frac{(2\sqrt{-y})^{1-b} J_{b-1} \left( \frac{2\pi n}{l} \sqrt{-y} \right)}{(2\sqrt{p})^{1-b} J_{b-1} \left( \frac{2\pi n}{l} \sqrt{p} \right) + (2\sqrt{q})^{1-b} J_{b-1} \left( \frac{2\pi n}{l} \sqrt{q} \right)}. \tag{16}$$

First let us consider the function series (13) and show its convergence. Functions  $\Phi_n(y)$  represent the quotient  $s$  of Bessel first kind modified functions  $I_\nu(z)$ .

As it is known, the functions  $I_\nu(z)$  are increasing monotonic functions, and the product  $(2\sqrt{y})^{1-b} I_{b-1} \left( \frac{2\pi n}{l} \sqrt{y} \right)$  is increasing as well, which equals  $\left( \frac{\pi n}{l} \right)^{b-1} / \Gamma(b)$  for  $y = 0$ . Therefore,

$$\frac{\left( \frac{\pi n}{l} \right)^{b-1}}{\Gamma(b)} \leq \sqrt{y}^{1-b} I_{b-1} \left( \frac{2\pi n}{l} \sqrt{y} \right) \leq \sqrt{p}^{1-b} I_{b-1} \left( \frac{2\pi n}{l} \sqrt{p} \right). \tag{17}$$

Thus, we get inequality

$$|\Phi_n(y)| < 1.$$

Therefore, for general term of series (13) we have the estimate

$$\left| \varphi_n \Phi_n(y) \sin \frac{\pi n}{l} x \right| < |\varphi_n|.$$

Since  $\varphi_n$  ( $n = 1, 2, \dots$ ) are the Fourier coefficients of the smooth boundary function  $\varphi(x)$ , then the series of numbers

$$\sum_{n=0}^{\infty} |\varphi_n|$$

is convergent. It is a majorant for the function series (13). This means that series (13) is uniformly and absolutely convergent and its sum  $u^+(x, y)$  is continuous in subdomain  $D^+$ .

Now, consider the general term of the series received by differentiation of the series (13) with respect to the variable  $y$ :

$$\frac{\partial}{\partial y} \left[ \varphi_n \Phi_n(y) \sin \frac{\pi n}{l} x \right] = \varphi_n \Phi'_n(y) \sin \frac{\pi n}{l} x.$$

Taking into account the well-known formula for differentiation of the Bessel function (see [3])

$$\frac{d}{dz} [z^{-n} I_n(z)] = z^{-n} I_{n+1}(z), \quad (18)$$

we obtain

$$\Phi'_n(y) = \frac{\sqrt{y}^{-b} I_b \left( \frac{2\pi n}{l} \sqrt{y} \right)}{\sqrt{p}^{1-b} I_{b-1} \left( \frac{2\pi n}{l} \sqrt{p} \right)} \cdot \frac{\pi n}{l}.$$

By virtue of the basic properties of the first kind modified Bessel functions, for the product  $\sqrt{y}^{-b} I_b \left( \frac{2\pi n}{l} \sqrt{y} \right)$  we have the following inequality

$$\frac{\left( \frac{\pi n}{l} \right)^b}{\Gamma(b+1)} \leq \sqrt{y}^{-b} I_b \left( \frac{2\pi n}{l} \sqrt{y} \right) \leq \sqrt{p}^{-b} I_b \left( \frac{2\pi n}{l} \sqrt{p} \right).$$

which is similar to (17).

In view of the following properties [3] of the first kind modified Bessel functions

$$I_n(z) \leq I_m(z), \quad n > m, \quad (19)$$

we get

$$\frac{\sqrt{y}^{-b} I_b \left( \frac{2\pi n}{l} \sqrt{y} \right)}{\sqrt{p}^{1-b} I_{b-1} \left( \frac{2\pi n}{l} \sqrt{p} \right)} \leq \frac{\sqrt{p}^{-b} I_b \left( \frac{2\pi n}{l} \sqrt{p} \right)}{\sqrt{p}^{1-b} I_{b-1} \left( \frac{2\pi n}{l} \sqrt{p} \right)} \leq 1.$$

Therefore, we have the estimate

$$\sum_{n=0}^{\infty} |\varphi_n \Phi'_n(y) \sin \frac{\pi n}{l} x| \leq \sum_{n=0}^{\infty} \frac{\pi n}{l} |\varphi_n|.$$

This estimate ensures uniform and absolute convergence of the series received by differentiation with respect to  $y$ , and also the continuity of its sum.

Thus, the following equality

$$u_y^+(x, y) = \sum_{n=0}^{\infty} \varphi_n \Phi'_n(y) \sin \frac{\pi n}{l} x \quad (20)$$

is valid.

Let us take into account the formula (18) and differentiate the series (20):

$$\sum_{n=0}^{\infty} \varphi_n \Phi_n''(y) \sin \frac{\pi n}{l} x, \tag{21}$$

where

$$\Phi_n''(y) = \frac{\sqrt{y}^{-(b+1)} I_{b+1} \left( \frac{2\pi n}{l} \sqrt{y} \right)}{\sqrt{p}^{1-b} I_{b-1} \left( \frac{2\pi n}{l} \sqrt{p} \right)} \cdot \left( \frac{\pi n}{l} \right)^2.$$

In this case for the expression  $\sqrt{y}^{-(b+1)} I_{b+1} \left( \frac{2\pi n}{l} \sqrt{y} \right)$  the inequality

$$\frac{\left( \frac{\pi n}{l} \right)^{b+1}}{\Gamma(b+q)} \leq \sqrt{y}^{-(b+1)} I_{b+1} \left( \frac{2\pi n}{l} \sqrt{y} \right) \leq \sqrt{p}^{-(b+1)} I_{b+1} \left( \frac{2\pi n}{l} \sqrt{p} \right)$$

is valid.

According to our request on smoothness of periodic functions  $\vartheta(x)$ , we have that

$$\sum_{n=0}^{\infty} n^2 |\varphi_n| \tag{22}$$

is convergent [4]. Due to inequality (19) for the series (20) we obtain

$$\sum_{n=0}^{\infty} \varphi_n \Phi_n''(y) \sin \frac{\pi n}{l} x \leq \sum_{n=0}^{\infty} \left( \frac{\pi n}{l} \right)^2 |\varphi_n|. \tag{23}$$

The majorant series of the function series (23) is convergent as well as series (22) is convergent. Therefore, we conclude that series (21) is absolutely and uniformly convergent. At the same time its sum is a continuous function solution and represents the second order derivative of  $u_{yy}^+(x, y)$ :

$$u_{yy}^+(x, y) = \sum_{n=0}^{\infty} \varphi_n \Phi_n''(y) \sin \frac{\pi n}{l} x.$$

The uniform and absolute convergence of the series received by differentiation of the series (13) twice on variable  $x$

$$u_x^+(x, y) = \sum_{n=0}^{\infty} \varphi_n \Phi_n(y) \frac{\pi n}{l} \cos \frac{\pi n}{l} x,$$

$$u_{xx}^+(x, y) = \sum_{n=0}^{\infty} \varphi_n \Phi_n(y) \left( \frac{\pi n}{l} \right)^2 \sin \frac{\pi n}{l} x,$$

can be easily proved as well. Therefore, the series (11) represents a regular solution in the subdomain  $D^+$ .

Now, consider the series (12) and let us show its uniform and absolute convergence in subdomain  $D^-$ . Let us rewrite the function (16) in the following form

$$\Psi_n(y) = \frac{(2\sqrt{-y})^{1-b} J_{b-1} \left( \frac{2\pi n}{l} \sqrt{-y} \right)}{\Delta(\rho, q)},$$

where

$$\Delta(\rho, q) = (2\sqrt{\rho})^{1-b} J_{b-1} \left( \frac{2\pi n}{l} \sqrt{\rho} \right) + (2\sqrt{q})^{1-b} J_{b-1} \left( \frac{2\pi n}{l} \sqrt{q} \right).$$

Let us transform denominator  $\Delta(\rho, q)$  as follows

$$\begin{aligned} \Delta(\rho, q) &= (2\sqrt{\rho})^{1-b} J_{b-1} \left( \frac{2\pi n}{l} \sqrt{\rho} \right) + (2\sqrt{q})^{1-b} J_{b-1} \left( \frac{2\pi n}{l} \sqrt{q} \right) \\ &= (2\sqrt{\rho})^{1-b} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(b+k)} \left( \frac{\pi n}{l} \sqrt{\rho} \right)^{b-1+2k} \\ &\quad + (2\sqrt{q})^{1-b} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(b+k)} \left( \frac{\pi n}{l} \sqrt{q} \right)^{b-1+2k} \\ &= (2\sqrt{\rho})^{1-b} \left[ \frac{\left( \frac{\pi n}{l} \sqrt{\rho} \right)^{b-1}}{\Gamma(b)} - \frac{\left( \frac{\pi n}{l} \sqrt{\rho} \right)^{b+1}}{1! \Gamma(b+1)} + \dots \right] \\ &\quad + (2\sqrt{q})^{1-b} \left[ \frac{\left( \frac{\pi n}{l} \sqrt{q} \right)^{b-1}}{\Gamma(b)} - \frac{\left( \frac{\pi n}{l} \sqrt{q} \right)^{b+1}}{1! \Gamma(b+1)} + \dots \right] \\ &= 2^{1-b} \left( \frac{\pi n}{l} \right)^{b-1} \left[ \frac{1}{\Gamma(b)} - \frac{\left( \frac{\pi n}{l} \sqrt{q} \right)^2}{1! \Gamma(b+1)} + \dots \right] \\ &\quad + 2^{1-b} \left( \frac{\pi n}{l} \right)^{b-1} \left[ \frac{1}{\Gamma(b)} - \frac{\left( \frac{\pi n}{l} \sqrt{q} \right)^2}{\Gamma(b+1)} + \dots \right] \\ &= 2^{1-b} \left( \frac{\pi n}{l} \right)^{b-1} \left\{ \left[ \frac{1}{\Gamma(b)} + R_1(\rho) \right] + \left[ \frac{1}{\Gamma(b)} + R_1(q) \right] \right\}, \end{aligned}$$

where  $R_1(\rho)$  and  $R_1(q)$  are the first remainders of the alternating-sign series in square brackets:

$$R_1(\rho) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \left( \frac{\pi n}{l} \sqrt{\rho} \right)^{2k+2}}{(k+1)! \Gamma(b+1+k)},$$

$$R_1(q) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\pi n}{l} \sqrt{q}\right)^{2k+2}}{(k+1)! \Gamma(b+1+k)}.$$

The terms of these series are monotonically decreasing by absolute value when  $n \rightarrow \infty$  and its general term converges to zero, therefore they belong to Leibnitz type series. It is well-known that the remainder of such series has a sign of its first term and by absolute value it is less then first term [5]. Taking into account this fact, we have inequalities

$$|R_1(\rho)| < \left| \frac{\left(\frac{\pi n}{l} \sqrt{\rho}\right)^2}{\Gamma(b+1)} \right|, \quad |R_1(q)| < \left| \frac{\left(\frac{\pi n}{l} \sqrt{q}\right)^2}{\Gamma(b+1)} \right|.$$

Therefore,

$$\begin{aligned} \Delta(\rho, q) &> 2^{1-b} \left(\frac{\pi n}{l}\right)^{b-1} \left\{ \left[ \frac{1}{\Gamma(b)} - \frac{\left(\frac{\pi n}{l} \sqrt{\rho}\right)^2}{1! \Gamma(b+1)} \right] + \left[ \frac{1}{\Gamma(b)} - \frac{\left(\frac{\pi n}{l} \sqrt{q}\right)^2}{1! \Gamma(b+1)} \right] \right\} \\ &= 2^{1-b} \left(\frac{\pi n}{l}\right)^{b-1} \left[ \frac{2}{\Gamma(b)} - \frac{\left(\frac{\pi n}{l} \sqrt{\rho}\right)^2 (\rho + q)}{\Gamma(b+1)} \right] \\ &= \frac{2^{1-b} (\pi n)^{b-1}}{b \Gamma(b) l^{b+1}} [2 b l^2 - (\pi n)^2 (\rho + q)]. \end{aligned} \tag{24}$$

The value of function  $\Psi_n(y)$  for  $y = 0$  is equal to  $(2\sqrt{-y})^{1-b} J_{b-1} \left(\frac{2\pi n}{l} \sqrt{-y}\right)$  and it converges to zero when  $n \rightarrow \infty$ . Hence, the numerator, as the first kind Bessel function, is bounded from top by concrete number

$$\inf_{\substack{n \in \mathbb{N} \\ y \in [0, -q]}} \left| \sqrt{-y}^{1-b} J_{b-1} \left(\frac{2\pi n}{l} \sqrt{-y}\right) \right| = B. \tag{25}$$

Therefore, due to inequality (24) for the series (14) we have the estimate

$$\sum_{n=0}^{\infty} \left| f_n \Psi_n(y) \sin \frac{\pi n}{l} x \right| \leq B_1 \sum_{n=0}^{\infty} \left| \frac{f_n}{(\pi n)^{b-1} [2 b l^2 - (\pi n)^2 (\rho + q)]} \right|, \tag{26}$$

where

$$B_1 = 2^{b-1} B b \Gamma(b) l^{b+1}.$$

The convergence of the majoring series (26) ensures the uniform and absolute convergence of the series (12) and the continuity of the function  $u^-(x, y)$  in subdomain  $D^-$ .



Let us differentiate the series (14) with respect to the variable  $y$

$$\sum_{n=0}^{\infty} f_n \Psi'_n(y) \sin \frac{\pi n}{l} x, \quad (27)$$

where, due to the formula (18),

$$\Psi'_n(y) = \frac{(2\sqrt{-y})^{-b} J_b\left(\frac{2\pi n}{l} \sqrt{-y}\right)}{\Delta(\rho, q)} \cdot \frac{\pi n}{l}.$$

The expression  $(2\sqrt{-y})^{-b} J_b\left(\frac{2\pi n}{l} \sqrt{-y}\right)$  equals to  $\left(\frac{\pi n}{l}\right)^b / \Gamma(b+1)$  when  $y = 0$  and it converges to zero when  $n \rightarrow \infty$ . Therefore, as for (25), it is bounded from top by the concrete number  $M$

$$\inf_{\substack{n \in \mathbb{N} \\ y \in [0, -q]}} \left| 2\sqrt{-y}^{-b} J_b\left(\frac{2\pi n}{l} \sqrt{-y}\right) \right| < M.. \quad (28)$$

As it appears for the series received by differentiation with respect to the variable  $y$  we have the estimate

$$\begin{aligned} & \sum_{n=0}^{\infty} \left| f_n \Psi'_n(y) \sin \frac{\pi n}{l} x \right| \\ & \leq M b \Gamma(b) 2^{b-1} l^{b+1} \sum_{n=0}^{\infty} \left| \frac{f_n}{(\pi n)^{-b-1} [2bl^2 - (\pi n)^2(\rho + q)]} \cdot \frac{\pi n}{l} \right|. \end{aligned}$$

The convergence of the majorant numerical series of the right-hand side ensures the uniform and absolute convergence of the differentiated series and the continuity of the function  $u_y^-(x, y)$ .

Therefore,

$$u_y^-(x, y) = \sum_{n=0}^{\infty} f_n \Psi'_n(y) \sin \frac{\pi n}{l} x.$$

Let us apply the formula (18) again and differentiate (27) with respect to the variable  $y$  once again, we get

$$\sum_{n=0}^{\infty} f_n \Psi''_n(y) \sin \frac{\pi n}{l} x, \quad (29)$$

where

$$\Psi''_n(y) = \frac{(2\sqrt{-y})^{-(b+1)} J_{b+1}\left(\frac{2\pi n}{l} \sqrt{-y}\right)}{\Delta(\rho, q)} \cdot \left(\frac{\pi n}{l}\right)^2.$$

For the expression  $(2\sqrt{-y})^{-(b+1)} J_{b+1} \left( \frac{2\pi n}{l} \sqrt{-y} \right)$  inequality

$$\inf_{\substack{n \in \mathbb{N} \\ y \in [0, -q]}} \left| 2\sqrt{-y}^{-(b+1)} J_{b+1} \left( \frac{2\pi n}{l} \sqrt{-y} \right) \right| < M_1$$

is valid, which is analogous to (28).

Thus, due to (24) for the series (29) we have the estimate

$$\sum_{n=0}^{\infty} \left| f_n \Psi_n''(y) \sin \frac{\pi n}{l} x \right| \leq M_2 \sum_{n=0}^{\infty} \left| \frac{f_n}{(\pi n)^{b-1} [2bl^2 - (\pi n)^2(\rho + q)]} \left( \frac{\pi n}{l} \right)^2 \right|,$$

where

$$M_2 = M_1 b \Gamma(b) 2^{b-1} l^{b+1}.$$

The right-hand side is a numerical series, whose convergence is provided by the smoothness of the function  $F(x)$ . This means that the series received from the series (12) by double differentiation with respect to the variable  $y$  is uniformly and absolutely convergent. Its sum is a continuous function:

$$u_{yy}^-(x, y) = \sum_{n=0}^{\infty} f_n \Psi_n''(y) \sin \frac{\pi n}{l} x.$$

The uniform and absolute convergence of the series (12) differentiated twice variable  $x$  can be easily proved. Therefore, the function  $u^-(x, y)$ , represented by the series (12), is the regular solution of the problem (1-2), (4-6) in subdomain  $D^-$ .

We note that the function  $\vartheta(x)$  constructed by the boundary function  $\varphi(x)$  by periodic continuation, is continuously differentiable up to the fourth order, while the function  $F(x)$  is continuously differentiable up to the second order.

Thus, the following **Theorem** is proved: *If condition (10) is valid, then nonlocal boundary value problem (2), (4-6) for equation (1) has a unique solution in the class of functions represented by Fourier-Bessel series (11), (12).*

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