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ON Φ-FIXED POINT FOR MAPS ON UNIFORM SPACES

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Abstract. The concept of fixed point is extended to Φ-fixed point for those maps on uniform spaces. Two results are presented, first for single-valued maps and second for set-valued maps.

1. Introduction and preliminaries

The fixed point theorem has applications in almost all branches of mathematics. The considering of the existence of fixed point for a mapping, is expressed in metric spaces, and some authors have extended this result in some other versions [1], [2], [3] and [4]. M.A. Khamsi and W.A. Kirk [6] have collected many results in fixed point theory which is a good source in this branch. Here, we would improve their results for single-valued and set-valued maps in uniform spaces, which is a generalization for metric space.

Definition 1.1. Let X be a nonempty set and $\Phi \subset 2^{X\times X}$ satisfies in the following
:
1) For any $u \in \Phi$, $\Delta = \{< x, x >: x \in X\} \subset u$.
2) If $u \in \Phi$ and $u \subset v$, then $v \in \Phi$.
3) If $u, v \in \Phi$, then $u \cap v \in \Phi$.
4) For any $u \in \Phi$, there exists $v \in \Phi$ such that, $vov \subset u$,
where, $vov = \{(x, z) : \exists y \in X; (x, y) \in \nu \quad and \quad (y, z) \in \nu\}$.
5) $u \in \Phi$ imply that, $u^{-1} \in \Phi$,
where, $u^{-1} = \{(x, y) : (y, x) \in u\}$.
Then, Φ is said to be a uniform structure for X and (X, Φ) a uniform space.

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Definition 1.2. Let (X, Φ) be a uniform space and $T : X \to X$ be a single-valued mapping, $x_0 \in X$ is said to be Φ-fixed point for T, if $(x_0, Tx_0) \in \bigcap_{u\in\Phi} u$.

Definition 1.3. Let (X, Φ) be a uniform space and $T : X \to 2^X$ a set-valued mapping, then $x_0 \in X$ is said to be a Φ-fixed point for T if there exists $z \in Tx_0$ such that $(x_0, z) \in \bigcap_{u\in\Phi} u$.

We set $u[x] = \{y \in X; < x, y >\in u\}$ for any $x \in X$, $u \in \Phi$.

2. Main results

Theorem 2.1. *Suppose that (X, Φ) is a uniform space and $T : X \to X$ a single-valued map. If there is $z \in X$ such that for any $\nu \in \Phi$, $\nu[z] \cap \nu[Tz] \neq \emptyset$, then T has at least one Φ-fixed point in X.*

Proof. To show that T has at least one Φ-fixed point in X, we must prove there exists at least one x_0 of X, such that $(x_0, Tx_0) \in \bigcap_{u\in\Phi} u$. Suppose on the contrary, assume that for any $x_0 \in X$ there exists $u_0 \in \Phi$, such that $(x_0, Tx_0) \notin u_0$. According to the property of uniform space, there exists $\nu \in \Phi$, such that $v o v \subset u_0$. Therefore, $(x_0, Tx_0) \notin v o v$. Hence, for any $y \in X$, $(x_0, y) \notin \nu$ or $(y, Tx_0) \notin \nu$, Then, for any $y \in X$, $y \notin \nu[x_0]$ or $y \notin \nu[Tx_0]$. Therefore, we obtain $\nu[x_0] \cap \nu[Tx_0] = \emptyset$, which is a contradiction by assumption. Hence, there exists $x_0 \in X$ such that for any $u \in \Phi$, $(x_0, Tx_0) \in \bigcap_{u\in\Phi} u$, i.e., x_0 is Φ-fixed point for T in X.

The following result is a direct consequence Following Theorem 2.1.

Corollary 2.2. *It should be noticed in Theorem 2.1, if (X, Φ) is a Hausdorff uniform space, then T has at least one fixed point in X.*

Theorem 2.3. *Suppose that (X, Φ) is a uniform space and $T : X \to 2^X$ is a set-valued mapping. If there exists at least one $x_0 \in X$ such that for any $u \in \Phi$ and for any $z \in Tx_0$, $u[x_0] \cap u[z] \neq \emptyset$, then x_0 is a Φ-fixed point for T.*

Proof. We will prove that, there is at least one $x_0 \in X$ and there exists $z \in Tx_0$ such that $(x_0, z) \in \bigcap_{u\in\Phi} u$. On the contrary, for any $x_0 \in X$, and for any $z \in Tx_0$, there exists $u_0 \in \Phi$ such that $(x_0, z) \notin u_0$. Hence, there is $\nu \in \Phi$ such that $v o v \subset u_0$, we have $(x_0, z) \notin v o v$. Therefore, for any $y \in X$, $(x_0, y) \notin \nu$ or $(y, z) \notin \nu$. Then for any $y \in X$, $y \notin \nu[x_0] \cap \nu[z]$, i.e., $\nu[x_0] \cap \nu[z] = \emptyset$ which is a contradiction. There is at least one $x_0 \in X$ which is a Φ-fixed point for T.

Following is a direct result of Theorem 2.3.

Corollary 2.4. *In Theorem 2.3, if (X, Φ) be a Hausdorff uniform space, then in fact x_0 is a fixed point for T.*

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