Bulletin of TICMI
Vol. 16, No. 1, 2012, 15–26

The Basic Boundary Value Problems of the Theory of Consolidation With Double Porosity for the Sphere

Basheleishvili Mikheil and Bitsadze Lamara*

Ilia State University,32,I.Chavchavadze Av., Tbilisi, 0179, Georgia

(*Received September 21, 2011; Accepted March 2, 2012*)

The purpose of this paper is to solve explicitly the basic first and second boundary value problems (BVPs) of the theory of consolidation with double porosity for the sphere and for the whole space with a spherical cavity. The obtained solutions are represented as absolutely and uniformly convergent series.

Keywords: Porous media, Double porosity.

AMS Subject Classification: 74G05, 74G10

Introduction

Theory of consolidation with double porosity has been proposed by Aifantis (see, e.g.,[1],[2]). In a material with two degrees of porosity, there are two pore systems, the primary and the secondary. For example in fissured rocks (i.e., a mass of porous blocks separated from each other by an interconnected and continuously distributed system of fissures) most of the porosity is provided by the pores of the blocks of the primary porosity, while most of permeability is provided by the fissures of the secondary porosity. When fluid flows and deformation processes occur simultaneously , three coupled partial differential equations can be derived [1],[2] to describe the relationships between governing pressure in the primary and secondary pores (and, therefore, the mass exchange between them) and the displacement of the solid. Inertia effects are neglected like Biot's theory.

The physical and mathematical foundations of the theory of double porosity were considered in the papers [1]-[3]. R. K. Wilson and E. C. Aifantis [1] gave detailed physical interpretations of the phenomenological coefficients appearing in the double porosity theory. They also solved several representative boundary value problems. Uniqueness and variational principle were established by D. E. Beskos and E. C. Aifantis [2] for the equations of double porosity, while Khaled, Beskos and Aifantis [3] provided a related finite element to consider the numerical solution of Aifantis' equations of double porosity (see [1],[2],[3] and the references therein). The basic results and the historical information on the theory of porous media were summarized by Boer [4].

The main goal of the present investigation is to construct explicitly, in the form of absolutely and uniformly convergent series, the solutions of the first

*Corresponding author. Email: lamarabits@yahoo.com

ISSN: 1512-0082 print

and the second boundary value problems (BVPs) of the theory of consolidation with double porosity for the sphere and for the whole space with spherical cave.

1. Formulation of Boundary Value Problems and uniqueness theorems

The basic Aifantis equations of statics of the theory of consolidation with double porosity are given by the partial differential equations in the following form [1], [2]

$$\mu\Delta u+(\lambda+\mu)graddivu-grad(\beta_1 p_1+\beta_2 p_2)=0, \tag{1.1}$$

$$(m_1\Delta-k)p_1+kp_2=0, \quad kp_1+(m_2\Delta-k)p_2=0, \tag{1.2}$$

where $u:=(u_1,u_2,u_3)$ is the displacement vector, p_1 and p_2 are the fluid pressures within the primary and the secondary pores, respectively. The constant λ is the Lame modulus, μ is the shear modulus and the constants β_1 and β_2 measure the change of porosities due to an applied volumetric strain. $m_j=\dfrac{k_j}{\mu^*}, j=1,2$. The constants k_1 and k_2 are the permeabilities of the primary and secondary systems of pores, the constant μ^* denotes the viscosity of the pore fluid and the constant k measures the transfer of fluid from the secondary pores to the primary pores. The quantities $\lambda, \quad \mu, \quad k, \quad \beta_j, \quad k_j \quad (j=1,2)$ and μ^* are positive constants. $\triangle$ is the Laplace operator.

Let D^+ be the ball, with the radius a , bounded by the spherical surface S. Denote by D^- the whole space with a spherical cave.

Definition 1. A vector-function $U(x)=(u_1,u_2,u_3,p_1,p_2)$ defined in the domain $D^+(D^-)$ is called regular if it has integrable continuous second derivatives in $D^+(D^-)$, and U itself and its first order derivatives are continuously extendable at every point of the boundary of $D^+(D^-)$, i.e., $U\in C^2(D^+)\bigcap C^1(\overline{D^+})$, $(U\in C^2(D^-)\bigcap C^1(\overline{D^-}))$. Note that for the infinite domain D^- the vector $U(x)$ additionally satisfies the following conditions at infinity:

$$U(x)=O(|x|^{-1}), \quad \frac{\partial U_k}{\partial x_j}=O(|x|^{-2}), \quad |x|^2=x_1^2+x_2^2+x_2^3, \quad j=1,2,3. \tag{1.3}$$

For system (1.1),(1.2) we pose the following boundary value problems:

Find a regular vector U, satisfying in $D^+(D^-)$ system (1.1),(1.2), and on the boundary S one of the following conditions:

Problem $(I)^{\pm}$ The displacement vector and the fluid pressures are given

$$u^{\pm}(z)=f(z)^{\pm}, \quad p_1^{\pm}(z)=f_4^{\pm}, \quad p_2^{\pm}(z)=f_5^{\pm}(z), \quad z\in S,$$

where $f^{\pm}(f_1,f_2,f_3)\in C^{1,\alpha}(S)$, $f_k^{\pm}\in C^{1,\alpha}(S)$, $0<\alpha\leq 1$, $k=4,5$, are prescribed functions;

Problem $(II)^{\pm}$ The stress vector and the normal derivatives of the pressure

$\dfrac{\partial p_j}{\partial n}$ are given

$$(Pu)^{\pm} = f(z)^{\pm}, \quad \left(\frac{\partial p_1(z)}{\partial n}\right)^{\pm} = f_4^{\pm}, \quad \left(\frac{\partial p_2(z)}{\partial n}\right)^{\pm} = f_5^{\pm}(z), \quad z \in S,$$

where $f^{\pm}(f_1, f_2, f_3) \in C^{1,\alpha}(S)$, $f_k^{\pm} \in C^{1,\alpha}(S)$, $0 < \alpha \leq 1$, $k = 4, 5$, are prescribed functions, Pu is a stress vector, which acts on an element of the S with the normal $n := (n_1, n_2, n_3)$,

$$P(\partial x, n)u := T(\partial x, n)u - n(\beta_1 p_1 + \beta_2 p_2), \tag{1.4}$$

here $T(\partial x, n)$ is a stress tensor [7]

$$T(\partial x, n) :=\| T_{kj}(\partial x, n) \|_{3\times 3},$$

$$T_{kj}(\partial x, n) := \mu\delta_{kj}\frac{\partial}{\partial n} + \lambda n_k \frac{\partial}{\partial x_j} + \mu n_j \frac{\partial}{\partial x_k}, \quad k, j, = 1, 2, 3. \tag{1.5}$$

Further we assume that p_j, $j = 1, 2$, are known, when $x \in D^+$ or $x \in D^-$ and rewrite (1.1) in the following form

$$\mu\Delta u + (\lambda + \mu) graddivu = grad(\beta_1 p_1 + \beta_2 p_2).$$

A particular solution of equation (1.1) is the following potential [7]

$$u_0(x) = -\frac{1}{4\pi}\iiint\limits_D \Gamma(x - y) grad(\beta_1 p_1 + \beta_2 p_2) dy, \tag{1.6}$$

where

$$\Gamma(x - y) = \frac{1}{4\mu(\lambda + 2\mu)} \left\| \frac{(\lambda + 3\mu)\delta_{kj}}{r} + \frac{(\lambda + \mu)(x_k - y_k)(x_j - y_j)}{r^3} \right\|_{3\times 3},$$

$$r^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2.$$

Substituting the volume potential u_0 into (1.1), we obtain

$$\mu\Delta u_0 + (\lambda + \mu) graddivu_0 = grad(\beta_1 p_1 + \beta_2 p_2). \tag{1.7}$$

Thus, we have proved that $u_0(x)$ is a particular solution of equation (1.1). In (1.6) D denotes either D^+ or D^-, $grad(\beta_1 p_1 + \beta_2 p_2)$ is a continuous vector along with its first derivatives in D^+. When $D = D^-$ that the vector $grad(\beta_1 p_1 + \beta_2 p_2)$ has to satisfy the following condition at infinity

$$grad(\beta_1 p_1 + \beta_2 p_2) = O(|x|^{-2-\alpha}), \alpha > 0.$$

Thus, the general solution of system (1.1) is representable in the form $u := V + u_0$, where

$$A(\partial x)V := \mu\Delta V + (\lambda+\mu) graddiv V = 0. \tag{1.8}$$

The last equation is the equation of an isotropic elastic body. So, we reduced the solution of basic BVPs of the theory of consolidation with double porosity to the solution of the basic BVPs for the equation of an isotropic elastic body.

2. Some Auxiliary Formulas

The spherical coordinates are defined by the equalities

$$\begin{array}{l} x_1 = \rho\sin\vartheta\cos\varphi, \quad x_2 = \rho\sin\vartheta\sin\varphi, \quad x_3 = \rho\cos\vartheta, \quad x \in D^+, \\ y_1 = a\sin\vartheta_0\cos\varphi_0, \quad y_2 = a\sin\vartheta_0\sin\varphi_0, \quad y_3 = a\cos\vartheta_0, \quad y \in S, \\ \rho^2 = x_1^2 + x_2^2 + x_3^2, \quad 0 \le \vartheta \le \pi, \quad 0 \le \varphi \le 2\pi. \end{array} \tag{2.1}$$

Let

$$f(\theta,\varphi) := \sum_{m=0}^{\infty} f_m(\vartheta,\varphi), \quad z \in S,$$

where f_m is the spherical function of order m :

$$f_m = \frac{2m+1}{4\pi a^2}\int_S P_m(\cos\gamma) f(y) dS_y,$$

P_m is the Legendre polynomial of the m-th order, γ is an angle between the radius-vectors Ox and Oy,

$$\cos\gamma = \frac{1}{|x||y|}\sum_{l=1}^{3} x_l y_l.$$

The general solutions of the equation $(\Delta - \lambda_0^2)\psi = 0$ in the domains D^+ and D^- have, correspondingly, the following forms ([6])

$$\begin{aligned} \psi(x) &= \sum_{n=0}^{\infty} \frac{J_{n+\frac{1}{2}}(i\lambda_0\rho)}{\sqrt{\rho}} Y_n(\vartheta,\varphi), \quad \rho < a, \\ \psi(x) &= \sum_{n=0}^{\infty} \frac{H^{(2)}_{n+\frac{1}{2}}(i\lambda_0\rho)}{\sqrt{\rho}} Y_n(\vartheta,\varphi), \quad \rho > a, \end{aligned} \tag{2.2}$$

where $J_{n+\frac{1}{2}}(i\lambda_0\rho)$ is the Legendre function, $H^{(2)}_{n+\frac{1}{2}}(i\lambda_0 a)$ is the second kind Hankel function, $Y_n(\vartheta,\varphi)$ is the spherical harmonic,

$$\lambda_0^2=\frac{k}{m_1}+\frac{k}{m_2}>0.$$

The general solutions of the equation $\Delta\phi=0$ in the domains D^+ and D^- have, correspondingly, the following forms (see [5], p.505)

$$\begin{aligned}\phi(x)&=\sum_{n=0}^{\infty}\frac{\rho^n}{(2n+1)a^{n-1}}Z_n(\vartheta,\varphi),\quad \rho<a,\\ \phi(x)&=\sum_{n=0}^{\infty}\frac{a^{n+2}}{(2n+1)\rho^{n+1}}Z_n(\vartheta,\varphi),\quad \rho>a,\end{aligned}\tag{2.3}$$

$Z_n(\theta,\phi)$ is the spherical harmonic.

It is easily seen that the general solution of system (1.2) is representable in the form

$$p_1=-m_2\psi+\phi,\quad p_2=m_1\psi+\phi,\tag{2.4}$$

where ψ and ϕ are arbitrary solutions of the following equations

$$(\Delta-\lambda_0^2)\psi=0,\quad \Delta\phi=0.$$

The following theorems are valid and we state them without proof.

Theorem 1. The first boundary value problem has at most one regular solution in the domains $D^+(D^-)$.

Theorem 2. A regular solution of the second boundary value problem is not unique in the domain D^+. Two regular solutions may differ by vector (u,p_1,p_2), where $u(x)=a+b\times x+c(\beta_1+\beta_2)x$, and $p_j(x)=c, j=1,2$, $x\in D^+$, a and b are constant vectors, while c is an arbitrary constant.

Theorem 3. The boundary value problem $(II)^-$ has a unique solution in the domain D^-.

3. Solution of the First Boundary Value Problem

Problem $(I)^+$. First of all we will construct a solution for system (1.2). A solution of the first boundary value problem ($p_1^+(z)=f_4^+,\ p_2^+(z)=f_5^+(z)$,) is sought in the following form:

$$\begin{aligned}p_1&=-m_2\sum_{n=0}^{\infty}\frac{J_{n+\frac12}(i\lambda_0\rho)}{\sqrt{\rho}}Y_n(\vartheta,\varphi)+\sum_{n=0}^{\infty}\frac{\rho^n}{(2n+1)a^{n-1}}Z_n(\vartheta,\varphi),\quad \rho<a,\\ p_2&=m_1\sum_{n=0}^{\infty}\frac{J_{n+\frac12}(i\lambda_0\rho)}{\sqrt{\rho}}Y_n(\vartheta,\varphi)+\sum_{n=0}^{\infty}\frac{\rho^n}{(2n+1)a^{n-1}}Z_n(\vartheta,\varphi),\quad \rho<a.\end{aligned}\tag{3.1}$$

Passing to the limit in (3.1) as $\rho \to a$, we have

$$-m_2 \sum_{n=0}^{\infty} \frac{J_{n+\frac{1}{2}}(i\lambda_0 a)}{\sqrt{a}} Y_n(\vartheta_0,\varphi_0) + a\sum_{n=0}^{\infty} \frac{1}{(2n+1)} Z_n(\vartheta_0,\varphi_0) = \sum_{n=0}^{\infty} \widehat{f}_{4n}(\vartheta_0,\varphi_0),$$

$$m_1 \sum_{n=0}^{\infty} \frac{J_{n+\frac{1}{2}}(i\lambda_0 a)}{\sqrt{a}} Y_n(\vartheta_0,\varphi_0) + a\sum_{n=0}^{\infty} \frac{1}{(2n+1)} Z_n(\vartheta_0,\varphi_0) = \sum_{n=0}^{\infty} \widehat{f}_{5n}(\vartheta_0,\varphi_0), \tag{3.2}$$

where

$$\widehat{f}_{kn}(\vartheta_0,\varphi_0) = \frac{2n+1}{4\pi a^2} \int_S P_n(\cos\gamma) f_k(y) dS_y, \quad k = 4, 5.$$

For coefficients Y_n and Z_n, (3.2) yield the following equations:

$$\begin{aligned} &-m_2 \frac{J_{n+\frac{1}{2}}(i\lambda_0 a)}{\sqrt{a}} Y_n(\vartheta_0,\varphi_0) + \frac{a}{2n+1} Z_n(\vartheta_0,\varphi_0) = \widehat{f}_{4n}(\vartheta_0,\varphi_0), \\ &m_1 \frac{J_{n+\frac{1}{2}}(i\lambda_0 a)}{\sqrt{a}} Y_n(\vartheta_0,\varphi_0) + \frac{a}{2n+1} Z_n(\vartheta_0,\varphi_0) = \widehat{f}_{5n}(\vartheta_0,\varphi_0), \quad n = 0, 1, ... \end{aligned} \tag{3.3}$$

By elementary calculation from (3.3) we obtain

$$\begin{aligned} Y_n(\vartheta_0,\varphi_0) &= \frac{\widehat{f}_{5n}(\vartheta_0,\varphi_0) - \widehat{f}_{4n}(\vartheta_0,\varphi_0)}{(m_1+m_2) J_{n+\frac{1}{2}}(i\lambda_0 a)} \sqrt{a}, \\ Z_n(\vartheta_0,\varphi_0) &= \frac{(2n+1)[m_1 \widehat{f}_{4n}(\vartheta_0,\varphi_0) + m_2 \widehat{f}_{5n}(\vartheta_0,\varphi_0)]}{a(m_1+m_2)}. \end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.1), we obtain a solution of the BVP in the form of series

$$\begin{aligned} p_1 &= \frac{-m_2\sqrt{a}}{(m_1+m_2)\sqrt{\rho}} \sum_{n=0}^{\infty} \frac{J_{n+\frac{1}{2}}(i\lambda_0 \rho)}{J_{n+\frac{1}{2}}(i\lambda_0 a)} (\widehat{f}_{5n}(\vartheta,\varphi) - \widehat{f}_{4n}(\vartheta,\varphi)) \\ &+ \frac{1}{(m_1+m_2)} \sum_{n=0}^{\infty} \frac{\rho^n}{a^n} [m_1 \widehat{f}_{4n}(\vartheta,\varphi) + m_2 \widehat{f}_{5n}(\vartheta,\varphi)], \\ p_2 &= \frac{m_1\sqrt{a}}{(m_1+m_2)\sqrt{\rho}} \sum_{n=0}^{\infty} \frac{J_{n+\frac{1}{2}}(i\lambda_0 \rho)}{J_{n+\frac{1}{2}}(i\lambda_0 a)} (\widehat{f}_{5n}(\theta,\phi) - \widehat{f}_{4n}(\vartheta,\varphi)) \\ &+ \frac{1}{(m_1+m_2)} \sum_{n=0}^{\infty} \frac{\rho^n}{a^n} [m_1 \widehat{f}_{4n}(\vartheta,\varphi) + m_2 \widehat{f}_{5n}(\vartheta,\varphi)], \quad \rho < a. \end{aligned} \tag{3.5}$$

Analogously, we construct a solution under boundary conditions

$$p_1^-(z) = f_4^-, \quad p_2^-(z) = f_5^-(z),$$

in the domain D^-

$$p_1 = \frac{-m_2\sqrt{a}}{(m_1+m_2)\sqrt{\rho}} \sum_{n=0}^{\infty} \frac{H^{(2)}_{n+\frac{1}{2}}(i\lambda_0\rho)}{H^{(2)}_{n+\frac{1}{2}}(i\lambda_0 a)} [\widehat{f}_{5n}(\vartheta,\varphi) - \widehat{f}_{4n}(\vartheta,\varphi)]$$

$$+\frac{1}{(m_1+m_2)} \sum_{n=0}^{\infty} \frac{a^{n+1}}{\rho^{n+1}} [m_1\widehat{f}_{4n}(\vartheta,\varphi) + m_2\widehat{f}_{5n}(\vartheta,\varphi)],$$

$$p_2 = \frac{m_1\sqrt{a}}{(m_1+m_2)\sqrt{\rho}} \sum_{n=0}^{\infty} \frac{H^{(2)}_{n+\frac{1}{2}}(i\lambda_0\rho)}{H^{(2)}_{n+\frac{1}{2}}(i\lambda_0 a)} [\widehat{f}_{5n}(\vartheta,\varphi) - \widehat{f}_{4n}(\vartheta,\varphi)] \tag{3.6}$$

$$+\frac{1}{(m_1+m_2)} \sum_{n=0}^{\infty} \frac{a^{n+1}}{\rho^{n+1}} [m_1\widehat{f}_{4n}((\vartheta,\varphi) + m_2\widehat{f}_{5n}(\vartheta,\varphi)], \quad \rho > a.$$

For absolutely and uniformly convergence of these series together with their first derivatives it is sufficient to assume that $f_k^{\pm} \in C^{1,\alpha}(S), \quad 0 < \alpha \leq 1, \quad k = 4,5.$ Solutions, obtained under such conditions, are regular in D^+.

For a ball the solution of system (1.8), when $V^{\pm} = F^{\pm}$ is constructed by Natroshvili [8] (this result can be found also in monograph [7]):

$$V(x) = \iint_S \overset{(1)}{K^+}(x,y)F^+(y)d_ys, \quad x \in D^+, \quad y \in S,$$

$$V(x) = \iint_S \overset{(1)}{K^-}(x,y)F^-(y)d_ys, \quad x \in D^-, \quad y \in S, \tag{3.7}$$

where $\overset{(1)}{K^+} :=\| \overset{(1)}{K^+_{kj}} \|_{3\times3}, \quad \overset{(1)}{K^-} :=\| \overset{(1)}{K^-_{kj}} \|_{3\times3},$

$$\overset{(1)}{K^+_{kj}} := \frac{1}{4\pi a}\left[\frac{a^2-\rho^2}{r^3}\delta_{ij} + \beta(a^2-\rho^2)\frac{\partial^2\Phi(x,y)}{\partial x_i \partial x_j}\right], \quad i,j = 1,2,3,$$

$$\Phi(x,y) := \int_0^1 \left[\frac{a^2-\rho^2t^2}{Q(t)} - \frac{1}{a} - \frac{3t\rho cos\gamma}{a^2}\right]\frac{dt}{t^{1+\alpha}},$$

$$Q(t) := (a^2 - 2a\rho t cos\gamma + \rho^2t^2)^{\frac{3}{2}},$$

$$\overset{(1)}{K}{}^{-}_{kj} := \frac{1}{4\pi a}\left[\frac{\rho^2-a^2}{r^3}\delta_{ij} + \beta(\rho^2-a^2)\frac{\partial^2\Phi^*(x,y)}{\partial x_i \partial x_j}\right], \quad i,j=1,2,3,$$

$$\Phi^*(x,y) := \int_0^1 \frac{\rho^2-a^2t^2}{Q^*(t)}t^\alpha dt, \quad Q^*(t) := (\rho^2 - 2a\rho t cos\gamma + a^2t^2)^{\frac{3}{2}},$$

$$cos\gamma = \frac{x_1y_1+x_2y_2+x_3y_3}{ar} = sin\theta sin\theta' cos(\phi-\phi') + cos\theta cos\theta',$$
$$r^2 = a^2 - 2atcos\gamma + \rho^2, \quad \beta = \frac{\lambda+\mu}{2(\lambda+3\mu)}, \quad \alpha = \frac{\lambda+2\mu}{2(\lambda+3\mu)} < 1, \quad F^\pm \in C^{1,\alpha}(S).$$

So we have proved the following

Theorem 4. The first BVP is uniquely solvable in the class of regular functions and the solution is represented in the form of absolutely and uniformly convergent series, if the boundary data are from the space $C^{1,\alpha}(S), \quad \alpha > \frac{1}{2}$.

4. Solution of the second Boundary Value Problem

Problem $(II)^+$. In this paragraph we will construct an explicit solution of the BVP for system (1.1),(1.2), when stresses and the normal derivatives of the pressure $\frac{\partial p_j}{\partial n}$ are assumed to be given on S

$$(Pu)^+ = f(z)^+, \quad \left(\frac{\partial p_1(z)}{\partial n}\right)^+ = f_4^+, \quad \left(\frac{\partial p_2(z)}{\partial n}\right)^+ = f_5^+(z), \quad z \in S,$$

where $f^+ \in C^{0,\alpha}(S), \quad f_k^+ \in C^{0,\alpha}(S), \quad 0 < \alpha \le 1, \quad k = 3,4.$

We seek the p_k in the domain D^+ in the form (3.1). Taking into account the fact that $\frac{\partial}{\partial n} = \frac{\partial}{\partial \rho}$, from (3.1) we obtain

$$\frac{\partial p_1}{\partial \rho} = -m_2\sum_{n=0}^\infty \frac{\partial}{\partial\rho}\frac{J_{n+\frac{1}{2}}(i\lambda_0\rho)}{\sqrt{\rho}}Y_n(\vartheta,\varphi) + \sum_{n=0}^\infty \frac{n\rho^{n-1}}{(2n+1)a^{n-1}}Z_n(\vartheta,\varphi), \quad \rho < a,$$

$$\frac{\partial p_2}{\partial \rho} = m_1\sum_{n=0}^\infty \frac{\partial}{\partial\rho}\frac{J_{n+\frac{1}{2}}(i\lambda_0\rho)}{\sqrt{\rho}}Y_n(\vartheta,\varphi) + \sum_{n=0}^\infty \frac{n\rho^{n-1}}{(2n+1)a^{n-1}}Z_n(\vartheta,\varphi), \quad \rho < a. \tag{4.1}$$

Let us rewrite (4.1) as

$$\frac{\partial p_1}{\partial \rho} = -m_2\sum_{n=0}^\infty H_n(\rho)Y_n(\vartheta,\varphi) + \sum_{n=0}^\infty \frac{n\rho^{n-1}}{(2n+1)a^{n-1}}Z_n(\vartheta,\varphi), \quad \rho < a,$$
$$\frac{\partial p_2}{\partial \rho} = m_1\sum_{n=0}^\infty H_n(\rho)Y_n(\vartheta,\varphi) + \sum_{n=0}^\infty \frac{n\rho^{n-1}}{(2n+1)a^{n-1}}Z_n(\vartheta,\varphi), \quad \rho < a, \tag{4.2}$$

where $H_n(\rho) := \dfrac{\partial}{\partial\rho}\dfrac{J_{n+\frac{1}{2}}(i\lambda_0\rho)}{\sqrt{\rho}}$.

Passing to the limit in (4.2) as $\rho \to a$, we have

$$\begin{aligned} &-m_2 \sum_{n=0}^{\infty} H_n(a)Y_n(\vartheta_0,\varphi_0)) + \sum_{n=0}^{\infty} \frac{n}{(2n+1)} Z_n(\vartheta_0,\varphi_0)) = \sum_{n=0}^{\infty} \widehat{f}_{4n}(\vartheta_0,\varphi_0), \\ &m_1 \sum_{n=0}^{\infty} H_n(a)Y_n(\vartheta_0,\varphi_0) + \sum_{n=0}^{\infty} \frac{n}{(2n+1)} Z_n(\vartheta_0,\varphi_0) = \sum_{n=0}^{\infty} \widehat{f}_{5n}(\vartheta_0,\varphi_0), \end{aligned} \tag{4.3}$$

where

$$\widehat{f}_{kn}(\vartheta_0,\varphi_0) = \frac{2n+1}{4\pi a^2} \int\limits_S P_n(\cos\gamma) f_k(y) dS_y, \quad k = 4,5.$$

For the coefficients Y_n and Z_n (4.3) yield the following equations:

$$\begin{aligned} &-m_2 H_n(a)Y_n(\vartheta_0,\varphi_0) + \frac{n}{(2n+1)} Z_n(\vartheta_0,\varphi_0) = \widehat{f}_{4n}(\vartheta_0,\varphi_0), \\ &m_1 H_n(a)Y_n(\vartheta_0,\varphi_0) + \frac{n}{(2n+1)} Z_n(\vartheta_0,\varphi_0) = \widehat{f}_{5n}(\vartheta_0,\varphi_0), \quad n = 0,1,2,.. \end{aligned} \tag{4.4}$$

By elementary calculation from (4.4) we define Y_n and Z_n for $n \geq 1$

$$\begin{aligned} &Y_n(\vartheta_0,\varphi_0) = \frac{\widehat{f}_{5n}(\vartheta_0,\varphi_0) - \widehat{f}_{4n}(\vartheta_0,\varphi_0)}{(m_1+m_2)H_n(a)}, \\ &Z_n(\vartheta_0,\varphi_0) = \frac{(2n+1)[m_1\widehat{f}_{4n}(\vartheta_0,\varphi_0) + m_2\widehat{f}_{5n}(\vartheta_0,\varphi_0)]}{n(m_1+m_2)}, \quad n = 1,2,... \end{aligned} \tag{4.5}$$

For the regularity of solutions $p_j, \quad j = 1,2$ it is sufficient that

$$\widehat{f}_{40}(0,0) = \frac{1}{4\pi a^2} \int\limits_S f_4 dS = 0, \quad \widehat{f}_{50}(0,0) = \frac{1}{4\pi a^2} \int\limits_S f_5 dS = 0.$$

Then, for coefficients Z_0 and Y_0 (4.4) yield the following equations:

$$\begin{aligned} &-m_2 H_0(a)Y_0(\vartheta_0,\varphi_0) + 0 \cdot Z_0(\vartheta_0,\varphi_0) = 0, \\ &m_1 H_0(a)Y_0(\vartheta_0,\varphi_0) + 0 \cdot Z_0(\vartheta_0,\varphi_0) = 0, \end{aligned}$$

whence, Z_0 is an arbitrary constant and $Y_0 = 0$.

Substituting (4.5) into (3.1), we obtain a solution of the second BVP in the form of series

$$p_1=\frac{-m_2}{(m_1+m_2)\sqrt{\rho}}\sum_{n=1}^{\infty}\frac{J_{n+\frac{1}{2}}(i\lambda_0\rho)}{H_n(a)}[\widehat{f}_{5n}(\vartheta,\varphi)-\widehat{f}_{4n}(\vartheta,\varphi)]$$

$$+\frac{1}{m_1+m_2}\sum_{n=1}^{\infty}\frac{\rho^n}{na^{n-1}}[m_1\widehat{f}_{4n}(\vartheta,\varphi)+m_2\widehat{f}_{5n}(\vartheta,\varphi)]+c,$$

$$p_2=\frac{m_1}{(m_1+m_2)\sqrt{\rho}}\sum_{n=1}^{\infty}\frac{J_{n+\frac{1}{2}}(i\lambda_0\rho)}{H_n(a)}[\widehat{f}_{5n}(\vartheta,\varphi)-\widehat{f}_{4n}(\vartheta,\varphi)] \tag{4.6}$$

$$+\frac{1}{m_1+m_2}\sum_{n=1}^{\infty}\frac{\rho^n}{na^{n-1}}[m_1\widehat{f}_{4n}(\vartheta,\varphi)+m_2\widehat{f}_{5n}(\vartheta,\varphi)]+c,\quad \rho<a.$$

Problem $(II)^-$ can be solved analogously

$$p_1=\frac{-m_2}{(m_1+m_2)\sqrt{\rho}}\sum_{n=1}^{\infty}\frac{H^{(2)}_{n+\frac{1}{2}}(i\lambda_0\rho)}{h_n(a)}[\widehat{f}_{5n}(\vartheta,\varphi)-\widehat{f}_{4n}(\vartheta,\varphi)]-$$

$$\frac{1}{m_1+m_2}\sum_{n=1}^{\infty}\frac{a^{n+2}}{(n+1)\rho^{n+1}}[m_1\widehat{f}_{4n}(\vartheta,\varphi)+m_2\widehat{f}_{5n}(\vartheta,\varphi)],$$

$$p_2=\frac{m_1}{(m_1+m_2)\sqrt{\rho}}\sum_{n=1}^{\infty}\frac{H^{(2)}_{n+\frac{1}{2}}(i\lambda_0\rho)}{h_n(a)}[\widehat{f}_{5n}(\vartheta,\varphi)-\widehat{f}_{4n}(\vartheta,\varphi)]- \tag{4.7}$$

$$\frac{1}{m_1+m_2}\sum_{n=1}^{\infty}\frac{a^{n+2}}{(n+1)\varrho^{n+1}}[m_1\widehat{f}_{4n}(\vartheta,\varphi)+m_2\widehat{f}_{5n}(\vartheta,\varphi)],\qquad \rho>a,$$

where $\quad h_n(\rho)=\dfrac{\partial}{\partial\rho}\dfrac{H^{(2)}_{n+\frac{1}{2}}(i\lambda_0\rho)}{\sqrt{\rho}}.$

$\dfrac{\partial p_k}{\partial n}$ can be calculated from (4.6)-(4.7).

The solution of the problem $(TV)^{\pm}=F^{\pm}$, for system (1.8) for a ball is constructed by Natroshvili [8] (this result can be found also in monograph [7]):

$$V(x)=\iint_S \overset{(2)}{K^+}(x,y)F^+(y)d_ys+a_1+[\omega,x]+\frac{c(\beta_1+\beta_2)}{3\lambda+2\mu}x,\quad x\in D^+,$$

$$TV=\frac{1}{4\pi\rho}\iint_S \parallel \frac{a^2-\rho^2}{r^3}\delta_{ij}+(a^2-\rho^2)\frac{\partial^2\Phi_4(x,y)}{\partial x_i\partial x_j}\parallel_{3x3}F^+(y)ds,\quad x\in D^+,$$

$$V(x) = \iint\limits_S \overset{(2)}{K}{}^{-}(x,y) F^-(y) d_y s, \quad x \in D^-,$$

$$TV = \frac{1}{4\pi\rho} \iint\limits_S \| \frac{\rho^2 - a^2}{r^3} \delta_{ij} + (\rho^2 - a^2) \frac{\partial^2 \Phi_4^*(x,y)}{\partial x_i \partial x_j} \|_{3x3} F^-(y) ds, \quad x \in D^-,$$

where $\overset{(2)}{K}{}^{+} := \| \overset{(2)}{K}{}^{+}_{kj} \|_{3\times 3}, \quad \overset{(2)}{K}{}^{-} := \| \overset{(2)}{K}{}^{-}_{kj} \|_{3\times 3},$

$$\overset{(2)}{K}{}^{+}_{kj} := \frac{1}{8\mu\pi} \left[(\Phi_1 + \Phi_2)\delta_{ij} + \frac{a^2 - 3\rho^2}{2} \frac{\partial^2 \Phi_3(x,y)}{\partial x_i \partial y_j} + x_j \frac{\partial}{\partial x_i}(\Phi_1 - \Phi_2) - 2x_i \frac{\partial \Phi_1}{\partial x_j} \right]$$

$$+ \frac{1}{8\mu\pi} \left[x_i \frac{\partial}{\partial x_j} (2\rho \frac{\partial \Phi_3}{\partial \rho} - \Phi_3) + \rho^2 \left(\frac{\partial^2 \Phi_2(x,y)}{\partial x_i \partial y_j} - \frac{\partial^2 \Phi_1(x,y)}{\partial x_i \partial y_j} \right) \right], \quad k,j = 1,2,3,$$

$$\Phi_1(x,y) := \int\limits_0^1 \left[\frac{a^2 - \rho^2 t^2}{Q(t)} - \frac{1}{a} \right] \frac{dt}{t}, \quad Q(t) := (a^2 - 2a\rho t cos\gamma + \rho^2 t^2)^{\frac{3}{2}},$$

$$\Phi_2(x,y) := \int\limits_0^1 \left[\frac{a^2 - \rho^2 t^2}{Q(t)} - \frac{1}{a} - \frac{3t\rho cos\gamma}{a^2} \right] \frac{dt}{t^2},$$

$$\Phi_0(x,y) := \int\limits_0^1 \left[\frac{a^2 - \rho^2 t^2}{Q(t)} - \frac{1}{a} \right] \frac{dt}{t^{1+\alpha_1}}, \quad \Phi_3 := \frac{1}{b_1} Im \Phi_0, \quad \Phi_4 := Re(b_2 \Phi_0),$$

$$\alpha_1 := b_0 + ib_1 = \frac{\mu + i\sqrt{2\lambda^2 + 6\lambda\mu + 3\mu^2}}{2(\lambda + \mu)}, \quad b_2 := \frac{1}{2} + \frac{3\lambda + 4\mu}{2\sqrt{2\lambda^2 + 6\lambda\mu + 3\mu^2}},$$

$$\overset{(2)}{K}{}^{-}_{kj} := \frac{1}{8\mu\pi} \left[-(\Phi_1^* + \Phi_2^*)\delta_{ij} + \frac{a^2 - 3\rho^2}{2} \frac{\partial^2 \Phi_3^*(x,y)}{\partial x_i \partial y_j} - x_j \frac{\partial}{\partial x_i}(\Phi_1 - \Phi_2) + 2x_i \frac{\partial \Phi_1^*}{\partial x_j} \right]$$

$$+ \frac{1}{8\mu\pi} \left[x_i \frac{\partial}{\partial x_j} (2\rho \frac{\partial \Phi_3^*}{\partial \rho} - \Phi_3^*) - \rho \left(\frac{\partial^2 \Phi_2(x,y)}{\partial x_i \partial y_j} - \frac{\partial^1 \Phi_1(x,y)}{\partial x_i \partial y_j} \right) \right], \quad k,j = 1,2,3,$$

$$\Phi_l^*(x,y) := \int\limits_0^1 \tfrac{\rho^2 - a^2 t^2}{Q^*(t)} t^{l-1} dt, \quad l = 1,2, \quad \Phi_3^* := \frac{2(\lambda + \mu)}{\sqrt{2\lambda^2 + 6\lambda\mu + 3\mu^2}} Im \int\limits_0^1 \frac{\rho^2 - a^2 t^2}{Q^*(t)} \frac{dt}{t^{\alpha_2}},$$

$$\Phi_4^*(x,y) := ReA \int\limits_0^1 \frac{\rho^2 - a^2 t^2}{Q^*(t)} \frac{dt}{t^{\alpha_2}}, \quad Q^*(t) := (\rho^2 - 2a\rho t cos\gamma + a^2 t^2)^{\frac{3}{2}},$$

$$\alpha_2 := \frac{-\mu + i\sqrt{2\lambda^2 + 6\lambda\mu + 3\mu^2}}{2(\lambda + \mu)}, \quad A := \frac{1}{2} - i \frac{3\lambda + 4\mu}{2\sqrt{2\lambda^2 + 6\lambda\mu + 3\mu^2}}.$$

Thus, we have proved the following

Theorem 5. For the solvability of the BVP $(II)^+$ it is necessary that the principal vector and the principal moment of external forces be equal to zero. The BVP $(II)^+$ is solvable in the class of regular functions and the solution is represented in the form of absolutely and uniformly convergent series if the boundary data are from space $C^{0,\alpha}(S)$, $\alpha > \frac{1}{2}$. Two regular solutions of BVP $(II)^+$ may differ only with an additive vector

$$a + [b, x] + \frac{c(\beta_1 + \beta_2)}{3\lambda + 2\mu},$$

where a, b, c are arbitrary real constant vectors, $x := x(x_1, x_2, x_3)$. The BVP $(II)^-$is solvable in the class of regular functions and the solution is represented in the form of absolutely and uniformly convergent series.

Acknowledgement.

The designated project has been fulfilled by financial support of the Shota Rustaveli National Science Foundation(Grant #GNSF/ST08/3-388).

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