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The Solutions of the Boundary Value Problems of the Theory of Thermoelasticity with Microtemperatures for an Elastic Circle

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In the present work, using absolutely and uniformly convergent series, the 2D boundary value problems of statics of the linear theory of thermoelasticity with microtemperatures for an elastic circle are solved explicitly. The question on the uniqueness of a solution of the problem is investigated.

Keywords: Thermoelasticity, Microtemperatura, Boundary value problem, Explicit solution.

AMS Subject Classification: 74F05, 74G30, 74G10.

1. Introduction

Together with generalization and development along several paths, the linear theory of thermoelasticity with microtemperatures has recently attracted considerable effort directed toward mathematical research and construction of explicit solutions for boundary value problems in specific domains. Of the publications devoted to such problems, we note [1,2], which also contain historical and bibliographic information.

2. Basic equations and boundary value problem

Consider a circle D of radius R with boundary S. Find a regular vector $U = (u_1, u_2, u_3, w_1, w_2)$, $(U \in C^1(\overline{D}) \cap C^2(D), \overline{D} = D \cup S)$ satisfying in the circle D a system of equations [1,2]:

$$\begin{aligned} &\mu\Delta u(x) + (\lambda+\mu) graddivu(x) = \beta gradu_3(x), \\ &k\Delta u_3(x) + k_1 divw(x) = 0, \qquad (1) \\ &k_6\Delta w(x) + (k_4+k_5) graddivw(x) - k_3 gradu_3(x) - k_2 w(x) = 0, \end{aligned}$$

and on the circumference S one of the following conditions:

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$$I.\ u(z)=f(z),\ u_3(z)=f_3(z),\ T''(\partial_z,n)w(z)=p(z);$$

$$II.\ u(z)=f(z),\ k\frac{\partial u_3(z)}{\partial n(z)}+k_1w(z)n(z)=f_3(z),\ T''(\partial_z,n)w(z)=p(z); \tag{2}$$

$$III.\ T'(\partial_z,n)u(z)-\beta u_3(z)n(z)=f(z),\ u_3(z)=f_3(z),\ T''(\partial_z,n)w(z)=p(z),$$

where u(x) is the displacement vector of the point x, $u=(u_1,u_2); w=(w_1,w_2)$ is the microtemperatures vector; u_3 is temperature measured from the constant absolute temperature T_0; n is the external unit normal vector to S; $f=(f_1,f_2)$, $p=(p_1,p_2)$, f_1,f_2,f_3-are the given functions on S; $\lambda,\mu,\beta,k,k_1,k_2,k_3,k_4,k_5,k_6$ are constitutive coefficients [1,2]; $T'u$ is the stress vector in the classical theory of elasticity; $T''w$ is stress vector for microtemperatures [2]:

$$\begin{aligned}
&T'(\partial_x,n)u(x)=\mu\frac{\partial u(x)}{\partial n}+\lambda n(x)divu(x)+\mu\sum_{i=1}^{2}n_i(x)gradu_i(x),\\
&T''(\partial_x,n)w(x)=(k_5+k_6)\frac{\partial w(x)}{\partial n}+k_4n(x)divw(x)+k_5\sum_{i=1}^{2}n_i(x)gradw_i(x).
\end{aligned} \tag{3}$$

Separately we will study the following problems:

1. Find in a circle D solution $u(x)$ of equation $(1)_1$, if on the circumference S there are given the values:
of the vector u (problem A_1);
of the vector $T'(\partial_x,n)u(x)-\beta u_3(x)n(x)$ (problem A_2).

2. Find in the circle D solutions $u_3(x)$ and $w(x)$ of the system of equations $(1)_2$ and $(1)_3$, if on the circumference S there are given the values:
of the function $u_3(z)$ and the vector$T''(\partial_z,n)w(z)$ (problem K_1);
of the function $k\dfrac{\partial u_3(z)}{\partial n(z)}+k_1w(z)n(z)$ and the vector $T''(\partial_z,n)w(z)$ (problem K_2).

Thus the above-formulated problems of thermoelasticity with microtemperatures can be considered as a union of two problems: I- (A_1,K_1), II- (A_1,K_2) and III- (A_2,K_1).

3. Uniqueness theorems

Let (u',u'_3,w') and (u'',u''_3,w'') be two different solutions of any of the problems I, II, III. Then the differences $u=u'-u''$, $u_3=u'_3-u''_3$ and $w=w'-w''$ of these solutions, obviously, satisfy the homogeneous system $(1)_0$ and zero boundary conditions $(2)_0$. For a regular solutions of equation $(1)_1$ and equations $(1)_2$ and $(1)_3$ the Green's formulas [2,3]:

$$\int_D [E_1(u(x),u(x)) - \beta u_3(x) divu(x)]dx = \int_S u(y)[T'(\partial_y,n)u(y) - \beta u_3(y)n(y)]d_yS,$$

$$\int_D [T_0E_2(w(x),w(x)) + k \mid gradu_3 \mid^2 + (k_1+k_3T_0)wgradu_3 + k_2T_0 \mid w(x) \mid^2]dx =$$

$$\int_S u_3(y)[k\frac{\partial u_3(y)}{\partial n(y)} + k_1w(y)n(y)] + T_0w(y)[T''(\partial_y,n)w(y)]d_yS, \quad (4)$$

is valid [2], where

$$E_1(u,u) = (\lambda+\mu)\Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\Big)^2 + \mu\Big(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\Big)^2 + \mu\Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\Big)^2;$$
$$E_2(w,w) = \tfrac{1}{2}(2k_4+k_5+k_6)\Big(\frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2}\Big)^2 + (k_6+k_5)\Big(\frac{\partial w_1}{\partial x_1} - \frac{\partial w_2}{\partial x_2}\Big)^2$$
$$+(k_6+k_5)\Big(\frac{\partial w_1}{\partial x_2} + \frac{\partial w_2}{\partial x_1}\Big)^2 + (k_6-k_5)\Big(\frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2}\Big)^2,$$

under the conditions that: $\lambda+\mu, \mu > 0$, $2k_4+k_5+k_6 > 0, k_6 \pm k_5 > 0$, E_1 and E_2 are nonnegative quadratic forms [3].

Taking into account formula $(4)_2$ and the homogeneous boundary conditions for the problems $K_i, (i=1,2)$, we obtain $E_2(w,w)=0, \quad gradu_3 = 0, \quad u_3 = 0, w = 0$. The solution of the above equations has the form: $u_3(x) = const, w = 0$.

The following theorems are valid.

Theorem 3.1 : *The difference of two arbitrary solutions of problem K_1 is equal to zero: $w(x) = 0, \quad u_3(x) = 0, \quad x \in D$.*

The difference of two arbitrary solutions of problem K_2 may differ only by an arbitrary constant: $w(x) = 0, \quad u_3(x) = const, \quad x \in D$.

Taking into account Theorem 3.1 and formula $(4)_1$, under the homogeneous boundary conditions for the problems I, II and III we obtain $E_1(u,u) - \beta u_3 divu = 0$. The solution of the above equation, when $u_3 = 0$ or $u_3 = const$, has the form

$$u_1(x) = -c_1x_2 + q_1, \quad u_2(x) = c_1x_1 + q_2, \quad (5)$$

where c_1, q_1 and q_2 are arbitrary constants.

The following theorems are valid.

Theorem 3.2 : *The difference of two arbitrary solutions of problem I is the vector $U(u_1(x), u_2(x), u_3(x), w_1(x), w_2(x))$, where $u_1 = u_2 = 0, \quad u_3 = 0, \quad w_1 = w_2 = 0$.*

Theorem 3.3 : *The difference of two arbitrary solutions of problem II is the vector $U(u_1(x), u_2(x), u_3(x), w_1(x), w_2(x))$, where $u_1 = u_2 = 0, \quad u_3 = c, \quad w_1 = w_2 = 0; \quad c$ is an arbitrary constant.*

Theorem 3.4 : *The difference of two arbitrary solutions of problem III is the vector $U(u_1(x), u_2(x), u_3(x), w_1(x), w_2(x))$, where u_1 and u_2 are expressed by formulas (5), and $u_3 = 0, \quad w_1 = w_2 = 0$.*

4. Solutions of the Problems

On the basis of the system $[(1)_2, (1)_3]$, we can write

$$\triangle(\triangle + s_1^2)u_3 = 0, \quad \triangle(\triangle + s_1^2)div w = 0.$$

Solutions of these equations are represented in the form [4]:

$$\begin{aligned} &u_3(x) = \varphi_1(x) + \varphi_2(x), \\ &w_1(x) = a_1\frac{\partial\varphi_1(x)}{\partial_{x_1}} + a_2\frac{\partial\varphi_2(x)}{\partial_{x_2}} - a_3\frac{\partial\varphi_3(x)}{\partial_{x_2}}, \\ &w_2(x) = a_1\frac{\partial\varphi_1(x)}{\partial_{x_2}} + a_2\frac{\partial\varphi_2(x)}{\partial_{x_1}} + a_3\frac{\partial\varphi_3(x)}{\partial_{x_1}}, \end{aligned} \tag{6}$$

where $\triangle\varphi_1 = 0, (\triangle + s_1^2)\varphi_2 = 0, (\triangle + s_2^2)\varphi_3 = 0, s_1^2 = -\dfrac{kk_2 - k_1k_3}{kk_7}$, $s_2^2 = -\dfrac{k_2}{k_6}, a_1 = -\dfrac{k_3}{k_2}, a_2 = -\dfrac{k}{k_1}, a_3 = \dfrac{k_6}{k_7}$; $k_7 = k_4+k_5+k_6$; $k, k_2, k_6, k_7 > 0$ [2].

Problem K_1. Taking into account formulas: $\dfrac{\partial}{\partial x_2} = n_2\dfrac{\partial}{\partial r} + \dfrac{n_1}{r}\dfrac{\partial}{\partial\psi}$, $\dfrac{\partial}{\partial x_1} = n_1\dfrac{\partial}{\partial r} - \dfrac{n_2}{r}\dfrac{\partial}{\partial\psi}$, we rewrite the representations (6) and the boundary conditions of the problemK_1 in the tangent and normal components:

$$\begin{aligned} &u_3(x) = \varphi_1(x) + \varphi_2(x), \\ &w_n(x) = a_1\frac{\partial}{\partial r}\varphi_1(x) + a_2\frac{\partial}{\partial r}\varphi_2(x) - a_3\frac{1}{r}\frac{\partial}{\partial\psi}\varphi_3(x), \\ &w_s(x) = a_1\frac{1}{r}\frac{\partial}{\partial\psi}\varphi_1(x) + a_2\frac{1}{r}\frac{\partial}{\partial\psi}\varphi_2(x) + a_3\frac{\partial}{\partial r}\varphi_3(x); \end{aligned} \tag{7}$$

$$\begin{aligned} &\varphi_1(z) + \varphi_2(z) = f_3(z), \quad k_7\left[\frac{\partial w_n}{\partial r}\right]_R + \frac{k_4}{R}\left[\frac{\partial w_s}{\partial\psi}\right]_R = p_n(z), \\ &k_6\left[\frac{\partial w_s}{\partial r}\right]_R + \frac{k_5}{R}\left[\frac{\partial w_n}{\partial\psi}\right]_R = p_s(z), \end{aligned} \tag{8}$$

where $w_n = (w \cdot n), w_s = (w \cdot s), p_n = (p \cdot n), p_s = (p \cdot s), n = (n_1, n_2), s = (-n_2, n_1)$, $\dfrac{\partial}{\partial n} = \dfrac{\partial}{\partial r}$.

The harmonic function φ_1 and metaharmonic functions φ_2 and φ_3 are represented in the form of series in the circle [5]:

$$\begin{aligned}
\varphi_1(x) &= \frac{1}{2}Y_{01} + \sum_{m=1}^{\infty} \left(\frac{r}{R}\right)^m (Y_{m1} \cdot \nu_m(\psi)), \\
\varphi_2(x) &= I_0(s_2 r)Y_{02} + \sum_{m=1}^{\infty} I_m(s_2 r)(Y_{m2} \cdot \nu_m(\psi)), \\
\varphi_3(x) &= I_0(s_3 r)Y_{03} + \sum_{m=1}^{\infty} I_m(s_3 r)(Y_{m3} \cdot \ s_m(\psi)),
\end{aligned} \tag{9}$$

respectively, where Y_{mk} are the unknown two-component constants vectors, $\nu_m(\psi) = (\cos m\psi, \sin m\psi), s_m(\psi) = (-\sin m\psi, \cos m\psi), k = 1, 2, m = 0, 1, ...$

Let the functions p_n, p_s and f_3 expand into the Fourier series:

$$\begin{aligned}
p_n(z) &= \frac{\alpha_0}{2} + \sum_{m=1}^{\infty}(\alpha_m \cdot \nu_m(\psi)), \quad p_s(z) = \frac{\beta_0}{2} + \sum_{m=1}^{\infty}(\beta_m \cdot s_m(\psi)), \\
f_3(z) &= \frac{\gamma_0}{2} + \sum_{m=1}^{\infty}(\gamma_m \cdot \nu_m(\psi)),
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
&\alpha_m = (\alpha_{m1}, \alpha_{m2}),\ \beta_m = (\beta_{m1}, \beta_{m2}),\ \gamma_m = (\gamma_{m1}, \gamma_{m2}), \\
&\alpha_{m1} = \frac{1}{\pi}\int_0^{2\pi} p_n(\theta)\cos(m\theta)d\theta, \\
&\alpha_{m2} = \frac{1}{\pi}\int_0^{2\pi} p_n(\theta)\sin(m\theta)d\theta,\ \beta_{m1} = \frac{1}{\pi}\int_0^{2\pi} p_s(\theta)\cos(m\theta)d\theta, \\
&\beta_{m2} = \frac{1}{\pi}\int_0^{2\pi} p_s(\theta)\sin(m\theta)d\theta, \\
&\gamma_{m1} = \frac{1}{\pi}\int_0^{2\pi} f_3(\theta)\cos(m\theta)d\theta, \quad \gamma_{m2} = \frac{1}{\pi}\int_0^{2\pi} f_3(\theta)\sin(m\theta)d\theta.
\end{aligned}$$

We substitute (9) into (7) and then the obtained expression and (10) into (8). Passing to the limit, as $r \to R$, for the unknowns Y_{mk} we obtain a system of algebraic equations:

$$\frac{1}{2}Y_{01} + I_0(s_2R)Y_{02} = \frac{\gamma_0}{2},\ k_7a_2s_2^2I_0''(s_2R)Y_{02} = \frac{\alpha_0}{2},\ k_6a_3s_3^2I_0''(s_3R)Y_{03} = \frac{\beta_0}{2}; \tag{11}$$

$$
\begin{aligned}
&Y_{m1}+I_m(s_2R)Y_{m2}=\gamma_m,\\
&a_1m[(k_7-k_4)m-k_7]Y_{m1}+a_2[k_7s_2^2I''_m(s_2R)R^2-k_4m^2a_1I_m(s_2R)]Y_{m2}\\
&+a_3m[k_7[s_3RI'_m(s_3R)-I_m(s_3R)]-k_4Rs_3I'_m(s_3R)]Y_{m3}=\alpha_mR^2,\\
&a_1m[(k_6+k_5)m-k_6]Y_{m1}\\
&+a_2m[k_6[s_2RI'_m(s_2R)-I_m(s_2R)]+k_5Rs_2I'_m(s_2R)]Y_{m2}\\
&+a_3[k_6R^2s_3^2I''_m(s_3R)+k_5a_3m^2I_m(s_3R)]=\beta_mR^2,\quad m=1,2,....
\end{aligned}
\tag{12}
$$

Relying on the theorem on the uniqueness of a solution of the problem we can conclude that the principal determinants of systems (11) and (12) are other than zero. Substituting the solutions of systems (11) and (12) into (9) and then into (6), we can find values of the functions $u_3(x), w_1(x)$ and $w_2(x)$.

Problem K_2. Taking into account formulas (6), the boundary conditions of the problem K_2 can be rewritten as:

$$
\begin{aligned}
&k\left[\frac{\partial u_3}{\partial r}\right]_R+k_1[w_n]_R=f_3(z),\quad k_7\left[\frac{\partial w_n}{\partial r}\right]_R+\frac{k_4}{R}\left[\frac{\partial w_s}{\partial\psi}\right]_R=p_n(z),\\
&k_6\left[\frac{\partial w_s}{\partial r}\right]_R+\frac{k_5}{R}\left[\frac{\partial w_n}{\partial\psi}\right]_R=p_s(z).
\end{aligned}
\tag{13}
$$

We substitute (9) into (7), then the obtained expression and (10) into (13). Passing to the limit, as $r\to R$, from (13) we obtain the system of linear algebraic equations with regard to the unknowns Y_{mk} for every value m:

$$
\begin{aligned}
&k_7a_2s_2^2I''_0(s_2R)Y_{02}=\frac{\alpha_0}{2},\quad k_6a_3s_3^2I''_0(s_3R)Y_{03}=\frac{\beta_0}{2},\\
&s_2I'_0(s_2R)(k+k_1a_2)Y_{02}=\frac{\gamma_0}{2};
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&m(k+k_1a_1)Y_{m1}+s_2I'_m(s_2R)(k+k_1a_2R)Y_{m2}+a_3mI_m(s_3R)Y_{m3}=\gamma_mR,\\
&a_1m[(k_7-k_4)m-k_7]Y_{m1}+a_2[k_7s_2^2I''_m(s_2R)R^2-k_4m^2a_1I_m(s_2R)]Y_{m2}\\
&+a_3m[k_7[s_3RI'_m(s_3R)-I_m(s_3R)]-k_4Rs_3I'_m(s_3R)]Y_{m3}=\alpha_mR^2,\\
&a_1m[(k_6+k_5)m-k_6]Y_{m1}\\
&+a_2m[k_6[s_2RI'_m(s_2R)-I_m(s_2R)]+k_5Rs_2I'_m(s_2R)]Y_{m2}\\
&+a_3[k_6R^2s_3^2I''_m(s_3R)+k_5a_3m^2I_m(s_3R)]=\beta_mR^2,\quad m=1,2,....
\end{aligned}
\tag{15}
$$

From equation $(1)_2$, taking into account the boundary conditions (2) and formulae (10) we can write

$$
\int_D[k\Delta u_3(x)+k_1divw(x)]dx=\int_S[k\frac{\partial u_3(y)}{\partial n(y)}+k_1w(y)n(y)]d_yS=0,
$$

$$\gamma_{01} = \frac{1}{\pi}\int\limits_0^{2\pi} f_3(\theta)d\theta = 0.$$

For the Y_{02} we obtain: $Y_{02} = 0$; then

$$\alpha_{01} = \frac{1}{\pi}\int\limits_0^{2\pi} p_n(\theta)d\theta = 0, \quad \beta_{01} = \frac{1}{\pi}\int\limits_0^{2\pi} p_s(\theta)d\theta = 0; \quad Y_{03} = 0, \quad Y_{01} = const.$$

Problem A_1. A solution $(1)_1$ is sought in the form

$$u(x) = v_0(x) + v(x), \tag{16}$$

where v_0 is a particular solution of equation $(1)_1$, and v is a general solution of the corresponding homogeneous equation $(1)_1$. Direct checking shows that v_0 has the form

$$v_0(x) = \frac{\beta}{\lambda+2\mu} grad[-\frac{1}{s_1^2}\varphi_2(x) + \varphi_0(x)], \tag{17}$$

where φ_0 is a biharmonic function: $\triangle\varphi_0 = \varphi_1$.

A solution $v(x) = (v_1(x), v_2(x))$ of the homogeneous equation corresponding to $(1)_1$:

$\mu\triangle v(x) + (\lambda+\mu) graddivv(x) = 0$

is sought in the form

$$\begin{aligned} v_1(x) &= \frac{\partial}{\partial x_1}[\Phi_1(x) + \Phi_2(x)] - \frac{\partial}{\partial x_2}\Phi_3(x), \\ v_2(x) &= \frac{\partial}{\partial x_2}[\Phi_1(x) + \Phi_2(x)] + \frac{\partial}{\partial x_1}\Phi_3(x), \end{aligned} \tag{18}$$

where

$$\begin{aligned} &\Delta\Phi_1(x) = 0, \quad \Delta\Delta\Phi_2(x) = 0, \quad \Delta\Delta\Phi_3(x) = 0, \\ &(\lambda+2\mu)\frac{\partial}{\partial x_1}\Delta\Phi_2(x) - \mu\frac{\partial}{\partial x_2}\Delta\Phi_3(x) = 0, \\ &(\lambda+2\mu)\frac{\partial}{\partial x_2}\Delta\Phi_2(x) + \mu\frac{\partial}{\partial x_1}\Delta\Phi_3(x) = 0, \end{aligned} \tag{19}$$

Φ_1, Φ_2, Φ_3 are the scalar functions.

Taking into account (16) condition $(2)_1$, we can write

$$v(z) = \Psi(z), \tag{20}$$

where $\Psi(z) = f(z) - v_0(z)$ is the known vector, v_0 is defined by formula (17), and φ_1 and φ_2 by equalities (9), where the value of the Y_{mk} vectors is defined by means of systems (11) and (12). The function φ_0 is a solution of the equation $\triangle\varphi_0 = \varphi_1$; it has the form

$$\varphi_0(x) = \frac{R^2}{4}\sum_{m=0}^{\infty}\frac{1}{m+1}\Big(\frac{r}{R}\Big)^{m+2}(Y_{m1}\cdot\nu_m(\psi)), \tag{21}$$

where Y_{m1} are defined from (11) and (12).

In view of (19), we can represent the harmonic function Φ_1 and biharmonic functions Φ_2 and Φ_3 in the form

$$\begin{aligned}&\Phi_1(x) = \sum_{m=0}^{\infty}\Big(\frac{r}{R}\Big)^{m}(X_{m1}\cdot\nu_m(\psi)), \quad \Phi_2(x) = \sum_{m=0}^{\infty}\Big(\frac{r}{R}\Big)^{m+2}(X_{m2}\cdot\nu_m(\psi)),\\ &\Phi_3(x) = \frac{R^2(\lambda+2\mu)}{\mu}\sum_{m=0}^{\infty}\Big(\frac{r}{R}\Big)^{m+2}(X_{m2}\cdot s_m(\psi)),\end{aligned} \tag{22}$$

where X_{mk} are the unknown two-component vectors, $k=1,2$.

Substituting (22) into (18), the obtained expressions into (20), we obtain the system of algebraic equations for every m, whose solution is written as follows:

$$\begin{aligned}&X_{01} = \frac{\eta_0 R}{4}, \quad X_{02} = \frac{\varsigma_0 R}{4(\lambda+2\mu)}, \quad X_{m1} = \frac{\eta_m R}{m} - \frac{(\varsigma_m-\eta_m)R}{2(\lambda+\mu)m},\\ &X_{m2} = \mu\frac{(\varsigma_m-\eta_m)R}{2(\lambda+\mu)m}, \quad m=1,2,\dots;\end{aligned} \tag{23}$$

where η_m and ς_m are the Fourier coefficients of the function $\Psi(z)$:

$$\begin{aligned}&\eta_m = (\eta_{m1},\eta_{m2}), \quad \eta_0 = (\eta_{01},0), \quad \varsigma_m = (\varsigma_{m1},\varsigma_{m2}), \quad \varsigma_0 = (\varsigma_{01},0),\\ &\eta_{m1} = \frac{1}{\pi}\int_0^{2\pi}\Psi_n(\theta)\cos m\theta d\theta, \qquad \gamma_{m2} = \frac{1}{\pi}\int_0^{2\pi}\Psi_s(\theta)\sin m\theta d\theta,\\ &\varsigma_{m1} = \frac{1}{\pi}\int_0^{2\pi}\Psi_s(\theta)\cos m\theta d\theta, \qquad \delta_{m2} = \frac{1}{\pi}\int_0^{2\pi}\Psi_n(\theta)\sin m\theta d\theta;\end{aligned} \tag{24}$$

Ψ_n and Ψ_s are normal and tangential components of the function $\Psi(z)$, respectively. Thus the solution of problem A_1 is represented by the sum (16) in which $v(x)$ is defined by means of formula (18), and $v_0(x)$ by formula (17).

Problem A_2. Taking into account (16) condition $(2)_3$, we can rewrite it as

$$T'(\partial_z,n)v(z) = \Psi(z), \tag{25}$$

where $\Psi(z) = f(z)+\beta u_3(z)n(z) - T'(\partial_z,n)v_0(z)$ is the known vector, $\Psi = (\Psi_1,\Psi_2)$.

We substitute (22) first into (18) and then into (25). For the unknowns X_{m1} and X_{m2} we obtain a system of algebraic equations whose solution has the form

$$X_{01}=\frac{\eta_0R^2}{4(\lambda+2\mu)},\quad X_{02}=\frac{\varsigma_0R^2}{4(\lambda+2\mu)},\quad X_{m1}=\frac{R^2}{c_3}\varsigma_m-\frac{c_4R^2}{c_2c_3-c_1c_4}(\mu\eta_m-c_1\varsigma_m),$$
$$X_{m2}=\frac{c_4R^2}{c_2c_3-c_1c_4}(\mu\eta_m-c_1\varsigma_m),$$
where $c_1=\mu[2(\lambda+\mu)m^2-(\lambda+2\mu)m]$, $c_2=2(\lambda+\mu)(\lambda+3\mu)m^2+(\lambda+2\mu)[(3\lambda+5\mu)m+2\mu]$, $c_3=m\mu(2\mu-1)$, $c_4=2(\lambda+3\mu)m(2m+3)+2(\lambda+2\mu), m=1,2,....$ η_m and ς_m are the Fourier coefficients of respectively normal and tangential components of the function $\Psi(z)$.

Having solved problems A_1, A_2, K_1 and K_2, we can write solutions of the initial problems I, II and III.

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