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Hankel and Berezin Type Operators on Weighted Besov Spaces of Holomorphic Functions on Polydisk

Anahit V. Harutyunyan* and George Marinescu

Yerevan State University, University of Cologne

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Assuming that S is the space of functions of regular variation and $\omega = (\omega_1, \dots, \omega_n)$, $\omega_j \in S$, by $B_p(\omega)$ we denote the class of all holomorphic functions defined on the polydisk U^n such that

$$\|f\|^p_{B_p(\omega)} = \int_{U^n} |Df(z)|^p \prod_{j=1}^{n} \frac{\omega_j(1-|z_j|)dm_{2n}(z)}{(1-|z_j|^2)^{2-p}} < +\infty,$$

where $dm_{2n}(z)$ is the $2n$-dimensional Lebesgue measure on U^n and D stands for a special fractional derivative of f defined here.

In this paper we consider the generalized little Hankel and Berezin type operators on $B_p(\omega)$ (and on $L_p(\omega)$) and prove some theorems about the boundedness of these operators.

Keywords: Weighted spaces, Polydisk, Little Hankel operator, Berezin operator, Multiplier.

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1. Introduction and auxiliary constructions

Numerous authors have contributed to holomorphic Besov spaces in the unit disc in $\mathbb{C}$ and in the unit ball in $\mathbb{C}^n$,(see Arazy-Fisher-Peetre [1], K. Stroethoff [16] O. Blasco [3], K. Zhu [18]). The investigation of holomorphic Besov space on the polydisk is of special interest. The polydisk is a product of n disks and one would expect that the natural generalisations of results from the one-dimensional case would be valid here, but it turns out that this is not true. The case of polydisk is different from the $n = 1$ case and from the case of the n-dimensional ball. For example, let us consider the classical theorem of Privalov: if $f \in \text{Lip}\ \alpha$, then $Kf \in \text{Lip}\ \alpha$, where Kf is a Cauchy type integral. It is known that the analogue of this theorem for multidimensional Lipschitz classes is not true ([9]), even though the analogue of this theorem for a sphere is valid ([13]). In many cases, especially, when the class is defined by means of derivatives, the generalisation of functional spaces in the polydisk is different from that on a unit ball. The generalisation of holomorphic Besov spaces on the polydisk see in [8].

Let

$$U^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n,\ |z_j| < 1,\ 1 \le j \le n\}$$

*Corresponding author. Email: anahit@ysu.am

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be the unit polydisk in the n-dimensional complex plane $\mathbb{C}^n$ and let

$$T^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n, \ |z_i| = 1, \ 1 \le i \le n\}$$

be its torus. We denote by $H(U^n)$ the set of holomorphic functions on U^n, by $L^\infty(U^n)$ the set of bounded measurable functions on U^n and by $H^\infty(U^n)$ the subspace of $L^\infty(U^n)$ consisting of holomorphic functions.

Let S be the class of all non-negative measurable functions ω on $(0,1)$, for which there exist positive numbers $M_\omega, \ q_\omega, \ m_\omega, \ (m_\omega, q_\omega \in (0,1))$, such that

$$m_\omega \le \frac{\omega(\lambda r)}{\omega(r)} \le M_\omega,$$

for all $r \in (0,1)$ and $\lambda \in [q_\omega, 1]$. Some properties of functions from S can be found in [14]. We set

$$-\alpha_\omega = \frac{\log m_\omega}{\log q_\omega^{-1}}; \qquad \beta_\omega = \frac{\log M_\omega}{\log q_\omega^{-1}}$$

and assume that $0 < \beta_\omega < 1$. For example, $\omega \in S$ if $\omega(t) = t^\alpha$, where $-1 < \alpha < \infty$.

Below, for convenience of notations, for $\zeta = (\zeta_1, ..., \zeta_n), \ z = (z_1, ..., z_n)$ we set

$$\omega(1-|z|) = \prod_{j=1}^n \omega_j(1-|z_j|), \ 1-|z| = \prod_{j=1}^n (1-|z_j|), \ 1-\overline{\zeta} z = \prod_{j=1}^n (1-\overline{\zeta}_j z_j).$$

Further, for $m = (m_1, ..., m_n)$ we set

$$(m+1) = (m_1+1)...(m_n+1), \ (m+1)! = (m_1+1)!...(m_n+1)!,$$
$$(1-|z|)^m = \prod_{j=1}^n (1-|z_j|)^{m_j}.$$

Throughout the paper let us assume $\omega_j \in S$, $1 \le j \le n$. The following definition gives the notion of the fractional differential.

Definition 1.1: *For a holomorphic function $f(z) = \sum_{(k)=(0)}^{(\infty)} a_k z^k$, $z \in U^n$, and for $\beta = (\beta_1, ..., \beta_n)$, $\beta_j > -1$, $(1 \le j \le n)$, we define the fractional differential D^β as follows*

$$D^\beta f(z) = \sum_{(k)=(0)}^{(\infty)} \prod_{j=1}^n \frac{\Gamma(\beta_j + 1 + k_j)}{\Gamma(\beta_j+1)\Gamma(k_j+1)} a_k z^k, \ \ k = (k_1, ..., k_n), \quad z \in U^n,$$

where $\Gamma(\cdot)$ is the Gamma function and $\sum_{(k)=(0)}^{(\infty)} = \sum_{k_1=0}^\infty \cdots \sum_{k_n=0}^\infty$.

If $\beta = (1, \dots, 1)$ then we put $D^\beta f(z) \equiv Df(z)$. Hence

$$Df(z_1, \dots, z_n) = \frac{\partial^n (f(z_1, \dots, z_n) z_1 \cdots z_n)}{\partial z_1 \dots \partial z_n}.$$

If $n = 1$ then Df is the usual derivative of the function $zf(z)$.

Let us define the weighted $L_p(\omega)$ spaces of holomorphic functions.

Definition 1.2: *Let $0 < p < +\infty 1$, $\beta_{\omega_j} < -1 (1 \leq j \leq n)$. We denote by $L_p(\omega)$ the set of all measurable functions on U^n, for which*

$$\|f\|^p_{L_p(\omega)} = \int_{U^n} |f(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^2} dm_{2n}(z) < +\infty.$$

Note that $L_p(\omega)$ is the L_p-space with respect to the measure $\omega(1-|z|)(1-|z|^2)^{-2} dm_{2n}(z)$. Using the conditions on ω ($\omega_j \in S$) we conclude that this measure is bounded.

Now we define holomorphic Besov spaces on the polydisk.

Definition 1.3: *Let $0 < p < +\infty$ and $f \in H(U^n)$. The function f is said to be in $B_p(\omega)$ if*

$$\|f\|^p_{B_p(\omega)} = \int_{U^n} |Df(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{2-p}} dm_{2n}(z) < +\infty\,.$$

From the definition of Df it follows that $||\cdot||_{B_p(\omega)}$ is indeed a norm. (We do not have to add $|f(0)|$). This follows from the fact that here $Df = 0$ implies $f = 0$ for a holomorphic f.

As in the one-dimensional case, $B_p(\omega)$ is a Banach space with respect to the norm $\|\cdot\|_{B_p(\omega)}$. For properties of holomorphic Besov spaces see [8].

To prove the main results we need the following auxiliary lemmas:

Lemma 1.4: *Let $m = (m_1, \ldots, m_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$, $\beta_j \geq 0, 1 \leq j \leq n$. Then If $f \in B_p(\omega)$ then*

$$|f(z)| \leq C \int_{U^n} \frac{(1-|\zeta|^2)^m}{|1-\bar{\zeta} z|^{m+1}} |Df(\zeta)| dm_{2n}(\zeta) \tag{1}$$

where $m_j \geq \alpha_{\omega_j} - 1$ $(1 \leq j \leq n)$.

The proof follows from [8, Lemma 2.5].

Lemma 1.5: *Let $n = 1$. Assume $a + 1 - \beta_\omega > 0$, $b > 1$ and $b - a - 2 > \alpha_\omega$. Then*

$$\int_U \frac{(1-|\zeta|^2)^a \omega(1-|\zeta|^2)}{|1-z\bar{\zeta}|^b} dm_2(\zeta) \leq \frac{\omega(1-|z|^2)}{(1-|z|^2)^{b-a-2}}\,. \tag{2}$$

For the proof see [6, Lemma 2].

2. Little Hankel and Berezin-type operators on $B_p(\omega)$

The investigation of Toeplitz operators are widely known (see for example [5, 6, 10, 17]). Some problems on the Toeplitz operators can be solved by means of Hankel operators and vice versa. In the classical theory of Hardy of holomorphic functions on the unit disk there is only one type of Hankel operator. In the $B_p(\omega)$ theory they are two: little Hankel operators and big Hankel operators. The analogue of the Hankel operators of the Hardy theory here are little Hankel operators, which were investigated by many authors (see for example [2, 8, 12]).

Let us define the little Hankel operators as follows: denote by $\overline{B}_p(\omega)$ the space of conjugate holomorphic functions on $B_p(\omega)$. For the integrable function f on U^n we define the generalized little Hankel operator with symbol $h \in L^\infty(U^n)$ by

$$h_g^\alpha(f)(z) = \overline{P}_\alpha(fg)(z) = \int_{U^n} \frac{(1-|\zeta|^2)^\alpha}{(1-\zeta\overline{z})^{\alpha+2}} f(\zeta)g(\zeta)dm_{2n}(\zeta),$$
$$\alpha = (\alpha_1, \dots, \alpha_n),\ \alpha_j > -1,\ 1 \le j \le n.$$

We denote the restriction of $||\cdot||_{L^p(\omega)}$ to $\overline{B}_p(\omega)$ by $||\cdot||_{\overline{B}_p(\omega)}$. We consider the little Hankel operators on $B_p(\omega)(0 < p < +\infty)$.

First off all we consider the case $0 < p < 1$.

Theorem 2.1: *Let* $0 < p < 1$, $f \in B_p(\omega)$ *(or* $f \in \overline{B}^p(\omega)$*)*, $g \in L^\infty(U^n)$. *Then* $h_g^\alpha(f) \in \overline{B}_p(\omega)$ *if and only if* $\alpha_j > \alpha_{\omega_j}/p - 2$, $1 \le j \le n$.

Corollary 2.2: *Let* $0 < p < 1$, $\alpha_j > \alpha_{\omega_j}/p - 2$, $1 \le j \le n$, $g \in L^\infty(U^n)$. *Then* h_g^α *is bounded on* $B_p(\omega)$, *(and on* $\overline{B}_p(\omega)$*). Moreover,* $\|h_g^\alpha\| \le C\|f\| \cdot \|g\|$

In the case if $p = 1$ we have

Theorem 2.3: *Let* $f \in B_1(\omega)$, $g \in L^\infty(U^n)$. *Then* $h_g^\alpha(f) \in \overline{B}_1(\omega)$ *if and only if* $\alpha_j > \alpha_{\omega_j} - 2$, $1 \le j \le n$.

Corollary 2.4: *Let* $\alpha_j > \alpha_{\omega_j}$, $1 \le j \le n$, $g \in L^\infty(U^n)$. *Then* h_g^α *is bounded on* $B_1(\omega)$ *and* $\|h_g^\alpha\| \le C\|f\| \cdot \|g\|$.

Now we consider the case of $1 < p < +\infty$.

Theorem 2.5: *Let* $1 < p < +\infty$, $f \in B_p(\omega)$ *(or* $f \in \overline{B}_p(\omega)$*)*, $g \in L^\infty(U^n)$. *Then if* $\alpha_j > \alpha_{\omega_j}$, $1 \le j \le n$ *then* $h_g^\alpha(f) \in \overline{B}_p(\omega)$

Corollary 2.6: *Let* $\alpha_j > \alpha_{\omega_j}$, $1 \le j \le n$, $g \in L^\infty(U^n)$. *Then* h_g^α *is bounded on* $B_p(\omega)$ *and* $\|h_g^\alpha\|t_{B_p(\omega)} \le C_3\|f\|_{B_p(\omega)} \cdot \|g\|_\infty$

The Berezin transform is the analogue of the Poisson transform in the $A^p(\alpha)$ (respectively, $(B_p(\omega))$) theory. It plays an important role especially in the study of Hankel and Toeplitz operators. In particular, some properties of those operators (for example, compactness, boundedness) can be proved by means of the Berezin transform (see [11, 16, 19]). The Berezin-type operators, on the other hand, are of independent interest.

We show, that some properties of Berezin-type operators of the one dimensional classical case also hold in the more general situation. For the integrable function f on U^n and for $g \in L^\infty(U^n)$ we define the Berezin-type operator in the following way

$$B_g^\alpha f(z) = \frac{(\alpha+1)}{\pi^n}(1-|z|^2)^{\alpha+2} \int_{U^n} \frac{(1-|\zeta|^2)^\alpha}{|1-z\overline{\zeta}|^{4+2\alpha}} f(\zeta)g(\zeta)dm_{2n}(\zeta).$$

In the case $\alpha = 0,\ g \equiv 1$ the operator B_g^α will be called the Berezin transform. Let us consider first the case $0 < p < 1$.

Theorem 2.7: *Let $0 < p < 1,\ f \in B_p(\omega)$ (or $f \in \overline{B}_p(\omega)$), $g \in L^\infty(U^n)$ and let $\alpha_j > \alpha_{\omega_j}/p - 2$, $1 \le j \le n$. Then $B_g^\alpha(f) \in L^p(\omega)$.*

Remark 1: The condition $\alpha_j + 2 > (\alpha_{\omega_j} + 2)/p, (1 \le j \le n)$ in Theorem 2.7 is necessary too. Moreover, if B_α is bounded on $L^p(\omega)$ then $\alpha_j + 2 > (\alpha_{\omega_j} + 2)/p, (1 \le j \le n)$.

Corollary 2.8: *Let $0 < p < 1,\ \alpha_j > (\alpha_{\omega_j} + 2)/p - 2,\ 1 \le j \le n,\ g \in L^\infty(U^n)$. Then B_g^α is bounded on $A^p(\omega)$ and on $\overline{A}^p(\omega)$.*

The case $1 < p < +\infty$ gives the next theorem

Theorem 2.9: *Let $1 < p < +\infty,\ f \in B_p(\omega)$ (or $f \in \overline{B}_p(\omega)$), $g \in L^\infty(U^n)$ and let $\alpha_j > (\alpha_{\omega_j}/p - 2$, $1 \le j \le n$. Then $B_g^\alpha(f) \in L_p(\omega)$.*

We consider now the case of $p = 1$.

Theorem 2.10: *Let $f \in B_1(\omega)$ (or $f \in \overline{B}_1(\omega)$), $g \in L^\infty(U^n)$. Then $B_g^\alpha(f) \in L_1(\omega)$ if and only if $\alpha_j > \alpha_{\omega_j}$, $1 \le j \le n$.*

Corollary 2.11: *Let $\alpha_j > \alpha_{\omega_j}$, $1 \le j \le n,\ g \in L^\infty(U^n)$. Then B_g^α is bounded on $L^1(\omega)$.*

In general, $h_g^\alpha(f)$ and B_g^α are not bounded.

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