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# On a Vibration Problem of Transversely Isotropic Bars

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In [1] transversely isotropic elastic piezoelectric nonhomogeneous bodies in the case when the poling axis coincides with one of the material symmetry axises is considered. The present paper is devoted to the dynamical problem of such materials when the constitutive coefficients depending on the body projection (i.e., on a domain lying in the plane of interest) variables may vanish either on a part or on the entire boundary of the projection

**Keywords:** Hierarchical models, Piezoelectrics, Partial differential equations and Systems with order degeneracy.

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## 1. Introduction

[1] is devoted to construction of hierarchical models for piezoelectric nonhomogeneous porous elastic and viscoelastic Kelvin-Voigt prismatic shells on the basis of linear theories [2]-[7], transversely isotropic elastic piezoelectric nonhomogeneous bodies in the case when the poling axis coincides with one of the material symmetry axes is considered. Namely, time-harmonic motion under conditions of anti-plane piezoelectric state is discussed. In [8] hierarchical models of piezoelectric transversely isotropic cusped bars are considered for static and oscillation problems in $(0;0)$ approximation. Peculiarities of nonclassical setting boundary conditions are analyzed.

Let a piezoelectric solid occupy a reference configuration $\Omega \in \mathbb{R}^3$. Under the quasi-static conditions, when the rate of change of the magnetic field is small and there is no electric current, i.e., the electric field $\mathbf{E}$ and magnetic field $\mathbf{M}$ are curl free, the governing equations have the following form

**Motion Equations**

$$X_{ji,j} + \Phi_i = \rho \ddot{u}_i(x_1, x_2, x_3, t), \quad (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3, \quad t > t_0, \;\; i = \overline{1,3}; \tag{1.1}$$

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$$H_{j,j} + H_0 = \rho_0 \ddot{\varphi} - \mathcal{F}, \tag{1.2}$$

$$D_{j,j} = f_e, \quad B_{j,j} = 0, \quad \Omega\times]0,T[, \tag{1.3}$$

where $X_{ij} \in C^1(\Omega)$ is the stress tensor; $\Phi_i$ are the volume force components; $\rho_0 := \rho k'$ ($k'$ is equilibrated inertia), $\rho$ is the reference mass density; $\varphi := \nu_0 - \nu \in C^2(\Omega)$ is the change of the volume fraction from the matrix reference volume fraction $\nu$ (clearly, the bulk reference density $\rho = \nu\gamma$, $0 < \nu \leq 1$, here $\gamma$ is the matrix reference density); $u_i \in C^2(\Omega)$ are the displacements; $H_j \in C^1(\Omega)$ is the component of the equilibrated stress vector, $H_0$ and $\mathcal{F}$ are the intrinsic and extrinsic equilibrated volume forces; Einstein's summation convention is used; indices after comma mean differentiation with respect to the corresponding variables of the Cartesian frame $Ox_1x_2x_3$ (throughout the work we assume existence of the indicated (continuous) derivatives unless otherwise stated); dots as superscripts of the symbols mean derivatives with respect to time $t$; $\chi : \Omega\times]0,T[\to \mathbb{R}^1$ and $\eta : \Omega\times]0,T[\to \mathbb{R}^1$ are electric and magnetic potentials, respectively, i.e., $\mathbf{E} = grad\chi$, $\mathbf{M} = grad\eta$, $f_e : \Omega\times]0,T[\to \mathbb{R}^1$ is electric charge density, $p_{kij}$ are the piezoelectric coefficients, $q_{kij}$ are the piezomagnetic coefficients,

$\varsigma_{jl}$ and $\xi_{jl}$ are the dielectric (permittivity) and magnetic permeability coefficients, respectively, $\tilde{a}_{jl}$ are the coupling coefficients connecting electric and magnetic fields. $\mathbf{D} := (D_1, D_2, D_3) : \Omega\times]0,T[\to \mathbb{R}^3$ is the electrical displacement vector, $\mathbf{B} := (B_1, B_2, B_3) : \Omega\times]0,T[\to \mathbb{R}^3$ is the magnetic induction vector.

**Constitutive Equations**

$$X_{ij} = E_{ijkl}e_{kl} + E^*_{ijkl}\dot{e}_{kl} + \tilde{b}\varphi\delta_{ij} + b^*\dot{\varphi}\delta_{ij} + p_{kij}\chi_{,k} + q_{kij}\eta_{,k}, \quad i,j = \overline{1,3}, \tag{1.4}$$

$$H_j = \tilde{\alpha}\varphi_{,j} + \alpha^*\dot{\varphi}_{,j}, \quad j = \overline{1,3}, \tag{1.5}$$

$$H_0 = -\tilde{b}e_{kk} - \tilde{\xi}\varphi - \nu^*\dot{e}_{kk} - \xi^*\dot{\varphi}, \tag{1.6}$$

$$D_j = p_{jkl}e_{kl} - \varsigma_{jl}\chi_{,l} - \tilde{a}_{jl}\eta_{,l}, \quad j = \overline{1,3}, \tag{1.7}$$

$$B_j = q_{jkl}e_{kl} - \tilde{a}_{jl}\chi_{,l} - \xi_{jl}\eta_{,l}, \quad j = \overline{1,3}, \tag{1.8}$$

where $e_{ij} \in C^1(\Omega)$ is the strain tensor; $E_{ijkl}$, $E^*_{ijkl}$, $\tilde{b}$, $b^*$, $\tilde{\alpha}$, $\alpha^*, \nu^*$, $\tilde{\xi}$, $\xi^*$, $p_{kij}$, $q_{kij}$, $\varsigma_{jl}$, $\tilde{a}_{jl}$, $\xi_{jl}$ are the constitutive coefficients, depending on $x_1$ and $x_2$;

**Kinematic Relations**

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i,j = \overline{1,3}. \tag{1.9}$$

Let us consider the transversely isotropic elastic piezoelectric material in the case when the poling axis coincides with one of the material symmetry axes [9]. A material behavior is said to be transversely isotropic if it is invariant with respect to an arbitrary rotation about a given axis. This material behavior is of special importance in the modelling of fibre-reinforced composite materials with a coordinate axis in the fibre direction and assumed isotropic in cross-sections orthogonal to fibre direction [10] (in our case to poling axis as well, since in the case under consideration they coincide). The transverse isotropic model is also suitable for biological applications because it adequately describes the elastic properties of bundled fibers aligned in one direction [11] (see also [12]).

Conditions of the antiplane state look like

(1) $u_1 \equiv 0, \quad u_2 \equiv 0, \quad u_3 \not\equiv 0;$
(2) $X_{13} \not\equiv 0, \quad X_{23} \not\equiv 0, \quad X_{\alpha\beta} \equiv 0, \quad \alpha, \beta = 1,2; \quad X_{33} \equiv 0;$
(3) $e_{13} \not\equiv 0, \quad e_{23} \not\equiv 0, \quad e_{\alpha\beta} \equiv 0, \quad \alpha, \beta = 1,2; \quad e_{33} \equiv 0;$
(4) $E_1 \not\equiv 0, \quad E_2 \not\equiv 0, \quad E_3 \equiv 0;$
(5) $D_1 \not\equiv 0, \quad D_2 \not\equiv 0, \quad D_3 \equiv 0.$

If we consider transversely isotropic piezoelectric materials, then (see [1])

$$E_{2323} = E_{1313} \not\equiv 0;$$

$$E_{2222} = E_{1111} \not\equiv 0, \;\; E_{1122} \not\equiv 0, \;\; E_{2233} = E_{1133} \not\equiv 0, \;\; E_{3333} \not\equiv 0; \tag{1.10}$$

$$p_{223} = p_{113} \not\equiv 0;$$

$$\varsigma_{22} = \varsigma_{11} \not\equiv 0, \;\; \varsigma_{33} \not\equiv 0.$$

Other elastic, piezoelectric, and dielectric permittivity constants are identically zero with regard to reciprocal symmetries.

## 2. Title problem

Let the closure of a domain of $R^3$, occupied by a piezoelectric elastic bar $\overline{V}$ with rectangular cross-sections (see [13], [14]) be:

$$\overline{V} := \Big\{(x_1, x_2, x_3) \in R^3 : 0 \le x_3 \le L; \; \overset{(-)}{h}_\alpha(x_3) \le x_\alpha \le \overset{(+)}{h}_\alpha(x_3),$$
$$\alpha = 1,2; \; L = const\Big\}$$

with

$$2h_\alpha(x_3) := \overset{(+)}{h}_\alpha(x_3) - \overset{(-)}{h}_\alpha(x_3) > 0,$$
$$\overset{(+)}{h}_\alpha(x_3),\ \overset{(-)}{h}_\alpha(x_3) \in C([0,L]),\ \ 0 \le x_3 \le L,\ \alpha = 1,2.$$

Therefore, we have (see [1], [8])

$$(E_{3333}u_{3,\alpha})_{,\alpha} + (p_{333}\chi_{,\alpha})_{,\alpha} - \Phi_3 = \rho\ddot{u}_{30}.$$

and

$$(p_{333}u_{3,\alpha})_{,\alpha} - (\varsigma_{33}\chi_{,\alpha})_{,\alpha} = f_e,$$

respectively.

Let

$$E_{3333} = E_0x_3^\kappa,\ \ p_{333} = p_0x_3^\kappa,\ \ \varsigma_{33} = \varsigma_0x_3^\kappa,\ \ E_0,\ p_0,\ \varsigma_0,\ \kappa = const > 0.$$

The last system can be rewritten as follows

$$(E_0x_3^\kappa u_{3,3})_{,3} + (p_0x_3^\kappa\chi_{,3})_{,3} - \rho\ddot{u}_3 = -\Phi_3, \tag{2.1}$$
$$(p_0x_3^\kappa u_{3,3})_{,3} - (\varsigma_0x_3^\kappa\chi_{,3})_{,3}- = f_e. \tag{2.2}$$

After multiplication (2.1) by $\varsigma_0$ and (2.2) by $p_0$ and summation we get

$$\Big((E_0 + \frac{p_0^2}{\varsigma_0})x_3^\kappa u_{3,3}\Big)_{,3} - \rho\ddot{u}_3 = F, \tag{2.3}$$

where $F := -\Phi_3 + f_e\frac{p_0}{\varsigma_0}$.

Let $\kappa < 1$. We solve (2.3) under the following boundary and initial conditions

$$u_3(0,t) = u_3(L,t) = 0,\ \ u_3(x_3,0) = \varphi_1(x_3),\ \ u_{3,t}(x_3,0) = \varphi_2(x_3), \tag{2.4}$$

where $\varphi_i(x_3) \in C^2(]0,L[)$, $i = 1,3$, are given functions such that

$$\varphi_i(0) = \varphi_i(L) = 0,\ \ i = 1,2.$$

Using the Fourier method, we will look for $u_3(x_3,t)$ in the following form

$$u_3(x_3,t) = X(x_3)T(t).$$

Let first $F \equiv 0$. Then from (2.3) we get

$$\frac{\Big((E_0 + \frac{p_0^2}{\varsigma_0})x_3^\kappa X'(x_3)\Big)'}{\rho X(x_3)} = \frac{T''(t)}{T(t)} = -\lambda = \text{const}.$$

Hence,

$$T''(t) + \lambda T(t) = 0, \tag{2.5}$$

$$\Big(E(x_3)X'(x_3)\Big)' = -\lambda\rho X(x_3). \tag{2.6}$$

From boundary conditions (2.4) we get

$$X(0) = X(L) = 0. \tag{2.7}$$

Now, we have to solve the following boundary value problem:

**Problem 1.** Find

$$X(x_3) \in C^2(]0, L[) \cap C([0, L])$$

which satisfies equation (2.6) and BCs (2.7).

After two times integration of (2.6) and using boundary conditions (2.7) we get

$$X(x_3) = -\lambda\rho \int\limits_{x_3}^{L} K(x_3, \xi)X(\xi)d\xi, \tag{2.8}$$

where

$$K(x_3, \xi) := \begin{cases} K_1(x_3, \xi),\ x_3 \le \xi \le L, \\ K_1(\xi, x_3),\ 0 \le \xi \le x_3, \end{cases}$$

$$:= \begin{cases} -\dfrac{\int\limits_0^{x_3} \frac{d\eta}{E(\eta)} \int\limits_{\xi}^{L} \frac{d\eta}{E(\eta)}}{\int\limits_0^{L} \frac{d\eta}{E(\eta)}},\ x_3 \le \xi \le L, \\ -\dfrac{\int\limits_0^{\xi} \frac{d\eta}{E(\eta)} \int\limits_{x_3}^{L} \frac{d\eta}{E(\eta)}}{\int\limits_0^{L} \frac{d\eta}{E(\eta)}},\ 0 \le \xi \le x_3, \end{cases} \tag{2.9}$$

**Proposition 2.1:** *$K(x_3, \xi)$ is symmetric with respect to $x_3$ and $\xi$.*

**Proof:** For $z_1$ and $z_2$, such that $0 < z_1, z_2 < L$ we get

$$K(z_1, z_2) = \begin{cases} K_1(z_1, z_2),\ 0 \le z_2 \le z_1, \\ K_1(z_2, z_1),\ z_2 \le z_1 \le L, \end{cases}$$

$$K(z_2, z_1) = \begin{cases} K_1(z_2, z_1),\ z_2 \le z_1 \le L, \\ K_1(z_1, z_2),\ 0 \le z_2 \le z_1, \end{cases}$$

i.e., $K(z_1, z_2) = K(z_2, z_1)$ for any $z_1, z_2 \in [0, L]$. □

**Proposition 2.2:** *The number of eigenvalues $\lambda_n$ of (2.9) is not finite.*

**Proof:** Let it be finite and $n = \overline{1, m}$, then $K(x_3, \xi)$ can be expressed as follows (see [15])

$$K(x_3, \xi) = \sum_{n=1}^{m} \frac{X_n(x_3)X_n(\xi)}{\lambda_n},$$

where

$$X_n(x_3) \in C^2(]0, l[) \;\Rightarrow\; K(x_3, \xi) \in C^1(]0; L] \times ]0, L[). \tag{2.10}$$

On the other hand, in view of (2.9),

$$K'_{x_3}(x_3, \xi)|_{\xi \to x_3 -} - K'_{x_3}(x_3, \xi)|_{\xi \to x_3 +} = \frac{1}{E(x_3)} \notin C^1(]0; L] \times ]0, L[), \tag{2.11}$$

(2.10) and (2.11) contradict to each others, thus the number of $\lambda_n$ is not finite. □

**Proposition 2.3:** *All of $\lambda_n$ are positive.*

**Proof:** Let denote by $X_n$ orthonormalized eigenfunctions of (2.9) (it can be assumed without loss of generality), then from

$$(E(x_3)X'_n(x_3))' = -\lambda\rho X_n(x_3)$$

after multiplication by $X_n(x_3)$ and integration the obtained expression from 0 to $L$ we obtain

$$\int_0^L (E(x_3)X'_n(x_3))' X_n(x_3) dx_3 = -\lambda_n \rho \int_0^L X_n(x_3)X_n(x_3) dx_3 = -\lambda_n \rho.$$

Further, following integration by parts of the left side of the last expression and taking into account of (2.7) we get

$$-\lambda_n \rho = -\int_0^L E(x_3)X'_n(x_3)' X_n(x_3) dx_3 \leq 0$$

i.e., $\lambda_n > 0$ for any $n$, since in the non trivial case $X_n \not\equiv 0$. □

The solution of (2.5) has the form

$$T_n(t) = b_1^n \sin(\sqrt{\lambda_n}\, t) + b_2^n \cos(\sqrt{\lambda_n}\, t), \quad b_i^n = const, \quad i = 1, 3.$$

Now, we can find a formal solution of the Problem 1 in the form as follows

$$u_3(x_3,t)=\sum_{n=1}^{\infty}X_n(x_3)(b_1^n\sin(\sqrt{\lambda_n}\,t)+b_2^n\cos(\sqrt{\lambda_n}\,t)). \qquad (2.12)$$

In view of initial condition (2.4), we formally have

$$\sum_{n=1}^{\infty}X_nb_2^n=\varphi_1(x_3) \qquad \sum_{n=1}^{\infty}\sqrt{\lambda_n}X_nb_1^n=\varphi_2(x_3). \qquad (2.13)$$

If

$$\Psi_i(x_3):=(E(x_3)\varphi_i'(x_3))'\in C[0,L],$$

then

$$\varphi_i=\int\limits_0^L K(x_3,\xi)\Psi_1(\xi)d\xi. \qquad (2.14)$$

Since $\Psi_i(\xi)\in C([0,L])$ and Propositions 2.1-2.3 we get absolutely and uniform convergence of the series

$$\varphi_i(x_3)=\sum_{n=1}^{\infty}\int_0^L\varphi_i(\xi)X_n(\xi)d\xi\cdot X_n(x_3),$$

i.e., of (2.13) on $[0,L]$ and

$$b_1^n=\frac{1}{\sqrt{\lambda_n}}\int_0^L X_n(x_3)\varphi_2(x_3)dx_3,$$

$$b_2^n=\int_0^L X_n(x_3)\varphi_1(x_3)dx_3.$$

Since there exists a positive minimum of eigenvalues, from the convergence of the second series of (2.13) follows absolutely and uniformly convergence of the series $\sum\limits_{n=1}^{\infty}X_n(x_3)b_1^n$on $[0,L]$. Therefore (2.12) is absolutely and uniformly convergent on $[0,L]$.

After formal differentiation of (2.12) we get

$$u_{3,t}(x_3,t)=\sum_{n=1}^{\infty}X_n(x_3)\sqrt{\lambda_n}(b_1^n\sin(\sqrt{\lambda_n}\,t)-b_2^n\cos(\sqrt{\lambda_n}\,t)), \qquad (2.15)$$

$$u_{3,tt}(x_3,t) = -\sum_{n=1}^{\infty} X_n(x_3)\lambda_n(b_1^n \sin(\sqrt{\lambda_n}\,t) + b_2^n \cos(\sqrt{\lambda_n}\,t)), \tag{2.16}$$

$$\frac{\partial^i}{\partial x_3^i} u_3(x_3,t) = \sum_{n=1}^{\infty} \frac{d^i}{dx_3^i} X_n(x_3)(b_1^n \sin(\sqrt{\lambda_n}\,t) + b_2^n \cos(\sqrt{\lambda_n}\,t)), \quad i = 1,3, \tag{2.17}$$

Analogously to prove the absolute and uniform convergence of (2.12) can be shown that

**Theorem 2.4:** *If*

$$\chi_i(x_3) := (E\Psi_i')', \quad i = 1,3 \ \text{ are integrable on } \ ]0;L[$$

*satisfying BCs*

$$\chi_i(0) = \chi_i(L) = 0$$

*(2.15), (2.16) and (2.17) convergent absolutely and uniformly on any* $[a,b] \in ]0,L[$.

Thus, (2.12) is the solution of Problem 1 for $F \equiv 0$.

Now, let us consider Problem 1 when $F(x_3,t) \not\equiv 0$, $\varphi_i = 0$, $i = 1,3$, and let $F(\cdot_2,t) \in L_2(0,l)$. Then $F(x_3,t)$ can be represented as a convergent series in $L_2(0,l)$:

$$F(x_3,t) = \sum_{n=1}^{\infty} (F(x_3,t), X_n)X_n,$$

hence,

$$F(x_3,t) = \sum_{n=1}^{\infty} X_n(x_3)F_n(t), \quad F_n(t) := \int_0^L F(x_3,t)X_n(x_3)dx_3.$$

Further, we look for the solution in the form

$$u_3(x_3,t) = \sum_{n=1}^{\infty} u_{3n}(x_3,t),$$

where $u_{3n}(x_3,t)$ is a solution of Problem 1 with $F(x_3,t)$ replaced by $X_n(x_3)F_n(t)$. Using the method of separation of variables, we can write

$$u_{3n}(x_3,t) = X_n(x_3)T_{1n}(t),$$

here

$$T_{1n}''(t) + \lambda_n T_{1n} t = F_n(t).$$

Therefore, $u_3(x_3,t)$ can be expressed as follows (see, e.g. [16])

$$u_3(x_3,t)=\sum_{n=1}^{\infty}\frac{1}{\sqrt{\lambda_n}}X_n\int_0^t sin(\sqrt{\lambda_n}(t-\tau))F_n(\tau)d\tau. \tag{2.18}$$

Now, similarly to the above reasoning if the following conditions are fulfilled

$$(E(x_3)F,_{x_3}(x_3,t)),_{x_3}\in C[0,L],$$

$$E(x_3)F,_{x_3}(x_3,t)|_{x_3=0}=E(x_3)F,_{x_3}(x_3,t)|_{x_3=l}=0,$$

we have the absolute and uniform convergence of series (2.21) on $[0,L]$, and the absolute and uniform convergence of the series

$$\frac{\partial^i}{\partial x_3^i}F(x_3,t)=\sum_{n=1}^{\infty}\frac{d^i}{dx_3^i}X_n(x_3)T_{1n},\quad i=1,3,$$

$$\frac{\partial^i}{\partial t^i}F(x_3,t)=\sum_{n=1}^{\infty}X_n(x_3)\frac{d^i}{dt^i}T_{1n},\quad i=1,3,$$

on any $[a,b]\subset]0,L[$.

**Remark 1:** Let $F(x_3,t),\ \varphi_i(x_3)\not\equiv 0$, then the solution of Problem 1 can be expresses as follows

$$u_3(x_3,t)=\sum_{n=1}^{\infty}u_{3n}(x_3,t),$$

where

$$u_{3n}(x_3,t)=X_n(x_3)(T_{1n}(t)+T_n(t)).$$

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