Bulletin of TICMI
Vol. 16, No. 1, 2012, 27–33

An Iteration Method for the Kirchhoff Static Beam

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(*Received December 26, 2011; Accepted March 5, 2012*)

The iteration method $u_k^{\mathrm{iv}} - (\lambda + 2/L\int_0^L u_{k-1}'^2 dx)u_k'' = f$, $k = 1, 2, \dots$, is used to solve the boundary value problem for the nonlinear differential equation $u^{\mathrm{iv}} - (\lambda+2/L\int_0^L u'^2 dx)u'' = f$. The approximation u_k is expressed as well-defined integrals of the functions u_{k-1} and f. The method error $u_k - u$ is estimated.

Keywords: Kirchhoff equation, Static beam, Iteration method, Algorithm error.

AMS Subject Classification: 65L10, 65L70.

1. Statement of the problem

We consider the following boundary value problem

$$u^{\mathrm{iv}}(x) - \left(\lambda + \frac{2}{L}\int_0^L u'^2(x)\,dx\right)u''(x) = f(x), \tag{1.1}$$

$$0 < x < L, \quad \lambda = const > 0,$$

$$u(0) = u(L) = 0, \quad u''(0) = u''(L) = 0, \tag{1.2}$$

where $f(x)$ is a given continuous function, and $u(x)$ is the sought solution.

Equation (1.1) is the stationary problem related to the equation

$$\frac{\partial^2 u}{\partial t^2} + \alpha_0\frac{\partial^4 u}{\partial x^4} - \left(\alpha_1 + \alpha_2\int_0^L\left(\frac{\partial u}{\partial x}\right)^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.3}$$

which was proposed by Woinowsky–Krieger [9] in 1950 as a model for the deflection of an extensible dynamic beam with hinged ends. The nonlinear term of this equation was for the first time used by Kirchhoff [4] who generalized D'Alembert's classical model. Therefore equations (1.1) and (1.3) are frequently called a Kirchhoff type equation for a dynamic and a static beam, respectively. The results of one of the initial mathematical studies of equations of (1.3) type are presented in [1] and [2].

For equation (1.1) and its generalizations, as well as for equations similar to (1.1), the problem of construction of numerical algorithms and estimation of their accuracy is studied in [3], [5]–[8]. Each of the algorithms used in these papers is a combination of two approximate methods, one of which reduces the problem to

ISSN: 1512-0082 print

the finite-dimensional one and the other is some iterative process of solution of the discrete system. In the present paper, a technique somewhat different from the above-mentioned one is proposed to solve problem (1.1),(1.2). The differential equation (1.1) is solved by an iteration method. At each iteration step, a boundary value problem is obtained for a linear differential equation whose solution is written in integrals. The algorithm accuracy is estimated by the method of a priori inequalities.

2. The algorithm

On choosing a function $u_0(x)$, $0 \le x \le L$, that together with its second derivative vanishes for $x = 0$ and $x = L$, we will seek for a solution of problem (1.1),(1.2) using the iteration process

$$u_k^{\mathrm{iv}}(x) - \left(\lambda + \frac{2}{L}\int_0^L u_{k-1}'^2(x)\,dx\right)u_k''(x) = f(x), \tag{2.4}$$
$$0 < x < L,$$
$$u_k(0) = u_k(L) = 0, \quad u_k''(0) = u_k''(L) = 0, \tag{2.5}$$
$$k = 1, 2, \dots\ ,$$

where $u_k(x)$ is the k-th approximation of the solution of problem (1.1),(1.2), $k = 0, 1, \dots$.

The considered algorithm makes it possible to express $u_k(x)$ through the preceding approximation in the integral form. Indeed, on denoting

$$\alpha_k = \lambda + \frac{2}{L}\int_0^L u_k'^2(x)\,dx,$$

we introduce the function $v_k(x) = u_k''(x)$, $k = 0, 1, \dots$.

Now, (2.4),(2.5) can be rewritten as relations

$$u_k''(x) = v_k(x), \quad 0 < x < L,$$
$$u_k(0) = u_k(L) = 0$$

and

$$v_k''(x) - \alpha_{k-1}v_k(x) = f(x), \quad 0 < x < L,$$
$$v_k(0) = v_k(L) = 0.$$

For $u_k(x)$ we have

$$u_k(x) = \frac{1}{L}\left((x - L)\int_0^x \xi v_k(\xi)\,d\xi + x\int_x^L (\xi - L)v_k(\xi)\,d\xi\right) \tag{2.6}$$

and $v_k(x)$ is representable in the form

$$\begin{aligned} v_k(x) = {} & \frac{1}{\sqrt{\alpha_{k-1}}\,\sinh(\sqrt{\alpha_{k-1}}\,L)} \\ & \times \biggl(\sinh\bigl(\sqrt{\alpha_{k-1}}\,(x-L)\bigr) \int_0^x \sinh\bigl(\sqrt{\alpha_{k-1}}\,\xi\bigr)\, f(\xi)\, d\xi \\ & + \sinh\bigl(\sqrt{\alpha_{k-1}}\,x\bigr) \int_x^L \sinh\bigl(\sqrt{\alpha_{k-1}}\,(\xi-L)\bigr)\, f(\xi)\, d\xi \biggr). \end{aligned} \tag{2.7}$$

Substituting (2.7) into (2.6) and applying the well-known equalities for hyperbolic functions, we obtain the desired formula

$$\begin{aligned} u_k(x) = {} & -\frac{1}{\alpha_{k-1}L}\biggl((x-L)\int_0^x \xi f(\xi)\, d\xi + x\int_x^L (\xi-L) f(\xi)\, d\xi \biggr) \\ & + \frac{1}{\alpha_{k-1}\sqrt{\alpha_{k-1}}\,\sinh(\sqrt{\alpha_{k-1}}\,L)} \biggl(\sinh\bigl(\sqrt{\alpha_{k-1}}(x-L)\bigr)\int_0^x \sinh\bigl(\sqrt{\alpha_{k-1}}\,\xi\bigr)\, f(\xi)\, d\xi \\ & + \sinh\bigl(\sqrt{\alpha_{k-1}}\,x\bigr) \int_x^L \sinh\bigl(\sqrt{\alpha_{k-1}}\,(\xi-L)\bigr)\, f(\xi)\, d\xi \biggr), \quad k = 1, 2, \dots , \end{aligned} \tag{2.8}$$

which makes it possible to obtain approximations $u_k(x)$ evading the differential problem (2.4),(2.5). Therefore approximations $u_k(x)$ are found by means of (2.8).

3. Error of the algorithm and the equation for it

Let us define the algorithm error as a difference

$$\Delta u_k(x) = u_k(x) - u(x), \quad k = 0, 1, \dots ,$$

between an approximate and an exact solutions.

Subtracting the respective relations in (1.1) and (1.2) from (2.4) and (2.5), we obtain

$$\begin{aligned} & \Delta u_k^{\mathrm{iv}}(x) - \biggl(\lambda + \frac{1}{L}\int_0^L \bigl(u_{k-1}'^2(x) + u'^2(x) \bigr)\, dx \biggr) \Delta u_k''(x) \\ & \qquad - \frac{1}{L}\biggl(\int_0^L \bigl(u_{k-1}'(x) + u'(x) \bigr)\, \Delta u_{k-1}'(x)\, dx \biggr) (u_k''(x) + u''(x)) = 0, \end{aligned} \tag{3.9}$$

$$\Delta u_k(0) = \Delta u_k(L) = 0, \quad \Delta u_k''(0) = \Delta u_k''(L) = 0. \tag{3.10}$$

We use (3.9), (10) to estimate the algorithm accuracy. For this, we need some a priori relations which are derived in the next paragraph.

4. Auxiliary inequalities

For the well-defined functions, sufficiently smooth for $0 \le x \le L$, we introduce the notation

$$(u(x), v(x)) = \int_0^L u(x)v(x)\,dx,$$

$$\|u(x)\|_p = \left(\int_0^L \left(\frac{d^p u(x)}{dx^p}\right)^2 dx\right)^{\frac{1}{2}}, \quad p = 0, 1, 2, \quad \|u(x)\| = \|u(x)\|_0 .$$

Lemma 4.1: *For a twice differentiable function $u(x)$, $0 \le x \le L$, that vanishes at $x = 0$ and $x = L$ the inequalities*

$$\frac{\sqrt{2}}{L}\,\|u(x)\| \le \|u(x)\|_1 \le \frac{L}{\sqrt{2}}\,\|u(x)\|_2 \tag{4.11}$$

are valid.

Proof: We have

$$u(x) = \int_0^x u'(\xi)\,d\xi.$$

Hence

$$|u(x)| \le \left(\int_0^x d\xi\right)^{\frac{1}{2}} \left(\int_0^x u'^2(\xi)\,d\xi\right)^{\frac{1}{2}} \le x^{\frac{1}{2}}\|u\|_1.$$

Therefore

$$\|u(x)\|^2 \le \frac{L^2}{2}\,\|u\|_1^2,$$

which implies the left inequality of (4.11). Using the latter and taking into account that

$$\|u(x)\|_1^2 = u(x)u'(x)\big|_0^L - (u(x), u''(x)) = -(u(x), u''(x)),$$

we complete the proof. □

Lemma 4.2: *For the solution of problem* (1.1), (1.2) *we have the inequality*

$$\|u(x)\|_1 \le c_1, \tag{4.12}$$

where c_1 is the constant calculated by the formula

$$c_1 = \frac{L}{2}\left(\frac{2}{L} + \lambda L\right)^{-\frac{1}{4}} \|f(x)\|^{\frac{1}{2}} . \tag{4.13}$$

Proof: We multiply equation (1.1) by $u(x)$ and then integrate the obtained equality with respect to x from 0 to L. Using (1.2), we get the relation

$$\|u(x)\|_2^2 + \left(\lambda + \frac{2}{L}\,\|u(x)\|_1^2\right)\|u(x)\|_1^2 = (f(x), u(x))\,,$$

which, together with (4.11), yields

$$\left(\lambda + \frac{2}{L^2} + \frac{2}{L}\,\|u(x)\|_1^2\right)\|u(x)\|_1^2 \le \frac{L}{\sqrt{2}}\,\|f(x)\|\,\|u(x)\|_1\,.$$

Thus we obtain the inequality

$$\frac{2}{L}\,\|u(x)\|_1^4 \le \frac{L^2}{8}\left(\frac{2}{L^2} + \lambda\right)^{-1}\|f(x)\|^2,$$

which implies estimate (4.12). □

Lemma 4.3: *The solution of problem* (2.4), (2.5) *satisfies the relation*

$$\|u_k(x)\|_1 \le c_2\,, \quad k = 1, 2, \dots\,, \tag{4.14}$$

where c_2 is a constant independent of k and defined by

$$c_2 = \frac{L^2}{\sqrt{2}}\left(\frac{2}{L} + \lambda L\right)^{-1}\|f(x)\|. \tag{4.15}$$

Proof: We multiply equation (2.4) by $u_k(x)$ and integrate with respect to x from 0 to L. Taking (2.5) into account, we see

$$\|u_k(x)\|_2^2 + \left(\lambda + \frac{2}{L}\,\|u_{k-1}(x)\|_1^2\right)\|u_k(x)\|_1^2 = (f(x), u_k(x))\,.$$

By applying (4.11) we find

$$\left(\lambda + \frac{2}{L^2} + \frac{2}{L}\,\|u_{k-1}(x)\|_1^2\right)\|u_k(x)\|_1^2 \le \frac{L}{\sqrt{2}}\,\|f(x)\|\,\|u_k(x)\|_1.$$

This implies (4.14). □

5. Estimation of the algorithm error

We multiply equation (3.9) by $\Delta u_k(x)$ and integrate the obtained equality with respect to x from 0 to L. Applying (3.10) we obtain

$$\|\Delta u_k(x)\|_2^2 + \left(\lambda + \frac{1}{L}\left(\|u_{k-1}(x)\|_1^2 + \|u(x)\|_1^2\right)\right)\|\Delta u_k(x)\|_1^2 + \frac{1}{L}\prod_{p=0}^{1}\left(u'_{k-p}(x) + u'(x), \Delta u'_{k-p}(x)\right) = 0.$$

By (4.11)

$$\left(\lambda + \frac{2}{L^2} + \frac{1}{L}\left(\|u_{k-1}(x)\|_1^2 + \|u(x)\|_1^2\right)\right)\|\Delta u_k(x)\|_1^2 \le \frac{1}{L}\prod_{p=0}^{1}\left(\|u_{k-p}(x)\|_1 + \|u(x)\|_1\right)\|\Delta u_{k-p}(x)\|_1.$$

Therefore

$$\|\Delta u_k(x)\|_1 \le \left(\frac{2}{L}\right)^{\frac{1}{2}}\left(\frac{1}{L}\left(\|u_{k-1}(x)\|_1^2 + \|u(x)\|_1^2\right)\right)^{\frac{1}{2}} \times \left(\lambda + \frac{2}{L^2} + \frac{1}{L}\left(\|u_{k-1}(x)\|_1^2 + \|u(x)\|_1^2\right)\right)^{-1} \times \left(\|u_k(x)\|_1 + \|u(x)\|_1\right)\|\Delta u_{k-1}(x)\|_1.$$

Since $\max\limits_{0\le y<\infty} \frac{y}{\alpha+y^2} = \frac{1}{2}\alpha^{-1/2}$ holds for $\alpha > 0$, we have

$$\|\Delta u_k(x)\|_1 \le \left(2\left(\frac{2}{L} + \lambda L\right)\right)^{-\frac{1}{2}}\left(\|u_k(x)\|_1 + \|u(x)\|_1\right)\|\Delta u_{k-1}(x)\|_1. \qquad (5.16)$$

Let the condition $q = \left(2\left(\frac{2}{L} + \lambda L\right)\right)^{-\frac{1}{2}}(c_1 + c_2) < 1$ be fulfilled, which, as follows from (4.13) and (4.15), is equivalent to the requirement

$$q = \frac{1}{4}\sum_{p=1}^{2}\left(L\sqrt{2}\left(\frac{2}{L} + \lambda L\right)^{-\frac{3}{4}}\|f(x)\|^{\frac{1}{2}}\right)^p < 1.$$

Then, by virtue of (5.16), (4.12), (4.14) and (4.11), we come to a conclusion that the iteration method (2.4),(2.5) or, which is the same, (2.8) reduces to the solution of problem (1.1),(1.2) and the estimate

$$\|\Delta u_k(x)\|_p \leq \left(\frac{L}{\sqrt{2}}\right)^{1-p} q^k \|\Delta u_0(x)\|_1,$$
$$k = 1, 2, \dots, \quad p = 0, 1,$$

holds for the error of the method.

Acknowledgement

The author acknowledges the support under grant 7353 of the Estonian Science Foundation.

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