J. Nonlinear Sci. Appl. 2 (2009), no. 2, 78–91

THE JOURNAL OF NONLINEAR SCIENCE AND APPLICATIONS
http://www.tjnsa.com

STRONG CONVERGENCE THEOREMS FOR EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS OF STRICT PSEUDO-CONTRACTION MAPPINGS

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ABSTRACT. The purpose of this paper is to introduce an iterative scheme for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a $k-$strict pseudo-contraction non-self mapping in Hilbert space. By the viscosity approximation algorithms, under suitable conditions , some strong convergence theorems for approximating to this common elements are proved. The results presented in the paper extend and improve some recent results of Marino and Xu [G.Marino,H.K.Xu, Weak and strong convergence theorems for $k-$strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl. 329 (2007) 336–349], Zhou [H.Zhou, Convergence theorems of fixed Points for $k-$strict pseudo-contractions in Hilbert spaces, Nonlinear Anal. 69 (2008) 456–462], Takahashi and Takahashi [S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 331 (2007) 506–515], Ceng,Homidan,etc [L. C. Ceng, S.A.Homidan, Q.H.Ansari, J. C. Yao, An iterative scheme for equilibrium problems and fixed point problems of strict pseudo-contraction mappings, J. Comput. Appl. Math. 223 (2009) 967–974].

1. INTRODUCTION

Throughout this paper, we assume that H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $||\cdot||$. C is a nonempty closed convex subset of H . Let $\phi : C\times C\to\mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem (for short, EP) is to find $x\in C$ such that

$$\phi(x,y)\geq 0,\qquad \forall\, y\in C. \tag{1.1}$$

Date: Received: 24 February 2009. Revised 8 March 2009.

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2000 *Mathematics Subject Classification.* 47H05. 47H09. 47H10.

Key words and phrases. Equilibrium problem, strict pseudo-contraction mapping, fixed point, strong convergence theorem.

The set of solutions of (1.1) is denoted by $EP(\phi)$. Given a mapping $T : C \to H$, let $\phi(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then $x \in EP(\phi)$ if and only if $x \in C$ is a solution of the variational inequality $\langle Tx, y - x\rangle \geq 0$ for all $y \in C$. In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an EP. In other words, the EP is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. In the last two decades, many papers have appeared in the literature on the existence of solutions of EP; see, for example [1,8,9] and references therein. Some solution methods have been proposed to solve the EP; see, for example [6,7,20,21] and references therein. Motivated by the work in [6,14,20],Takahashi and Takahashi [21] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of the EP(1.1) and the set of the fixed points of a nonexpansive mapping in the setting of Hilbert spaces. They also studied the strong convergence of the sequences generated by their algorithm for a solution of the EP which is also a fixed point of a nonexpansive mapping defined on a closed convex subset of a Hilbert space.

Very recently, Ceng, Homidan, Ansari and Yao [4] introduced an iterative scheme for finding a common element of the set of solutions of the EP(1.1) and the set of the fixed points of a $k-$strict pseudo-contraction self-mapping in the setting of real Hilbert spaces. They proved some weak and strong convergence theorems of the sequences generated by their proposed scheme.

Recall that a mapping $f : H \to H$ is said to be contractive if there exists a constant $\alpha \in (0, 1)$ such that for all $x, y \in H$

$$\|f(x) - f(y)\| \leq \alpha||x - y||.$$

Let A be a strongly positive bounded linear operator on H, that is,there exists a constant $\widetilde{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \widetilde{\gamma}||x||^2, \quad \forall\, x \in H.$$

A mapping $T : H \to H$ is called nonexpansive, if such that

$$||Tx - Ty|| \leq ||x - y||, \quad \forall\, x, y \in H.$$

We denote by $F(T)$the set of all fixed points of T,that is $F(T) = \{x \in H : Tx = x\}$. The mapping $T : C \to H$ is called a $k-$strict pseudo-contraction if there exists a constant $k \in [0, 1)$ such that

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||(I - T)x - (I - T)y||^2 \tag{1.2}$$

for all $x, y \in C$. When $k = 0$, T is said to be nonexpansive, and it is said to be pseudo-contractive if $k = 1$. T is said to be strongly pseudo-contractive if there exists a positive constant $\lambda \in (0, 1)$ such that $T-\lambda I$ is pseudo-contractive. Cleary, the class of $k-$strict pseudo-contractions falls into the one between classes of nonexpansive mappings and pseudo-contractions. We remark also that the class of strongly pseudo-contractive mappings is independent of the class of $k-$strict pseudo-contractions (see [2,3]).

It is very clear that,in a real Hilbert space H,(1.2) is equivalent to

$$\langle Tx - Ty, x - y\rangle \le ||x-y||^2 - \frac{1-k}{2}||(x-Tx)-(y-Ty)||^2 \tag{1.3}$$

for all $x, y \in C$.

Recall that the normal Mann's iterative algorithm was introduced by Mann[12] in 1953. Since then,construction of fixed points for nonexpansive mappings and $k-$strict pseudo-contractions via the normal Mann's iterative algorithm has been extensively investigated by many authors (see, e.g.,[2,3,12,13,15,22]). Reich [17] showed that the conclusion also holds good in the setting of uniformly convex Banach spaces with a *Fréchet* differentiable norm.It is well known that Reich's result is one of the fundamental convergence results.Recently, Marino and Xu [13] extended Reich's result [17] to strict pseudo-contraction mappings in the setting of Hilbert spaces.

Very recently, Zhou [25] modified normal Mann's iterative process for non-self $k-$strict pseudo-contractions to have strong convergence in Hilbert spaces.

Motivated and inspired by Ceng, Homidan, Ansari and Yao [4], Marino and Xu [13], Takahashi and Takahashi [21], Zhou [25], the purpose of this paper is to introduce an iterative scheme for finding a common element of the set of solutions of equilibrium problem (1.1) and the set of fixed points of a $k-$strict pseudo-contraction non-self mapping in Hilbert space. By the viscosity approximation algorithms, under suitable conditions , some strong convergence theorems for approximating to this common elements are proved. The results presented in the paper extend and improve some recent results of Ceng, Homidan, Ansari and Yao [4], Kim and Xu [11], Marino and Xu [13], Moudafi [14], Takahashi and Takahashi [21], Wittmann [23], Zhou [25].

2. PRELIMINARIES

In the sequel, we use $x_n \rightharpoonup x$ and $x_n \to x$ to denote the weak convergence and strong convergence of the sequence $\{x_n\}$ in H, respectively. Let H be a real Hilbert space, C be a nonempty closed convex subset of H. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$||x - P_C x|| \le ||x - y||, \quad \forall\, y \in C.$$

Such a mapping P_C from H onto C is called the *metric projection*.

Remark 1 It is wellknown that the metric projection P_C has the following properties:

(1) P_C is firmly nonexpansive. i.e.,

$$||P_C x - P_C y||^2 \le \langle P_C x - P_C y,\ x - y\rangle, \quad \forall\, x, y \in H,$$

(2) For each $x \in H$,

$$z = P_C(x) \Leftrightarrow \langle x - z,\ z - y\rangle \ge 0, \quad \forall\, y \in C.$$

A spave X is said to satisfy the Opial condition if for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y||, \quad \forall\, y \in X,\ y \ne x.$$

Lemma 2.1. *([13]) Let H be a real Hilbert space. There hold the following identities:*

$$||tx + (1-t)y||^2 = t||x||^2 + (1-t)||y||^2 - t(1-t)||x-y||^2,\ \forall\ t \in [0,1],$$

for all $x, y \in H$.

Lemma 2.2. *([19]) Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that*

$$x_{n+1} = (1-\beta_n)z_n + \beta_n x_n,$$

for all integers $n \geq 1$ and

$$\limsup_{n\to\infty}(||z_{n+1} - z_n|| - ||x_{n+1} - x_n||) \leq 0.$$

Then, $\lim_{n\to\infty} ||z_n - x_n|| = 0$.

Lemma 2.3. *([24]) Assume $\{a_n\}$is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\varrho_n)a_n + \epsilon_n, \quad \forall\, n \geq n_0,$$

where n_0 is some nonnegative integer, $\{\varrho_n\}$ is a sequence in $(0,1)$ and $\{\epsilon_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \varrho_n = \infty$;
(2) $\limsup_{n\to\infty} \epsilon_n/\varrho_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\epsilon_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. *([13,25]) If T is a $k-$strict pseudo-contraction on closed convex subset C of a real Hilbert space H , then the fixed point set $F(T)$ is closed convex so that the projection $P_{F(T)}$ is well defined.*

Lemma 2.5. *([3]) Let $T : C \to H$ be a $k-$strict pseudo-contraction. Define $S : C \to H$ by $Sx = \lambda x + (1-\lambda)Tx$ for each $x \in C$. Then, as $\lambda \in [k,1)$,S is a nonexpansive mapping such that $F(S) = F(T)$.*

Lemma 2.6. *([5]) let E be a real Banach space, $J : E \to 2^{E^*}$ be the normalized duality mapping , Then for any $x, y \in E$. the following conclusion holds:*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x+y)\rangle, \quad \forall\, j(x+y) \in J(x+y)$$

Especially, if $E = H$ is a real Hilbert space, then

$$||x+y||^2 \leq ||x||^2 + 2\langle y, x+y\rangle, \quad \forall\, x, y \in H.$$

For solving the equilibrium problem, we assume that the bifunction $\phi : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $\phi(x,x) = 0,\ \ \forall\, x \in C$,
(A2) ϕ is monotone, that is, $\phi(x,y) + \phi(y,x) \leq 0,\ \forall\, x, y \in C$,
(A3) For all $x, y, z \in C$,

$$\limsup_{t\downarrow 0} \phi(tz + (1-t)x, y) \leq \phi(x,y),\ \forall\, x, y, z \in C,$$

(A4) For all $x \in C$, the function $y \mapsto \phi(x, y)$ is convex and lower semicontinuous.

Lemma 2.7. *([6,21]) Let C be a nonempty closed convex subset of a H and let $\phi : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$\phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall\, y \in C.$$

Lemma 2.8. *([6]) Assume that $\phi : C \times C \to \mathbb{R}$ satisfying $(A1) - (A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : \phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall\, y \in C\}.$$

Then,

(1) *T_r is single-valued,*
(2) *T_r is firmly nonexpansive, that is, $\forall\, x, y \in H$,*

$$||T_r x - T_r y||^2 \leq \langle T_r x - T_r y, x - y\rangle,$$

(3) *$F(T_r) = EP(\phi)$,*
(4) *$EP(\phi)$ is nonempty, closed and convex.*

3. Main results

Theorem 3.1. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H. Let $\phi : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$. Let A be a strongly positive linear bounded operator on H with coefficient $\widetilde{\gamma} > 0$ such that $0 < \gamma < \widetilde{\gamma}/\alpha$. $T : C \to H$ be a $k-$strictly pseudo-contractive nonself-mapping such that $F(T) \cap EP(\phi) \neq \emptyset$, and $f : H \to H$ be a contractive mapping with a contractive constant $\alpha \in (0, 1)$. For any given $x_1 \in H$, let $\{x_n\}$ and $\{u_n\}$ be the iterative sequence defined by*

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall\, y \in C, \\ y_n = \delta_n u_n + (1 - \delta_n)T u_n, & \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)y_n, & \forall\, n \geq 1. \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$ are three sequences in $[0, 1]$ and $r_n \subset (0, \infty)$. If the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$; $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $k \leq \delta_n \leq \lambda < 1$ *for all* $n \geq 1$ *and* $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$,
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(iv) $\liminf_{n\to\infty} r_n > 0$, $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Then $\{x_n\}$ and $\{u_n\}$ cnoverge strongly to $p \in F(T) \cap EP(\phi)$, where $p = P_{F(T)\cap EP(\phi)}(I - A + \gamma f)(p)$.

Proof. We divide the proof of Theorem 3.1 into seven steps:

(I) First prove that there exists $x^* \in C$, such that $x^* = P_{F(T)\cap EP(\phi)}(I - A + \gamma f)(x^*)$.

Note that for the control conditions (i) and (iii), we may assume, without loss of generality,that $\alpha_n \leq (1-\beta_n)||A||^{-1}$. Since A is linear bounded self-adjoint operator on H, then

$$||A|| = \sup\{|\langle Au, u\rangle| : u \in H, ||u|| = 1\}.$$

Observe that

$$\begin{aligned}\langle((1-\beta_n)I - \alpha_n A)u, u\rangle &= 1 - \beta_n - \alpha_n\langle Au, u\rangle \\ &\geq 1 - \beta_n - \alpha_n||A|| \\ &\geq 0.\end{aligned}$$

that is to say $(1-\beta_n)I - \alpha_n A$ is positive. It follows that

$$\begin{aligned}||(1-\beta_n)I - \alpha_n A|| &= \sup\{\langle((1-\beta_n)I - \alpha_n A)u, u\rangle : u \in H, ||u|| = 1\} \\ &= \sup\{1 - \beta_n - \alpha_n\langle Au, u\rangle : u \in H, ||u|| = 1\} \\ &\leq 1 - \beta_n - \alpha_n\widetilde{\gamma}.\end{aligned}$$

Since f is a contraction with coefficient $\alpha \in (0, 1)$. Then, we have

$$\begin{aligned}&||P_{F(T)\cap EP(\phi)}(I - A + \gamma f)(x) - P_{F(T)\cap EP(\phi)}(I - A + \gamma f)(y)|| \\ &\qquad \leq ||(I - A + \gamma f)(x) - (I - A + \gamma f)(y)|| \\ &\qquad \leq ||I - A||\|x - y\| + \gamma||f(x) - f(y)|| \\ &\qquad \leq (1 - \widetilde{\gamma})||x - y|| + \gamma\alpha||x - y|| \\ &\qquad = (1 - (\widetilde{\gamma} - \gamma\alpha))||x - y||.\end{aligned}$$

for all $x, y \in H$. Therefore, $P_{F(T)\cap EP(\phi)}(I - A + \gamma f)$ is also a contraction, By the Banach theorem, there exists a unique element $x^* \in C$ such that

$$x^* = P_{F(T)\cap EP(\phi)}(I - A + \gamma f)(x^*).$$

(II) Now we prove that the sequences $\{x_n\}$ and $\{u_n\}$ is bounded.

Let $p \in F(T) \cap EP(\phi)$. From the definition of T_r, we note that $u_n = T_{r_n}x_n$. It follows that

$$\begin{aligned}||u_n - p|| &= ||T_{r_n}x_n - T_{r_n}p|| \\ &\leq ||x_n - p||.\end{aligned} \tag{3.2}$$

From (3.1) and (3.2) we obtain

$$\begin{aligned}||y_n - p||^2 &= ||P_C[\delta_n u_n + (1-\delta_n)Tu_n] - p||^2 \\ &\leq ||\delta_n(u_n - p) + (1-\delta_n)(Tu_n - p)||^2 \\ &= \delta_n||u_n - p||^2 + (1-\delta_n)||Tu_n - p||^2 - \delta_n(1-\delta_n)||Tu_n - u_n||^2 \\ &\leq ||u_n - p||^2 - (1-\delta_n)(\delta_n - k)||Tu_n - u_n||^2 \\ &\leq ||u_n - p||^2 \leq ||x_n - p||^2.\end{aligned} \tag{3.3}$$

Hence from (3.1) and (3.3) we have

$$\begin{aligned}
||x_{n+1}-p|| &= ||\alpha_n(\gamma f(x_n)-Ap)+\beta_n(x_n-p)+((1-\beta_n)I-\alpha_n A)(y_n-p)|| \\
&\le (1-\beta_n-\alpha_n\widetilde{\gamma})||y_n-p||+\beta_n||x_n-p||+\alpha_n||\gamma f(x_n)-p|| \\
&\le (1-\beta_n-\alpha_n\widetilde{\gamma})||x_n-p||+\beta_n||x_n-p|| \\
&\quad +\alpha_n\gamma||f(x_n)-f(p)||+\alpha_n||\gamma f(p)-Ap|| \\
&\le (1-\alpha_n\widetilde{\gamma})||x_n-p||+\alpha_n\gamma\alpha||x_n-p||+\alpha_n||\gamma f(p)-Ap|| \\
&\le (1-(\widetilde{\gamma}-\gamma\alpha)\alpha_n)||x_n-p||+\alpha_n||\gamma f(p)-Ap|| \\
&\le \max\{||x_n-p||,\frac{1}{\widetilde{\gamma}-\gamma\alpha}||\gamma f(p)-Ap||\} \\
&\le \cdots \\
&\le \max\{||x_1-p||,\frac{1}{\widetilde{\gamma}-\gamma\alpha}||\gamma f(p)-Ap||\}, \quad \forall\, n\ge 1.
\end{aligned}$$

This implies that $\{x_n\}$ is a bounded sequence in H, and so $\{u_n\}$, $\{Tu_n\}$, $\{y_n\}$, $\{Ay_n\}$ and $\{f(x_n)\}$ are bounded sequences in H.

(III) Next we prove that $||x_{n+1}-x_n||\to 0$.

In fact, from $u_n=T_{r_n}x_n$ and $u_{n+1}=T_{r_{n+1}}x_{n+1}$, we have

$$\phi(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0,\quad \forall\, y\in C, \tag{3.4}$$

and

$$\phi(u_{n+1},y)+\frac{1}{r_{n+1}}\langle y-u_{n+1},u_{n+1}-x_{n+1}\rangle\ge 0,\ \forall\, y\in C, \tag{3.5}$$

Putting $y=u_{n+1}$ in (3.4) and $y=u_n$ in (3.5), we have

$$\phi(u_n,u_{n+1})+\frac{1}{r_n}\langle u_{n+1}-u_n,u_n-x_n\rangle\ge 0,$$

and

$$\phi(u_{n+1},u_n)+\frac{1}{r_{n+1}}\langle u_n-u_{n+1},u_{n+1}-x_{n+1}\rangle\ge 0.$$

It follows from (A2) that

$$\langle u_{n+1}-u_n,\frac{u_n-x_n}{r_n}-\frac{u_{n+1}-x_{n+1}}{r_{n+1}}\rangle\ge 0.$$

That is

$$\langle u_{n+1}-u_n,u_n-u_{n+1}+u_{n+1}-x_n-\frac{r_n}{r_{n+1}}(u_{n+1}-x_{n+1})\rangle\ge 0.$$

This implies that

$$\begin{aligned}
||u_{n+1}-u_n||^2 &\le \langle u_{n+1}-u_n,x_{n+1}-x_n+\left(1-\frac{r_n}{r_{n+1}}\right)(u_{n+1}-x_{n+1})\rangle \\
&\le ||u_{n+1}-u_n||\left\{||x_{n+1}-x_n||+|1-\frac{r_n}{r_{n+1}}|\cdot||u_{n+1}-x_{n+1}||\right\}.
\end{aligned}$$

Since $\liminf_{n\to\infty} r_n > 0$, without loss of generality, we may assume that there exists a real number h such that $r_n > h > 0$, for all$n \geq 1$.Then, we have

$$\begin{aligned} ||u_{n+1} - u_n|| &\leq ||x_{n+1} - x_n|| + |1 - \frac{r_n}{r_{n+1}}| \cdot ||u_{n+1} - x_{n+1}|| \\ &\leq ||x_{n+1} - x_n|| + \frac{M}{h}|r_{n+1} - r_n|. \end{aligned} \tag{3.6}$$

where $M = \sup_{n\geq 1}\{||u_n - x_n||\}$.

Define a mapping $T_n x := \delta_n x + (1 - \delta_n)Tx$ for each $x \in C$. Then $T_n : C \to H$ is nonexpansive. Indeed,by using (1.2), (3.1), Lemma 2.1 and condition (ii),we have for all $x, y \in C$ that

$$\begin{aligned} &||T_n x - T_n y||^2 \\ &\leq ||\delta_n(x - y) + (1 - \delta_n)(Tx - Ty)||^2 \\ &= \delta_n||x - y||^2 + (1 - \delta_n)||Tx - Ty||^2 - \delta_n(1 - \delta_n)||x - Tx - (y - Ty)||^2 \\ &\leq \delta_n||x - y||^2 + (1 - \delta_n)\left[||x - y||^2 + k||x - Tx - (y - Ty)||^2\right] \\ &\quad - \delta_n(1 - \delta_n)||x - Tx - (y - Ty)||^2 \\ &= ||x - y||^2 - (1 - \delta_n)(\delta_n - k)||x - Tx - (y - Ty)||^2 \\ &\leq ||x - y||^2, \end{aligned}$$

which implies that $T_n : C \to H$ is nonexpansive.

By using (3.1) and noting that T_n is nonexpansive, we have

$$\begin{aligned} ||y_{n+1} - y_n|| &= ||T_{n+1}u_{n+1} - T_n u_n|| \\ &= ||T_{n+1}u_{n+1} - T_{n+1}u_n + T_{n+1}u_n - T_n u_n|| \\ &\leq ||u_{n+1} - u_n|| + ||T_{n+1}u_n - T_n u_n|| \\ &= ||u_{n+1} - u_n|| + ||\delta_{n+1}u_n + (1 - \delta_{n+1})Tu_n \\ &\quad - (\delta_n u_n + (1 - \delta_n)Tu_n)|| \\ &\leq ||u_{n+1} - u_n|| + |\delta_{n+1} - \delta_n|||u_n - Tu_n|| \\ &\leq ||u_{n+1} - u_n|| + M_1|\delta_{n+1} - \delta_n|, \end{aligned} \tag{3.7}$$

where $M_1 = \sup_{n\geq 1}\{||u_n - Tu_n||\}$.

Letting $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n,\ n \geq 1$. Then we have

$$\begin{aligned} z_{n+1} - z_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}\gamma f(x_{n+1}) + ((1 - \beta_{n+1})I - \alpha_{n+1}A)y_{n+1}}{1 - \beta_{n+1}} \\ &\quad - \frac{\alpha_n\gamma f(x_n) + ((1 - \beta_n)I - \alpha_n A)y_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(\gamma f(x_{n+1}) - Ay_{n+1}) \\ &\quad + \frac{\alpha_n}{1 - \beta_n}(Ay_n - \gamma f(x_n)) + y_{n+1} - y_n. \end{aligned}$$

From (3.6) and (3.7) ,we get

$$\begin{aligned}
&||z_{n+1}-z_n||-||x_{n+1}-x_n|| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(||\gamma f(x_{n+1})||+||Ay_{n+1}||)+\frac{\alpha_n}{1-\beta_n}(||Ay_n||+||\gamma f(x_n)||) \\
&\quad +||y_{n+1}-y_n||-||x_{n+1}-x_n|| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(||\gamma f(x_{n+1})||+||Ay_{n+1}||)+\frac{\alpha_n}{1-\beta_n}(||Ay_n||+||\gamma f(x_n)||) \\
&\quad +||u_{n+1}-u_n||+M_1|\delta_{n+1}-\delta_n|-||x_{n+1}-x_n|| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(||\gamma f(x_{n+1})||+||Ay_{n+1}||)+\frac{\alpha_n}{1-\beta_n}(||Ay_n||+||\gamma f(x_n)||) \\
&\quad +M_1|\delta_{n+1}-\delta_n|+\frac{M}{h}|r_{n+1}-r_n|.
\end{aligned}$$

By conditions (i)-(iv) and $\{Ay_n\}$,$\{f(x_n)\}$ are bounded, we have

$$\limsup_{n\to\infty}(||z_{n+1}-z_n||-||x_{n+1}-x_n||)\le 0.$$

Hence by Lemma 2.2 we have

$$\lim_{n\to\infty}||z_n-x_n||=0.$$

Consequently

$$\lim_{n\to\infty}||x_{n+1}-x_n||=\lim_{n\to\infty}(1-\beta_n)||z_n-x_n||=0. \tag{3.8}$$

Hence from (3.1), we can obtain

$$\begin{aligned}
||x_n-y_n|| &\le ||x_n-x_{n+1}||+||x_{n+1}-y_n|| \\
&\le ||x_n-x_{n+1}||+||\alpha_n(\gamma f(x_n)-Ay_n)+\beta_n(x_n-y_n)|| \\
&\le ||x_n-x_{n+1}||+\alpha_n(||\gamma f(x_n)||+||Ay_n||)+\beta_n||x_n-y_n||,
\end{aligned}$$

that is

$$||x_n-y_n||\le \frac{1}{1-\beta_n}||x_n-x_{n+1}||+\frac{\alpha_n}{1-\beta_n}(||\gamma f(x_n)||+||Ay_n||),$$

which together with condition (i),(iii) and (3.8) implies

$$\lim_{n\to\infty}||x_n-y_n||=0. \tag{3.9}$$

(IV) Next we prove that $||x_n-u_n||\to 0$.

Indeed, for any given $z\in F(T)\cap EP(\phi)$, since T_r is firmly nonexpansive, then we have

$$\begin{aligned}
||u_n-z||^2 &= ||T_{r_n}x_n-T_{r_n}z||^2 \\
&\le \langle T_{r_n}x_n-T_{r_n}z, x_n-z\rangle \\
&= \langle u_n-z, x_n-z\rangle \\
&= \frac{1}{2}(||u_n-z||^2+||x_n-z||^2-||x_n-u_n||^2).
\end{aligned}$$

It follows that

$$||u_n-z||^2\le ||x_n-z||^2-||x_n-u_n||^2. \tag{3.10}$$

Using Lemma2.6, (3.1), (3.3) and (3.10), we have

$$
\begin{aligned}
&||x_{n+1}-z||^2\\
&\quad = ||\alpha_n(\gamma f(x_n)-Az)+\beta_n(x_n-y_n)+(I-\alpha_n A)(y_n-z)||^2\\
&\quad \le ||(I-\alpha_n A)(y_n-z)+\beta_n(x_n-y_n)||^2+2\alpha_n\langle \gamma f(x_n)-Az, x_{n+1}-z\rangle\\
&\quad \le [||I-\alpha_n A\|\|y_n-z||+\beta_n||x_n-y_n||]^2+2\alpha_n||\gamma f(x_n)-Az\|\|x_{n+1}-z\|\\
&\quad \le [(1-\alpha_n\widetilde{\gamma})\|u_n-z||+\beta_n||x_n-y_n||]^2+2\alpha_n||\gamma f(x_n)-Az\|\|x_{n+1}-z\|\\
&\quad \le (1-\alpha_n\widetilde{\gamma})^2||u_n-z||^2+2(1-\alpha_n\widetilde{\gamma})\beta_n||u_n-z||||x_n-y_n||\\
&\qquad +\beta_n^2||x_n-y_n||^2+2\alpha_n||\gamma f(x_n)-Az\|\|x_{n+1}-z\|\\
&\quad \le (1-\alpha_n\widetilde{\gamma})^2[||x_n-z||^2-||x_n-u_n||^2]+2(1-\alpha_n\widetilde{\gamma})\beta_n||u_n-z||||x_n-y_n||\\
&\qquad +\beta_n^2||x_n-y_n||^2+2\alpha_n||\gamma f(x_n)-Az\|\|x_{n+1}-z\|\\
&\quad \le (1-2\alpha_n\widetilde{\gamma}+(\alpha_n\widetilde{\gamma})^2)||x_n-z||^2-(1-\alpha_n\widetilde{\gamma})^2||x_n-u_n||^2+\beta_n^2||x_n-y_n||^2\\
&\qquad +2(1-\alpha_n\widetilde{\gamma})\beta_n||u_n-z||||x_n-y_n||+2\alpha_n||\gamma f(x_n)-Az\|\|x_{n+1}-z\|\\
&\quad \le ||x_n-z||^2+\alpha_n\widetilde{\gamma}^2||x_n-z||^2-(1-\alpha_n\widetilde{\gamma})^2||x_n-u_n||^2+\beta_n^2||x_n-y_n||^2\\
&\quad +2(1-\alpha_n\widetilde{\gamma})\beta_n||u_n-z||||x_n-y_n||+2\alpha_n||\gamma f(x_n)-Az\|\|x_{n+1}-z\|.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&(1-\alpha_n\widetilde{\gamma})^2||x_n-u_n||^2\\
&\quad \le ||x_n-z||^2-||x_{n+1}-z||^2+\alpha_n\widetilde{\gamma}^2||x_n-z||^2+\beta_n^2||x_n-y_n||^2\\
&\qquad +2(1-\alpha_n\widetilde{\gamma})\beta_n||u_n-z||||x_n-y_n||+2\alpha_n||\gamma f(x_n)-Az\|\|x_{n+1}-z\|\\
&\quad \le ||x_n-x_{n+1}||(||x_n-z||+||x_{n+1}-z||)+\alpha_n\widetilde{\gamma}^2||x_n-z||^2+\beta_n^2||x_n-y_n||^2\\
&\qquad +2(1-\alpha_n\widetilde{\gamma})\beta_n||u_n-z||||x_n-y_n||+2\alpha_n||\gamma f(x_n)-Az\|\|x_{n+1}-z\|.
\end{aligned}
$$

By virtue of comdition (i) $\alpha_n \to 0$, (3.8) and (3.9), note that $\{f(x_n)\}$, $\{x_n\}$, $\{u_n\}$ are bounded, these imply that

$$||x_n-u_n|| \to 0, \quad (as \quad n\to\infty). \tag{3.11}$$

(V) Next we prove that $||u_n-Su_n|| \to 0$.

From condition (ii), we have $\delta_n \to \lambda$ as $n\to\infty$, where $\lambda\in[k,1)$. Define $S: C\to H$ by $Sx=\lambda x+(1-\lambda)Tx$. Then, S is nonexpansive with $F(S)=F(T)$ by Lemma 2.5. Notice that

$$
\begin{aligned}
||x_n-Su_n|| &\le ||x_n-y_n||+||y_n-Su_n||\\
&= ||x_n-y_n||+||\delta_n u_n+(1-\delta_n)Tu_n-(\lambda u_n+(1-\lambda)Tu_n)||\\
&\le ||x_n-y_n||+|\delta_n-\lambda|||u_n-Tu_n]||,
\end{aligned}
$$

which combines with (3.9) yielding that

$$\lim_{n\to\infty}||x_n-Su_n||=0. \tag{3.12}$$

Observe that

$$||u_n-Su_n|| \le ||u_n-x_n||+||x_n-Su_n||.$$

From (3.11) and (3.12), we have

$$\lim_{n\to\infty} ||u_n - Su_n|| = 0. \tag{3.13}$$

(VI) Next we prove that

$$\limsup_{n\to\infty} \langle (\gamma f - A)p, x_n - p\rangle \le 0, \tag{3.14}$$

where $p = P_{F(T)\cap EP(\phi)}(I - A + \gamma f)(p)$. To show this inequality, we can choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\limsup_{n\to\infty} \langle (\gamma f - A)(p), Su_n - p\rangle = \lim_{i\to\infty} \langle (\gamma f - A)p, Su_{n_i} - p\rangle. \tag{3.15}$$

Since $\{u_{n_i}\}$ is bounded in C,without loss of generality, we can assume that $u_{n_i} \rightharpoonup w \in C$ as $i \to \infty$. Now we prove that $w \in F(T) \cap EP(\phi)$. Now we show that $w \in EP(\phi)$. In fact, since $u_n = T_{r_n} x_n$, we have

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall\, y \in C.$$

By condition (A2), we have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge \phi(y, u_n).$$

and hence

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \ge \phi(y, u_{n_i}).$$

Since $||u_{n_i} - x_{n_i}|| \to 0$ and $u_{n_i} \rightharpoonup w$, from condition (A4), we have

$$\phi(y, w) \le 0, \quad \forall\, y \in C.$$

For any $t \in (0, 1]$ and $y \in C$, let $y_t = ty + (1-t)w$, then $y_t \in C$ and $\phi(y_t, w) \le 0$. From (A1) and (A4), we have

$$\begin{aligned}
0 &= \phi(y_t, y_t) = \phi(y_t, ty + (1-t)w)\\
&= t\phi(y_t, y) + (1-t)\phi(y_t, w)\\
&\le t\phi(y_t, y),
\end{aligned}$$

and hence

$$\phi(y_t, y) \ge 0.$$

By condition (A3), we have $\phi(w, y) \ge 0$, $\forall\, y \in C$. Hence $w \in EP(\phi)$.

We shall show $w \in F(T)$. Since Hilbert spaces are Opial's spaces, suppose the contrary, $w \notin F(S)$, i.e., $w \ne Sw$. Since $u_{n_i} \rightharpoonup w$, from Opial's condition and (3.13), we have

$$\begin{aligned}
\liminf_{i\to\infty} ||u_{n_i} - w|| &< \liminf_{i\to\infty} ||u_{n_i} - Sw||\\
&\le \liminf_{i\to\infty} (||u_{n_i} - Su_{n_i}|| + ||Su_{n_i} - Sw||)\\
&\le \liminf_{i\to\infty} ||Su_{n_i} - Sw||\\
&\le \liminf_{i\to\infty} ||u_{n_i} - w||.
\end{aligned}$$

This is a contradiction. We get $w \in F(S)$. Again by Lemma 2.5, we have $w \in F(S) = F(T)$. Therefore $w \in F(T) \cap EP(\phi)$.

Since $p = P_{F(T)\cap EP(\phi)}(I - A + \gamma f)(p)$. It follows from (3.12),(3.13),(3.15) and Remark 1 that

$$\begin{aligned}
\limsup_{n\to\infty}\langle(\gamma f - A)(p), x_n - p\rangle &= \limsup_{n\to\infty}\langle(\gamma f - A)(p), x_n - Su_n + Su_n - p\rangle \\
&\le \limsup_{n\to\infty}\langle(\gamma f - A)(p), Su_n - p\rangle \\
&= \lim_{i\to\infty}\langle(\gamma f - A)(p), Su_{n_i} - p\rangle \\
&= \lim_{i\to\infty}\langle(\gamma f - A)(p), Su_{n_i} - u_{n_i} + u_{n_i} - p\rangle \\
&= \langle(\gamma f - A)(p), w - p\rangle \le 0.
\end{aligned}$$

(VII) Finally, we prove that $\{x_n\}$ and $\{u_n\}$ converge strongly to p. In fact, from (3.1),(3.3) and Lemma 2.6, we have

$$\begin{aligned}
&||x_{n+1} - p||^2 \\
&= ||\alpha_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) + ((1-\beta_n)I - \alpha_n A)(y_n - p)||^2 \\
&\le ||((1-\beta_n)I - \alpha_n A)(y_n - p) + \beta_n(x_n - p)||^2 + 2\alpha_n\langle\gamma f(x_n) - Ap, x_{n+1} - p\rangle \\
&\le [||((1-\beta_n)I - \alpha_n A)(y_n - p)|| + ||\beta_n(x_n - p)||]^2 \\
&\quad + 2\alpha_n\gamma\langle f(x_n) - f(p), x_{n+1} - p\rangle + 2\alpha_n\langle\gamma f(p) - Ap, x_{n+1} - p\rangle \\
&\le [(1-\beta_n-\alpha_n\widetilde{\gamma})||y_n - p|| + \beta_n||x_n - p||]^2 + 2\alpha_n\gamma\alpha||x_n - p||||x_{n+1} - p|| \\
&\quad + 2\alpha_n\langle\gamma f(p) - Ap, x_{n+1} - p\rangle \\
&\le [(1-\beta_n-\alpha_n\widetilde{\gamma})||x_n - p|| + \beta_n||x_n - p||]^2 + 2\alpha_n\gamma\alpha||x_n - p||||x_{n+1} - p|| \\
&\quad + 2\alpha_n\langle\gamma f(p) - Ap, x_{n+1} - p\rangle \\
&\le (1-\alpha_n\widetilde{\gamma})^2||x_n - p||^2 + \alpha_n\gamma\alpha\{||x_n - p||^2 + ||x_{n+1} - p||^2\} \\
&\quad + 2\alpha_n\langle\gamma f(p) - Ap, x_{n+1} - p\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
||x_{n+1} - p||^2 &\le \frac{(1-\alpha_n\widetilde{\gamma})^2 + \alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}||x_n - p||^2 + \frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle f(p) - p, x_{n+1} - p\rangle \\
&\le \left(1 - \frac{2\alpha_n(\widetilde{\gamma} - \gamma\alpha)}{1-\alpha_n\gamma\alpha}\right)||x_n - p||^2 + \frac{(\alpha_n\widetilde{\gamma})^2}{1-\alpha_n\gamma\alpha}||x_n - p||^2 \\
&\quad + \frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle f(p) - p, x_{n+1} - p\rangle \\
&\le \left(1 - \frac{2\alpha_n(\widetilde{\gamma} - \gamma\alpha)}{1-\alpha_n\gamma\alpha}\right)||x_n - p||^2 + \frac{2\alpha_n(\widetilde{\gamma} - \gamma\alpha)}{1-\alpha_n\gamma\alpha} \\
&\quad \times \left(\frac{\alpha_n\widetilde{\gamma}^2 M_2}{2(\widetilde{\gamma} - \gamma\alpha)} + \frac{1}{\widetilde{\gamma} - \gamma\alpha}\langle f(p) - p, x_{n+1} - p\rangle\right) \\
&= (1 - \varrho_n)||x_n - p||^2 + \varrho_n\sigma_n,
\end{aligned}$$

where $M_2 = \sup_{n\geq 1}\{||x_n - p||^2\}$,

$$\varrho_n = \frac{2\alpha_n(\widetilde{\gamma} - \gamma\alpha)}{1 - \alpha_n\gamma\alpha} \quad \text{and} \quad \sigma_n = \frac{\alpha_n\widetilde{\gamma}^2 M_2}{2(\widetilde{\gamma} - \gamma\alpha)} + \frac{1}{\widetilde{\gamma} - \gamma\alpha}\langle f(p) - p, x_{n+1} - p\rangle.$$

From condition (i) and (3.14) that $\varrho_n \to 0$, $\sum_{n=1}^{\infty} \varrho_n = \infty$ and $\limsup_{n\to\infty} \sigma_n \leq 0$. Hence, by Lemma 2.3, the sequence $\{x_n\}$ converges strongly to p. Consequently, we can obtain that $\{u_n\}$ also converges strongly to p. □

Taking $\gamma = 1$ and $A = I$ is an identity mapping in Theorem 3.1, we can obtain the following results immediately.

Theorem 3.2. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H. let $\phi : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$. Let $T : C \to H$ be a $k-$strictly pseudo-contractive nonself-mapping such that $F(T) \cap EP(\phi) \neq \emptyset$, and $f : H \to H$ be a contractive mapping with a contractive constant $\alpha \in (0, 1)$. For any given $x_1 \in H$, let $\{x_n\}$ and $\{u_n\}$ be the iterative sequence defined by*

$$\begin{cases} \phi(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall\, y \in C, \\ y_n = \delta_n u_n + (1 - \delta_n)Tu_n, & \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \beta_n - \alpha_n)y_n, & \forall\, n \geq 1. \end{cases} \tag{3.16}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$ are three sequences in $[0, 1]$ and $r_n \subset (0, \infty)$. If the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$; $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $k \leq \delta_n \leq \lambda < 1$ *for all* $n \geq 1$ *and* $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$,

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(iv) $\liminf_{n\to\infty} r_n > 0$, $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Then $\{x_n\}$ and $\{u_n\}$ cnoverge strongly to $p \in F(T) \cap EP(\phi)$, where $p = P_{F(T)\cap EP(\phi)} f(p)$.

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