AN ELASTIC GREEN MATRIX FOR A SEMISTRIP

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Abstract. A Green matrix is constructed for one mixed two–dimensional problem of the elasticity theory. The results of work [1] are thereby defined more correctly.

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Assume that that we are seeking for a solution of the system of equations of the elasticity theory

$$\begin{aligned}(\lambda+2\mu)\,\frac{\partial^2 u_1}{\partial x^2}+\mu\,\frac{\partial^2 u_1}{\partial y^2}+(\lambda+\mu)\,\frac{\partial^2 u_2}{\partial x\partial y}&=f_1(x,y),\\ (\lambda+\mu)\,\frac{\partial^2 u_1}{\partial x\partial y}+\mu\,\frac{\partial^2 u_2}{\partial x^2}+(\lambda+2\mu)\,\frac{\partial^2 u_2}{\partial y^2}&=f_2(x,y)\end{aligned}\qquad(1)$$

in a domain $-\infty<x\le 0$, $0\le y\le b$, under the following boundary conditions, interesting from the standpoint of application

$$\mu\left(\frac{\partial u_1}{\partial y}\,(x,\alpha b)+\frac{\partial u_2}{\partial x}\,(x,\alpha b)\right)=0,\quad u_2(x,\alpha b)=0,$$
$$-\infty<x\le 0,\quad \alpha=0,1,$$
$$u_1(0,y)=0,\quad \mu\left(\frac{\partial u_1}{\partial y}\,(0,y)+\frac{\partial u_2}{\partial x}\,(0,y)\right)=0,\quad 0\le y\le b,$$

and that for $x\to-\infty$ a solution is bounded.

Suppose that $f_i(x,y)\in L_1((-\infty,0\,],[0,b\,]\,)$ and $\int_{-\infty}^{x}\int_0^b f_i(\xi,\eta)d\xi d\eta=O((1-x)^{-1})$ as $x\to-\infty,\ i=1,2$.

To obtain the Green matrix of the posed problem we will use the method and notation of the paper [1]. Let us define the vectors $\mathbf{U}(x,y)=(u_1(x,y),u_2(x,y))$ and $\mathbf{F}(x,y)=(f_1(x,y),f_2(x,y))$ (here and in what follows the vector transposition sign is omitted). Assume that

$$\mathbf{U}(x,y)=\textstyle\sum_{n=0}^{\infty}Q_n(y)\mathbf{U}_n(x),\quad \mathbf{F}(x,y)=\sum_{n=0}^{\infty}Q_n(y)\mathbf{F}_n(x),$$
$$Q_n(y)=\mathrm{diag}(\cos\nu y,\sin\nu y),\quad \mathbf{U}_n(x)=(u_{1n}(x),u_{2n}(x))\,,\qquad(3)$$
$$\mathbf{F}_n(x)=(f_{1n}(x),f_{2n}(x))\,,\quad \nu=\tfrac{\pi n}{b}\,.$$

Owing to such a representation, conditions (2.1) are fulfilled. From (1)–(3) we conclude that to find a pair of functions u_{1n}, u_{2n}, $n=1,2,\ldots$, it is required to solve the system of ordinary differential equations

$$\begin{aligned} (\lambda+2\mu)u''_{1n}-\mu\nu^2 u_{1n}+(\lambda+\mu)\nu u'_{2n} &= f_{1n}, \\ -(\lambda+\mu)\nu u'_{1n}+\mu u''_{2n}-(\lambda+2\mu)\nu^2 u_{2n} &= f_{2n} \end{aligned} \tag{4}$$

under the boundary conditions by which

$$u_{1n}(0)=0, \quad u'_{2n}(0)=0, \tag{5}$$

must be fulfilled at the point $x=0$, while for $x\to-\infty$, u_{1n} and u_{2n} are bounded.

As to the case $n=0$, here we have only one sought for function $u_{10}(x)$ satisfying the first equation in (4) and the first equality in (5) and bounded for $x\to-\infty$.

To solve problem (4),(5), we use the well-known method [2]. First we consider the homogeneous system corresponding to (4). The system of its fundamental solutions forms the rectangular matrix $\Phi_n=(\phi^n_{ij})$, $i=1,2$, $j=1,2,3,4$, whose elements are

$$\begin{aligned} &\phi^n_{11}=-\phi^n_{21}=e^{\nu x}, \quad \phi^n_{12}=\phi^n_{22}=e^{-\nu x}, \\ &\phi^n_{i3}=\left[(-1)^i(\lambda+\mu)\nu x+(i-1)(\lambda+3\mu)\right]e^{\nu x}, \\ &\phi^n_{i4}=\left[(\lambda+\mu)\nu x-(i-1)(\lambda+3\mu)\right]e^{-\nu x}, \quad i=1,2. \end{aligned}$$

Using fundamental solutions, we come to the conclusion that the general solution of system (4) has the form

$$\mathbf{U}_n(x)=\int_{-\infty}^{x} S_n(x,\xi)\mathbf{F}_n(\xi)\,d\xi+\Phi_n(x)D_n, \quad n=1,2,\dots, \tag{6}$$

where $S_n(x,\xi)=(s^n_{ij})$ is the second order matrix, $i,j=1,2$, whose elements are defined by the relations

$$\begin{aligned} s^n_{ii}(x,\xi) &= \frac{m}{2}\Big(\frac{1}{\nu}(\lambda+3\mu)\,\mathrm{sh}\nu(\mathrm{x}-\xi) \\ &\qquad +(-1)^i(\lambda+\mu)(x-\xi)\mathrm{ch}\nu(\mathrm{x}-\xi)\Big), \ \ \mathrm{i}=1,2, \\ -s^n_{12}(x,\xi) &= s^n_{21}(x,\xi)=\frac{m}{2}(\lambda+\mu)(x-\xi)\,\mathrm{sh}\nu(\mathrm{x}-\xi), \\ m &= (\mu(\lambda+2\mu))^{-1}, \end{aligned} \tag{7}$$

and $D_n=(d_i)$ is the column-matrix of arbitrary constants, $i=1,2,3,4$.

Taking into account (6), let us choose elements D_n so that (5) and the condition of the boundedness of u_{1n} and u_{2n} for $x\to-\infty$ be fulfilled. As a result, we obtain

$$\begin{aligned} d_1 &= \frac{m}{2}\int_{-\infty}^{0}\Big[\Big(\frac{1}{\nu}(\lambda+3\mu)\,\mathrm{sh}\nu\xi-(\lambda+\mu)\xi\mathrm{ch}\nu\xi\Big)f_1(\xi) \\ &\qquad +(\lambda+\mu)\xi\mathrm{sh}\nu\xi\mathrm{f}_2(\xi)\Big]\,\mathrm{d}\xi, \\ d_2 &= d_4=0, \quad d_3=\frac{m}{2\nu}\int_{-\infty}^{0}(\mathrm{sh}\nu\xi\mathrm{f}_1(\xi)-\mathrm{ch}\nu\xi\mathrm{f}_2(\xi))\,d\xi. \end{aligned} \tag{8}$$

Formulas (6)–(8) imply

$$\mathbf{U}_n(x) = \int_{-\infty}^{0} g_n(x,\xi)\mathbf{F}_n(\xi)\, d\xi, \tag{9}$$

where $g_n(x,\xi) = (g_{ij}^n(x,\xi))$ is the second order matrix, $i,j = 1,2$, moreover,

$$g_{ij}^n(x,\xi) = \begin{cases} \gamma_{ij}^n(x,\xi) + s_{ij}^n(x,\xi) & \text{for } \mathrm{x} \geq \xi, \\ \gamma_{ij}^n(x,\xi) & \text{for } \mathrm{x} \leq \xi, \end{cases} \tag{10}$$

and

$$\begin{aligned} \gamma_{11}^n(x,\xi) &= m\left(p(x-\xi) - q(x-\xi) - p(x+\xi) + q(x+\xi)\right), \\ \gamma_{12}^n(x,\xi) &= m\left(p(x-\xi) + p(x+\xi)\right), \\ \gamma_{21}^n(x,\xi) &= m\left(p(x+\xi) - p(x-\xi)\right), \\ \gamma_{22}^n(x,\xi) &= -m\left(p(x-\xi) + q(x-\xi) + p(x+\xi) + q(x+\xi)\right), \end{aligned} \tag{11}$$

with the notation

$$p(u) = \frac{1}{4}\,(\lambda+\mu)ue^{\nu u}, \quad q(u) = \frac{1}{4\nu}\,(\lambda+3\mu)e^{\nu u}.$$

Formula (9) holds for $n = 0$ too, where $g_0(x,\xi) = (g_{ij}^0(x,\xi))$, $i,j = 1,2$, and in the cases $x \geq \xi$ and $x \leq \xi$ g_{11}^0 is equal respectively to $m\mu x$ and $m\mu\xi$, while for other elements of the matrix $g_0(x,\xi)$ we have $g_{12}^0 = g_{21}^0 = g_{22}^0 = 0$ in both cases.

By virtue of (3) and (9)

$$\mathbf{U}(x,y) = \int_{-\infty}^{0}\int_{0}^{b} G(x,y,\xi,\eta)\mathbf{F}(\xi,\eta)\, d\xi\, d\eta,$$

where the sought Green matrix $G(x,y,\xi,\eta)$ is defined by the relation

$$G(x,y,\xi,\eta) = \frac{1}{b}\sum_{n=0}^{\infty} \varepsilon_n Q_n(y) g_n(x,\xi) Q_n(\eta). \tag{12}$$

Here ε_n is equal to 1 for $n = 0$, and to 2 in other cases. To obtain an explicit form of the elements $G_{ij}(x,y,\xi,\eta)$ of the matrix $G(x,y,\xi,\eta)$, $i,j = 1,2$, we use (7), (10)–(12), the well-known formulas [3]

$$\begin{aligned} &\sum_{n=1}^{\infty} t^n \cos n\theta = (1 - t\cos\theta)(1 - 2t\cos\theta + t^2)^{-1}, \\ &\sum_{n=1}^{\infty} t^n n^{-1} \cos n\theta = -\frac{1}{2}\,\ln(1 - 2t\cos\theta + t^2), \\ &t^2 < 1, \quad 0 < \theta < 2\pi, \end{aligned}$$

and the notation

$$z=x+iy,\;\; \zeta=\xi+i\eta,\;\; \omega(u)=e^{\frac{\pi u}{b}},\;\; P(u)=\mathrm{Re}(1-\omega(\mathrm{u})),\;\; \mathrm{S(u)}=\mathrm{Im}\omega(\mathrm{u}),$$
$$E(u)=|1-\omega(u)|^2,\;\; Q(u)=P(u)E^{-1}(u),\;\; T(u)=S(u)E^{-1}(u).$$

When $x\le\xi$, for the diagonal elements we have

$$G_{ii}(x,y,\xi,\eta)=\tfrac{2-i}{b}\,m\mu\mathrm{Re}\zeta+\tfrac{\mathrm{m}}{8\pi}\,(\lambda+3\mu)\,\ln\Big[\mathrm{E(z-\zeta)E(z-\overline{\zeta})}\Big(\mathrm{E(z+\zeta)}\times$$
$$\times E(z+\overline{\zeta})\Big)^{(-1)^i}\Big]-\tfrac{m}{4b}\,(\lambda+\mu)\textstyle\sum_{k=-1,+1}k^i\mathrm{Re(z+k\zeta)}\left(\mathrm{Q(z+k\zeta)+Q(z+k\overline{\zeta})}\right),$$
$$i=1,2,$$

while the nondiagonal ones are defined by the formula

$$G_{ij}(x,y,\xi,\eta)=\frac{m}{4b}\,(\lambda+\mu)\sum_{k=-1,+1}k^i\mathrm{Re(z+k\zeta)}\left(\mathrm{T(z+k\zeta)-T(z+k\overline{\zeta})}\right),$$
$$i,j=1,2,\quad i\ne j.$$

For $x\ge\xi$ these relations readily imply formulas for $G_{ij}(x,y,\xi,\eta)$ if we use the well-known property of symmetry: in the diagonal and nondiagonal elements we should make the replacement $x\leftrightarrow\xi$ and, in addition to this, the nondiagonal elements should be interchanged. We obtain

$$G_{ii}(x,y,\xi,\eta)=\frac{2-i}{b}\,m\mu\mathrm{Rez}+\frac{\mathrm{m}}{8\pi}\,(\lambda+3\mu)\,\ln\Big[\mathrm{E(-z+\zeta)E(-z+\overline{\zeta})}\;\times$$
$$\times\Big(E(z+\zeta)E(z+\overline{\zeta})\Big)^{(-1)^i}\Big]-\frac{m}{4b}\,(\lambda+\mu)\sum_{k=-1,+1}k^{i+1}\,\mathrm{Re(z+k\zeta)(Q(kz+\zeta)}+$$
$$+Q(kz+\overline{\zeta})),\qquad i=1,2,$$
$$G_{ij}(x,y,\xi,\eta)=\frac{m}{4b}\,(\lambda+\mu)\sum_{k=-1,+1}k^{i+1}\mathrm{Re(z+k\zeta)}\left(\mathrm{T(kz+\zeta)-T(kz+\overline{\zeta})}\right),$$
$$i,j=1,2,\quad i\ne j.$$

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