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LOCAL CONVERGENCE ANALYSIS FOR A CERTAIN CLASS OF INEXACT METHODS

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Abstract. We provide a local convergence analysis for a certain class inexact methods in a Banach space setting, in order to approximate a solution of a nonlinear equation [6]. The assumptions involve center–Lipschitz–type and radius–Lipschitz–type conditions [15], [8], [5]. Our results have the following advantages (under the same computational cost): larger radii, and finer error bounds on the distances involved than in [8], [15] in many interesting cases.

Numerical examples further validating the theoretical results are also provided in this study.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet–differentiable operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time–invariant system is driven by the equation $\dot{x} = Q(x)$, for some suitable operator Q, where x is the state. Then the equilibrium states are

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determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative–when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

We study the convergence of inexact Newton method (INMB):

$$\begin{aligned}
&For \;\; n = 0 \;\; step\, 1 \;\; until \;\; convergence \;\; do\\
&Find \;\; the \;\; step \;\; \Delta_n \;\; with \;\; satisfies:\\
&\mathcal{B}_n \, \Delta_n = -F(x_n) + r_n
\end{aligned}$$

where,

$$\frac{\parallel \mathcal{P}_n \, r_n \parallel}{\parallel \mathcal{P}_n \, F(x_n) \parallel} \leq \eta_n \leq 1;$$

Set:

$$x_{n+1} = x_n + \Delta_n, \quad (n \geq 0), \tag{1.2}$$

Here, $\mathcal{P}_n$ is an invertible operator, and $\mathcal{B}_n^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ for each n.

The (INMB) was considered by Morini in [13], whereas, if $\mathcal{P}_n = I$ $(n \geq 0)$, the method has been studied extensively in [1]–[3], [6], [7], [9]–[15] under various Lipschitz–type hypotheses. A survey of such results can be found in [5] (see, also [4], [8], [15]). The advantages of introducing operators $\mathcal{P}_n$ have been explained in [13]. In case $\mathcal{B}_n = F'(x_n)$ $(n \geq 0)$, we will denote (INMB) by (INMF).

In this study, we are motivated by the work in [8], [15], where, radius Lipschitz–type conditions are used (see (2.3)) to provide a local as well as a semilocal convergence for Newton's method. We use weaker and needed center–Lipschitz–type conditions (see (2.2)) to find upper bounds on the distances $\parallel F'(x)^{-1} \, F'(x^\star) \parallel$ $(x \in \mathcal{D})$ instead of the stronger (2.3) used in [8], [15] for Newton's method. It turns out that this approach leads to a local convergence analysis not only for Newton's method, but also for (INMB) and (INMF), with the following advantages, and under the same computational cost (see Remark 2.5):

(a) larger convergence radii,
(b) finer estimates for the distances $\parallel x_n - x^\star \parallel$ $(n \geq 0)$.

Numerical examples further validating the theoretical results are also provided.

2. Local convergence analysis of (INMF) and (INMB)

We provide four local convergence results for (INMB) and (INMF):

Theorem 2.1. *Let $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be a Fréchet–differentiable operator.*

Assume:

there exist $x^\star \in \mathcal{D}$, satisfying equation (1.1), such that $F'(x^\star)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, $r_0 > 0$, with

$$U(x^\star, r_0) = \{x \in \mathcal{X} : \| x - x^\star \| \leq r_0\} \subseteq \mathcal{D}, \tag{2.1}$$

positive integrable functions L_0, and L, satisfying center–Lipschitz condition, and radius Lipschitz–type condition:

$$\| F'(x^\star)^{-1} \, [F'(x) - F'(x^\star)] \| \leq \int_0^{\rho(x)} L_0(t) \, dt, \tag{2.2}$$

$$\| F'(x^\star)^{-1} \, [F'(x) - F'(x_\theta)] \| \leq \int_{\theta\,\rho(x)}^{\rho(x)} L(t) \, dt, \tag{2.3}$$

respectively, for all $x \in U(x^\star, r_0)$, $x_\theta = x^\star + \theta \, (x - x^\star)$, $\rho(x) = \| x - x^\star \|$, $\theta \in [0, 1]$,

$$v_n = \theta_n \;\; \| (\mathcal{P}_n \; F'(x_n))^{-1} \| \; \| \mathcal{P}_n \; F'(x_n) \| = \theta_n \; \text{cond} \; (\mathcal{P}_n \, F'(x_n)) \leq v < 1, \tag{2.4}$$

and

$$(1 - v) \int_0^{r_0} L_0(t) \, dt + (1 + v) \int_0^{r_0} L(t) \, dt \leq 1 - v. \tag{2.5}$$

Then, sequence $\{x_n\}$ $(n \geq 0)$ generated by (INMF) is well defined, remains in $U(x^\star, r_0)$ for all $n \geq 0$, and converges to $x^\star$, provided that $x_0 \in U(x^\star, r_0)$.

Moreover, the following estimate holds:

$$\| x_{n+1} - x^\star \| \leq \alpha \;\; \| x_n - x^\star \|, \tag{2.6}$$

where,

$$\alpha = (1 + v) \, \frac{\displaystyle\int_0^{\rho(x_0)} L(t) \, dt}{1 - \displaystyle\int_0^{\rho(x_0)} L_0(t) \, dt} + v \leq 1. \tag{2.7}$$

Proof. By hypothesis $x_0 \in U(x^\star, r_0)$, and $\alpha \in [0, 1)$, since, by (2.5), and the positivity of L_0, L:

$$\alpha < (1 + v) \, \frac{\displaystyle\int_0^{r_0} L(t) \, dt}{1 - \displaystyle\int_0^{r_0} L_0(t) \, dt} + v \leq 1.$$

Let us assume $x_m \in U(x^\star, r_0)$, $m \leq n$, we shall show (2.6), and $x_{m+1} \in U(x^\star, r_0)$ hold, for all m.

In view of (2.2), (2.5), and the induction hypotheses, we get:

$$\| F'(x^\star)^{-1} \, (F(x_m) - F(x_0)) \| \leq \int_0^{\rho(x_m)} L_0(t) \, dt \leq \int_0^{r_0} L_0(t) \, dt < 1. \tag{2.8}$$

It follows from the Banach lemma on invertible operators [1], [5], that $F'(x_m)^{-1}$ exists, and

$$\| F'(x_m)^{-1} \, F'(x^\star) \| \leq \left(1 - \int_0^{\rho(x_m)} L_0(t) \, dt\right)^{-1} \leq \left(1 - \int_0^{r_0} L_0(t) \, dt\right)^{-1}. \tag{2.9}$$

Using (INMF), we obtain the approximation:

$$\begin{aligned} x_{m+1} - x^\star &= x_m - x^\star - F'(x_m)^{-1} \, (F(x_m) - F(x^\star) - r_m) \\ &= F'(x_m)^{-1} \, F'(x^\star) \int_0^1 F'(x^\star)^{-1} \, (F(x_m) - F(x_\theta)) \, (x_m - x^\star) \, d\theta + \\ &\quad F'(x_m) \, \mathcal{P}_m^{-1} \, \mathcal{P}_m \, r_m. \end{aligned} \tag{2.10}$$

By (2.3), (2.4), (2.7), (2.9), (2.10), and the induction hypotheses, we obtain in turn:

$$\begin{aligned} \| x_{m+1} - x^\star \| \leq\ & \| F'(x_m)^{-1} \, F'(x^\star) \| \times \\ & \int_0^1 \| F'(x^\star)^{-1} \, (F(x_m) - F(x_\theta)) \| \, \| x_m - x^\star \| \, d\theta + \\ & \theta_m \, \| (\mathcal{P}_m \, F'(x_m))^{-1} \| \, \| \mathcal{P}_m \, F(x_m) \| \\ \leq\ & \frac{1}{1 - \int_0^{\rho(x_m)} L_0(t) \, dt} \int_0^1 \int_{\theta\,\rho(x_m)}^{\rho(x_m)} L(t) \, dt \, \rho(x_m) \, d\theta + \\ & \theta_m \, \| (\mathcal{P}_m \, F'(x_m))^{-1} \| \, \| \mathcal{P}_m \, F'(x_m) \, F'(x_m)^{-1} \, F(x_m) \| \\ \leq\ & \frac{\int_0^{\rho(x_m)} L(t) \, t \, dt}{1 - \int_0^{\rho(x_m)} L_0(t) \, dt} + \\ & \theta_m \, \text{cond} \, (\mathcal{P}_m \, F'(x_m)) \left(\| x_m - x^\star \| + \frac{\int_0^{\rho(x_m)} L(t) \, dt}{1 - \int_0^{\rho(x_m)} L_0(t) \, dt} \right). \end{aligned} \tag{2.11}$$

Then

$$\begin{aligned}
\| x_{m+1} - x^\star \| &\leq (1+v_m) \frac{\displaystyle\int_0^{\rho(x_m)} L(t)\ t\ dt}{1 - \displaystyle\int_0^{\rho(x_m)} L_0(t)\ dt} + v_m\ \rho(x_m) \\
&\leq \left((1+v_m) \frac{\displaystyle\int_0^{\rho(x_m)} L(t)\ t\ dt}{1 - \displaystyle\int_0^{\rho(x_m)} L_0(t)\ dt} + v_m \right) \rho(x_m) \\
&\leq \alpha\ \| x_m - x^\star \| < \| x_m - x^\star \|,
\end{aligned} \tag{2.12}$$

which implies $x_{m+1} \in U(x^\star, r_0)$, and $\lim\limits_{m \longrightarrow \infty} x_m = x^\star$.

That completes the proof of Theorem 2.1. □

Proposition 2.2. *Under hypotheses (2.2)–(2.4) of Theorem 2.1, further assume:*

function

$$L_c(t) = t^{1-c}\ L(t) \tag{2.13}$$

is nondecresing for some $c \in [0,1]$;

there exists $r_1 > 0$, such that:

$$U(x^\star, r_1) \subseteq \mathcal{D}, \tag{2.14}$$

$$\frac{(1+v) \displaystyle\int_0^{r_1} L(t)\ t\ dt}{r_1 \left(1 - \displaystyle\int_0^{r_1} L_0(t)\ dt \right)} + v \leq 1. \tag{2.15}$$

Then, sequence $\{x_n\}$ $(n \geq 0)$ generated by (INMF) is well defined, remains in $U(x^\star, r_1)$ for all $n \geq 0$, and converges to $x^\star$, provided that $x_0 \in \overline{U}(x^\star, r_1)$.

Moreover, the following estimate holds:

$$\| x_{n+1} - x^\star \| \leq \beta_n\ \ \| x_n - x^\star \|, \tag{2.16}$$

where,

$$\begin{aligned}
\beta_n &= (1+v) \frac{\displaystyle\int_0^{\rho(x_0)} L(t)\ t\ dt\ \rho(x_n)^c}{\rho(x_0)^{1+c} \left(1 - \displaystyle\int_0^{\rho(x_0)} L_0(t)\ dt \right)} + v \\
&\leq \beta = (1+v) \frac{\displaystyle\int_0^{\rho(x_0)} L(t)\ t\ dt}{\rho(x_0) \left(1 - \displaystyle\int_0^{\rho(x_0)} L_0(t)\ dt \right)} + v < 1.
\end{aligned} \tag{2.17}$$

Proof. We follow the proof of Theorem 2.1 until (2.10). Define function $f_{d,c}$, $(d \geq 0)$, by:

$$f_{d,c}(s) = \frac{1}{s^{c+d}} \int_0^s t^d \ L(t) \ dt. \tag{2.18}$$

In view of Lemma 2.2 in [15], function $f_{d,c}$ is non–decreasing. It then follows in turn:

$$\begin{aligned}
\| x_{m+1} - x^\star \| &\leq (1+v_m) \frac{\displaystyle\int_0^{\rho(x_m)} L(t) \ t \ dt}{1 - \displaystyle\int_0^{\rho(x_m)} L_0(t) \ dt} + v_m \ \rho(x_m) \\
&= (1+v_m) \frac{f_{1,c}(\rho(x_m))}{1 - \displaystyle\int_0^{\rho(x_m)} L_0(t) \ dt} \ \rho(x_m)^{1+c} + v_m \ \rho(x_m) \\
&\leq 1+v_m) \frac{f_{1,c}(\rho(x_0))}{1 - \displaystyle\int_0^{\rho(x_0)} L_0(t) \ dt} \ \rho(x_m)^{1+c} + v_m \ \rho(x_m) \\
&\leq \beta_m \ \| x_m - x^\star \| \leq \beta \ \| x_m - x^\star \| < \| x_m - x^\star \|,
\end{aligned} \tag{2.19}$$

which implies $x_{m+1} \in U(x^\star, r_1)$, and $\lim\limits_{m \longrightarrow \infty} x_m = x^\star$.

That completes the proof of Proposition 2.2. □

Theorem 2.3. *Under hypotheses (2.1)–(2.4) of Thorem 2.1 (for $r_2 > 0$, replacing r_0), further assume:*

$$\| \mathcal{B}(x)^{-1} \ F'(x) \| \leq w_1, \tag{2.20}$$

$$\| \mathcal{B}(x)^{-1} \ (F'(x) - \mathcal{B}(x)) \| \leq w_2, \tag{2.21}$$

hold for all $x \in U(x^\star, r_2)$,

and

$$(1+v) \ w_1 \int_0^{r_2} L(t) \ dt + (1 - w_2 - w_1 \ v) \int_0^{r_2} L_0(t) \ dt \leq 1 - w_2 - w_1 \ v. \tag{2.22}$$

Then, sequence $\{x_n\}$ $(n \geq 0)$ generated by (INMB) is well defined, remains in $U(x^\star, r_2)$ for all $n \geq 0$, and converges to $x^\star$, provided that $x_0 \in U(x^\star, r_2)$.

Moreover, the following estimate holds:

$$\| x_{n+1} - x^\star \| \leq \gamma \ \| x_n - x^\star \|, \tag{2.23}$$

where,

$$\gamma = (1+v) \frac{w_1 \displaystyle\int_0^{\rho(x_0)} L(t) \ dt}{1 - \displaystyle\int_0^{\rho(x_0)} L_0(t) \ dt} + w_2 + w_1 \ v < 1. \tag{2.24}$$

Proof. Using the properties of functions L_0, L, (2.12), and (2.24), we obtain $\gamma \in (0,1)$. By (INMB), if $x_m \in U(x^\star, r_2)$, we have the approximation:

$$\begin{aligned} x_{m+1} - x^\star &= x_m - x^\star - \mathcal{B}_m^{-1}\ (F(x_m) - F(x^\star)) + \mathcal{B}_m^{-1}\ r_m \\ &= -\mathcal{B}_m^{-1}\ F'(x_m) \int_0^1 F'(x_m)^{-1}\ F'(x^\star)\ F'(x^\star)^{-1}\ (F'(x_m) - F'(x_\theta)) \\ &\quad (x_m - x^\star)\ d\theta + \mathcal{B}_m^{-1}\ (F'(x_m) - \mathcal{B}_m)\ (x_m - x^\star) + \mathcal{B}_m^{-1}\ \mathcal{P}_m^{-1}\ \mathcal{P}_m\ r_m. \end{aligned} \tag{2.25}$$

In view of (2.2)–(2.4), (2.9), (2.20), and the induction hypotheses, we obtain in turn:

$$\begin{aligned} \| x_{m+1} - x^\star \| &\leq \| \mathcal{B}_m^{-1}\ F'(x_m) \| \times \\ &\quad \int_0^1 \| F'(x_m)^{-1}\ F'(x^\star) \|\ \| F'(x^\star)^{-1}\ (F'(x_m) - F'(x_\theta)) \| \\ &\quad \| x_m - x^\star \|\ \ d\theta + \| \mathcal{B}_m^{-1}\ (F'(x_m) - \mathcal{B}_m) \|\ \| x_m - x^\star \| + \\ &\quad \theta_m\ \ \| (\mathcal{P}_m\, \mathcal{B}_m)^{-1} \|\ \| \mathcal{B}_m\, F(x_m) \| \\ &\leq \frac{w_1}{1 - \displaystyle\int_0^{\rho(x_m)} L_0(t)\ dt} \int_0^1 \int_{\theta\,\rho(x_m)}^{\rho(x_m)} L(t)\ dt\ \rho(x_m)\ d\theta + w_2\ \rho(x_m) + \\ &\quad \theta_m\ \ \| \mathcal{P}_m^{-1}\, F'(x_m) \|\ \| (\mathcal{P}_m\, F'(x_m))^{-1} \| \\ &\quad \| \mathcal{P}_m\ F'(x_m) \|\ \| F'(x_m)^{-1}\ F(x_m) \| \\ &\leq \frac{w_1 \displaystyle\int_0^{\rho(x_m)} L(t)\ t\ dt}{1 - \displaystyle\int_0^{\rho(x_m)} L_0(t)\ dt} + w_2\ \rho(x_m) + \\ &\quad w_1\ v_m \left(\rho(x_m) + \frac{\displaystyle\int_0^{\rho(x_m)} L(t)\ t\ dt}{1 - \displaystyle\int_0^{\rho(x_m)} L_0(t)\ dt} \right) \\ &\leq (1 + v_m)\ \frac{w_1 \displaystyle\int_0^{\rho(x_m)} L(t)\ t\ dt}{1 - \displaystyle\int_0^{\rho(x_m)} L_0(t)\ dt} + (w_2 + w_1\ v_m)\ \rho(x_m) \\ &\leq \left((1 + v_m)\ \frac{w_1 \displaystyle\int_0^{\rho(x_m)} L(t)\ dt}{1 - \displaystyle\int_0^{\rho(x_m)} L_0(t)\ dt} + w_2 + w_1\ v_m \right) \rho(x_m) \\ &\leq \gamma\ \ \| x_m - x^\star \| < \| x_m - x^\star \|, \end{aligned} \tag{2.26}$$

which implies $x_{m+1} \in \overline{U}(x^\star, r_2)$, and $\lim\limits_{m \longrightarrow \infty} x_m = x^\star$.

That completes the proof of Theorem 2.3. □

Proposition 2.4. *Under hypotheses (2.1)–(2.4) (for $r_3 > 0$, replacing r_0), (2.13), (2.20), (2.21) (for r_3, replacing r_2), further assume r_3 satisfies:*

$$(1+v)\ \frac{w_1 \displaystyle\int_0^{r_3} L(t)\ t\ dt}{r_3 \left(1 - \displaystyle\int_0^{r_3} L_0(t)\ dt\right)} + w_2 + w_1\ v \leq 1. \tag{2.27}$$

Then, sequence $\{x_n\}$ $(n \geq 0)$ generated by (INMB) is well defined, remains in $U(x^\star, r_3)$ for all $n \geq 0$, and converges to $x^\star$.

Moreover, the following estimate holds:

$$\| x_{n+1} - x^\star \| \leq \delta\ \ \| x_n - x^\star \|, \tag{2.28}$$

where,

$$\delta = (1+v)\ \frac{w_1 \displaystyle\int_0^{\rho(x_0)} L(t)\ t\ dt}{\rho(x_0) \left(1 - \displaystyle\int_0^{\rho(x_0)} L_0(t)\ dt\right)} + w_2 + w_1\ v < 1. \tag{2.29}$$

Proof. Using the properties of functions L, L_0, (2.27), and (2.29), we deduce $\delta \in (0,1)$. If $x_m \in U(x^\star, r_3)$, as in Proposition 2.2, using (2.26), we get in turn:

$$\begin{aligned}
\| x_{m+1} - x^\star \| &\leq (1+v_m)\ \frac{w_1 \displaystyle\int_0^{\rho(x_m)} L(t)\ t\ dt}{1 - \displaystyle\int_0^{\rho(x_m)} L_0(t)\ dt} + (w_2 + w_1\ v_m)\ \rho(x_m) \\
&\leq (1+v)\ \frac{w_1\ f_{1,c}(\rho(x_m))}{1 - \displaystyle\int_0^{\rho(x_m)} L_0(t)\ dt}\ \rho(x_m)^{1+c} + (w_2 + w_1\ v)\ \rho(x_m) \\
&\leq (1+v)\ \frac{w_1\ f_{1,c}(\rho(x_0))}{1 - \displaystyle\int_0^{\rho(x_0)} L_0(t)\ dt}\ \rho(x_m)^{1+c} + (w_2 + w_1\ v)\ \rho(x_m) \\
&\leq \delta\ \ \| x_m - x^\star \| < \| x_m - x^\star \|,
\end{aligned}$$

which implies $x_{m+1} \in U(x^\star, r_3)$, and $\lim\limits_{m \longrightarrow \infty} x_m = x^\star$.

That completes the proof of Proposition 2.4. □

Remark 2.5. Note that in general

$$L_0(t) \leq L(t) \qquad t \leq 0, \tag{2.30}$$

and $\dfrac{L(t)}{L_0(t)}$ can be arbitrarily large [4], [6].

In order for us to compare our results with relevant ones already in the literature, let us consider the case $\mathcal{B}_n = F'(x_n)$ $(n \geq 0)$, $v = w_1 = w_2 = 0$. Then the convergence radii in [15], and [8] (for $c = 1$) are given by

$$\int_0^r L(t)\ dt \leq \frac{1}{2}, \tag{2.31}$$

and

$$\frac{1}{\overline{r}} \int_0^{\overline{r}} (\overline{r} + t)\ L(t)\ dt \leq 1, \tag{2.32}$$

respectively.

In view of (2.5), (2.15), (2.30), (2.31), and (2.32), we obtain:

$$r \leq r_0, \tag{2.33}$$

and

$$\overline{r} \leq r_1. \tag{2.34}$$

In case strict inequality holds in (2.30), then so does in (2.33), and (2.34).

Moreover, the error estimates (2.6), and (2.16) are finer (smaller) than the corresponding ones in [8], and [15]. Note that these advantages are obtained under the same computational cost, since in practice, the computation of function L requires that of L_0.

Remark 2.6. The local results obtained here can be used for projection methods such as Arnoldi's, the generalized minimum residual method (GMRES), the generalized conjugate residual method (GCR), for combined Newton/finite projection methods and in connection with the mesh independence principle to develop the cheapest and most efficient mesh refinement strategies [5].

Remark 2.7. The local results can also be used to solve equations of the form $F(x) = 0$, where F' satisfies the autonomous differential equation [5]:

$$F'(x) = P(F(x)), \tag{2.35}$$

where, $P:\ \mathcal{Y} \to \mathcal{X}$ is a known continuous operator. We have $F'(x^\star) = P(F(x^\star)) = P(0)$, so we can apply our results without actually knowing the solution $x^\star$ of equation (1.1).

Example 2.8. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, and define function F on $\mathcal{D} = U(0,1)$ by:

$$F(x) = e^x - 1. \tag{2.36}$$

Then we can set $P(x) = x + 1$ in (2.35). Using (2.2), (2.3), and (2.36), we have:

$$L_0(t) = \ell_0 = e - 1, \quad \text{and} \quad L(t) = \ell = e. \tag{2.37}$$

In view of (2.5), (2.31), and (2.37), we get:

$$r = \frac{1}{2\ \ell} < r_0 = \frac{1}{\ell_0 + \ell}, \tag{2.38}$$

and, in particular

$$r = .183939721 < r_0 = .225399674.$$

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