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Vectorial Hardy type fractional inequalities

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Here we present vectorial integral inequalities for products of multivariate convex and increasing functions applied to vectors of functions. As applications we derive a wide range of vectorial fractional inequalities of Hardy type. They involve the left and right Riemann-Liouville fractional integrals and their generalizations, in particular the Hadamard fractional integrals. Also inequalities for left and right Riemann-Liouville, Caputo, Canavati and their generalizations fractional derivatives. These application inequalities are of L_p type, $p \geq 1$, and exponential type.

Keywords: multivariate Jensen inequality, fractional integral, fractional derivative, Hardy fractional inequality, Hadamard fractional integral.

AMS Subject Classification: 26A33, 26D10, 26D15.

1. Introduction

We start with some facts about fractional derivatives needed in the sequel, for more details see, for instance [1], [10].

Let $a < b$, $a, b \in \mathbb{R}$. By $C^N([a,b])$, we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order N, and $AC([a,b])$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^N([a,b])$, we denote the space of all functions g with $g^{(N-1)} \in AC([a,b])$. For any $\alpha \in \mathbb{R}$, we denote by $[\alpha]$ the integral part of α (the integer k satisfying $k \leq \alpha < k+1$), and $\lceil \alpha \rceil$ is the ceiling of α ($\min\{n \in \mathbb{N},\, n \geq \alpha\}$). By $L_1(a,b)$, we denote the space of all functions integrable on the interval (a,b), and by $L_\infty(a,b)$ the set of all functions measurable and essentially bounded on (a,b). Clearly, $L_\infty(a,b) \subset L_1(a,b)$

We start with the definition of the Riemann-Liouville fractional integrals, see [13]. Let $[a,b]$, $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional integrals $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by

$$\left(I_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt, \quad (x > a), \tag{1.1}$$

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$$\left(I_{b-}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}f(t)(t-x)^{\alpha-1}dt,\quad (x<b), \tag{1.2}$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function. These integrals are called the left-sided and the right-sided fractional integrals. We mention some properties of the operators $I_{a+}^{\alpha}f$ and $I_{b-}^{\alpha}f$ of order $\alpha>0$, see also [15]. The first result yields that the fractional integral operators $I_{a+}^{\alpha}f$ and $I_{b-}^{\alpha}f$ are bounded in $L_p(a,b)$, $1\le p\le\infty$, that is

$$\left\|I_{a+}^{\alpha}f\right\|_p\le K\left\|f\right\|_p,\quad \left\|I_{b-}^{\alpha}f\right\|_p\le K\left\|f\right\|_p \tag{1.3}$$

where

$$K=\frac{(b-a)^{\alpha}}{\alpha\Gamma(\alpha)}. \tag{1.4}$$

Inequality (1.3), that is the result involving the left-sided fractional integral, was proved by H. G. Hardy in one of his first papers, see [11]. He did not write down the constant, but the calculation of the constant was hidden inside his proof.

Next we are motivated by [12]. We produce a wide range of vectorial integral inequalities related to integral operators, with applications to vectorial Hardy type fractional inequalities.

2. Main Results

Let $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ be measure spaces with positive σ-finite measures, and let $k:\Omega_1\times\Omega_2\to\mathbb{R}$ be a nonnegative measurable function, $k(x,\cdot)$ measurable on Ω_2 and

$$K(x)=\int_{\Omega_2}k(x,y)\,d\mu_2(y),\quad x\in\Omega_1. \tag{2.1}$$

We suppose that $K(x)>0$ a.e. on Ω_1, and by a weight function (shortly: a weight), we mean a nonnegative measurable function on the actual set. Let the measurable functions $g_i:\Omega_1\to\mathbb{R}$, $i=1,...,n$, with the representation

$$g_i(x)=\int_{\Omega_2}k(x,y)\,f_i(y)\,d\mu_2(y), \tag{2.2}$$

where $f_i:\Omega_2\to\mathbb{R}$ are measurable functions, $i=1,...,n$.

Denote by $\overrightarrow{x}=x:=(x_1,...,x_n)\in\mathbb{R}^n$, $\overrightarrow{g}:=(g_1,...,g_n)$ and $\overrightarrow{f}:=(f_1,...,f_n)$.

We consider here $\Phi:\mathbb{R}_+^n\to\mathbb{R}$ a convex function, which is increasing per coordinate, i.e. if $x_i\le y_i$, $i=1,...,n$, then $\Phi(x_1,...,x_n)\le\Phi(y_1,...,y_n)$.

Examples for Φ :

1) Given g_i is convex and increasing on $\mathbb{R}_+$, then $\Phi(x_1,...,x_n):=\sum_{i=1}^{n}g_i(x_i)$ is convex on $\mathbb{R}_+^n$, and increasing per coordinate; the same properties hold for:

2) $\|x\|_p = \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}}$, $p \geq 1$,
3) $\|x\|_\infty = \max\limits_{i\in\{1,\ldots,n\}} x_i$,
4) $\sum_{i=1}^{n} x_i^2$,
5) $\sum_{i=1}^{n} \left(i \cdot x_i^2\right)$,
6) $\sum_{i=1}^{n} \sum_{j=1}^{i} x_j^2$,
7) $\ln\left(\sum_{i=1}^{n} e^{x_i}\right)$,
8) let g_j are convex and increasing per coordinate on $\mathbb{R}_+^n$, then so is $\sum_{j=1}^{m} e^{g_j(x)}$, and so is $\ln\left(\sum_{j=1}^{m} e^{g_j(x)}\right)$, $x \in \mathbb{R}_+^n$.

It is a well known fact that, if $C \subseteq \mathbb{R}^n$ is an open and convex set, and $f : C \to \mathbb{R}$ is a convex function, then f is continuous on C.

Proposition 2.1: *Let $\Phi : \mathbb{R}_+^n \to \mathbb{R}$ be a convex function which is increasing per coordinate. Then Φ is continuous.*

Proof: The set $(0,\infty)^n$ is an open and convex subset of $\mathbb{R}^n$. Thus Φ is continuous there. So we need to prove only that Φ is continuous at the origin $0 = (0,\ldots,0)$. By $B(0,r)$ we denote the open ball in $\mathbb{R}^n$, $r > 0$. Let $x \in B(0,r) \cap \mathbb{R}_+^n$, $x \neq 0$; that is $0 < \|x\| < r$.

Define $g : [0,r] \to \mathbb{R}$ by $g(t) := \Phi\left(t \cdot \frac{x}{\|x\|}\right)$, $t \in [0,r]$. For $t_1, t_2 \in [0,r]$, $\lambda \in (0,1)$, we observe that $g(\lambda t_1 + (1-\lambda)t_2) = \Phi\left((\lambda t_1 + (1-\lambda)t_2)\frac{x}{\|x\|}\right) = \Phi\left(\lambda\left(t_1\frac{x}{\|x\|}\right) + (1-\lambda)\left(t_2\frac{x}{\|x\|}\right)\right) \leq \lambda\Phi\left(t_1\frac{x}{\|x\|}\right) + (1-\lambda)\Phi\left(t_2\frac{x}{\|x\|}\right) = \lambda g(t_1) + (1-\lambda)g(t_2)$, that is g is a convex function on $[0,r]$.

Next let $t_1 \leq t_2$, $t_1, t_2 \in [0,r]$, then
$g(t_1) = \Phi\left(t_1\frac{x}{\|x\|}\right) = \Phi\left(t_1\frac{x_1}{\|x\|}, t_1\frac{x_2}{\|x\|}, \ldots, t_1\frac{x_n}{\|x\|}\right) \leq \Phi\left(t_2\frac{x_1}{\|x\|}, t_2\frac{x_2}{\|x\|}, \ldots, t_2\frac{x_n}{\|x\|}\right) = \Phi\left(t_2\frac{x}{\|x\|}\right) = g(t_2)$, hence $g(t_1) \leq g(t_2)$, that is g is increasing on $[0,r]$. Of course g is continuous on $(0,r)$.

We first prove that g is continuous at zero. Consider the line (l) through $(0, g(0))$ and $(r, g(r))$. It has slope $\frac{g(r)-g(0)}{r} \geq 0$, and equation $y = l(z) = \left(\frac{g(r)-g(0)}{r}\right)z + g(0)$. If $g(r) = g(0)$, then $g(t) = g(0)$, for all $t \in [0,r]$, so trivially g is continuous at zero and r.

We treat the other case of $g(r) > g(0)$. By convexity of g we have that for any $0 < z < r$, it is $g(z) \leq l(z)$, equivalently, $g(z) \leq \left(\frac{g(r)-g(0)}{r}\right)z + g(0)$, equivalently $0 \leq g(z) - g(0) \leq \left(\frac{g(r)-g(0)}{r}\right)z$; here $\frac{g(r)-g(0)}{r} > 0$. Letting $z \to 0$, then $g(z) - g(0) \to 0$. That is $\lim\limits_{z\to 0} g(z) = g(0)$, proving continuity of g at zero. So that g is continuous on $[0,r)$.

Clearly Φ is continuous at $r \cdot \frac{x}{\|x\|} \in (0,\infty)^n$. So we choose $(r_n)_{n\in\mathbb{N}}$ such that $0 < \|x\| < r_n \leq r$, with $r_n \to r$, then $\Phi\left(r_n\frac{x}{\|x\|}\right) \to \Phi\left(r\frac{x}{\|x\|}\right)$, proving continuity of g at r.

Therefore g is continuous on $[0,r]$.

Hence there exists $M > 0$ such that $|g(t)| < M$, $\forall\, t \in [0,r]$. Since g is convex on $[0,r]$ it has an increasing slope, therefore $\frac{g(\|x\|)-g(0)}{\|x\|} \leq \frac{g(r)-g(0)}{r} < \frac{M-g(0)}{r}$, that is $g(\|x\|) - \Phi(0) < \left(\frac{M-\Phi(0)}{r}\right)\|x\|$. Equivalently, we have $0 \leq \Phi(x) - \Phi(0) < \left(\frac{M-\Phi(0)}{r}\right)\|x\|$. Clearly $\lim_{x\to 0}\Phi(x) = \Phi(0)$, proving continuity of Φ at $x = 0$. ■

We need also

Theorem 2.2: *(multivariate Jensen inequality, see also [8, p. 76], [14]) Let f be a convex function defined on a convex subset $C \subseteq \mathbb{R}^n$, and let $X = (X_1, ..., X_n)$ be a random vector such that $P(X \in C) = 1$. Assume also $E(|X|)$, $E(|f(X)|) < \infty$. Then $EX \in C$, and*

$$f(EX) \leq Ef(X). \tag{2.3}$$

We give our first main result.

Theorem 2.3: *Let u be a weight function on Ω_1, and k, K, g_i, f_i, $i = 1, ..., n \in \mathbb{N}$, and Φ defined as above. Assume that the function $x \to u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$. Define v on Ω_2 by*

$$v(y) := \int_{\Omega_1} u(x)\frac{k(x,y)}{K(x)}d\mu_1(x) < \infty. \tag{2.4}$$

Then

$$\int_{\Omega_1} u(x)\Phi\left(\frac{|g_1(x)|}{K(x)}, ..., \frac{|g_n(x)|}{K(x)}\right)d\mu_1(x) \leq$$

$$\int_{\Omega_2} v(y)\Phi(|f_1(y)|, ..., |f_n(y)|)\,d\mu_2(y), \tag{2.5}$$

under the assumptions:
(i) f_i, $\Phi(|f_1|, ..., |f_n|)$, are $k(x,y)\,d\mu_2(y)$ -integrable, μ_1 -a.e. in $x \in \Omega_1$, for all $i = 1, ..., n$,
(ii) $v(y)\Phi(|f_1(y)|, ..., |f_n(y)|)$ is μ_2 -integrable.

Proof: Here we use Proposition 2.1, Jensen's inequality, Tonelli's theorem, Fubini's theorem, and that Φ is increasing per coordinate. We have

$$\int_{\Omega_1} u(x)\Phi\left(\left|\frac{\overrightarrow{g}(x)}{K(x)}\right|\right)d\mu_1(x) =$$

$$\int_{\Omega_1} u(x)\Phi\left(\frac{1}{K(x)}\left|\int_{\Omega_2} k(x,y) f_1(y)\,d\mu_2(y)\right|,\right.$$

$$\left.\frac{1}{K(x)}\left|\int_{\Omega_2} k(x,y) f_2(y)\,d\mu_2(y)\right|, ..., \frac{1}{K(x)}\left|\int_{\Omega_2} k(x,y) f_n(y)\,d\mu_2(y)\right|\right)d\mu_1(x)$$

$$\leq \int_{\Omega_1} u(x)\,\Phi\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\,|f_1(y)|\,d\mu_2(y),...,\right.$$

$$\left.\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\,|f_n(y)|\,d\mu_2(y)\right)d\mu_1(x) \leq$$

(by Jensen's inequality)

$$\int_{\Omega_1} \frac{u(x)}{K(x)}\left(\int_{\Omega_2} k(x,y)\,\Phi(|f_1(y)|,...,|f_n(y)|)\,d\mu_2(y)\right)d\mu_1(x) =$$

$$\int_{\Omega_1} \frac{u(x)}{K(x)}\left(\int_{\Omega_2} k(x,y)\,\Phi\left(\left|\overrightarrow{f}(y)\right|\right)d\mu_2(y)\right)d\mu_1(x) =$$

$$\int_{\Omega_1}\left(\int_{\Omega_2} \frac{u(x)}{K(x)}k(x,y)\,\Phi\left(\left|\overrightarrow{f}(y)\right|\right)d\mu_2(y)\right)d\mu_1(x) =$$

$$\int_{\Omega_2}\left(\int_{\Omega_1} \frac{u(x)}{K(x)}k(x,y)\,\Phi\left(\left|\overrightarrow{f}(y)\right|\right)d\mu_1(x)\right)d\mu_2(y) =$$

$$\int_{\Omega_2}\Phi\left(\left|\overrightarrow{f}(y)\right|\right)\left(\int_{\Omega_1} \frac{u(x)}{K(x)}k(x,y)\,d\mu_1(x)\right)d\mu_2(y) =$$

$$\int_{\Omega_2}\Phi\left(\left|\overrightarrow{f}(y)\right|\right)v(y)\,d\mu_2(y) =$$

$$\int_{\Omega_2}\Phi(|f_1(y)|,...,|f_n(y)|)\,v(y)\,d\mu_2(y),$$

proving the claim. ■

Notation 2.4: *From now on we may write*

$$\overrightarrow{g}(x) = \int_{\Omega_2} k(x,y)\,\overrightarrow{f}(y)\,d\mu_2(y), \tag{2.6}$$

which means

$$(g_1(x),...,g_n(x)) = \left(\int_{\Omega_2} k(x,y)\,f_1(y)\,d\mu_2(y),...,\int_{\Omega_2} k(x,y)\,f_n(y)\,d\mu_2(y)\right). \tag{2.7}$$

Similarly, we may write

$$\left|\overrightarrow{g}(x)\right| = \left|\int_{\Omega_2} k(x,y)\overrightarrow{f}(y)\,d\mu_2(y)\right|, \tag{2.8}$$

and we mean

$$(|g_1(x)|,...,|g_n(x)|) =$$

$$\left(\left|\int_{\Omega_2} k(x,y) f_1(y)\,d\mu_2(y)\right|,...,\left|\int_{\Omega_2} k(x,y) f_n(y)\,d\mu_2(y)\right|\right). \tag{2.9}$$

We also can write that

$$\left|\overrightarrow{g}(x)\right| \leq \int_{\Omega_2} k(x,y)\left|\overrightarrow{f}(y)\right| d\mu_2(y), \tag{2.10}$$

and we mean the fact that

$$|g_i(x)| \leq \int_{\Omega_2} k(x,y)|f_i(y)|\,d\mu_2(y), \tag{2.11}$$

for all $i = 1,...,n$,*etc.*

Notation 2.5: *Next let* $(\Omega_1, \Sigma_1, \mu_1)$ *and* $(\Omega_2, \Sigma_2, \mu_2)$ *be measure spaces with positive* σ*-finite measures, and let* $k_j : \Omega_1 \times \Omega_2 \to \mathbb{R}$ *be a nonnegative measurable function,* $k_j(x,\cdot)$ *measurable on* Ω_2 *and*

$$K_j(x) = \int_{\Omega_2} k_j(x,y)\,d\mu_2(y), \quad x \in \Omega_1, j = 1,...,m. \tag{2.12}$$

We suppose that $K_j(x) > 0$ *a.e. on* Ω_1*. Let the measurable functions* $g_{ji} : \Omega_1 \to \mathbb{R}$ *with the representation*

$$g_{ji}(x) = \int_{\Omega_2} k_j(x,y) f_{ji}(y)\,d\mu_2(y), \tag{2.13}$$

where $f_{ji} : \Omega_2 \to \mathbb{R}$ *are measurable functions,* $i = 1,...,n$ *and* $j = 1,...,m$.

Denote the function vectors $\overrightarrow{g_j} := (g_{j1}, g_{j2}, ..., g_{jn})$ *and* $\overrightarrow{f_j} := (f_{j1},...,f_{jn})$, $j = 1,...,m$.

We say $\overrightarrow{f_j}$ *is integrable with respect to measure* μ*, iff all* f_{ji} *are integrable with respect to* μ.

We also consider here $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1,...,m$, *convex functions that are increasing per coordinate. Again* u *is a weight function on* Ω_1.

Our second main result is when $m = 2$.

Theorem 2.6: *Here all as in Notation 2.5. Assume that the function*

$x\mapsto\left(\frac{u(x)k_1(x,y)k_2(x,y)}{K_1(x)K_2(x)}\right)$ *is integrable on* Ω_1*, for each* $y\in\Omega_2$*. Define* λ_2 *on* Ω_2 *by*

$$\lambda_2\left(y\right):=\int_{\Omega_1}\frac{u\left(x\right)k_1\left(x,y\right)k_2\left(x,y\right)}{K_1\left(x\right)K_2\left(x\right)}d\mu_1\left(x\right)<\infty. \tag{2.14}$$

Then

$$\int_{\Omega_1}u\left(x\right)\Phi_1\left(\left|\frac{\overrightarrow{g_1}\left(x\right)}{K_1\left(x\right)}\right|\right)\Phi_2\left(\left|\frac{\overrightarrow{g_2}\left(x\right)}{K_2\left(x\right)}\right|\right)d\mu_1\left(x\right)\le$$

$$\left(\int_{\Omega_2}\Phi_2\left(\left|\overrightarrow{f_2}\left(y\right)\right|\right)d\mu_2\left(y\right)\right)\left(\int_{\Omega_2}\Phi_1\left(\left|\overrightarrow{f_1}\left(y\right)\right|\right)\lambda_2\left(y\right)d\mu_2\left(y\right)\right), \tag{2.15}$$

under the assumptions:
(i) $\{f_{1i},\Phi_1\left(\left|f_{11}\right|,...,\left|f_{1n}\right|\right)\}$, $\{f_{2i},\Phi_2\left(\left|f_{21}\right|,...,\left|f_{2n}\right|\right)\}$ *are* $k_j\left(x,y\right)d\mu_2\left(y\right)$ *-integrable,* μ_1 *-a.e. in* $x\in\Omega_1$*,* $j=1,2$ *(respectively), for all* $i=1,...,n$*.*
(ii) $\lambda_2\Phi_1\left(\left|\overrightarrow{f_1}\right|\right)$*,* $\Phi_2\left(\left|\overrightarrow{f_2}\right|\right)$*, are both* μ_2 *-integrable.*

Proof: Acting, similarly as in the proof of Theorem 2.3 we have

$$\int_{\Omega_1}u\left(x\right)\Phi_1\left(\left|\frac{\overrightarrow{g_1}\left(x\right)}{K_1\left(x\right)}\right|\right)\Phi_2\left(\left|\frac{\overrightarrow{g_2}\left(x\right)}{K_2\left(x\right)}\right|\right)d\mu_1\left(x\right)=$$

$$\int_{\Omega_1}u\left(x\right)\Phi_1\left(\left|\frac{1}{K_1\left(x\right)}\int_{\Omega_2}k_1\left(x,y\right)\overrightarrow{f_1}\left(y\right)d\mu_2\left(y\right)\right|\right)\cdot \tag{2.16}$$

$$\Phi_2\left(\left|\frac{1}{K_2\left(x\right)}\int_{\Omega_2}k_2\left(x,y\right)\overrightarrow{f_2}\left(y\right)d\mu_2\left(y\right)\right|\right)d\mu_1\left(x\right)\le$$

$$\int_{\Omega_1}u\left(x\right)\Phi_1\left(\frac{1}{K_1\left(x\right)}\int_{\Omega_2}k_1\left(x,y\right)\left|\overrightarrow{f_1}\left(y\right)\right|d\mu_2\left(y\right)\right)\cdot$$

$$\Phi_2\left(\frac{1}{K_2\left(x\right)}\int_{\Omega_2}k_2\left(x,y\right)\left|\overrightarrow{f_2}\left(y\right)\right|d\mu_2\left(y\right)\right)d\mu_1\left(x\right)\le$$

$$\int_{\Omega_1}u\left(x\right)\frac{1}{K_1\left(x\right)}\left(\int_{\Omega_2}k_1\left(x,y\right)\Phi_1\left(\left|\overrightarrow{f_1}\left(y\right)\right|\right)d\mu_2\left(y\right)\right)\cdot$$

$$\frac{1}{K_2\left(x\right)}\left(\int_{\Omega_2}k_2\left(x,y\right)\Phi_2\left(\left|\overrightarrow{f_2}\left(y\right)\right|\right)d\mu_2\left(y\right)\right)d\mu_1\left(x\right)= \tag{2.17}$$

(calling $\gamma_1(x) := \int_{\Omega_2} k_1(x,y)\,\Phi_1\left(\left|\overrightarrow{f_1}(y)\right|\right) d\mu_2(y)$)

$$\int_{\Omega_1}\int_{\Omega_2} \frac{u(x)\,\gamma_1(x)}{K_1(x)\,K_2(x)} k_2(x,y)\,\Phi_2\left(\left|\overrightarrow{f_2}(y)\right|\right) d\mu_2(y)\,d\mu_1(x) =$$

$$\int_{\Omega_2}\int_{\Omega_1} \frac{u(x)\,\gamma_1(x)}{K_1(x)\,K_2(x)} k_2(x,y)\,\Phi_2\left(\left|\overrightarrow{f_2}(y)\right|\right) d\mu_1(x)\,d\mu_2(y) =$$

$$\int_{\Omega_2} \Phi_2\left(\left|\overrightarrow{f_2}(y)\right|\right)\left(\int_{\Omega_1} \frac{u(x)\,\gamma_1(x)}{K_1(x)\,K_2(x)} k_2(x,y)\,d\mu_1(x)\right) d\mu_2(y) =$$

$$\int_{\Omega_2} \Phi_2\left(\left|\overrightarrow{f_2}(y)\right|\right)\cdot \tag{2.18}$$

$$\left(\int_{\Omega_1} \frac{u(x)\,k_2(x,y)}{K_1(x)\,K_2(x)}\left(\int_{\Omega_2} k_1(x,y)\,\Phi_1\left(\left|\overrightarrow{f_1}(y)\right|\right) d\mu_2(y)\right) d\mu_1(x)\right) d\mu_2(y) =$$

$$\int_{\Omega_2} \Phi_2\left(\left|\overrightarrow{f_2}(y)\right|\right)\cdot$$

$$\left[\int_{\Omega_1}\left(\int_{\Omega_2} \frac{u(x)\,k_1(x,y)\,k_2(x,y)}{K_1(x)\,K_2(x)}\Phi_1\left(\left|\overrightarrow{f_1}(y)\right|\right) d\mu_2(y)\right) d\mu_1(x)\right] d\mu_2(y) =$$

$$\left(\int_{\Omega_2} \Phi_2\left(\left|\overrightarrow{f_2}(y)\right|\right) d\mu_2(y)\right)\cdot$$

$$\left[\int_{\Omega_1}\left(\int_{\Omega_2} \frac{u(x)\,k_1(x,y)\,k_2(x,y)}{K_1(x)\,K_2(x)}\Phi_1\left(\left|\overrightarrow{f_1}(y)\right|\right) d\mu_2(y)\right) d\mu_1(x)\right] =$$

$$\left(\int_{\Omega_2} \Phi_2\left(\left|\overrightarrow{f_2}(y)\right|\right) d\mu_2(y)\right)\cdot$$

$$\left[\int_{\Omega_2}\left(\int_{\Omega_1} \frac{u(x)\,k_1(x,y)\,k_2(x,y)}{K_1(x)\,K_2(x)}\Phi_1\left(\left|\overrightarrow{f_1}(y)\right|\right) d\mu_1(x)\right) d\mu_2(y)\right] =$$

$$\left(\int_{\Omega_2} \Phi_2\left(\left|\overrightarrow{f_2}(y)\right|\right) d\mu_2(y)\right) \cdot \tag{2.19}$$

$$\left[\int_{\Omega_2} \Phi_1\left(\left|\overrightarrow{f_1}(y)\right|\right)\left(\int_{\Omega_1} \frac{u(x)\, k_1(x,y)\, k_2(x,y)}{K_1(x)\, K_2(x)} d\mu_1(x)\right) d\mu_2(y)\right] =$$

$$\left(\int_{\Omega_2} \Phi_2\left(\left|\overrightarrow{f_2}(y)\right|\right) d\mu_2(y)\right)\left[\int_{\Omega_2} \Phi_1\left(\left|\overrightarrow{f_1}(y)\right|\right) \lambda_2(y)\, d\mu_2(y)\right],$$

proving the claim. ■

When $m = 3$, the corresponding result follows.

Theorem 2.7 : *Here all as in Notation 2.5. Assume that the function* $x \mapsto \left(\frac{u(x)k_1(x,y)k_2(x,y)k_3(x,y)}{K_1(x)K_2(x)K_3(x)}\right)$ *is integrable on* Ω_1*, for each* $y \in \Omega_2$*. Define* λ_3 *on* Ω_2 *by*

$$\lambda_3(y) := \int_{\Omega_1} \frac{u(x)\, k_1(x,y)\, k_2(x,y)\, k_3(x,y)}{K_1(x)\, K_2(x)\, K_3(x)} d\mu_1(x) < \infty. \tag{2.20}$$

Then

$$\int_{\Omega_1} u(x) \prod_{j=1}^{3} \Phi_j\left(\left|\frac{\overrightarrow{g_j}(x)}{K_j(x)}\right|\right) d\mu_1(x) \leq \tag{2.21}$$

$$\left(\prod_{j=2}^{3} \int_{\Omega_2} \Phi_j\left(\left|\overrightarrow{f_j}(y)\right|\right) d\mu_2(y)\right)\left(\int_{\Omega_2} \Phi_1\left(\left|\overrightarrow{f_1}(y)\right|\right) \lambda_3(y)\, d\mu_2(y)\right),$$

under the assumptions:

(i) $\overrightarrow{f_j}$, $\Phi_j\left(\left|\overrightarrow{f_j}\right|\right)$, *are* $k_j(x,y)\, d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$, $j = 1, 2, 3,$

(ii) $\lambda_3 \Phi_1\left(\left|\overrightarrow{f_1}\right|\right), \Phi_2\left(\left|\overrightarrow{f_2}\right|\right), \Phi_3\left(\left|\overrightarrow{f_3}\right|\right)$*, are all* μ_2 *-integrable.*

Proof : We also have

$$\int_{\Omega_1} u(x) \prod_{j=1}^{3} \Phi_j\left(\left|\frac{\overrightarrow{g_j}(x)}{K_j(x)}\right|\right) d\mu_1(x) =$$

$$\int_{\Omega_1} u(x) \prod_{j=1}^{3} \Phi_j\left(\left|\frac{1}{K_j(x)} \int_{\Omega_2} k_j(x,y)\, \overrightarrow{f_j}(y)\, d\mu_2(y)\right|\right) d\mu_1(x) \leq \tag{2.22}$$

$$\int_{\Omega_1} u(x) \prod_{j=1}^{3} \Phi_j \left(\frac{1}{K_j(x)} \int_{\Omega_2} k_j(x,y) \left| \overrightarrow{f_j}(y) \right| d\mu_2(y) \right) d\mu_1(x) \leq$$

$$\int_{\Omega_1} u(x) \prod_{j=1}^{3} \left(\frac{1}{K_j(x)} \int_{\Omega_2} k_j(x,y) \Phi_j \left(\left| \overrightarrow{f_j}(y) \right| \right) d\mu_2(y) \right) d\mu_1(x) =$$

$$\int_{\Omega_1} \left(\frac{u(x)}{\prod_{j=1}^{3} K_j(x)} \right) \left(\prod_{j=1}^{3} \int_{\Omega_2} k_j(x,y) \Phi_j \left(\left| \overrightarrow{f_j}(y) \right| \right) d\mu_2(y) \right) d\mu_1(x) =$$

(calling $\theta(x) := \frac{u(x)}{\prod_{j=1}^{3} K_j(x)}$)

$$\int_{\Omega_1} \theta(x) \left(\prod_{j=1}^{3} \int_{\Omega_2} k_j(x,y) \Phi_j \left(\left| \overrightarrow{f_j}(y) \right| \right) d\mu_2(y) \right) d\mu_1(x) = \qquad (2.23)$$

$$\int_{\Omega_1} \theta(x) \left[\int_{\Omega_2} \left(\prod_{j=1}^{2} \int_{\Omega_2} k_j(x,y) \Phi_j \left(\left| \overrightarrow{f_j}(y) \right| \right) d\mu_2(y) \right) \right.$$

$$\left. k_3(x,y) \Phi_3 \left(\left| \overrightarrow{f_3}(y) \right| \right) d\mu_2(y) \right] d\mu_1(x) =$$

$$\int_{\Omega_1} \left(\int_{\Omega_2} \theta(x) \left(\prod_{j=1}^{2} \int_{\Omega_2} k_j(x,y) \Phi_j \left(\left| \overrightarrow{f_j}(y) \right| \right) d\mu_2(y) \right) \right.$$

$$\left. k_3(x,y) \Phi_3 \left(\left| \overrightarrow{f_3}(y) \right| \right) d\mu_2(y) \right) d\mu_1(x) =$$

$$\int_{\Omega_2} \left(\int_{\Omega_1} \theta(x) \left(\prod_{j=1}^{2} \int_{\Omega_2} k_j(x,y) \Phi_j \left(\left| \overrightarrow{f_j}(y) \right| \right) d\mu_2(y) \right) \right.$$

$$k_3(x,y)\,\Phi_3\left(\left|\overrightarrow{f_3}(y)\right|\right)d\mu_1(x)\Big)\,d\mu_2(y)=$$

$$\int_{\Omega_2}\Phi_3\left(\left|\overrightarrow{f_3}(y)\right|\right)\left(\int_{\Omega_1}\theta(x)\,k_3(x,y)\left(\prod_{j=1}^{2}\int_{\Omega_2}k_j(x,y)\,\Phi_j\left(\left|\overrightarrow{f_j}(y)\right|\right)d\mu_2(y)\right)\right. \tag{2.24}$$

$$d\mu_1(x))\,d\mu_2(y)=$$

$$\int_{\Omega_2}\Phi_3\left(\left|\overrightarrow{f_3}(y)\right|\right)\left[\int_{\Omega_1}\theta(x)\,k_3(x,y)\left(\int_{\Omega_2}\left\{\int_{\Omega_2}k_1(x,y)\,\Phi_1\left(\left|\overrightarrow{f_1}(y)\right|\right)d\mu_2(y)\right\}\cdot\right.\right.$$

$$\left.\left.k_2(x,y)\,\Phi_2\left(\left|\overrightarrow{f_2}(y)\right|\right)d\mu_2(y)\right)d\mu_1(x)\right]d\mu_2(y)=$$

$$\int_{\Omega_2}\Phi_3\left(\left|\overrightarrow{f_3}(y)\right|\right)\left[\int_{\Omega_1}\left(\int_{\Omega_2}\theta(x)\,k_2(x,y)\,k_3(x,y)\,\Phi_2\left(\left|\overrightarrow{f_2}(y)\right|\right)\cdot\right.\right. \tag{2.25}$$

$$\left.\left.\left\{\int_{\Omega_2}k_1(x,y)\,\Phi_1\left(\left|\overrightarrow{f_1}(y)\right|\right)d\mu_2(y)\right\}d\mu_2(y)\right)d\mu_1(x)\right]d\mu_2(y)=$$

$$\left(\int_{\Omega_2}\Phi_3\left(\left|\overrightarrow{f_3}(y)\right|\right)d\mu_2(y)\right)\left[\int_{\Omega_1}\left(\int_{\Omega_2}\theta(x)\,k_2(x,y)\,k_3(x,y)\,\Phi_2\left(\left|\overrightarrow{f_2}(y)\right|\right)\cdot\right.\right.$$

$$\left.\left.\left\{\int_{\Omega_2}k_1(x,y)\,\Phi_1\left(\left|\overrightarrow{f_1}(y)\right|\right)d\mu_2(y)\right\}d\mu_2(y)\right)d\mu_1(x)\right]=$$

$$\left(\int_{\Omega_2}\Phi_3\left(\left|\overrightarrow{f_3}(y)\right|\right)d\mu_2(y)\right)\left[\int_{\Omega_2}\left(\int_{\Omega_1}\theta(x)\,k_2(x,y)\,k_3(x,y)\,\Phi_2\left(\left|\overrightarrow{f_2}(y)\right|\right)\cdot\right.\right.$$

$$\left.\left.\left\{\int_{\Omega_2}k_1(x,y)\,\Phi_1\left(\left|\overrightarrow{f_1}(y)\right|\right)d\mu_2(y)\right\}d\mu_1(x)\right)d\mu_2(y)\right]= \tag{2.26}$$

$$\left(\int_{\Omega_2}\Phi_3\left(\left|\overrightarrow{f_3}(y)\right|\right)d\mu_2(y)\right)\left[\int_{\Omega_2}\Phi_2\left(\left|\overrightarrow{f_2}(y)\right|\right)\left(\int_{\Omega_1}\theta(x)\,k_2(x,y)\,k_3(x,y)\cdot\right.\right.$$

$$\left(\int_{\Omega_2} k_1(x,y)\,\Phi_1\left(\left|\overrightarrow{f_1}(y)\right|\right) d\mu_2(y)\right) d\mu_1(x)\Big) d\mu_2(y)\Big] =$$

$$\left(\int_{\Omega_2} \Phi_3\left(\left|\overrightarrow{f_3}(y)\right|\right) d\mu_2(y)\right)\left[\int_{\Omega_2} \Phi_2\left(\left|\overrightarrow{f_2}(y)\right|\right)\left\{\int_{\Omega_1}\left(\int_{\Omega_2} \theta(x)\prod_{j=1}^{3} k_j(x,y)\,\cdot\right.\right.\right.$$

$$\left.\left.\left.\Phi_1\left(\left|\overrightarrow{f_1}(y)\right|\right) d\mu_2(y)\right) d\mu_1(x)\right\} d\mu_2(y)\right] =$$

$$\left(\int_{\Omega_2} \Phi_3\left(\left|\overrightarrow{f_3}(y)\right|\right) d\mu_2(y)\right)\left(\int_{\Omega_2} \Phi_2\left(\left|\overrightarrow{f_2}(y)\right|\right) d\mu_2(y)\right)\cdot$$

$$\left(\int_{\Omega_1}\left(\int_{\Omega_2} \theta(x)\prod_{j=1}^{3} k_j(x,y)\,\Phi_1\left(\left|\overrightarrow{f_1}(y)\right|\right) d\mu_2(y)\right) d\mu_1(x)\right) = \tag{2.27}$$

$$\left(\prod_{j=2}^{3}\int_{\Omega_2} \Phi_j\left(\left|\overrightarrow{f_j}(y)\right|\right) d\mu_2(y)\right)\cdot$$

$$\left(\int_{\Omega_2}\left(\int_{\Omega_1} \theta(x)\prod_{j=1}^{3} k_j(x,y)\,\Phi_1\left(\left|\overrightarrow{f_1}(y)\right|\right) d\mu_1(x)\right) d\mu_2(y)\right) =$$

$$\left(\prod_{j=2}^{3}\int_{\Omega_2} \Phi_j\left(\left|\overrightarrow{f_j}(y)\right|\right) d\mu_2(y)\right)\cdot$$

$$\left(\int_{\Omega_2} \Phi_1\left(\left|\overrightarrow{f_1}(y)\right|\right)\left(\int_{\Omega_1} \theta(x)\prod_{j=1}^{3} k_j(x,y)\,d\mu_1(x)\right) d\mu_2(y)\right) =$$

$$\left(\prod_{j=2}^{3}\int_{\Omega_2} \Phi_j\left(\left|\overrightarrow{f_j}(y)\right|\right) d\mu_2(y)\right)\left(\int_{\Omega_2} \Phi_1\left(\left|\overrightarrow{f_1}(y)\right|\right)\lambda_3(y)\,d\mu_2(y)\right), \tag{2.28}$$

proving the claim. ∎

For general $m \in \mathbb{N}$, the following result is valid.

Theorem 2.8: *Again here we follow Notation 2.5. Assume that the function* $x \mapsto \left(\frac{u(x) \prod_{j=1}^{m} k_j(x,y)}{\prod_{j=1}^{m} K_j(x)} \right)$ *is integrable on* Ω_1*, for each* $y \in \Omega_2$*. Define* λ_m *on* Ω_2 *by*

$$\lambda_m(y) := \int_{\Omega_1} \left(\frac{u(x) \prod_{j=1}^{m} k_j(x,y)}{\prod_{j=1}^{m} K_j(x)} \right) d\mu_1(x) < \infty. \tag{2.29}$$

Then

$$\int_{\Omega_1} u(x) \prod_{j=1}^{m} \Phi_j \left(\left| \frac{\overrightarrow{g_j}(x)}{K_j(x)} \right| \right) d\mu_1(x) \leq \tag{2.30}$$

$$\left(\prod_{j=2}^{m} \int_{\Omega_2} \Phi_j \left(\left| \overrightarrow{f_j}(y) \right| \right) d\mu_2(y) \right) \left(\int_{\Omega_2} \Phi_1 \left(\left| \overrightarrow{f_1}(y) \right| \right) \lambda_m(y) \, d\mu_2(y) \right),$$

under the assumptions:
(i) $\overrightarrow{f_j}, \Phi_j \left(\left| \overrightarrow{f_j} \right| \right)$*, are* $k_j(x,y) \, d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$*,* $j = 1, ..., m$*,*
(ii) $\lambda_m \Phi_1 \left(\left| \overrightarrow{f_1} \right| \right), \Phi_2 \left(\left| \overrightarrow{f_2} \right| \right), \Phi_3 \left(\left| \overrightarrow{f_3} \right| \right), ..., \Phi_m \left(\left| \overrightarrow{f_m} \right| \right)$*, are all* μ_2 *-integrable.*

When $k(x,y) = k_1(x,y) = k_2(x,y) = ... = k_m(x,y)$, then $K(x) := K_1(x) = K_2(x) = ... = K_m(x)$. Then from Theorem 2.8 we get:

Corollary 2.9: *Assume that the function* $x \mapsto \left(\frac{u(x) k^m(x,y)}{K^m(x)} \right)$ *is integrable on* Ω_1*, for each* $y \in \Omega_2$*. Define* U_m *on* Ω_2 *by*

$$U_m(y) := \int_{\Omega_1} \left(\frac{u(x) k^m(x,y)}{K^m(x)} \right) d\mu_1(x) < \infty. \tag{2.31}$$

Then

$$\int_{\Omega_1} u(x) \prod_{j=1}^{m} \Phi_j \left(\left| \frac{\overrightarrow{g_j}(x)}{K(x)} \right| \right) d\mu_1(x) \leq \tag{2.32}$$

$$\left(\prod_{j=2}^{m} \int_{\Omega_2} \Phi_j \left(\left| \overrightarrow{f_j}(y) \right| \right) d\mu_2(y) \right) \left(\int_{\Omega_2} \Phi_1 \left(\left| \overrightarrow{f_1}(y) \right| \right) U_m(y) \, d\mu_2(y) \right),$$

under the assumptions:

(i) $\overrightarrow{f_j}$, $\Phi_j\left(\left|\overrightarrow{f_j}\right|\right)$, *are* $k(x,y)\,d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x\in\Omega_1$, $j=1,...,m$,

(ii) $U_m\Phi_1\left(\left|\overrightarrow{f_1}\right|\right), \Phi_2\left(\left|\overrightarrow{f_2}\right|\right), \Phi_3\left(\left|\overrightarrow{f_3}\right|\right), ..., \Phi_m\left(\left|\overrightarrow{f_m}\right|\right)$, *are all* μ_2 *-integrable.*

When $m=2$ from Corollary 2.9 we obtain

Corollary 2.10: *Assume that the function* $x\mapsto\left(\frac{u(x)k^2(x,y)}{K^2(x)}\right)$ *is integrable on* Ω_1*, for each* $y\in\Omega_2$*. Define* U_2 *on* Ω_2 *by*

$$U_2(y):=\int_{\Omega_1}\left(\frac{u(x)\,k^2(x,y)}{K^2(x)}\right)d\mu_1(x)<\infty. \tag{2.33}$$

Then

$$\int_{\Omega_1}u(x)\,\Phi_1\left(\left|\frac{\overrightarrow{g_1}(x)}{K(x)}\right|\right)\Phi_2\left(\left|\frac{\overrightarrow{g_2}(x)}{K(x)}\right|\right)d\mu_1(x)\le \tag{2.34}$$

$$\left(\int_{\Omega_2}\Phi_2\left(\left|\overrightarrow{f_2}(y)\right|\right)d\mu_2(y)\right)\left(\int_{\Omega_2}\Phi_1\left(\left|\overrightarrow{f_1}(y)\right|\right)U_2(y)\,d\mu_2(y)\right),$$

under the assumptions:
(i) $\overrightarrow{f_1}$, $\overrightarrow{f_2}$, $\Phi_1\left(\left|\overrightarrow{f_1}\right|\right)$, $\Phi_2\left(\left|\overrightarrow{f_2}\right|\right)$ *are all* $k(x,y)\,d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x\in\Omega_1$,
(ii) $U_2\Phi_1\left(\left|\overrightarrow{f_1}\right|\right), \Phi_2\left(\left|\overrightarrow{f_2}\right|\right)$, *are both* μ_2 *-integrable.*

For $m\in\mathbb{N}$, the following more general result is also valid.

Theorem 2.11: *Let* $\rho\in\{1,...,m\}$ *be fixed. Assume that the function* $x\mapsto\left(\frac{u(x)\prod_{j=1}^{m}k_j(x,y)}{\prod_{j=1}^{m}K_j(x)}\right)$ *is integrable on* Ω_1*, for each* $y\in\Omega_2$*. Define* λ_m *on* Ω_2 *by*

$$\lambda_m(y):=\int_{\Omega_1}\left(\frac{u(x)\prod_{j=1}^{m}k_j(x,y)}{\prod_{j=1}^{m}K_j(x)}\right)d\mu_1(x)<\infty. \tag{2.35}$$

Then

$$I:=\int_{\Omega_1}u(x)\prod_{j=1}^{m}\Phi_j\left(\left|\frac{\overrightarrow{g_j}(x)}{K_j(x)}\right|\right)d\mu_1(x)\le \tag{2.36}$$

$$\left(\prod_{\substack{j=1\\j\neq\rho}}^{m}\int_{\Omega_2}\Phi_j\left(\left|\overrightarrow{f_j}(y)\right|\right)d\mu_2(y)\right)\left(\int_{\Omega_2}\Phi_\rho\left(\left|\overrightarrow{f_\rho}(y)\right|\right)\lambda_m(y)\,d\mu_2(y)\right):=I_\rho,$$

under the assumptions:
(i) $\overrightarrow{f_j}$, $\Phi_j\left(\left|\overrightarrow{f_j}\right|\right)$*, are* $k_j(x,y)\,d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x\in\Omega_1$, $j=1,...,m$,
(ii) $\lambda_m\Phi_\rho\left(\left|\overrightarrow{f_\rho}\right|\right);\Phi_1\left(\left|\overrightarrow{f_1}\right|\right),\Phi_2\left(\left|\overrightarrow{f_2}\right|\right),\Phi_3\left(\left|\overrightarrow{f_3}\right|\right),...,\widehat{\Phi_\rho\left(\left|\overrightarrow{f_\rho}\right|\right)},...,\Phi_m\left(\left|\overrightarrow{f_m}\right|\right)$,
are all μ_2 *-integrable, where* $\widehat{\Phi_\rho\left(\left|\overrightarrow{f_\rho}\right|\right)}$ *means a missing item.*

We make

Remark 1 : In the notations and assumptions of Theorem 2.11, replace assumption (ii) by the assumption,
(iii) $\Phi_1\left(\left|\overrightarrow{f_1}\right|\right),...,\Phi_m\left(\left|\overrightarrow{f_m}\right|\right);\lambda_m\Phi_1\left(\left|\overrightarrow{f_1}\right|\right),...,\lambda_m\Phi_m\left(\left|\overrightarrow{f_m}\right|\right)$, are all μ_2 -integrable functions.

Then, clearly it holds,

$$I\leq\frac{\sum_{\rho=1}^{m}I_\rho}{m}. \tag{2.37}$$

Two general applications of Theorem 2.11 follow for specific Φ_j.

Theorem 2.12 : *Here all as in Theorem 2.11. It holds*

$$\int_{\Omega_1}u(x)\prod_{j=1}^{m}\left(\sum_{i=1}^{n}e^{\left|\frac{g_{ji}(x)}{K_j(x)}\right|}\right)d\mu_1(x)\leq \tag{2.38}$$

$$\left(\prod_{\substack{j=1\\j\neq\rho}}^{m}\int_{\Omega_2}\left(\sum_{i=1}^{n}e^{|f_{ji}(y)|}\right)d\mu_2(y)\right)\left(\int_{\Omega_2}\left(\sum_{i=1}^{n}e^{|f_{\rho i}(y)|}\right)\lambda_m(y)\,d\mu_2(y)\right),$$

under the assumptions:
(i) $\overrightarrow{f_j}$, $\left(\sum_{i=1}^{n}e^{|f_{ji}(y)|}\right)$*, are* $k_j(x,y)\,d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x\in\Omega_1$, $j=1,...,m$,
(ii) $\lambda_m(y)\left(\sum_{i=1}^{n}e^{|f_{\rho i}(y)|}\right)$ *and* $\left(\sum_{i=1}^{n}e^{|f_{ji}(y)|}\right)$ *for* $j\neq\rho$, $j=1,...,m$*, are all* μ_2 *-integrable.*

Proof: Apply Theorem 2.11 with $\Phi_j(x_1,...,x_n) = \sum_{i=1}^{m} e^{x_i}$, for all $j = 1,...,m$. ■

We continue with

Theorem 2.13: *Here all as in Theorem 2.11 and* $p \geq 1$. *It holds*

$$\int_{\Omega_1} u(x) \prod_{j=1}^{m} \left\| \frac{\overrightarrow{g_j}(x)}{K_j(x)} \right\|_p d\mu_1(x) \leq \tag{2.39}$$

$$\left(\prod_{\substack{j=1 \\ j\neq\rho}}^{m} \int_{\Omega_2} \left\| \overrightarrow{f_j}(y) \right\|_p d\mu_2(y) \right) \left(\int_{\Omega_2} \left\| \overrightarrow{f_\rho}(y) \right\|_p \lambda_m(y)\, d\mu_2(y) \right),$$

under the assumptions:

(i) $\left\| \overrightarrow{f_j} \right\|_p$ *is* $k_j(x,y)\, d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$, $j = 1,...,m$,

(ii) $\lambda_m \left\| \overrightarrow{f_\rho} \right\|_p ; \left\| \overrightarrow{f_j} \right\|_p$ $j \neq \rho$, $j = 1,...,m$, *are all* μ_2 *-integrable.*

Proof: Apply Theorem 2.11 with $\Phi_j(x_1,...,x_n) = \left\| \overrightarrow{x} \right\|_p$, $\overrightarrow{x} = (x_1,...,x_n)$, for all $j = 1,...,m$. ■

We make

Remark 2: Let f_{ji} be Lebesgue measurable functions from (a,b) into $\mathbb{R}$, such that $\left(I_{a+}^{\alpha_j}(|f_{ji}|)\right)(x) \in \mathbb{R}$, $\forall\, x \in (a,b)$, $\alpha_j > 0$, $j = 1,...,m$, $i = 1,...,n$, e.g. when $f_{ji} \in L_\infty(a,b)$.

Consider here

$$g_{ji}(x) = \left(I_{a+}^{\alpha_j} f_{ji}\right)(x), \quad x \in (a,b), \quad j = 1,...,m, i = 1,...,n,$$

we remind

$$\left(I_{a+}^{\alpha_j} f_{ji}\right)(x) = \frac{1}{\Gamma(\alpha_j)} \int_a^x (x-t)^{\alpha_j - 1} f_{ji}(t)\, dt. \tag{2.40}$$

Notice that $g_{ji}(x) \in \mathbb{R}$ and it is Lebesgue measurable.

We pick $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, the Lebesgue measure.

We see that

$$\left(I_{a+}^{\alpha_j} f_{ji}\right)(x) = \int_a^b \frac{\chi_{(a,x]}(t)\,(x-t)^{\alpha_j - 1}}{\Gamma(\alpha_j)} f_{ji}(t)\, dt, \tag{2.41}$$

where χ stands for the characteristic function.

So, we pick here

$$k_j(x,t) := \frac{\chi_{(a,x]}(t)(x-t)^{\alpha_j-1}}{\Gamma(\alpha_j)}, j = 1, ..., m. \tag{2.42}$$

In fact

$$k_j(x,y) = \begin{cases} \frac{(x-y)^{\alpha_j-1}}{\Gamma(\alpha_j)}, & a < y \leq x, \\ 0, & x < y < b. \end{cases} \tag{2.43}$$

Clearly it holds

$$K_j(x) = \int_{(a,b)} \frac{\chi_{(a,x]}(y)(x-y)^{\alpha_j-1}}{\Gamma(\alpha_j)} dy = \frac{(x-a)^{\alpha_j}}{\Gamma(\alpha_j+1)}, \tag{2.44}$$

$a < x < b$, $j = 1, ..., m$.

Notice that

$$\prod_{j=1}^{m} \frac{k_j(x,y)}{K_j(x)} = \prod_{j=1}^{m} \left(\frac{\chi_{(a,x]}(y)(x-y)^{\alpha_j-1}}{\Gamma(\alpha_j)} \cdot \frac{\Gamma(\alpha_j+1)}{(x-a)^{\alpha_j}} \right) =$$

$$\prod_{j=1}^{m} \left(\frac{\chi_{(a,x]}(y)(x-y)^{\alpha_j-1}\alpha_j}{(x-a)^{\alpha_j}} \right) = \frac{\chi_{(a,x]}(y)(x-y)^{\left(\sum_{j=1}^{m}\alpha_j - m\right)} \left(\prod_{j=1}^{m}\alpha_j\right)}{(x-a)^{\left(\sum_{j=1}^{m}\alpha_j\right)}}. \tag{2.45}$$

Calling

$$\alpha := \sum_{j=1}^{m} \alpha_j > 0, \ \gamma := \prod_{j=1}^{m} \alpha_j > 0, \tag{2.46}$$

we have that

$$\prod_{j=1}^{m} \frac{k_j(x,y)}{K_j(x)} = \frac{\chi_{(a,x]}(y)(x-y)^{\alpha-m}\gamma}{(x-a)^{\alpha}}. \tag{2.47}$$

Therefore, for (2.29), we get for appropiate weight u that

$$\lambda_m(y) = \gamma \int_y^b u(x) \frac{(x-y)^{\alpha-m}}{(x-a)^{\alpha}} dx < \infty, \tag{2.48}$$

for all $a < y < b$.

Let now

$$u(x) = (x-a)^{\alpha}, \; x \in (a,b). \tag{2.49}$$

Then

$$\lambda_m(y) = \gamma \int_y^b (x-y)^{\alpha-m} dx = \frac{\gamma (b-y)^{\alpha-m+1}}{\alpha-m+1}, \tag{2.50}$$

$y \in (a,b)$, where $\alpha > m-1$.

By Theorem 2.12 we get

$$\int_a^b (x-a)^{\alpha} \prod_{j=1}^{m} \left(\sum_{i=1}^{n} e^{\left(\frac{\left| \left(I_{a+}^{\alpha_j} f_{ji} \right)(x) \right| \Gamma(\alpha_j+1)}{(x-a)^{\alpha_j}} \right)} \right) dx \leq \tag{2.51}$$

$$\left(\frac{\gamma}{\alpha-m+1} \right) \left(\prod_{\substack{j=1 \\ j \neq \rho}}^{m} \int_a^b \left(\sum_{i=1}^{n} e^{|f_{ji}(y)|} \right) dy \right) \cdot$$

$$\left(\int_a^b (b-y)^{\alpha-m+1} \left(\sum_{i=1}^{n} e^{|f_{\rho i}(y)|} \right) dy \right) \leq$$

$$\left(\frac{\gamma (b-a)^{\alpha-m+1}}{\alpha-m+1} \right) \prod_{j=1}^{m} \left(\int_a^b \left(\sum_{i=1}^{n} e^{|f_{ji}(y)|} \right) dy \right), \tag{2.52}$$

under the assumptions:
(i) $\alpha > m-1$,
(ii) $\left(\sum_{i=1}^{n} e^{|f_{ji}(y)|} \right)$ is $\frac{\chi_{(a,x]}(y)(x-y)^{\alpha_j-1}}{\Gamma(\alpha_j)} dy$ -integrable, a.e. in $x \in (a,b)$, $j = 1, ..., m$,
(iii) $\left(\sum_{i=1}^{n} e^{|f_{ji}(y)|} \right)$, $j = 1, ..., m$, are all Lebesgue integrable on (a,b).

Let $p \geq 1$, by Theorem 2.13 we get

$$\int_a^b (x-a)^{\alpha} \left(\prod_{j=1}^{m} \left\| \frac{\left(\overrightarrow{I_{a+}^{\alpha_j} f_j} \right)(x)}{(x-a)^{\alpha_j}} \right\|_p \right) \left(\prod_{j=1}^{m} \Gamma(\alpha_j+1) \right) dx \leq$$

$$\left(\frac{\gamma (b-a)^{\alpha-m+1}}{\alpha-m+1} \right) \prod_{j=1}^{m} \left(\int_a^b \left\| \overrightarrow{f_j}(y) \right\|_p dy \right). \tag{2.53}$$

Above $\overrightarrow{I_{a+}^{\alpha_j} f_j} := \left(I_{a+}^{\alpha_j} f_{j1}, ..., I_{a+}^{\alpha_j} f_{jn}\right)$, $j = 1, ..., m$, etc.

But we see that

$$\prod_{j=1}^{m} \left\| \frac{\left(\overrightarrow{I_{a+}^{\alpha_j} f_j}\right)(x)}{(x-a)^{\alpha_j}} \right\|_p = \left(\frac{1}{(x-a)^{\alpha}}\right) \prod_{j=1}^{m} \left\| \overrightarrow{I_{a+}^{\alpha_j} f_j}(x) \right\|_p . \tag{2.54}$$

We have proved that

$$\int_a^b \left(\prod_{j=1}^{m} \left\| \left(\overrightarrow{I_{a+}^{\alpha_j} f_j}\right)(x) \right\|_p \right) dx \le$$

$$\left(\frac{\gamma (b-a)^{\alpha-m+1}}{(\alpha - m + 1)\left(\prod_{j=1}^{m} \Gamma(\alpha_j + 1)\right)} \right) \prod_{j=1}^{m} \left(\int_a^b \left\| \overrightarrow{f_j}(y) \right\|_p dy \right). \tag{2.55}$$

Thus we derive that

$$\left\| \prod_{j=1}^{m} \left\| \left(\overrightarrow{I_{a+}^{\alpha_j} f_j}\right) \right\|_p \right\|_{1,(a,b)} \le$$

$$\left(\frac{\gamma (b-a)^{\alpha-m+1}}{(\alpha - m + 1)\left(\prod_{j=1}^{m} \Gamma(\alpha_j + 1)\right)} \right) \prod_{j=1}^{m} \left\| \left\| \overrightarrow{f_j} \right\|_p \right\|_{1,(a,b)}, \tag{2.56}$$

under the assumptions:

(i) $\alpha > m - 1$, $p \ge 1$,

(ii) $\left\| \overrightarrow{f_j} \right\|_p$ is $\frac{\chi_{(a,x]}(y)(x-y)^{\alpha_j - 1}}{\Gamma(\alpha_j)} dy$-integrable, a.e. in $x \in (a,b)$, $j = 1, ..., m$,

(iii) $\left\| \overrightarrow{f_j} \right\|_p$, $j = 1, ..., m$, are all Lebesgue integrable on (a,b).

Using the last condition (iii), we derive that $f_{ji} \in L_1(a,b)$, for all $j = 1, ..., m$; $i = 1, ..., n$ and by assuming $\alpha_j \ge 1$, we obtain that $I_{a+}^{\alpha_j}(|f_{ji}|)$ is finite on (a,b).

We continue with

Remark 3: Let f_{ji} be Lebesgue measurable functions : $(a,b) \to \mathbb{R}$, such that $I_{b-}^{\alpha_j}(|f_{ji}|)(x) < \infty$, $\forall\ x \in (a,b)$, $\alpha_j > 0$, $j = 1, ..., m$, $i = 1, ..., n$, e.g. when $f_{ji} \in L_\infty(a,b)$.

Consider here

$$g_{ji}(x) = \left(I_{b-}^{\alpha_j} f_{ji}\right)(x), \quad x \in (a,b), \quad j = 1, ..., m, i = 1, ..., n, \tag{2.57}$$

we remind

$$\left(I_{b-}^{\alpha_j} f_{ji}\right)(x) = \frac{1}{\Gamma(\alpha_j)} \int_x^b f_{ji}(t)(t-x)^{\alpha_j - 1} dt, \tag{2.58}$$

$(x < b)$.

Notice that $g_{ji}(x) \in \mathbb{R}$ and it is Lebesgue measurable.

We pick $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, the Lebesgue measure.

We see that

$$\left(I_{b-}^{\alpha_j} f_{ji}\right)(x) = \int_a^b \chi_{[x,b)}(t) \frac{(t-x)^{\alpha_j - 1}}{\Gamma(\alpha_j)} f_{ji}(t) dt. \tag{2.59}$$

So, we pick here

$$k_j(x,t) := \chi_{[x,b)}(t) \frac{(t-x)^{\alpha_j-1}}{\Gamma(\alpha_j)}, \quad j = 1, ..., m. \tag{2.60}$$

In fact

$$k_j(x,y) = \begin{cases} \frac{(y-x)^{\alpha_j-1}}{\Gamma(\alpha_j)}, & x \le y < b, \\ 0, \quad a < y < x. \end{cases} \tag{2.61}$$

Clearly it holds

$$K_j(x) = \int_{(a,b)} \chi_{[x,b)}(y) \frac{(y-x)^{\alpha_j-1}}{\Gamma(\alpha_j)} dy = \frac{(b-x)^{\alpha_j}}{\Gamma(\alpha_j+1)}, \tag{2.62}$$

$a < x < b$, $j = 1, ..., m$.

Notice that

$$\prod_{j=1}^m \frac{k_j(x,y)}{K_j(x)} = \prod_{j=1}^m \left(\chi_{[x,b)}(y) \frac{(y-x)^{\alpha_j-1}}{\Gamma(\alpha_j)} \cdot \frac{\Gamma(\alpha_j+1)}{(b-x)^{\alpha_j}} \right) =$$

$$\prod_{j=1}^m \left(\chi_{[x,b)}(y) \frac{(y-x)^{\alpha_j-1} \alpha_j}{(b-x)^{\alpha_j}} \right) = \chi_{[x,b)}(y) \frac{(y-x)^{\left(\sum_{j=1}^m \alpha_j - m\right)} \left(\prod_{j=1}^m \alpha_j\right)}{(b-x)^{\left(\sum_{j=1}^m \alpha_j\right)}}. \tag{2.63}$$

Calling

$$\alpha := \sum_{j=1}^m \alpha_j > 0, \; \gamma := \prod_{j=1}^m \alpha_j > 0, \tag{2.64}$$

we have that

$$\prod_{j=1}^{m}\frac{k_j(x,y)}{K_j(x)}=\frac{\chi_{[x,b)}(y)(y-x)^{\alpha-m}\gamma}{(b-x)^{\alpha}}. \tag{2.65}$$

Therefore, for (2.29), we get for appropiate weight u that

$$\lambda_m(y)=\gamma\int_a^y u(x)\frac{(y-x)^{\alpha-m}}{(b-x)^{\alpha}}dx<\infty, \tag{2.66}$$

for all $a<y<b$.

Let now

$$u(x)=(b-x)^{\alpha},\ x\in(a,b). \tag{2.67}$$

Then

$$\lambda_m(y)=\gamma\int_a^y (y-x)^{\alpha-m}dx=\frac{\gamma(y-a)^{\alpha-m+1}}{\alpha-m+1}, \tag{2.68}$$

$y\in(a,b)$, where $\alpha>m-1$.

By Theorem 2.12 we get

$$\int_a^b (b-x)^{\alpha}\prod_{j=1}^{m}\left(\sum_{i=1}^{n}e^{\left(\frac{\left|\left(I_{b-}^{\alpha_j}f_{ji}\right)(x)\right|\Gamma(\alpha_j+1)}{(b-x)^{\alpha_j}}\right)}\right)dx\leq \tag{2.69}$$

$$\left(\frac{\gamma}{\alpha-m+1}\right)\left(\prod_{\substack{j=1\\ j\neq\rho}}^{m}\int_a^b\left(\sum_{i=1}^{n}e^{|f_{ji}(y)|}\right)dy\right)\cdot$$

$$\left(\int_a^b (y-a)^{\alpha-m+1}\left(\sum_{i=1}^{n}e^{|f_{\rho i}(y)|}\right)dy\right)\leq$$

$$\left(\frac{\gamma(b-a)^{\alpha-m+1}}{\alpha-m+1}\right)\prod_{j=1}^{m}\left(\int_a^b\left(\sum_{i=1}^{n}e^{|f_{ji}(y)|}\right)dy\right), \tag{2.70}$$

under the assumptions:

(i) $\alpha>m-1$,

(ii) $\left(\sum_{i=1}^{n}e^{|f_{ji}(y)|}\right)$ is $\frac{\chi_{[x,b)}(y)(y-x)^{\alpha_j-1}}{\Gamma(\alpha_j)}dy$ -integrable, a.e. in $x\in(a,b)$, $j=1,...,m$,

(iii) $\left(\sum_{i=1}^{n}e^{|f_{ji}(y)|}\right)$, $j=1,...,m$, are all Lebesgue integrable on (a,b).

Let $p \geq 1$, by Theorem 2.13 we get

$$\int_a^b (b-x)^{\alpha} \left(\prod_{j=1}^{m} \left\| \frac{\left(\overrightarrow{I_{b-}^{\alpha_j} f_j}\right)(x)}{(b-x)^{\alpha_j}} \right\|_p \right) \left(\prod_{j=1}^{m} \Gamma(\alpha_j+1) \right) dx \leq$$

$$\left(\frac{\gamma (b-a)^{\alpha-m+1}}{\alpha-m+1} \right) \prod_{j=1}^{m} \left(\int_a^b \left\| \overrightarrow{f_j}(y) \right\|_p dy \right). \tag{2.71}$$

But we see that

$$\prod_{j=1}^{m} \left\| \frac{\left(I_{b-}^{\alpha_j} f_{j_i}\right)(x)}{(b-x)^{\alpha_j}} \right\|_p = \left(\frac{1}{(b-x)^{\alpha}} \right) \prod_{j=1}^{m} \left\| I_{b-}^{\alpha_j} f_{j_i}(x) \right\|_p. \tag{2.72}$$

We have proved that

$$\int_a^b \left(\prod_{j=1}^{m} \left\| \left(I_{b-}^{\alpha_j} f_{ji}\right)(x) \right\|_p \right) dx \leq$$

$$\left(\frac{\gamma (b-a)^{\alpha-m+1}}{(\alpha-m+1)\left(\prod_{j=1}^{m} \Gamma(\alpha_j+1)\right)} \right) \prod_{j=1}^{m} \left(\int_a^b \left\| \overrightarrow{f_j}(y) \right\|_p dy \right). \tag{2.73}$$

Thus we derive that

$$\left\| \prod_{j=1}^{m} \left\| \left(I_{b-}^{\alpha_j} f_{ji}\right) \right\|_p \right\|_{1,(a,b)} \leq$$

$$\left(\frac{\gamma (b-a)^{\alpha-m+1}}{(\alpha-m+1)\left(\prod_{j=1}^{m} \Gamma(\alpha_j+1)\right)} \right) \prod_{j=1}^{m} \left\| \left\| \overrightarrow{f_j} \right\|_p \right\|_{1,(a,b)}, \tag{2.74}$$

under the assumptions:
(i) $\alpha > m-1$, $p \geq 1$,
(ii) $\left\| \overrightarrow{f_j} \right\|_p$ is $\frac{\chi_{[x,b)}(y)(y-x)^{\alpha_j-1}}{\Gamma(\alpha_j)} dy$-integrable, a.e. in $x \in (a,b)$, $j = 1,...,m$,
(iii) $\left\| \overrightarrow{f_j} \right\|_p$, $j = 1,...,m$, are all Lebesgue integrable on (a,b).

Using the last assumption (iii), we derive again that $f_{ji} \in L_1(a,b)$, for all $j = 1,...,m$; $i = 1,...,n$, and by assuming $\alpha_j \geq 1$, we obtain that $I_{b-}^{\alpha_j}(|f_{ji}|)$ is finite on (a,b).

We mention

Definition 2.14: ([1], p. 448) The left generalized Riemann-Liouville fractional derivative of f of order $\beta > 0$ is given by

$$D_a^{\beta} f(x) = \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{dx}\right)^n \int_a^x (x-y)^{n-\beta-1} f(y)\, dy, \tag{2.75}$$

where $n = [\beta] + 1$, $x \in [a, b]$.

For $a, b \in \mathbb{R}$, we say that $f \in L_1(a, b)$ has an L_∞ fractional derivative $D_a^{\beta} f$ $(\beta > 0)$ in $[a, b]$, if and only if

(1) $D_a^{\beta-k} f \in C([a, b])$, $k = 2, ..., n = [\beta] + 1$,
(2) $D_a^{\beta-1} f \in AC([a, b])$
(3) $D_a^{\beta} f \in L_\infty(a, b)$.

Above we define $D_a^0 f := f$ and $D_a^{-\delta} f := I_{a+}^{\delta} f$, if $0 < \delta \leq 1$.

From [1, p. 449] and [10] we mention and use

Lemma 2.15: *Let $\beta > \alpha \geq 0$ and let $f \in L_1(a, b)$ have an L_∞ fractional derivative $D_a^{\beta} f$ in $[a, b]$ and let $D_a^{\beta-k} f(a) = 0$, $k = 1, ..., [\beta] + 1$, then*

$$D_a^{\alpha} f(x) = \frac{1}{\Gamma(\beta-\alpha)} \int_a^x (x-y)^{\beta-\alpha-1} D_a^{\beta} f(y)\, dy, \tag{2.76}$$

for all $a \leq x \leq b$.

Here $D_a^{\alpha} f \in AC([a, b])$ for $\beta - \alpha \geq 1$, and $D_a^{\alpha} f \in C([a, b])$ for $\beta - \alpha \in (0, 1)$.

Notice here that

$$D_a^{\alpha} f(x) = \left(I_{a+}^{\beta-\alpha}\left(D_a^{\beta} f\right)\right)(x), \quad a \leq x \leq b. \tag{2.77}$$

We give

Theorem 2.16: *Let $f_{ji} \in L_1(a, b)$, $\alpha_j, \beta_j : \beta_j > \alpha_j \geq 0$, $j = 1, ..., m$; $i = 1, ..., n$. Here $(f_{ji}, \alpha_j, \beta_j)$ fulfill terminology and assumptions of Definition 2.14 and Lemma 2.15. Let $\overline{\alpha} := \sum_{j=1}^{m} (\beta_j - \alpha_j)$, $\overline{\gamma} := \prod_{j=1}^{m} (\beta_j - \alpha_j)$, assume $\overline{\alpha} > m - 1$. Then*

$$\int_a^b (x-a)^{\overline{\alpha}} \prod_{j=1}^{m} \left(\sum_{i=1}^{n} e^{\left(\frac{\left| \left(D_a^{\alpha_j} f_{ji}\right)(x) \right| \Gamma(\beta_j - \alpha_j + 1)}{(x-a)^{\left(\beta_j - \alpha_j\right)}} \right)} \right) dx \leq$$

$$\left(\frac{\overline{\gamma}(b-a)^{\overline{\alpha}-m+1}}{\overline{\alpha} - m + 1} \right) \left(\prod_{j=1}^{m} \left(\int_a^b \left(\sum_{i=1}^{n} e^{\left| \left(D_a^{\beta_j} f_{ji}\right)(y) \right|} \right) dy \right) \right). \tag{2.78}$$

Proof: Use of (2.51)-(2.52). ■

We also give

Theorem 2.17: *All here as in Theorem 2.16, plus $p \geq 1$. Then*

$$\left\| \prod_{j=1}^{m} \left\| \overrightarrow{D_a^{\alpha_j} f_j} \right\|_p \right\|_{1,(a,b)} \leq \tag{2.79}$$

$$\left(\frac{\overline{\gamma}\,(b-a)^{(\overline{\alpha}-m+1)}}{(\overline{\alpha}-m+1)\prod_{j=1}^{m}\left(\Gamma\left(\beta_j-\alpha_j+1\right)\right)} \right) \left(\prod_{j=1}^{m} \left\| \left\| \overrightarrow{D_a^{\beta_j} f_j} \right\|_p \right\|_{1,(a,b)} \right).$$

Above $\overrightarrow{D_a^{\beta_j} f_j} := \left(D_a^{\beta_j} f_{j1}, ..., D_a^{\beta_j} f_{jn} \right)$, $j = 1, ..., m, etc.$

Proof: By (2.56). ■

We need

Definition 2.18: ([6], p. 50, [1], p. 449) Let $\nu \geq 0$, $n := \lceil \nu \rceil$, $f \in AC^n([a,b])$. Then the left Caputo fractional derivative is given by

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt$$

$$= \left(I_{a+}^{n-\nu} f^{(n)} \right)(x), \tag{2.80}$$

and it exists almost everywhere for $x \in [a,b]$, in fact $D_{*a}^{\nu} f \in L_1(a,b)$, ([1], p. 394).

We have $D_{*a}^{n} f = f^{(n)}$, $n \in \mathbb{Z}_+$.

We also need

Theorem 2.19: *([4]) Let $\nu \geq \rho + 1$, $\rho > 0$, $\nu, \rho \notin \mathbb{N}$. Call $n := \lceil \nu \rceil$, $m^* := \lceil \rho \rceil$. Assume $f \in AC^n([a,b])$, such that $f^{(k)}(a) = 0$, $k = m^*, m^*+1, ..., n-1$, and $D_{*a}^{\nu} f \in L_\infty(a,b)$. Then $D_{*a}^{\rho} f \in AC([a,b])$ (where $D_{*a}^{\rho} f = \left(I_{a+}^{m^*-\rho} f^{(m^*)} \right)(x)$), and*

$$D_{*a}^{\rho} f(x) = \frac{1}{\Gamma(\nu-\rho)} \int_a^x (x-t)^{\nu-\rho-1} D_{*a}^{\nu} f(t)\, dt$$

$$= \left(I_{a+}^{\nu-\rho} \left(D_{*a}^{\nu} f \right) \right)(x), \tag{2.81}$$

$\forall\, x \in [a,b]$.

We give

Theorem 2.20: *Let* (f_{ji}, ν_j, ρ_j), $j = 1, ..., m$, $m \geq 2$, $i = 1, ..., n$, *as in the assumptions of Theorem 2.19. Set* $\alpha^* := \sum_{j=1}^{m} (\nu_j - \rho_j)$, $\gamma^* := \prod_{j=1}^{m} (\nu_j - \rho_j)$. *Here* $a, b \in \mathbb{R}$, $a < b$. *Then*

$$\int_a^b (x-a)^{\alpha^*} \prod_{j=1}^{m} \left(\sum_{i=1}^{n} e^{\left(\left| D_{*a}^{\rho_j} f_{ji}(x) \right| \left(\frac{\Gamma(\nu_j - \rho_j + 1)}{(x-a)^{(\nu_j - \rho_j)}} \right) \right)} \right) dx \leq$$

$$\left(\frac{\gamma^* (b-a)^{\alpha^* - m + 1}}{(\alpha^* - m + 1)} \right) \left(\prod_{j=1}^{m} \left(\int_a^b \left(e^{\left| D_{*a}^{\nu_j} f_{ji}(y) \right|} \right) dy \right) \right). \quad (2.82)$$

Proof: Use of (2.51), (2.52). See here that $\alpha^* \geq m > m - 1$. ■

We continue with

Theorem 2.21: *All as in Theorem 2.20, plus* $p \geq 1$. *Then*

$$\left\| \prod_{j=1}^{m} \left\| \overrightarrow{D_{*a}^{\rho_j} f_j} \right\|_p \right\|_{1,(a,b)} \leq \quad (2.83)$$

$$\left(\frac{\gamma^* (b-a)^{(\alpha^* - m + 1)}}{(\alpha^* - m + 1) \left(\prod_{j=1}^{m} (\Gamma(\nu_j - \rho_j + 1)) \right)} \right) \prod_{j=1}^{m} \left\| \left\| \overrightarrow{D_{*a}^{\nu_j} f_j} \right\|_p \right\|_{1,(a,b)}.$$

Proof: By (2.56). ■

We need

Definition 2.22: ([2], [7], [9]) Let $\alpha \geq 0$, $n := \lceil \alpha \rceil$, $f \in AC^n([a, b])$. We define the right Caputo fractional derivative of order $\alpha \geq 0$, by

$$\overline{D}_{b-}^{\alpha} f(x) := (-1)^n I_{b-}^{n-\alpha} f^{(n)}(x), \quad (2.84)$$

we set $\overline{D}_{-}^{0} f := f$, i.e.

$$\overline{D}_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(n-\alpha)} \int_x^b (J-x)^{n-\alpha-1} f^{(n)}(J) \, dJ. \quad (2.85)$$

Notice that $\overline{D}_{b-}^{n} f = (-1)^n f^{(n)}$, $n \in \mathbb{N}$.

We need

Theorem 2.23: *([4]) Let $f \in AC^n([a,b])$, $\alpha > 0$, $n \in \mathbb{N}$, $n := \lceil \alpha \rceil$, $\alpha \geq \rho + 1$, $\rho > 0$, $r = \lceil \rho \rceil$, $\alpha, \rho \notin \mathbb{N}$. Assume $f^{(k)}(b) = 0$, $k = r, r+1, ..., n-1$, and $\overline{D}_{b-}^{\alpha} f \in L_\infty([a,b])$. Then*

$$\overline{D}_{b-}^{\rho} f(x) = \left(I_{b-}^{\alpha-\rho} \left(\overline{D}_{b-}^{\alpha} f \right) \right)(x) \in AC([a,b]), \tag{2.86}$$

that is

$$\overline{D}_{b-}^{\rho} f(x) = \frac{1}{\Gamma(\alpha-\rho)} \int_x^b (t-x)^{\alpha-\rho-1} \left(\overline{D}_{b-}^{\alpha} f \right)(t)\, dt, \tag{2.87}$$

$\forall\ x \in [a,b]$.

We give

Theorem 2.24: *Let $(f_{ji}, \alpha_j, \rho_j)$, $j = 1, ..., m$, $m \geq 2$, $i = 1, ..., n$, as in the assumptions of Theorem 2.23. Set $A := \sum_{j=1}^{m} (\alpha_j - \rho_j)$, $B := \prod_{j=1}^{m} (\alpha_j - \rho_j)$. Here $a, b \in \mathbb{R}$, $a < b$, $p \geq 1$. Then*

$$\left\| \prod_{j=1}^{m} \left\| \overrightarrow{\overline{D}_{b-}^{\rho_j} f_j} \right\|_p \right\|_{1,(a,b)} \leq \tag{2.88}$$

$$\left(\frac{B(b-a)^{(A-m+1)}}{(A-m+1) \left(\prod_{j=1}^{m} (\Gamma(\alpha_j - \rho_j + 1)) \right)} \right) \prod_{j=1}^{m} \left\| \left\| \overrightarrow{\overline{D}_{b-}^{\nu_j} f_j} \right\|_p \right\|_{1,(a,b)}.$$

Proof: By (2.56), plus $A \geq m > m - 1$. ■

We continue with

Theorem 2.25: *All here as in Theorem 2.24. Then*

$$\int_a^b (b-x)^A \left(\prod_{j=1}^{m} \ln \left(\sum_{i=1}^{n} e^{\left(\left| \overline{D}_{b-}^{\rho_j} f_{ji}(x) \right| \left(\frac{\Gamma(\alpha_j-\rho_j+1)}{(b-x)^{(\alpha_j-\rho_j)}} \right) \right)} \right) \right) dx \leq$$

$$\left(\frac{B(b-a)^{A-m+1}}{(A-m+1)} \right) \left(\prod_{j=1}^{m} \left(\int_a^b \ln \left(\sum_{i=1}^{n} e^{\left| \overline{D}_{b-}^{\alpha_j} f_{ji}(y) \right|} \right) dy \right) \right). \tag{2.89}$$

Proof: Using Theorem 2.11. ■

We give

Definition 2.26: Let $\nu > 0$, $n := [\nu]$, $\alpha := \nu - n$ $(0 \leq \alpha < 1)$. Let $a, b \in \mathbb{R}$, $a \leq x \leq b$, $f \in C([a,b])$. We consider $C_a^{\nu}([a,b]) := \{f \in C^n([a,b]) : I_{a+}^{1-\alpha} f^{(n)} \in C^1([a,b])\}$. For $f \in C_a^{\nu}([a,b])$, we define the left generalized ν-fractional derivative of f over $[a,b]$ as

$$\Delta_a^{\nu} f := \left(I_{a+}^{1-\alpha} f^{(n)} \right)', \tag{2.90}$$

see [1], p. 24, and Canavati derivative in [5].

Notice here $\Delta_a^{\nu} f \in C([a,b])$.

So that

$$(\Delta_a^{\nu} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f^{(n)}(t)\, dt, \tag{2.91}$$

$\forall\, x \in [a,b]$.

Notice here that

$$\Delta_a^n f = f^{(n)}, \;\; n \in \mathbb{Z}_+. \tag{2.92}$$

We need

Theorem 2.27: *([4]) Let $f \in C_a^{\nu}([a,b])$, $n = [\nu]$, such that $f^{(i)}(a) = 0$, $i = r, r+1, ..., n-1$, where $r := [\rho]$, with $0 < \rho < \nu$. Then*

$$(\Delta_a^{\rho} f)(x) = \frac{1}{\Gamma(\nu-\rho)} \int_a^x (x-t)^{\nu-\rho-1} (\Delta_a^{\nu} f)(t)\, dt, \tag{2.93}$$

i.e.

$$(\Delta_a^{\rho} f) = I_{a+}^{\nu-\rho} (\Delta_a^{\nu} f) \in C([a,b]). \tag{2.94}$$

Thus $f \in C_a^{\rho}([a,b])$.

We present

Theorem 2.28: *Let (f_{ji}, ν_j, ρ_j), $j = 1, ..., m$, $m \geq 2$; $i = 1, ..., n$, as in the assumptions of Theorem 2.27. Set $A := \sum_{j=1}^{m} (\nu_j - \rho_j)$, $B := \prod_{j=1}^{m} (\nu_j - \rho_j)$. Here $a, b \in \mathbb{R}$, $a < b$, $p \geq 1$, and $A > m - 1$. Then*

$$\left\| \prod_{j=1}^{m} \left\| \overrightarrow{\Delta_a^{\rho_j} f_j} \right\|_p \right\|_{1,(a,b)} \leq \tag{2.95}$$

$$\left(\frac{B\left(b-a\right)^{(A-m+1)}}{\left(A-m+1\right)\left(\prod_{j=1}^{m}\left(\Gamma\left(\nu_j-\rho_j+1\right)\right)\right)}\right)\prod_{j=1}^{m}\left\|\left\|\overrightarrow{\Delta_a^{\nu_j}f_j}\right\|_p\right\|_{1,(a,b)}.$$

Proof: By (2.56). ■

We continue with

Theorem 2.29: *All here as in Theorem 2.28. Then*

$$\int_a^b (x-a)^A \left(\prod_{j=1}^{m}\ln\left(\sum_{i=1}^{n} e^{\left(\left|\Delta_a^{\rho_j}f_{ji}(x)\right|\frac{\Gamma(\nu_j-\rho_j+1)}{(x-a)^{(\nu_j-\rho_j)}}\right)}\right)\right)dx \leq$$

$$\left(\frac{B\left(b-a\right)^{A-m+1}}{A-m+1}\right)\left(\prod_{j=1}^{m}\left(\int_a^b \ln\left(\sum_{i=1}^{n} e^{\left|\Delta_a^{\nu_j}f_{ji}(y)\right|}\right)\right)dy\right). \tag{2.96}$$

Proof: Using Theorem 2.11. ■

We need

Definition 2.30: ([2]) Let $\nu > 0$, $n := [\nu]$, $\alpha = \nu - n$, $0 < \alpha < 1$, $f \in C([a,b])$. Consider

$$C_{b-}^{\nu}([a,b]) := \{f \in C^n([a,b]) : I_{b-}^{1-\alpha}f^{(n)} \in C^1([a,b])\}. \tag{2.97}$$

Define the right generalized ν-fractional derivative of f over $[a,b]$, by

$$\Delta_{b-}^{\nu}f := (-1)^{n-1}\left(I_{b-}^{1-\alpha}f^{(n)}\right)'. \tag{2.98}$$

We set $\Delta_{b-}^{0}f = f$. Notice that

$$\left(\Delta_{b-}^{\nu}f\right)(x) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b (J-x)^{-\alpha} f^{(n)}(J)\,dJ, \tag{2.99}$$

and $\Delta_{b-}^{\nu}f \in C([a,b])$.

We also need

Theorem 2.31: *([4]) Let $f \in C_{b-}^{\nu}([a,b])$, $0 < \rho < \nu$. Assume $f^{(i)}(b) = 0$, $i = r, r+1, ..., n-1$, where $r := [\rho]$, $n := [\nu]$. Then*

$$\Delta_{b-}^{\rho}f(x) = \frac{1}{\Gamma(\nu-\rho)}\int_x^b (J-x)^{\nu-\rho-1}\left(\Delta_{b-}^{\nu}f\right)(J)\,dJ, \tag{2.100}$$

$\forall\ x \in [a,b]$, *i.e.*

$$\Delta_{b-}^{\rho} f = I_{b-}^{\nu-\rho}\left(\Delta_{b-}^{\nu} f\right) \in C\left([a,b]\right), \tag{2.101}$$

and $f \in C_{b-}^{\rho}\left([a,b]\right)$.

We give

Theorem 2.32: *Let* (f_{ji}, ν_j, ρ_j), $j = 1, ..., m$, $m \geq 2$; $i = 1, ..., n$, *as in the assumptions of Theorem 2.31. Set* $A := \sum_{j=1}^{m} (\nu_j - \rho_j)$, $B := \prod_{j=1}^{m} (\nu_j - \rho_j)$. *Here* $a, b \in \mathbb{R}$, $a < b$, $p \geq 1$, *and* $A > m - 1$. *Then*

$$\left\| \prod_{j=1}^{m} \left\| \overrightarrow{\Delta_{b-}^{\rho_j} f_j} \right\|_p \right\|_{1,(a,b)} \leq \tag{2.102}$$

$$\left(\frac{B\,(b-a)^{(A-m+1)}}{(A-m+1)\left(\prod_{j=1}^{m} \left(\Gamma\left(\nu_j - \rho_j + 1\right)\right)\right)} \right) \prod_{j=1}^{m} \left\| \left\| \overrightarrow{\Delta_{b-}^{\nu_j} f_j} \right\|_p \right\|_{1,(a,b)}.$$

Proof: By (2.56). ■

Theorem 2.33: *All here as in Theorem 2.32. Then*

$$\int_a^b (b-x)^A \left(\prod_{j=1}^{m} \ln \left(\sum_{i=1}^{n} e^{\left(\left|\Delta_{b-}^{\rho_j} f_{ji}(x)\right| \frac{\Gamma(\nu_j - \rho_j + 1)}{(b-x)^{(\nu_j - \rho_j)}} \right)} \right) \right) dx \leq$$

$$\left(\frac{B\,(b-a)^{A-m+1}}{A-m+1} \right) \left(\prod_{j=1}^{m} \left(\int_a^b \ln \left(\sum_{i=1}^{n} e^{\left|\Delta_{b-}^{\nu_j} f_{ji}(y)\right|} \right) dy \right) \right). \tag{2.103}$$

Proof: Using Theorem 2.11. ■

We make

Definition 2.34: [13, p. 99] The fractional integrals of a function f with respect to given function g are defined as follows:

Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$. Here g is a strictly increasing function on $[a,b]$ and $g \in C^1\left([a,b]\right)$. The left- and right-sided fractional integrals of a function f with respect to another function g in $[a,b]$ are given by

$$\left(I_{a+;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t)\, f(t)\, dt}{\left(g(x) - g(t)\right)^{1-\alpha}},\ x > a, \tag{2.104}$$

$$\left(I_{b-;g}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}\frac{g'(t)\,f(t)\,dt}{(g(t)-g(x))^{1-\alpha}},\ x<b, \tag{2.105}$$

respectively.

We make

Remark 4: Let f_{ji} be Lebesgue measurable functions from (a,b) into $\mathbb{R}$, such that $\left(I_{a+;g}^{\alpha_j}(|f_{ji}|)\right)(x)\in\mathbb{R}$, $\forall\, x\in(a,b)$, $\alpha_j>0$, $j=1,...,m$, $i=1,...,n$.

Consider

$$g_{ji}(x):=\left(I_{a+;g}^{\alpha_j}f_{ji}\right)(x),\ x\in(a,b),\ \ j=1,...,m,\ \ i=1,...,n. \tag{2.106}$$

where

$$\left(I_{a+;g}^{\alpha_j}f_{ji}\right)(x)=\frac{1}{\Gamma(\alpha_j)}\int_{a}^{x}\frac{g'(t)\,f_{ji}(t)\,dt}{(g(x)-g(t))^{1-\alpha_j}},\ \ x>a. \tag{2.107}$$

Notice that $g_{ji}(x)\in\mathbb{R}$ and it is Lebesgue measurable.

We pick $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(x)=dx$, $d\mu_2(y)=dy$, the Lebesgue measure.

We see that

$$\left(I_{a+;g}^{\alpha_j}f_{ji}\right)(x)=\int_{a}^{b}\frac{\chi_{(a,x]}(t)\,g'(t)\,f_{ji}(t)}{\Gamma(\alpha_j)(g(x)-g(t))^{1-\alpha_j}}dt, \tag{2.108}$$

where χ is the characteristic function.

So, we pick here

$$k_j(x,t):=\frac{\chi_{(a,x]}(t)\,g'(t)}{\Gamma(\alpha_j)(g(x)-g(t))^{1-\alpha_j}}\quad j=1,...,m. \tag{2.109}$$

In fact

$$k_j(x,y)=\begin{cases}\frac{g'(y)}{\Gamma(\alpha_j)(g(x)-g(y))^{1-\alpha_j}} & a<y\le x,\\ 0, \quad x<y<b.\end{cases} \tag{2.110}$$

Clearly it holds

$$K_j(x)=\int_{a}^{b}\frac{\chi_{(a,x]}(y)\,g'(y)}{\Gamma(\alpha_j)(g(x)-g(y))^{1-\alpha_j}}dy=$$

$$\int_{a}^{x}\frac{g'(y)}{\Gamma(\alpha_j)(g(x)-g(y))^{1-\alpha_j}}dy=\frac{1}{\Gamma(\alpha_j)}\int_{a}^{x}(g(x)-g(y))^{\alpha_j-1}\,dg(y)= \tag{2.111}$$

$$\frac{1}{\Gamma(\alpha_j)}\int_{g(a)}^{g(x)}(g(x)-z)^{\alpha_j-1}\,dz=\frac{(g(x)-g(a))^{\alpha_j}}{\Gamma(\alpha_j+1)}.$$

So for $a < x < b$, $j = 1, ..., m$, we get

$$K_j(x) = \frac{(g(x) - g(a))^{\alpha_j}}{\Gamma(\alpha_j + 1)}. \tag{2.112}$$

Notice that

$$\prod_{j=1}^{m} \frac{k_j(x,y)}{K_j(x)} = \prod_{j=1}^{m} \left(\frac{\chi_{(a,x]}(y)\, g'(y)}{\Gamma(\alpha_j)(g(x) - g(y))^{1-\alpha_j}} \cdot \frac{\Gamma(\alpha_j + 1)}{(g(x) - g(a))^{\alpha_j}} \right) =$$

$$\frac{\chi_{(a,x]}(y)\,(g(x) - g(y))^{\left(\sum_{j=1}^{m} \alpha_j - m\right)} (g'(y))^m \left(\prod_{j=1}^{m} \alpha_j\right)}{(g(x) - g(a))^{\left(\sum_{j=1}^{m} \alpha_j\right)}}. \tag{2.113}$$

Calling

$$\alpha := \sum_{j=1}^{m} \alpha_j > 0, \quad \gamma := \prod_{j=1}^{m} \alpha_j > 0, \tag{2.114}$$

we have that

$$\prod_{j=1}^{m} \frac{k_j(x,y)}{K_j(x)} = \frac{\chi_{(a,x]}(y)\,(g(x) - g(y))^{\alpha - m} (g'(y))^m \gamma}{(g(x) - g(a))^{\alpha}}. \tag{2.115}$$

Therefore, for (2.35), we get for appropiate weight u that (denote λ_m by λ_m^g)

$$\lambda_m^g(y) = \gamma \left(g'(y)\right)^m \int_y^b u(x) \frac{(g(x) - g(y))^{\alpha - m}}{(g(x) - g(a))^{\alpha}} dx < \infty, \tag{2.116}$$

for all $a < y < b$.

Let now

$$u(x) = (g(x) - g(a))^{\alpha} g'(x), \quad x \in (a,b). \tag{2.117}$$

Then

$$\lambda_m^g(y) = \gamma \left(g'(y)\right)^m \int_y^b (g(x) - g(y))^{\alpha - m} g'(x)\, dx =$$

$$\gamma \left(g'(y)\right)^m \int_{g(y)}^{g(b)} (z - g(y))^{\alpha - m} dz = \tag{2.118}$$

$$\gamma \left(g' \left(y \right) \right)^{m} \frac{\left(g \left(b \right) - g \left(y \right) \right)^{\alpha - m + 1}}{\alpha - m + 1},$$

with $\alpha > m - 1$. That is

$$\lambda_{m}^{g} \left(y \right) = \gamma \left(g' \left(y \right) \right)^{m} \frac{\left(g \left(b \right) - g \left(y \right) \right)^{\alpha - m + 1}}{\alpha - m + 1}, \tag{2.119}$$

$\alpha > m - 1$, $y \in (a, b)$.

By Theorem 2.11 we get, for $p \geq 1$, that

$$\int_{a}^{b} \left(g \left(x \right) - g \left(a \right) \right)^{\alpha} g' \left(x \right) \prod_{j=1}^{m} \left\| \frac{\left(\overrightarrow{I_{a+;g}^{\alpha_j} f_j \left(x \right)} \right) \Gamma \left(\alpha_j + 1 \right)}{\left(g \left(x \right) - g \left(a \right) \right)^{\alpha_j}} \right\|_{p} dx \leq$$

$$\left(\frac{\gamma \left\| g' \right\|_{\infty}^{m} \left(g \left(b \right) - g \left(x \right) \right)^{\alpha - m + 1}}{\alpha - m + 1} \right) \left(\prod_{j=1}^{m} \int_{a}^{b} \left\| \overrightarrow{f_j} \left(y \right) \right\|_{p} dy \right). \tag{2.120}$$

So we have proved that

$$\int_{a}^{b} g' \left(x \right) \prod_{j=1}^{m} \left\| \left(\overrightarrow{I_{a+;g}^{\alpha_j} f_j} \left(x \right) \right) \right\|_{p} dx \leq$$

$$\left(\frac{\gamma \left\| g' \right\|_{\infty}^{m} \left(g \left(b \right) - g \left(a \right) \right)^{\alpha - m + 1}}{\left(\alpha - m + 1 \right) \prod_{j=1}^{m} \left(\Gamma \left(\alpha_j + 1 \right) \right)} \right) \left(\prod_{j=1}^{m} \int_{a}^{b} \left\| \overrightarrow{f_j} \left(y \right) \right\|_{p} dy \right), \tag{2.121}$$

under the asumptions:

(i) $p \geq 1$, $\alpha > m - 1$, f_{ji} with $I_{a+;g}^{\alpha_j} \left(|f_{ji}| \right)$ finite, $j = 1, ..., m$; $i = 1, ..., n$,

(ii) $\left\| \overrightarrow{f_j} \right\|_{p}$ are $\frac{\chi_{(a,x]}(y) g'(y) dy}{\Gamma(\alpha_j)(g(x) - g(y))^{1 - \alpha_j}}$-integrable, a.e. in $x \in (a, b)$, $j = 1, ..., m$,

(iii) $\left\| \overrightarrow{f_j} \right\|_{p}$ are Lebesgue integrable, $j = 1, ..., m$.

We need

Definition 2.35: ([12]) Let $0 < a < b < \infty$, $\alpha > 0$. The left- and right-sided Hadamard fractional integrals of order α are given by

$$\left(J_{a+}^{\alpha} f \right) \left(x \right) = \frac{1}{\Gamma \left(\alpha \right)} \int_{a}^{x} \left(\ln \frac{x}{y} \right)^{\alpha - 1} \frac{f \left(y \right)}{y} dy, \quad x > a, \tag{2.122}$$

and

$$\left(J_{b-}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\left(\ln\frac{y}{x}\right)^{\alpha-1}\frac{f(y)}{y}dy,\quad x<b, \tag{2.123}$$

respectively.

Notice that the Hadamard fractional integrals of order α are special cases of left- and right-sided fractional integrals of a function f with respect to another function, here $g(x)=\ln x$ on $[a,b]$, $0<a<b<\infty$.

Above f is a Lebesgue measurable function from (a,b) into $\mathbb{R}$, such that $\left(J_{a+}^{\alpha}(|f|)\right)(x)$ and/or $\left(J_{b-}^{\alpha}(|f|)\right)(x)\in\mathbb{R}$, $\forall\, x\in(a,b)$.

We give

Theorem 2.36 : *Let (f_{ji},α_j), $j=1,...,m$, $i=1,...,n$, and $J_{a+}^{\alpha_j}f_{ji}$ as in Definition 2.35. Set $\alpha:=\sum_{j=1}^m\alpha_j$, $\gamma:=\prod_{j=1}^m\alpha_j$; $p\geq1$, $\alpha>m-1$. Then*

$$\int_a^b\prod_{j=1}^m\left\|\overrightarrow{J_{a+}^{\alpha_j}f_j}(x)\right\|_p dx\leq \tag{2.124}$$

$$\left(\frac{b\gamma\left(\ln\left(\frac{b}{a}\right)\right)^{\alpha-m+1}}{a^m(\alpha-m+1)\prod_{j=1}^m(\Gamma(\alpha_j+1))}\right)\left(\prod_{j=1}^m\int_a^b\left\|\overrightarrow{f_j}(y)\right\|_p dy\right),$$

under the assumptions:
(i) $\left(J_{a+}^{\alpha_j}|f_{ji}|\right)$ *finite,* $j=1,...,m$; $i=1,...,n$,
(ii) $\left\|\overrightarrow{f_j}\right\|_p$ *are* $\left(\frac{\chi_{(a,x]}(y)dy}{\Gamma(\alpha_j)y\left(\ln\left(\frac{x}{y}\right)\right)^{1-\alpha_j}}\right)$ *-integrable, a.e. in* $x\in(a,b)$, $j=1,...,m$,
(iii) $\left\|\overrightarrow{f_j}\right\|_p$ *are Lebesgue integrable,* $j=1,...,m$.

Proof : By (2.121). ■

We make

Remark 5 : Let f_{ji} be Lebesgue measurable functions from (a,b) into $\mathbb{R}$, such that $\left(I_{b-;g}^{\alpha_j}(|f_{ji}|)\right)(x)\in\mathbb{R}$, $\forall\, x\in(a,b)$, $\alpha_j>0$, $j=1,...,m$, $i=1,...,n$.

Consider

$$g_{ji}(x):=\left(I_{b-;g}^{\alpha_j}f_{ji}\right)(x),\quad x\in(a,b),\quad j=1,...,m,\quad i=1,...,n, \tag{2.125}$$

where

$$\left(I_{b-;g}^{\alpha_j} f_{ji}\right)(x) = \frac{1}{\Gamma(\alpha_j)} \int_x^b \frac{g'(t) f_{ji}(t)\, dt}{(g(t) - g(x))^{1-\alpha_j}}, \quad x < b. \tag{2.126}$$

Notice that $g_{ji}(x) \in \mathbb{R}$ and it is Lebesgue measurable.

We pick $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, the Lebesgue measure.

We see that

$$\left(I_{b-;g}^{\alpha_j} f_{ji}\right)(x) = \int_a^b \frac{\chi_{[x,b)}(t)\, g'(t) f_{ji}(t)\, dt}{\Gamma(\alpha_j)(g(t) - g(x))^{1-\alpha_j}}, \tag{2.127}$$

where χ is the characteristic function.

So, we pick here

$$k_j(x, y) := \frac{\chi_{[x,b)}(y)\, g'(y)}{\Gamma(\alpha_j)(g(y) - g(x))^{1-\alpha_j}}, \quad j = 1, ..., m. \tag{2.128}$$

In fact

$$k_j(x, y) = \begin{cases} \frac{g'(y)}{\Gamma(\alpha_j)(g(y)-g(x))^{1-\alpha_j}}, & x \le y < b, \\ 0, \quad a < y < x. \end{cases} \tag{2.129}$$

Clearly it holds

$$K_j(x) = \int_a^b \frac{\chi_{[x,b)}(y)\, g'(y)\, dy}{\Gamma(\alpha_j)(g(y) - g(x))^{1-\alpha_j}} =$$

$$\frac{1}{\Gamma(\alpha_j)} \int_x^b g'(y)(g(y) - g(x))^{\alpha_j - 1} dy = \tag{2.130}$$

$$\frac{1}{\Gamma(\alpha_j)} \int_{g(x)}^{g(b)} (z - g(x))^{\alpha_j - 1} dg(y) = \frac{(g(b) - g(x))^{\alpha_j}}{\Gamma(\alpha_j + 1)}.$$

So for $a < x < b$, $j = 1, ..., m$, we get

$$K_j(x) = \frac{(g(b) - g(x))^{\alpha_j}}{\Gamma(\alpha_j + 1)}. \tag{2.131}$$

Notice that

$$\prod_{j=1}^{m} \frac{k_j(x, y)}{K_j(x)} = \prod_{j=1}^{m} \left(\frac{\chi_{[x,b)}(y)\, g'(y)}{\Gamma(\alpha_j)(g(y) - g(x))^{1-\alpha_j}} \cdot \frac{\Gamma(\alpha_j + 1)}{(g(b) - g(x))^{\alpha_j}} \right) =$$

$$\frac{\chi_{[x,b)}\left(y\right)\left(g'\left(y\right)\right)^{m}\left(g\left(y\right)-g\left(x\right)\right)^{\left(\sum_{j=1}^{m}\alpha_j-m\right)}\prod_{j=1}^{m}\alpha_j}{\left(g\left(b\right)-g\left(x\right)\right)^{\sum_{j=1}^{m}\alpha_j}}. \tag{2.132}$$

Calling

$$\alpha:=\sum_{j=1}^{m}\alpha_j>0,\quad \gamma:=\prod_{j=1}^{m}\alpha_j>0, \tag{2.133}$$

we have that

$$\prod_{j=1}^{m}\frac{k_j\left(x,y\right)}{K_j\left(x\right)}=\frac{\chi_{[x,b)}\left(y\right)\left(g'\left(y\right)\right)^{m}\left(g\left(y\right)-g\left(x\right)\right)^{\alpha-m}\gamma}{\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}}. \tag{2.134}$$

Therefore, for (2.35), we get for appropiate weight u that (denote λ_m by λ_m^g)

$$\lambda_m^g\left(y\right)=\gamma\left(g'\left(y\right)\right)^{m}\int_a^y u\left(x\right)\frac{\left(g\left(y\right)-g\left(x\right)\right)^{\alpha-m}}{\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}}dx<\infty, \tag{2.135}$$

for all $a<y<b$.

Let now

$$u\left(x\right)=\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}g'\left(x\right),\quad x\in\left(a,b\right). \tag{2.136}$$

Then

$$\lambda_m^g\left(y\right)=\gamma\left(g'\left(y\right)\right)^{m}\int_a^y g'\left(x\right)\left(g\left(y\right)-g\left(x\right)\right)^{\alpha-m}dx=$$

$$\gamma\left(g'\left(y\right)\right)^{m}\int_a^y\left(g\left(y\right)-g\left(x\right)\right)^{\alpha-m}dg\left(x\right)=\gamma\left(g'\left(y\right)\right)^{m}\int_{g(a)}^{g(y)}\left(g\left(y\right)-z\right)^{\alpha-m}dz= \tag{2.137}$$

$$\gamma\left(g'\left(y\right)\right)^{m}\frac{\left(g\left(y\right)-g\left(a\right)\right)^{\alpha-m+1}}{\alpha-m+1},$$

with $\alpha>m-1$. That is

$$\lambda_m^g\left(y\right)=\gamma\left(g'\left(y\right)\right)^{m}\frac{\left(g\left(y\right)-g\left(a\right)\right)^{\alpha-m+1}}{\alpha-m+1}, \tag{2.138}$$

$\alpha>m-1$, $y\in\left(a,b\right)$.

By Theorem 2.11 we get, for $p \geq 1$, that

$$\int_a^b (g(b) - g(x))^{\alpha} g'(x) \prod_{j=1}^m \left\| \frac{\left(\overrightarrow{I_{b-;g}^{\alpha_j} f_j}(x)\right) \Gamma(\alpha_j + 1)}{(g(b) - g(x))^{\alpha_j}} \right\|_p dx \leq$$

$$\left(\frac{\gamma \|g'\|_{\infty}^m (g(b) - g(a))^{\alpha - m + 1}}{\alpha - m + 1} \right) \left(\prod_{j=1}^m \int_a^b \left\| \overrightarrow{f_j}(y) \right\|_p dy \right). \tag{2.139}$$

So we have proved that

$$\int_a^b g'(x) \prod_{j=1}^m \left\| \left(\overrightarrow{I_{b-;g}^{\alpha_j} f_j}(x) \right) \right\|_p dx \leq$$

$$\left(\frac{\gamma \|g'\|_{\infty}^m (g(b) - g(a))^{\alpha - m + 1}}{(\alpha - m + 1) \prod_{j=1}^m \Gamma(\alpha_j + 1)} \right) \left(\prod_{j=1}^m \int_a^b \left\| \overrightarrow{f_j}(y) \right\|_p dy \right). \tag{2.140}$$

under the assumptions:
(i) $p \geq 1$, $\alpha > m - 1$, f_{ij} with $I_{b-;g}^{\alpha_j} |f_{ji}|$ finite, $j = 1, ..., m$; $i = 1, ..., n$,
(ii) $\left\| \overrightarrow{f_j} \right\|_p$ are $\left(\frac{\chi_{[x,b)}(y) g'(y) dy}{\Gamma(\alpha_j)(g(y) - g(x))^{1-\alpha_j}} \right)$ -integrable, a.e. in $x \in (a, b)$, $j = 1, ..., m$,
(iii) $\left\| \overrightarrow{f_j} \right\|_p$ are Lebesgue integrable, $j = 1, ..., m$.

Theorem 2.37: *Let (f_{ji}, α_j), $j = 1, ..., m$, $i = 1, ..., n$, and $J_{b-}^{\alpha_j} f_{ji}$ as in Definition 2.35. Set $\alpha := \sum_{j=1}^m \alpha_j$, $\gamma := \prod_{j=1}^m \alpha_j$; $p \geq 1$, $\alpha > m - 1$. Then*

$$\int_a^b \prod_{j=1}^m \left\| \overrightarrow{J_{b-}^{\alpha_j} f_j}(x) \right\|_p dx \leq \tag{2.141}$$

$$\left(\frac{b\gamma \left(\ln\left(\frac{b}{a}\right) \right)^{\alpha - m + 1}}{a^m (\alpha - m + 1) \prod_{j=1}^m \Gamma(\alpha_j + 1)} \right) \left(\prod_{j=1}^m \int_a^b \left\| \overrightarrow{f_j}(y) \right\|_p dy \right),$$

under the assumptions:

(i) $\left(J_{b-}^{\alpha_j}|f_{ji}|\right)$ *finite,* $j=1,...,m$; $i=1,...,n$,

(ii) $\left\|\overrightarrow{f_j}\right\|_p$ *are* $\left(\frac{\chi_{[x,b)}(y)dy}{\Gamma(\alpha_j)y\left(\ln\left(\frac{y}{x}\right)\right)^{1-\alpha_j}}\right)$ *-integrable, a.e. in* $x\in(a,b)$, $j=1,...,m$,

(iii) $\left\|\overrightarrow{f_j}\right\|_p$ *are Lebesgue integrable,* $j=1,...,m$.

Proof: By (2.140). ∎

References

[1] G.A. Anastassiou, *Fractional Differentiation Inequalities*, Research Monograph, Springer, New York, 2009

[2] G.A. Anastassiou, *On right fractional calculus*, Chaos, Solitons and Fractals, **42** (2009), 365-376

[3] G.A. Anastassiou, *Balanced fractional opial inequalities*, Chaos, Solitons and Fractals, **42**, No. 3 (2009), 1523-1528

[4] G.A. Anastassiou, *Fractional representation formulae and right fractional inequalities*, Mathematical and Computer Modelling, **54** (11-12) (2011), 3098-3115

[5] J.A. Canavati, *The Riemann-Liouville Integral*, Nieuw Archief Voor Wiskunde, **5** (1) (1987), 53-75

[6] Kai Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, 2004, 1st edition, Springer, New York, Heidelberg, 2010

[7] A.M.A. El-Sayed and M. Gaber, *On the finite Caputo and finite Riesz derivatives*, Electronic Journal of Theoretical Physics, **3**, No. 12 (2006), 81-95

[8] T.S. Ferguson, *Mathematical statistics*, Academic Press, New York, 1967

[9] R. Gorenflo and F. Mainardi, *Essentials of Fractional Calculus*, 2000, Maphysto Center, http://www.maphysto.dk/oldpages/ events/LevyCAC2000/MainardiNotes/fm2k0a.ps

[10] G.D. Handley, J.J. Koliha and J. Pečarić, *Hilbert-Pachpatte type integral inequalities for fractional derivatives*, Fractional Calculus and Applied Analysis, **4**, No. 1 (2001), 37-46

[11] H.G. Hardy, *Notes on some points in the integral calculus*, Messenger of Mathematics, **47**, No. 10 (1918), 145-150

[12] S. Iqbal, K. Krulic and J. Pecaric, *On an inequality of H.G. Hardy*, J. of Inequalities and Applications, Volume 2010, Article ID 264347, 23 pages

[13] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, **204** of North-Holland Mathematics Studies, Elsevier, New York, NY, USA, 2006

[14] M. Perlman, *Jensen's Inequality for a Convex Vector-valued function on an infinite-dimensional space*, J. of Multivariate Analysis, **4** (1974), 52-65

[15] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integral and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, Yverdon, Switzerland, 1993