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The Behavior of Small Sets under the Product Operation

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For invariant (quasi-invariant) σ-finite measures on an uncountable group $(G, \cdot)$, the behavior of measure zero sets with respect to the product operation is studied.

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Dedicated to the 100-th Anniversary of Professor Sh. Pkhakadze.

Let $(G, \cdot)$ be an arbitrary group and let μ be a nonzero σ-finite G-invariant (more generally, G-quasiinvariant) measure defined on some σ-algebra of subsets of G. We recall that the symbol $I(\mu)$ denotes the σ-ideal of subsets of G, generated by the family of all μ-measure zero sets. Members of $I(\mu)$ are usually called small sets with respect to the given measure μ.

Throughout this article, we use the following standard notation:
R is the set of all real numbers;
N is the set of all natural numbers;
$dom(\mu)$ is the domain of a given measure μ;
$ran(f)$ is the range of a given function f;
ω_1 is the first uncountable ordinal number;
$X \cdot Y$ is the product of two sets X and Y in $(G, \cdot)$, i.e.,

$$X \cdot Y = \{x \cdot y : x \in X, y \in Y\};$$

$\mu_1 \supset \mu$ - a measure μ_1 is extension of the given measure μ.

In the present paper an approach to some questions of the theory of invariant (quasi-invariant) measures is discussed. Such an approach is useful in certain situations, where given groups are equipped with various nonzero σ-finite left invariant (left quasi-invariant) measures.

We would like to consider some general method which is oriented to the study of invariant (quasi-invariant) measures. This method is purely algebraic and turns out to be helpful in various questions of measure theory. We call it the method of surjective homomorphisms.

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Let $(G_1, \cdot)$ and $(G_2, \cdot)$ be arbitrary uncountable groups. Let the group G_2 be equipped with a σ-finite G_2-left-invariant (G_2-left-quasi-invariant) measure μ and let

$$\varphi : G_1 \to G_2$$

be a surjective homomorphism. Consider the family of sets

$$S = \{\varphi^{-1}(Y) : Y \in dom(\mu)\}$$

and define a functional ν on this family by putting

$$\nu(\varphi^{-1}(Y)) = \mu(Y),$$

where $Y \in dom(\mu)$.

Then this functional ν is a measure satisfying the following relations:

(a) $S = dom(\nu)$ is a G_1-left-invariant σ-algebra of subsets of G_1;

(b) ν is a non-atomic σ-finite G_1-left-invariant (left-quasi-invariant) measure on S.

We need several auxiliary statements which play an essential role for our future purpose.

The following statement is valid.

Lemma 1: *Let $(G_1, \cdot)$ and $(G_2, \cdot)$ be arbitrary uncountable groups. Let the group G_2 be equipped with a σ-finite G_2-left-invariant (G_2-left-quasi-invariant) measure μ and let*

$$\varphi : G_1 \to G_2$$

be a surjective homomorphism.

If a measure μ' is some σ-finite G_2-left-invariant (G_2-left-quasi-invariant) extension of measure μ on G_2, then ν' is σ-finite G_1-left-invariant (G_1-left-quasi-invariant) extension of the measure ν on G_1, where ν and ν' are measures respectively corresponding to μ and μ' under the surjective homomorphism φ (as described above).

The proof of Lemma 1 can be found in [13].

It is well known that the class of small sets stable under the operation of countable unions. It is natural to investigate the analogous question under the product operation.

Proposition: *If μ is an arbitrary measure on $(G, \cdot)$, then the following two assertions are equivalent:*

a) there exist two sets $X \in I(\mu)$ and $Y \in I(\mu)$ such that $X \cdot Y = G$;

b) there exists a set $X \in I(\mu)$ such that $X \cdot X = G$.

If $\mathbf{R}$ is the real line, then for $G = \mathbf{R}$ and for $\mu = \lambda$, where λ denotes the standard Lebesgue measure on $\mathbf{R}$, the validity of assertion a) is well known. Starting with a) and applying a Hamel basis of $\mathbf{R}$, W. Sierpinski has established that there exist two sets $X \subset \mathbf{R}$ and $Y \subset \mathbf{R}$, satisfying the relations

$$X \in I(\lambda),\ Y \in I(\lambda),\ X + Y \notin dom(\lambda),$$

where

$$X + Y = \{x + y : x \in X, y \in Y\}.$$

For more details, see [5], [6]. Some generalization of this result for uncountable vector spaces over **Q** and for quasi-invariant extensions of measures on such spaces can be found in [5]. Similar properties of algebraic sums of subsets of the real line **R** are also discussed in [9], [10].

It is reasonable to ask whether similar statements hold in more general situations when no topology is considered on a given group. Namely, it is natural to pose the following question:

let $(G, \cdot)$ be an uncountable group equipped with a nonzero σ-finite complete G-invariant (G-quasiinvariant) measure μ.

Do there exist two sets $X \in I(\mu)$ and $Y \in I(\mu)$ whose algebraic sum $X \cdot Y$ does not belong to $dom(\mu)$.

Notice that, for an arbitrary uncountable commutative group $(G, +)$ and for a nonzero σ-finite complete G-invariant (G-quasiinvariant) measure μ on G, we do not have a direct analogue of this question.

Therefore, the formulation of the question posed above should be replaced by another one. Namely, the following problem is of interest from the measure-theoretical point of view.

Let $(G, \cdot)$ be an uncountable group and let μ be a nonzero σ-finite left G-invariant (left G-quasiinvariant) measure on G.

Does there exist a left G-invariant (left G-quasiinvariant) measure μ' on G extending μ and such that for some sets $X \in I(\mu')$ and $Y \in I(\mu')$, the relation

$$X \cdot Y \notin dom(\mu')$$

is satisfied?

In this question for an uncountable commutative group $(G, +)$ the following statement is valid.

Theorem 1: *Let $(G, +)$ be an uncountable commutative group and let μ be a nonzero σ-finite G-invariant (G-quasi-invariant) measure on G. There exists a G-invariant (G-quasi-invariant) complete measure $\hat{\mu}$ on G extending μ and such that, for some two sets $X \in I(\hat{\mu})$ and $Y \in I(\hat{\mu})$, the relation*

$$X + Y \notin dom(\hat{\mu})$$

is satisfied.

For a proof of Theorem 1, see [7], [9].

Let $(G, \cdot)$ be an arbitrary uncountable group.

Lemma 2: *Let $(H, \cdot)$ be an uncountable group (commutative or noncommutative) and let μ be a nonzero σ-finite H-invariant measure on H. If*

$$\varphi : G \to H$$

is a surjective homomorphism and there exist a nonzero σ-finite H-left invariant measure $\mu' \supset \mu$ and two sets $X \in I(\mu')$ and $Y \in I(\mu')$ on H such that

$$X \otimes Y \notin dom(\mu'),$$

then there exist measures ν and ν' on G and two sets $X' \in I(\nu')$ and $Y' \in I(\nu')$ on G for which the following relations are satisfied:
(a)$\nu' \supset \nu$;
(b) $X' \cdot Y' \notin dom(\nu')$;
(c) ν and ν' are G-left invariant measures on G.

Proof: Let $(G, \cdot)$ be an arbitrary uncountable group and let $(H, \otimes)$ be an uncountable group satisfying the above-mentioned conditions.
Consider the families of sets

$$S_1 = \{\varphi^{-1}(Z) : Z \in dom(\mu)\},$$

$$S_2 = \{\varphi^{-1}(Z') : Z' \in dom(\mu')\}$$

and define the functional ν and ν' by the formulas:

$$\nu(\varphi^{-1}(Z)) = \mu(Z) \qquad (Z \in dom(\mu)),$$

$$\nu'(\varphi^{-1}(Z')) = \mu'(Z') \qquad (Z' \in dom(\mu')).$$

It is clear that:
(a) the definitions of measure ν and ν' are correct, because φ is a surjection;
(b) S_1 and S_1 are G-invariant σ-algebras of subsets of G;
(c) ν and ν' are σ-finite measures on S_1 and S_1, respectively;
(d) $\nu' \supset \nu$.
Let $X' = \varphi^{-1}(X)$ and $Y' = \varphi^{-1}(Y)$.
Since we have the following relations

$$X' \cdot Y' = \varphi^{-1}(X) \cdot \varphi^{-1}(Y) = \varphi^{-1}(X \cdot Y),$$

and taking into account our assumption on X and Y, we infer that there exist two sets X' and Y' on G such that

$$X' \cdot Y' \notin dom(\nu')$$

which ends the proof of Lemma 2. □

Let us introduce one important notion from the theory of left-invariant (left-quasi-invariant) measures(cf.[4], [7], [14]).

Let $(G, \cdot)$ be an arbitrary group and let $Y \subset G$. We say that Y is G-absolutely negligible in G if, for any σ-finite G-left-invariant (left-quasi-invariant) measure μ on G, there exists a G-left-invariant (left-quasi-invariant) measure μ' on G extending μ and such that $\mu'(Y) = 0$.

Lemma 3: *Let $(G_1, \cdot)$ and $(G_2, \cdot)$ be two groups, let*

$$\varphi : G_1 \to G_2$$

be a surjective homomorphism and let Y be a G_2-absolutely negligible subset of G_2. Then the set $X = \varphi^{-1}(Y)$ is G_1-absolutely negligible in G_1.

For a proof of Lemma 3, see [10].

Remark 1: Let $\mathbf{M}(\mathbf{R}^n)$ be the class of all those nonzero σ-finite measures on $\mathbf{R}^n$, which are invariant (quasi-invariant) under the group of all translations of $\mathbf{R}^n$. In the paper [8] it was proved that there exists a Mazurkiewicz set which is a Hamel basis of $\mathbf{R}^2$ and, consequently, is absolutely negligible with respect to $\mathbf{M}(\mathbf{R}^n)$. According to this fact we deduce that the projection of Mazurkiewicz set on the real line $\mathbf{R}$ coincides with $\mathbf{R}$. This fact shows that a homomorphic image of an absolutely negligible set, in general, is not an absolutely negligible set.

We need an auxiliary proposition which yields a purely algebraic characterization of absolutely negligible sets and plays an essential role in the investigation of various properties of such sets.

Remark 2: Let $(G, \cdot)$ be an arbitrary uncountable group and let Y be a subset of G. Then the following two relations are equivalent:

1) Y is a G-absolutely negligible set in G;

2) for each countable family $\{g_i : i \in I\}$ of elements from G, there exists a countable family $\{h_j : j \in J\}$ of elements from G, satisfying the equality

$$\cap_{j \in J}(h_j \cdot (\cup_{i \in I}(g_i \cdot Y)) = \emptyset.$$

For the proof of this equivalence, see e.g. [6] or [7].

Lemma 4: *Let $(G, \cdot)$ be an uncountable group and let*

$$G = G_1 \times G_2 \qquad (G_1 \cap G_2 = \{e\})$$

be a representation of G in the form of the direct product of its two subgroups G_1 and G_2. Suppose also that a set $Y \subset G_1$ is G_1-absolutely negligible in G_1. Then the set $Y \times G_2$ turns out to be G-absolutely negligible in G.

Proof: Consider the canonical projection pr_1 of the group G onto group G_1 and denote it by φ. Then Lemma 3 gives at once the required result.

From the above lemmas we readily obtain the following statement. □

Theorem 2: *Let $(G, \cdot)$ an $(H, \cdot)$ be arbitrary uncountable groups and let*

$$\varphi : G \to H$$

be a surjective homomorphism. Let μ be a nonzero σ-finite H-left invariant measure on H. If there exist a nonzero σ-finite H-left invariant (H-left-quasi-invariant) *measure* $\mu' \supset \mu$ *on* H *and two absolutely negligible sets* X *and* Y *such that*

$$X \cdot Y \notin dom(\mu'),$$

then there exist nonzero σ*-finite* G*-left invariant (*G*-left-quasi-invariant) measures* ν *and* ν' *satisfying the following relations:*

(1) ν' *is a nonzero* σ*-finite* G*-left invariant (*G*-left-quasi-invariant) measure on* G*;*

(2) $\nu' \supset \nu$*;*

(3) there exist two absolutely negligible sets X' *and* Y' *such that* $X' \cdot Y' \notin dom(\nu')$.

Remark 3: If $(G, \cdot)$ is an uncountable commutative group, then the existence of two absolutely negligible sets X and Y such that $X + Y \notin dom(\mu')$ is guaranteed by Theorem 1.

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