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Oscillation Criteria for Differential and Discrete Equation with Several Delays

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Consider the differential and difference equations

$$x'(t) + \sum_{i=1}^{m} p_i(t)\, x\big(\tau_i(t)\big) = 0 \tag{1}$$

and

$$\Delta u(k) + \sum_{i=1}^{m} q_i(k)\, u(\sigma_i(k)) = 0, \tag{2}$$

where $p_i \in C(R_+, R_+)$, $\tau_i \in C(R_+, R)$, $\tau(t) \le t$, $\lim\limits_{t\to+\infty} \tau_i(t) = +\infty$; $\Delta u(k) = u(k+1) - u(k)$, $q_i : N \to R_+$, $\sigma_i : N \to N$, $\sigma_i(k) \le k-1$ and $\lim\limits_{t\to+\infty} \sigma_i(k) = +\infty$ $i = 1, \dots, m)$.

Sufficient oscillation conditions are presented for differential (1) and difference (2) equations.

Key words: Oscillation, Differential equations, Difference equations.

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1. Differential equations

Consider the differential equation

$$x'(t) + \sum_{i=1}^{m} p_i(t)\, x\big(\tau_i(t)\big) = 0, \quad t \ge t_0, \tag{1.1}$$

where the functions $p_i; \tau_i \in C([t_0, +\infty); \mathbb{R}^+)$, for every $i = 1, 2, \dots, m$ (here $R^+ = [0, +\infty)$),

$$\tau_i(t) \le t \ \text{ for } \ t \ge 0, \quad \lim_{t\to+\infty} \tau_i(t) = +\infty.$$

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Let $t_* \in [t_0, +\infty)$, $\tau(t) = \min\{\tau_i(t) : i = 1, \dots, m\}$ and $\tau_{(-1)}(t) = \sup\{s : \tau(s) \le t\}$. Under a solution of the equation we understand $u \in C([t_0, +\infty); \mathbb{R})$ function a continuously differentiable on $[\tau_{(-1)}(t_*), +\infty)$ and satisfying (1.1) for $t \ge \tau_{(-1)}(t_*)$. Such a solution is called oscillatory if it has arbitrary large zeros, and otherwise it is called nonoscillatory.

In the special case, where $m = 1$, equation (1.1) is reduced to the equation

$$x'(t) + p(t)\, x\big(\tau(t)\big) = 0. \tag{1.2}$$

The first systematic study for the oscillation of all solutions to the equation (1.2) was made by Myshkis [1]. He proved that any solutions of equation (1.2) oscillate if

$$\limsup_{t\to+\infty} \big(t - \tau(t)\big) < +\infty \quad \text{and} \quad \liminf_{t\to+\infty} \big(t - \tau(t)\big) \liminf_{t\to+\infty} p(t) > \frac{1}{e}\,.$$

In 1972, Ladas, Lakshmikantan and Papadakis [2] proved that if τ is a non-decreasing function and

$$\limsup_{t\to+\infty} \int_{\tau(t)}^{t} p(s)\, ds > 1,$$

then all solutions of equation (1.2) oscillate.

In 1979, Ladas [3] proved that, if $\tau(t) = t - \Delta$ and

$$\liminf_{t\to+\infty} \int_{t-\Delta}^{t} p(s)\, ds > \frac{1}{e}\,,$$

then all solutions of equation (1.2) oscillate, while in 1982, Koplatadze and Chanturia [4] established the following

Theorem 1.1: *If*

$$\liminf_{t\to+\infty} \int_{\tau(t)}^{t} p(s)\, ds > \frac{1}{e}\,,$$

then all solutions of equation (1.2) *oscillate and if there exists* $t_0 \ge 0$ *such that*

$$\int_{\tau(t)}^{t} p(s)\, ds \le \frac{1}{e} \quad \textit{for} \quad t \ge t_0,$$

then equation (1.2) *has a non-oscillatory solution.*

In 1993, Koplatadze and Kvinikadze [5] proved

Theorem 1.2: *Let for some* $k \in N$

$$\limsup_{t\to+\infty} \int_{\tau(t)}^{t} p(s) \exp\left\{ \int_{\tau(s)}^{\tau(t)} p(\xi)\, \psi_k(\xi)\, d\xi \right\} > 1,$$

where $\psi_1(t) = 0$,

$$\psi_i(t) = \exp\left\{ \int_{\tau(t)}^{t} p(\xi)\, \psi_{i-1}(\xi)\, d\xi \right\} \quad (i = 1, \dots, k).$$

Then all solutions of equation (1.2) *oscillate.*

Corollary 1.3: *Let*

$$\limsup_{t\to+\infty} \int_{\tau(t)}^{t} p(s)\, ds > 1 - \alpha(p_*).$$

Then all solutions of equation (1.2) *oscillate, where*

$$p_* = \liminf_{t\to+\infty} \int_{\tau(t)}^{t} p(s)\, ds \quad \text{and} \quad \alpha(p_*) = \frac{1 - p_* - \sqrt{1 - 2p_* - p_*^2}}{2}\,,$$

$$0 \le p_* \le \frac{1}{e}\,.$$

Corollary 1.4: *Let*

$$\liminf_{t\to+\infty} \int_{\tau(t)}^{t} p(s)\, ds > \frac{1}{e}\,.$$

Then all solutions of equation (1.2) *oscillate.*

Concerning the constants 1 and $\frac{1}{e}$ which appear in the above conditions Berezansky and Brawerman [7] established the following

Theorem 1.5: *For any* $\alpha \in \left(\frac{1}{e}, 1\right)$ *there exists a nonoscillatory equation*

$$x'(t) + p(t)\, x(t - \tau) = 0,$$

where $\tau > 0$, $p(t) \ge 0$ *and*

$$\limsup_{t\to+\infty} \int_{t-\tau}^{t} p(s)\, ds = \alpha.$$

Also, Brawerman and Karpuz [8] proved that for any $k \ge 0$ there exists equation (1.2) such that

$$\limsup_{t\to+\infty} \int_{t-\tau}^{t} p(s)\, ds > k,$$

but equation (1.2) has a non-oscillatory solution.

In 2004, Berikelashvili, Jokhadze and Koplatadze [6] proved

Theorem 1.6 : *Let there exist a function* $\mu \in C([t_0, +\infty), (0, +\infty))$ *such that*

$$\frac{1}{\mu(t)} \int_{\tau(t)}^{t} \exp\big(\mu(s)\big)\, p(s)\, ds \le 1 \quad \text{for} \quad t \ge t_0,$$

then equation (1.2) *has a positive solution. Let there a exist function* $\mu \in C(R_+; (0, +\infty))$ *such that*

$$\liminf_{t\to+\infty} \mu(t) > 0, \quad \limsup_{t\to+\infty} \mu(t) < +\infty$$

and

$$\liminf_{t\to+\infty} \frac{1}{\mu(t)} \int_{\tau(t)}^{t} \mu(s)\, p(s)\, ds > \frac{1}{e}$$

then all solutions of equation (1.2) *oscillate.*

Now consider the differential equation with several delays

$$x'(t) + \sum_{i=1}^{m} p_i(t)\, x\big(\tau_i(t)\big) = 0. \tag{1.3}$$

In 2000 Koplatadze, Grammatikpoulos and Stavroulakis [9] proved

Theorem 1.7 : *If*

$$\int_0^{+\infty} \big|p_i(t) - p_j(t)\big| dt < +\infty, \quad i, j = 1, \ldots, m$$

and

$$\sum_{i=1}^{m} \liminf_{t\to+\infty} \int_{\tau_i(t)}^{t} p_i(s)\, ds > \frac{1}{e}$$

then all solutions of equation (1.3) *oscillate.*

Theorem 1.8 : *Let there exist non-decreasing functions* σ_i *such that* $\tau_i(t) \le \sigma_i(t) \le t$ *and*

$$\limsup_{t\to+\infty} \prod_{j=1}^{m} \Bigg[\prod_{i=1}^{m} \int_{\sigma_i(t)}^{t} p_i(s) \exp\Bigg(\int_{\tau_i(s)}^{\sigma_i(t)} \sum_{i=1}^{m} p_i(\xi) \times \\ \times \exp\Bigg(\int_{\tau_i(\xi)}^{\xi} \sum_{i=1}^{m} p_i(u) du \Bigg) d\xi \Bigg) ds \Bigg]^{\frac{1}{m}} > \frac{1}{m^m}.$$

Then all solutions of equation (1.3) *oscillate.*

Theorem 1.9: *Let there exists non-decreasing functions* σ_i *such that* $\tau_i(t) \leq \sigma_i(t) \leq t$ $(i = 1, \dots, m)$ *and*

$$\limsup_{\varepsilon\to 0+}\Bigg(\limsup_{t\to+\infty}\prod_{j=1}^{m}\Bigg(\prod_{i=1}^{m}\int_{\sigma_i(t)}^{t} p_i(s)\times \times \exp\Bigg(\int_{\tau_i(s)}^{\sigma_i(t)}\sum_{i=1}^{m}(\lambda_i^* - \varepsilon)p_i(\xi)\,d\xi\Bigg)ds\Bigg)^{\frac{1}{m}}\Bigg) > \frac{1}{m^m}\,.$$

Then all solutions of equation (1.3) *oscillate, where* λ_i^* *is the smaller root of the equation*

$$e^{p_i\lambda} = \lambda,$$

and

$$p_i = \liminf_{t\to+\infty}\int_{\tau_i(t)}^{t} p_i(s)\,ds.$$

Corollary 1.10: *Let* τ_i *be non-decreasing functions and*

$$\limsup_{t\to+\infty}\prod_{j=1}^{m}\Bigg(\prod_{i=1}^{m}\int_{\tau_j(t)}^{t} p_i(s)ds\Bigg)^{\frac{1}{m}} > \frac{1}{m^m}\,.$$

Then all solutions of equation (1.3) *oscillate.*

Corollary 1.11: *Let* τ_i *be non-decreasing functions* $p_i(t) \geq p(t) \geq 0$, $i = 1, \dots, m$ *and*

$$\limsup_{t\to+\infty}\prod_{j=1}^{m}\int_{\tau_j(t)}^{t} p(s)ds > \frac{1}{m^m}\,.$$

Then all solutions of equation (1.3) *oscillate.*

Corollary 1.12: *Let* $p_i \geq p = \text{const}$ *and*

$$p^m \limsup_{t\to+\infty}\prod_{i=1}^{m}\big(t - \tau_i(t)\big) > \frac{1}{m^m}\,.$$

Then all solutions of equation (1.3) *oscillate.*

Theorem 1.13: *Let*

$$\int_0^{+\infty}\Bigg(\frac{1}{m}\sum_{i=1}^{m}p_i(t) - \Big(\prod_{i=1}^{m}p_i(t)\Big)^{\frac{1}{m}}\Bigg)dt < +\infty$$

and

$$\liminf_{t\to+\infty}\sum_{i=1}^{m}\int_{\tau_i(t)}^{t} p^*(s)ds > \frac{m}{e}\,.$$

Then all solutions of equation (1.3) *oscillate, where* $p^* = \sum\limits_{i=1}^{m} p_i(t)$.

Corollary 1.14: *Let*

$$\int_0^{+\infty} \big|p_i(t)-p_j(t)\big|dt < +\infty, \quad i,j=1,\dots,m$$

and

$$\liminf_{t\to+\infty}\sum_{i=1}^{m}\int_{\tau_i(t)}^{t} p_i(s)\,ds > \frac{1}{e}\,, \tag{1.4}$$

then all solution of equation (1.3) *oscillate.*

Example. Let

$$\tau_i(t)=\alpha_i\,t \quad \big(\tau_i(t)=t^{\alpha_i}\big), \quad i=1,\dots,m, \quad 0<\alpha_i<1,$$
$$p_i(t)=\frac{\lambda}{t\sum\limits_{i=1}^{m}\alpha_i^{-\lambda}} \quad \left(p_i(t)=\frac{\lambda}{t\,(\ln t)^{\lambda+1}\sum\limits_{i=1}^{m}\alpha_i^{-\lambda}}\right).$$

Then the function $x(t)=t^{-\lambda}$ ($x(t)=\ln^{-\lambda}t$) is the solution of equation (1.3). On the other hand, for any $\varepsilon>0$ there exists $\delta>0$ such that if

$$|\alpha_i-\alpha_1|<\delta \quad (i=1,\dots,m)$$

then

$$\frac{1-\varepsilon}{e} \le \liminf_{t\to+\infty}\sum_{i=1}^{m}\int_{\tau_i(t)}^{t} p_i(s)\,ds \le \frac{1}{e}\,,$$

i.e. condition (1.4) is an optimal condition.

2. Difference Equations

Consider the difference equation

$$\Delta u(k)+p(k)\,u(\tau(k))=0, \tag{2.1}$$

where $\Delta u(k)=u(k+1)-u(k)$, $p:N\to R_+$, $\tau:N\to N$, $\tau(k)\le k-1$ and $\lim\limits_{k\to+\infty}\tau(k)=+\infty$.

Theorem 2.1 [10]**:** *Let* $\tau(k) = k - n$ *and*

$$\liminf_{k\to+\infty} \sum_{i=k-n}^{k-1} p(i) > \Big(\frac{n}{n+1}\Big)^{n+1},$$

then all solutions of equation (2.1) *oscillate.*

Theorem 2.2 [11]**:** *Let*

$$\liminf_{k\to+\infty} \sum_{i=\tau(k)}^{k-1} p(i) = \alpha \le 1$$

and

$$\limsup_{k\to+\infty} \sum_{i=\sigma(k)}^{k} p(i) > 1 - \big(1 - \sqrt{1-\alpha}\big)^2.$$

Then all solutions of equation (2.1) *oscillate, where*

$$\sigma(k) = \max\big\{\tau(s) : 1 \le s \le k,\ s \in N\big\}.$$

Theorem 2.3 [12]**:** *Let*

$$\liminf_{k\to+\infty} \sum_{i=\tau(k)}^{k-1} p(i) > \frac{1}{e}.$$

Then all solutions of equation (2.1) *oscillate.*

Now consider the difference equation with several delays

$$\Delta u(k) + \sum_{i=1}^{m} p_i(k)\, u(\tau_i(k)) = 0. \tag{2.2}$$

Theorem 2.4 : *Let*

$$\sum_{k=1}^{+\infty} \Big(\frac{1}{m} \sum_{i=1}^{m} p_i(k) - \Big(\prod_{i=1}^{m} p_i(k)\Big)^{\frac{1}{m}}\Big) < +\infty$$

and

$$\liminf_{k\to+\infty} \sum_{i=1}^{m} \Big(\sum_{s=\tau_i(k)}^{k-1} p_*(s)\Big) > \frac{m}{e}\,.$$

Then all solutions of equation (2.2) *oscillate, where* $p_*(k) = \sum_{i=1}^{m} p_i(k)$.

Theorem 2.5 : *Let*

$$\sum_{k=1}^{+\infty} \big| p_i(k) - p_j(k) \big| < +\infty \quad (j, i = \overline{1, m})$$

and

$$\liminf_{k \to +\infty} \sum_{i=1}^{m} \Bigg(\sum_{j=\tau_i(k)}^{k-1} p_i(j) \Bigg) > \frac{1}{e} \,,$$

then all solutions of equation (2.2) *oscillate.*

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